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A Piecewise Line-Search Technique for Maintaining the Positive Definiteness of the Matrices in the SQP Method

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Abstract. A technique for maintaining the positive definiteness of the matrices in the quasi-Newton version of the SQP algorithm is proposed. In our algorithm, matrices approximating the Hessian of the augmented Lagrangian are updated. The positive definiteness of these matrices in the space tangent to the constraint manifold is ensured by a so-called piecewise line-search technique, while their positive definiteness in a complementary subspace is obtained by setting the augmentation parameter. In our experiment, the combination of these two ideas leads to a new algorithm that turns out to be more robust and often improves the results obtained with other approaches.

Keywords: BFGS formula, equality constrained optimization, piecewise line-search, quasi-Newton algorithm, successive quadratic programming

1. Introduction

We consider from a numerical point of view the nonlinear equality constrained optimization problem

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & c(x) = 0, \quad x \in \Omega, \end{array} \tag{1.1}$$

where Ω is an open set of $\mathbb{R}^n$ and the two functions $f : \Omega \to \mathbb{R}$ and $c : \Omega \to \mathbb{R}^m$ $(1 \leq m < n)$ are sufficiently smooth. Although constrained optimization problems often present inequality constraints also, studying (1.1) is of interest. Such a study can indeed be considered as a first stage in the analysis of more general settings. Also, the nonlinear interior point approach for dealing with inequalities often transform the original problem in a series of problems with equality constraints only by introducing and penalizing slack variables (see [5] and the references therein). The algorithms presented in this paper could be adapted to this approach. Finally, it happens that problems with equality constraints only are encountered.

Since the set Ω in (1.1) is open, it does not define general constraints. It is just the set on which good properties of f and c hold. For example, we will always suppose that the following assumption hold.

General Assumption: for all $x \in \Omega$, the $m \times n$ Jacobian matrix of c at x, denoted by $A(x) = \nabla c(x)^\top$, has full rank.

The Lagrangian function associated with problem (1.1) is defined on $\Omega \times \mathbb{R}^m$ by

$$\ell(x, \lambda) = f(x) + \lambda^\top c(x), \tag{1.2}$$

where $\lambda \in \mathbb{R}^m$ is called the *Lagrange multiplier*. The first order optimality conditions of problem (1.1) at a local solution x_* with associated multiplier λ_* can be written

$$\nabla f_* + A_*^\top \lambda_* = 0 \quad \text{and} \quad c_* = 0. \tag{1.3}$$

Throughout the paper, we use the notation $f_* = f(x_*)$, $\nabla f_* = \nabla f(x_*)$, $A_* = A(x_*)$, etc. Also, we assume that the second order sufficient conditions of optimality hold at (x_*, λ_*). Using the notation $L_* = \nabla^2_{xx} \ell(x_*, \lambda_*)$, this can be written

$$h^\top L_* h > 0, \quad \text{for all } h \neq 0 \text{ such that } A_* h = 0. \tag{1.4}$$

The sequential quadratic programming (SQP) algorithm is a Newton-like method in (x, λ) applied to the first order optimality conditions (1.3) (see for example Fletcher [14], Bonnans et al. [4], or the survey paper [2]). The kth iteration of the algorithm can be described as follows. Given an iterate pair $(x_k, \lambda_k) \in \Omega \times \mathbb{R}^m$, approximating a solution pair (x_*, λ_*), the following *quadratic subproblem* is solved for $d \in \mathbb{R}^n$:

$$\begin{array}{ll} \min & \nabla f_k^\top d + \frac{1}{2} d^\top M_k d \\ \text{s.t.} & c_k + A_k d = 0. \end{array} \tag{1.5}$$

We adopt the notation $f_k = f(x_k)$, $\nabla f_k = \nabla f(x_k)$, $c_k = c(x_k)$, $A_k = A(x_k)$, etc. In (1.5), it is suitable to take for M_k the Hessian of the Lagrangian or an approximation to it. Let us denote by $(d_k, \lambda_k^{\text{QP}})$ a primal-dual solution of (1.5), i.e., a solution of its optimality conditions

$$\nabla f_k + M_k d_k + A_k^\top \lambda_k^{\text{QP}} = 0 \quad \text{and} \quad c_k + A_k d_k = 0. \tag{1.6}$$

The link between (1.3) and (1.5) is that, when M_k is the Hessian of the Lagrangian, $(d_k, \lambda_k^{\text{QP}} - \lambda_k)$ is the Newton step for the system (1.3) at (x_k, λ_k).

Close to a solution, it is then desirable that the algorithm updates (x_k, λ_k) by $x_{k+1} = x_k + d_k$ and $\lambda_{k+1} = \lambda_k^{\text{QP}}$. To ensure convergence from remote starting points, however, the direction d_k is often used as a search direction, along which a stepsize $\alpha_k > 0$ is chosen. The stepsize is adjusted such that the next iterate

$$x_{k+1} = x_k + \alpha_k d_k,$$

reduces sufficiently the value of some merit function.

This paper deals with the quasi-Newton version of the SQP algorithm: the matrix M_k is updated at each iteration by a formula, whose aim is to make M_k closer to some Hessian, typically the Hessian of the Lagrangian or the Hessian of the augmented Lagrangian. In such a method, it is convenient to force M_k to be positive definite, so that d_k is a descent direction of the merit function. This can be achieved by using the BFGS formula: for some vectors γ_k and δ_k in $\mathbb{R}^n$,

$$M_{k+1} = M_k - \frac{M_k \delta_k \delta_k^\top M_k}{\delta_k^\top M_k \delta_k} + \frac{\gamma_k \gamma_k^\top}{\gamma_k^\top \delta_k}. \tag{1.7}$$

This formula is interesting because, as this is well known, the positive definiteness of the updated matrices is sustained from M_k to M_{k+1} if and only if the following *curvature condition* holds:

$$\gamma_k^\top \delta_k > 0. \tag{1.8}$$

Therefore, if the initial matrix M_0 is positive definite and if (1.8) is realized at each iteration, all the matrices M_k will be positive definite. One of the aims of this paper is to propose a technique to realize this curvature condition in a consistent manner at each iteration. This is not an easy task, as we now explain.

When M_k is taken as an approximation of the Hessian of the Lagrangian, it makes sense to take for γ_k in (1.7) the vector

$$\gamma_k^\ell = \nabla_x \ell(x_{k+1}, \lambda) - \nabla_x \ell(x_k, \lambda), \tag{1.9}$$

where λ is some multiplier, usually λ_k^{QP}, since this change in the gradient of the Lagrangian gives information on the Hessian of the Lagrangian. However, the lack of positive definiteness of L_* makes this approach difficult. Indeed, with this choice of γ_k, the curvature condition may never be realized for any displacement $\delta_k = x_{k+1} - x_k$ along d_k, because the Lagrangian may have negative curvature along this direction, even close to the solution.

The idea of modifying the vector γ_k^ℓ to force satisfaction of the curvature condition goes back at least to Powell [30], who has suggested choosing γ_k as a convex combination of γ_k^ℓ and $M_k \delta_k$:

$$\gamma_k^{\text{P}} = \theta \gamma_k^\ell + (1 - \theta) M_k \delta_k. \tag{1.10}$$

In (1.10), θ is the number in $(0, 1]$, the closest to 1, such that the inequality

$$\left(\gamma_k^{\text{P}}\right)^\top \delta_k \geq \eta\, \delta_k^\top M_k \delta_k$$

is satisfied. The constant η is set to 0.2 in [30] and to 0.1 in [31]. Powell's correction of γ_k^ℓ is certainly the most widely used technique in practice. Its success is due to its appealing simplicity and its usually good numerical performance. The fact that it may encounter difficulties partly motivates the present study (see [31] or [32, p. 125]). Another motivation is that the best known result obtained so far on the speed of convergence with Powell's correction (precisely, the r-superlinear convergence, see [29]) is not as good as one can reasonably expect, which is the q-superlinear convergence.

Another modification of γ_k^ℓ consists in taking for γ_k an approximation of the change in the gradient of the augmented Lagrangian, which is the function (1.2) to which the augmentation term $\frac{r}{2}\|c(x)\|^2$ is added ($\|\cdot\|$ denotes the ℓ_2-norm). This idea, proposed by Tapia [36], has roots in the work of Han [22] and Tapia [35] and was refined later by Byrd, Tapia, and Zhang [8] (BTZ for short). In the augmented Lagrangian-based method of BTZ, the vector γ_k is set to

$$\gamma_k^{\mathrm{S}} = \gamma_k^\ell + r_k A_{k+1}^\top A_{k+1}\delta_k, \tag{1.11}$$

where the augmentation parameter r_k is the smallest nonnegative number such that

$$\left(\gamma_k^{\mathrm{S}}\right)^\top \delta_k \geq \max\left\{\left|\left(\gamma_k^\ell\right)^\top \delta_k\right|, \nu_{\mathrm{BTZ}}\|A_{k+1}\delta_k\|^2\right\}. \tag{1.12}$$

In the implementation of the algorithm given in [8], the authors set $\nu_{\mathrm{BTZ}} = 0.01$. It is clear that this approach does not work when $A_{k+1}\delta_k = 0$ and $(\gamma_k^\ell)^\top \delta_k < 0$. In this case, the authors propose the following "back-up strategy": when (1.12) does not hold with $r_k = 0$ and

$$\|A_{k+1}\delta_k\| < \min\{\beta_{\mathrm{BTZ}}, \|\delta_k\|\}\|\delta_k\|, \tag{1.13}$$

where β_{BTZ} is a small positive number (the value 0.01 is chosen in [8]), then the vector $A_{k+1}^\top A_{k+1}\delta_k$ in (1.11) is replaced by δ_k and r_k is set such that (1.12) is satisfied. In [8], assuming the convergence of (x_k, λ_k) to (x_*, λ_*), a unit stepsize, and a matrix updated by the BFGS formula, it is shown that the sequence $\{x_k\}$ converges r-superlinearly to x_*. Furthermore, numerical experiments demonstrate that the technique is competitive with Powell's correction. The back-up strategy used in the BTZ algorithm is, however, not very attractive. Indeed, when it is active, the pair $(\gamma_k^{\mathrm{S}}, \delta_k)$ refers to a matrix different from the Hessian of the Lagrangian or the Hessian of the augmented Lagrangian, modifying the search direction d_k in a way that is not supported by the theory.

The present paper develops a new approach, which uses a stepsize determination process to realize the curvature condition (1.8). Therefore, the new method can be viewed as an extension of a well established technique in unconstrained optimization, which uses the Wolfe line-search to obtain (1.8) (see [12, 24, 37, 38]).

Our algorithm combines two ideas. On the one hand, the Wolfe line-search technique used in unconstrained optimization and the experience acquired with reduced Hessian methods for equality constrained problems [18, 20] have shown that an appropriate move along the constraint manifold $\{x : c(x) = c_k\}$ can take care of the positive definiteness of the "tangent part" of the matrices M_k. The tangent or longitudinal component of the displacement (i.e., along the constraint manifold) is determined by a specific algorithm, that we call *piecewise line-search* (PLS for short). Its aim is to realized a so-called *reduced Wolfe condition*, which will be introduced in Section 3 (see formula (3.4)). By this, we mimic what is done in unconstrained optimization, where the curvature condition is fulfilled by a line-search algorithm satisfying the Wolfe conditions (see for instance [24, Chap. II,

§ 3.3]). In constrained optimization, this approach is more delicate, however, since the curvature condition (1.8) with $\gamma_k = \gamma_k^\ell$ may never hold along a straight line (recall that the Lagrangian may have negative curvature along the SQP direction). Nevertheless, we will show that there is a path defined by a particular differential equation along which the reduced Wolfe condition can be realized. The PLS technique consists in following a piecewise linear approximation of this "guiding path", each discretization point being successively chosen by means of an Armijo line-search process along intermediate search directions. An important aspect of this approach is that it is readily adapted to a global framework, being able to force convergence from remote starting points.

On the other hand, for the "transversal part" of the matrices M_k, we follow the BTZ approach and incorporate in the vector γ_k a term of the form $r_k A_k^\top A_k \delta_k$:

$$\gamma_k = \tilde{\gamma}_k + r_k A_k^\top A_k \delta_k, \quad \text{where } \tilde{\gamma}_k \simeq L_* \delta_k.$$

The explicit form of γ_k will be stated in Section 3.1. The specificity of our algorithm is that, because of the realization of the reduced Wolfe condition and the structure of the vector $\tilde{\gamma}_k$, the curvature condition (1.8) can always be satisfied, even when $A_k \delta_k = 0$. This is quite different from the BTZ approach, for which a back-up strategy is necessary.

This is the third paper describing how the PLS technique can be used to maintain the positive definiteness of the updated matrices in quasi-Newton algorithms for solving equality constrained problems. The approach was first introduced in [18] in the framework of reduced quasi-Newton algorithms, in which the updated matrices approach the reduced Hessian of the Lagrangian. In that paper, the vector γ_k is a difference of reduced gradients, both evaluated asymptotically in the same tangent space. In a way, this is the reference algorithm, since it may be viewed as a straightforward generalization of the Wolfe line-search to equality constrained problems. It has however the inconvenience to require at least two linearizations of the constraints per iteration. For some problems, this may be an excessive extra cost. For this reason, an approach that requires asymptotically only one linearization of the constraints per iteration is developed in [20]. In that paper also, the technique is introduced for reduced quasi-Newton algorithms. The present paper shows how the PLS technique can be used within the SQP (or full Hessian) algorithm.

To conclude this introduction, let us mention that there are other ways of implementing quasi-Newton algorithms for equality constraint minimization. Coleman and Fenyes [11], Biegler, Nocedal, and Schmid [1], and Gurwitz [21] update separately approximations of different parts of the Hessian of the Lagrangian. Another possibility is to use quasi-Newton versions of the reduced Hessian algorithm, in which approximations of the reduced Hessian of the Lagrangian are updated. This is attractive since this matrix is positive definite near a regular solution. Also, by contrast to the full Hessian method presented in this paper, which is appropriate for small or medium scale problems, this approach can also be used for large scale problems so long as the order $n - m$ of the reduced Hessian remains small. The reduced Hessian approach has, however, its own drawbacks: either the constraints have to be linearized twice per iteration, which is sometimes an important extra cost, or an update criterion has to be introduced, which does not always give good numerical results. We refer the reader to [7, 10, 17, 18, 20, 27, 39] for further developments along that line.

The paper is organized as follows. In Section 2, we make precise our notation, the form of the SQP direction, and our choice of merit function. Section 3 presents our approach to satisfying the curvature condition (1.8) and outlines the PLS technique. Section 4 shows the finite termination of the search algorithm. In comparison with the material presented in [18], this result is proved in a more straightforward manner and it is less demanding on the way the intermediate stepsizes are determined. The overall minimization algorithm is given in Section 5. As we have mentioned, this algorithm is immediately expressed in a global setting. Therefore, in this section, we can concentrate on its global convergence rather than on its local properties. We also give conditions assuring the admissibility of the unit stepsize, which will lead to a criterion selecting the iterations at which the PLS has to be launched. Section 6 gives more details on implementation issues and Section 7 relates the results of numerical tests. The paper terminates with a conclusion section.

2. Background material and notation

Let us first introduce two decompositions of $\mathbb{R}^n$ that will be useful throughout the paper. Each of them decomposes the variable space into two complementary subspaces and is characterized by a triple $(Z^-(x), A^-(x), Z(x))$. The columns of the matrices $Z^-(x)$ and $A^-(x)$ span the two complementary subspaces and $Z(x)$ is deduced from $Z^-(x)$, $A^-(x)$, and $A(x)$.

In the first decomposition, $Z^-(x)$ and $A^-(x)$ can be considered as an additional data on the structure of the problem. These operators and $Z(x)$ have to satisfy the following properties.

- $Z^-(x)$ is an $n \times (n-m)$ matrix, whose columns form a basis of the null space $\mathcal{N}(A(x))$ of $A(x)$:

$$A(x)Z^-(x) = O_{m\times(n-m)}. \tag{2.1}$$

- $A^-(x)$ is an $n \times m$ right inverse of $A(x)$:

$$A(x)A^-(x) = I_m. \tag{2.2}$$

  In particular, the columns of $A^-(x)$ form a basis of a subspace complementary to $\mathcal{N}(A(x))$.
- $Z(x)$ is the unique $(n-m)\times n$ matrix such that

$$Z(x)Z^-(x) = I_{n-m} \quad \text{and} \quad Z(x)A^-(x) = 0_{(n-m)\times m}. \tag{2.3}$$

From these properties, we can deduce the following identity:

$$A^-(x)A(x) + Z^-(x)Z(x) = I_n. \tag{2.4}$$

Using (2.2) and (2.3), we see that $A^-(x)A(x)$ and $Z^-(x)Z(x)$ are slant projectors on the range space of $A^-(x)$ and the complementary subspace $\mathcal{N}(A(x))$. For a motivation of this choice of notation and for practical examples of operators $A^-(x)$ and $Z^-(x)$, see Gabay [15].

From these operators, the notions of reduced gradient and Lagrange multiplier estimate can be introduced. The *reduced gradient* of f at x is defined by

$$g(x) = Z^-(x)^\top \nabla f(x). \tag{2.5}$$

Using (2.1), we have $g(x) = Z^-(x)^\top \nabla_x \ell(x, \lambda_*)$, so that

$$\nabla g_*^\top = Z_*^{-\top} L_*. \tag{2.6}$$

The first equation in (1.3) and (2.2) imply that $\lambda_* = -A_*^{-\top} \nabla f_*$. Therefore, the vector

$$\lambda(x) = -A^-(x)^\top \nabla f(x). \tag{2.7}$$

can be considered as a *Lagrange multiplier estimate*. Similarly, using (2.2), we have $\lambda(x) = -A^-(x)^\top \nabla_x \ell(x, \lambda_*) + \lambda_*$, so that

$$\nabla \lambda_*^\top = -A_*^{-\top} L_*. \tag{2.8}$$

The second useful decomposition of $\mathbb{R}^n$ differs from the first one by the choice of the subspace complementary to $\mathcal{N}(A(x))$. It comes from the form of the solution of the quadratic subproblem (1.5) and therefore it depends only on the problem data. Let M_k be the current Hessian approximation with the property that $Z^-(x)^\top M_k Z^-(x)$ is positive definite, and define

$$H_k(x) = (Z^-(x)^\top M_k Z^-(x))^{-1}.$$

Let x be a point in Ω and consider the quadratic subproblem in $d \in \mathbb{R}^n$:

$$\begin{aligned} \min \quad & \nabla f(x)^\top d + \tfrac{1}{2} d^\top M_k d \\ \text{s.t.} \quad & c(x) + A(x)d = 0. \end{aligned} \tag{2.9}$$

Let us denote by $(d_k^{\text{QP}}(x), \lambda_k^{\text{QP}}(x))$ the primal-dual solution of (2.9). Using the first decomposition of $\mathbb{R}^n$ at x, it is not difficult to see that the primal solution can be written (see also Gabay [16])

$$d_k^{\text{QP}}(x) = -Z^-(x)H_k(x)g(x) - (I - Z^-(x)H_k(x)Z^-(x)^\top M_k)A^-(x)c(x). \tag{2.10}$$

Using (2.1) and (2.2), we find that the factor of $c(x)$ in (2.10) satisfies

$$A(x)[(I - Z^-(x)H_k(x)Z^-(x)^\top M_k)A^-(x)] = I_m.$$

Hence, the product of matrices inside the square brackets forms a right inverse of $A(x)$, which is denoted by $\hat{A}_k^-(x)$. Defining

$$\hat{Z}_k(x) = H_k(x)Z^-(x)^\top M_k, \tag{2.11}$$

we have

$$\hat{A}_k^-(x) = (I - Z^-(x)\hat{Z}_k(x))A^-(x), \tag{2.12}$$

and thus (2.10) can be rewritten

$$d_k^{\text{QP}}(x) = -Z^-(x)H_k(x)g(x) - \hat{A}_k^-(x)c(x). \tag{2.13}$$

We have built a triple $(Z^-(x), \hat{A}_k^-(x), \hat{Z}_k(x))$ satisfying conditions (2.1), (2.2), and (2.3), hence defining suitably a second decomposition of $\mathbb{R}^n$. In particular, we have

$$\hat{A}_k^-(x)A(x) + Z^-(x)\hat{Z}_k(x) = I_n. \tag{2.14}$$

Note that despite the fact that $A^-(x)$ and $Z^-(x)$ are used in formula (2.12), the operator $\hat{A}_k^-(x)$ does not depend on the choice of right inverse and tangent basis. Indeed, $-\hat{A}_k^-(x)c(x)$ is also defined as the solution of the quadratic subproblem (2.9) in which $\nabla f(x)$ is set to zero (see also formula (4.7) in the proof of Lemma 4.3 below).

In order to simplify the notation, we denote by $\hat{Z}_k$ and $\hat{A}_k^-$ the matrices $\hat{Z}_k(x_k)$ and $\hat{A}_k^-(x_k)$. With this convention, the solution d_k of the quadratic subproblem (1.5) can be written

$$d_k = -Z_k^- H_k g_k - \hat{A}_k^- c_k. \tag{2.15}$$

The vector $\hat{Z}_k d_k = -H_k g_k$ is called the *reduced tangent direction*.

Using $\hat{Z}_k(x)\hat{A}_k^-(x) = 0$ and the nonsingularity of $H_k(x)$, we have from (2.11) the following useful identity

$$Z^-(x)^\top M_k \hat{A}_k^-(x) = 0_{(n-m)\times m}, \tag{2.16}$$

which means that the tangent space $\mathcal{N}(A(x))$ and the range space of $\hat{A}_k^-(x)$ are perpendicular for the scalar product associated with M_k. In particular, if L_* is used in place of M_k in the previous equality and if $x = x_*$, we obtain

$$Z_*^{-\top} L_* \hat{A}_*^- = 0. \tag{2.17}$$

With (2.6), this shows that the columns of $\hat{A}_*^-$ form a basis of the space tangent to the reduced gradient manifold $\{g = 0\}$ at x_*. Therefore, from (2.15), we see that the SQP direction d_k has a *longitudinal component* $-Z_k^- H_k g_k$, tangent to the manifold $\{c = c_k\}$, and a *transversal component* $-\hat{A}_k^- c_k$, which tends to be tangent to the manifold $\{g = g_k\}$ when the pair $(x_k, Z_k^{-\top} M_k)$ is close to $(x_*, Z_*^{-\top} L_*)$.

In this paper, the globalization of the SQP method follows the approach of Bonnans [3]. We take as merit function the nondifferentiable augmented Lagrangian

$$\Theta_{\mu,\sigma}(x) = f(x) + \mu^\top c(x) + \sigma \|c(x)\|_P, \tag{2.18}$$

in which $\mu \in \mathbb{R}^m$, σ is a positive number, and $\|\cdot\|_P$ is an arbitrary (primal) norm on $\mathbb{R}^m$. This norm may differ from the ℓ_2-norm and it is not squared in $\Theta_{\mu,\sigma}$. We denote by $\|\cdot\|_D$ the dual norm associated with $\|\cdot\|_P$ with respect to the Euclidean scalar product:

$$\|u\|_D = \sup_{\|v\|_P=1} u^\top v.$$

The penalty function $\Theta_{\mu,\sigma}$ is convenient for globalizing the SQP method for at least two reasons. On the one hand, the penalization is exact, provided the *exactness condition*

$$\|\mu - \lambda_*\|_D < \sigma \tag{2.19}$$

holds (see for example Han and Mangasarian [23], and Bonnans [3]). On the other hand, the Armijo inequality using this function accepts the unit stepsize asymptotically, under some natural conditions (this is analyzed in Section 5, see also [3]).

We recall that $(\psi \circ \phi)$ has directional derivatives at a point x, if ψ is Lipschitz continuous in a neighborhood of $\phi(x)$ and has directional derivatives at $\phi(x)$, and if ϕ has directional derivatives at x. Furthermore, $(\psi \circ \phi)'(x; h) = \psi'(\phi(x); \phi'(x; h))$. In particular, due to its convexity, a norm has the properties of the function ψ above, and since f and c are supposed smooth, $\Theta_{\mu,\sigma}$ has directional derivatives.

We conclude this section by giving formulae for the directional derivatives of $\Theta_{\mu,\sigma}$ and by giving conditions for having descent directions. Let d be a vector of $\mathbb{R}^n$ satisfying the linear constraints $c(x) + A(x)d = 0$. The directional derivative of $\Theta_{\mu,\sigma}$ at x in the direction d is given by

$$\Theta'_{\mu,\sigma}(x; d) = \nabla f(x)^\top d - \mu^\top c(x) - \sigma\|c(x)\|_P \tag{2.20}$$

(for the differentiation of the term with the norm, use the very definition of directional derivative, see for example [20]). For any multiplier λ, we then have

$$\Theta'_{\mu,\sigma}(x; d) = \nabla_x \ell(x, \lambda)^\top d + (\lambda - \mu)^\top c(x) - \sigma\|c(x)\|_P. \tag{2.21}$$

Therefore, if d is a descent direction of the Lagrangian function at (x, λ), in the sense that $\nabla_x \ell(x, \lambda)^\top d < 0$ (which in particular holds for the direction d_k when $(x, \lambda) = (x_k, \lambda_k^{\text{QP}})$), then d is also a descent direction of $\Theta_{\mu,\sigma}$ at x provided the *descent condition*

$$\|\lambda - \mu\|_D \le \sigma \tag{2.22}$$

holds (compare with the exactness condition (2.19)).

3. The approach

This section describes our quasi-Newton version of the SQP algorithm in a global framework. Our aim is to develop a consistent way of updating the positive definite matrix M_k, using adequate vectors γ_k and δ_k.

3.1. *Computation of γ_k*

As we said in the introduction, we force M_k to be an approximation of the Hessian of the augmented Lagrangian. This is equivalent to considering the problem

$$\begin{array}{ll} \min & f(x) + \frac{r}{2}\|c(x)\|^2 \\ \text{s.t.} & c(x) = 0, \quad x \in \Omega, \end{array}$$

for some $r \geq 0$. This problem has the same solutions as problem (1.1) and has a Lagrangian whose Hessian at (x_*, λ_*) is

$$L_*^r = L_* + r A_*^\top A_*.$$

It is well known that when (1.4) holds, L_*^r is positive definite when r is larger than some threshold value. Therefore, it makes sense to force M_k to approach L_*^r for some sufficiently large r and to keep its positive definiteness.

For this purpose, we would like to have for some $r_k \geq 0$:

$$\gamma_k \simeq L_*^{r_k}\delta_k \simeq L_*\delta_k + r_k A_k^\top A_k \delta_k.$$

Using successively (2.4), (2.6), and (2.8), we get

$$\begin{aligned} L_*\delta_k &= Z_k^\top Z_k^{-\top} L_*\delta_k + A_k^\top A_k^{-\top} L_*\delta_k \\ &\simeq Z_k^\top \nabla g_*^\top \delta_k - A_k^\top \nabla\lambda_*^\top \delta_k \\ &\simeq Z_k^\top (g_{k+1} - g_k) - A_k^\top(\lambda_{k+1} - \lambda_k), \end{aligned} \tag{3.1}$$

provided

$$\delta_k \simeq x_{k+1} - x_k.$$

This approximate computation motivates our choice of γ_k, which is

$$\gamma_k = Z_k^\top(g_{k+1} - g_k) - A_k^\top(\lambda_{k+1} - \lambda_k) + r_k A_k^\top A_k \delta_k. \tag{3.2}$$

This formula is very close to the one used by Byrd, Tapia, and Zhang (BTZ for short), given by (1.11). The main difference is that γ_k^ℓ is split into two terms for reasons that are discussed now. For this, let us look at the form of the scalar product $\gamma_k^\top \delta_k$, which we want to have positive:

$$\gamma_k^\top \delta_k = (g_{k+1} - g_k)^\top Z_k \delta_k - (\lambda_{k+1} - \lambda_k)^\top A_k \delta_k + r_k \|A_k \delta_k\|^2. \tag{3.3}$$

When $A_k\delta_k \neq 0$, it is clear that the curvature condition (1.8) can be satisfied by choosing r_k sufficiently large. Remember that when $A_k\delta_k$ is close to zero, the BTZ approach needs a back-up strategy. For our form of γ_k, $A_k\delta_k = 0$ implies that

$$\gamma_k^\top \delta_k = (g_{k+1} - g_k)^\top Z_k \delta_k.$$

A possible way of satisfying the curvature condition in this case would be to choose the next iterate x_{k+1} such that $g_{k+1}^\top Z_k\delta_k > g_k^\top Z_k\delta_k$. We believe, however, that this may not be possible at iteration where $A_k\delta_k \neq 0$, because $Z_k\delta_k$ may not be a reduced descent direction (meaning that $g_k^\top Z_k\delta_k$ may not be negative). Now when $A_k\delta_k = 0$ and δ_k is parallel to the SQP direction d_k, we have $c_k = 0$ and, from (2.15), d_k reduces to $d_k = -Z_k^- H_k g_k = Z_k^- \hat{Z}_k d_k$, which implies that $Z_k d_k = \hat{Z}_k d_k$. Therefore, by forcing the inequality

$$g_{k+1}^\top \hat{Z}_k d_k > g_k^\top \hat{Z}_k d_k,$$

the curvature condition can be fulfilled when $A_k\delta_k = 0$. The important point is that, as we shall see, it is always possible to realize this inequality, even when $c_k \neq 0$, because $\hat{Z}_k d_k = -H_k g_k$ is a reduced descent direction ($g_k^\top \hat{Z}_k d_k < 0$). The piecewise line-search (PLS) technique introduced for reduced quasi-Newton methods in [18] and extended in [20] is designed for realizing this inequality.

3.2. Guiding path

From the discussion above, a central point of our algorithm is to find the next iterate x_{k+1} in order to get, in particular, the following *reduced Wolfe condition*

$$g_{k+1}^\top \hat{Z}_k d_k \geq \omega_2 g_k^\top \hat{Z}_k d_k, \tag{3.4}$$

for some constant $\omega_2 \in (0, 1)$.

Contrary to the unconstrained case, condition (3.4) may fail, whatever point x_{k+1} is taken along the SQP direction d_k. On the other hand, Proposition 3.2 below shows that along the path p_k defined by the following differential equation

$$\begin{cases} p_k'(\xi) = Z^-(p_k(\xi))\hat{Z}_k d_k - \hat{A}_k^-(p_k(\xi))c(p_k(\xi)) \\ p_k(0) = x_k, \end{cases} \tag{3.5}$$

one can find a stepsize ξ_k, such that the merit function $\Theta_{\mu,\sigma}$ decreases and the reduced Wolfe condition (3.4) holds:

$$\Theta_{\mu,\sigma}(p_k(\xi_k)) \leq \Theta_{\mu,\sigma}(x_k) \quad \text{and} \quad g(p_k(\xi_k))^\top \hat{Z}_k d_k \geq \omega_2 g_k^\top \hat{Z}_k d_k. \tag{3.6}$$

Note that d_k is tangent to the path p_k at $\xi = 0$. Note also that the reduced tangent component of $p_k'(\xi)$ keeps the constant value $\hat{Z}_k d_k$ along the path. This is further motivated in [20].

In the proof of Proposition 3.2, we will need the following lemma. We say that a function ϕ is *locally Lipschitz continuous* on an open set X if any point of X has a neighborhood on which ϕ is Lipschitz continuous.

Lemma 3.1. *Let $\alpha > 0$ and $\phi : [0, \alpha] \to \Omega$ be a continuous function having right derivatives on $(0, \alpha)$. Suppose that f and c are locally Lipschitz continuous on Ω and have* *directional derivatives on* $\phi((0, \alpha))$. *Then there exists* $\bar{\alpha} \in (0, \alpha)$ *such that*

$$\Theta_{\mu,\sigma}(\phi(\alpha)) - \Theta_{\mu,\sigma}(\phi(0)) \le \alpha \Theta'_{\mu,\sigma}(\phi(\bar{\alpha}); \phi'(\bar{\alpha}; 1)).$$

Proof: Since c is locally Lipschitz continuous on Ω, so is $\|c(\cdot)\|_P$. Furthermore, by the hypotheses on c and the convexity of the norm, $\|c(\cdot)\|_P$ has directional derivatives on $\phi((0, \alpha))$. Therefore, with the hypotheses, we deduce that $\Theta_{\mu,\sigma}$ is locally Lipschitz continuous on Ω and has directional derivatives on $\phi((0, \alpha))$. Now with the properties of ϕ, we see that $\Theta_{\mu,\sigma} \circ \phi$ has right derivatives on $(0, \alpha)$ and that for any $\bar{\alpha} \in (0, \alpha)$:

$$(\Theta_{\mu,\sigma} \circ \phi)'(\bar{\alpha}; 1) = \Theta'_{\mu,\sigma}(\phi(\bar{\alpha}); \phi'(\bar{\alpha}; 1)).$$

On the other hand, the function $\Theta_{\mu,\sigma} \circ \phi$ is continuous on $[0, \alpha]$ and, since it has right derivatives on $(0, \alpha)$, there exists $\bar{\alpha} \in (0, \alpha)$ such that

$$(\Theta_{\mu,\sigma} \circ \phi)(\alpha) - (\Theta_{\mu,\sigma} \circ \phi)(0) \le \alpha(\Theta_{\mu,\sigma} \circ \phi)'(\bar{\alpha}; 1).$$

(see for instance Schwartz [34, Chap. III, § 2, Remarque 11]).

Combining this inequality with the preceding equality gives the result. □

Proposition 3.2. *Suppose that the path* $\xi \mapsto p_k(\xi)$ *defined by* (3.5) *exists for sufficiently large stepsize* $\xi \ge 0$. *Suppose also that* f *and* c *are continuously differentiable*, *that* $\Theta_{\mu,\sigma}$ *is bounded from below along the path* p_k, *that* $\|\lambda_k^{\text{QP}}(p_k(\xi)) - \mu\|_D \le \sigma$ *whenever* $p_k(\xi)$ *exists*, *that* M_k *is positive definite*, *and that* $\omega_2 \in (0, 1)$. *Then, the inequalities in* (3.6) *are satisfied for some stepsize* $\xi_k > 0$.

Proof: To lighten the notation in the proof, we

$$(d(\xi), \lambda(\xi)) = \left(d_k^{\text{QP}}(p_k(\xi)), \lambda_k^{\text{QP}}(p_k(\xi))\right)$$

the primal-dual solution of the quadratic subproblem

$$\begin{aligned} &\min \quad \nabla f(p_k(\xi))^\top d + \tfrac{1}{2} d^\top M_k d \\ &\text{s.t.} \quad c(p_k(\xi)) + A(p_k(\xi)) d = 0. \end{aligned}$$

The first order optimality conditions give

$$\nabla_x \ell(p_k(\xi), \lambda(\xi)) = -M_k d(\xi),$$

and $d(\xi)$ can be written (see (2.13))

$$d(\xi) = -Z^-(p_k(\xi)) H_k(p_k(\xi)) g(p_k(\xi)) - \hat{A}_k^-(p_k(\xi)) c(p_k(\xi)).$$

Let us show that, when the second inequality in (3.6) or reduced Wolfe condition does not hold for $\xi_k = \xi$, then

$$\Theta'_{\mu,\sigma}(p_k(\xi); p'_k(\xi)) < -\omega_2 g_k^\top H_k g_k. \tag{3.7}$$

Using successively (2.21), the hypothesis $\|\lambda(\xi)-\mu\|_D \leq \sigma$, the optimality condition above, the form of $d(\xi)$ and $p_k'(\xi)$, the identity (2.16), and the positive definiteness of M_k, we get

$$
\begin{aligned}
&\Theta'_{\mu,\sigma}(p_k(\xi);\, p_k'(\xi)) \\
&\quad = \nabla_x \ell(p_k(\xi), \lambda(\xi))^\top p_k'(\xi) + (\lambda(\xi) - \mu)^\top c(p_k(\xi)) - \sigma \|c(p_k(\xi))\|_P \\
&\quad \leq \nabla_x \ell(p_k(\xi), \lambda(\xi))^\top p_k'(\xi) \\
&\quad = -d(\xi)^\top M_k p_k'(\xi) \\
&\quad = -g(p_k(\xi))^\top H_k g_k - c(p_k(\xi))^\top \hat{A}_k^-(p_k(\xi))^\top M_k \hat{A}_k^-(p_k(\xi)) c(p_k(\xi)) \\
&\quad \leq -g(p_k(\xi))^\top H_k g_k.
\end{aligned}
$$

Therefore, when the reduced Wolfe condition does not hold, we have (3.7).

On the other hand, we see by using Lemma 3.1 with $\phi = p_k$ that, as long as the path p_k exists, for any $\xi > 0$, one can find $\bar{\xi} \in (0, \xi)$ such that

$$\Theta_{\mu,\sigma}(p_k(\xi)) - \Theta_{\mu,\sigma}(x_k) \leq \xi \Theta'_{\mu,\sigma}(p_k(\bar{\xi});\, p_k'(\bar{\xi})).$$

Therefore, if the reduced Wolfe condition is never realized along the path p_k, we would have by (3.7)

$$\Theta_{\mu,\sigma}(p_k(\xi)) - \Theta_{\mu,\sigma}(x_k) < -\xi \omega_2 g_k^\top H_k g_k, \tag{3.8}$$

which would imply the unboundedness of the merit function along this path and would contradict the hypotheses.

At the first stepsize $\xi_k > 0$ at which the reduced Wolfe condition is satisfied, by continuity, (3.8) is still verified with a nonstrict inequality. This shows that, for this stepsize, the merit function $\Theta_{\mu,\sigma}$ has decreased. □

The inequality $\|\lambda_k^{\text{QP}}(p_k(\xi)) - \mu\|_D \leq \sigma$ used as hypothesis in the previous proposition can be compared with the descent condition (2.22).

3.3. Outline of the PLS algorithm

The success of the path p_k defined by (3.5) suggests searching for the next iterate x_{k+1} along a discretized version of this path. Taking a precise discretization may not succeed and would be computationally expensive. Therefore, we propose to take as often as possible a unit stepsize along the directions obtained by an explicit Euler discretization of the differential Equation (3.5). With this technique, the search path becomes piecewise linear. It is proved in Section 4 that the search along this path succeeds in a finite number of trials.

The piecewise line-search (PLS) algorithm generates intermediate points $x_{k,i}$, for $i = 0, \ldots, i_k$, with $x_{k,0} = x_k$ and $x_{k,i_k} = x_{k+1}$. We adopt the notation $f_{k,i} = f(x_{k,i})$, $\nabla f_{k,i} = \nabla f(x_{k,i})$, $c_{k,i} = c(x_{k,i})$, $Z_{k,i}^- = Z^-(x_{k,i})$, $A_{k,i}^- = A^-(x_{k,i})$, $\hat{A}_{k,i}^- = \hat{A}_k^-(x_{k,i})$, and $\hat{Z}_{k,i} = \hat{Z}_k(x_{k,i})$. The iterations of the PLS algorithm, computing $x_{k,i+1}$ from $x_{k,i}$, are called *inner iterations* and their number is denoted by i_k.

The point $x_{k,i+1}$ is obtained from $x_{k,i}$ by

$$x_{k,i+1} = x_{k,i} + \alpha_{k,i} d_{k,i}, \tag{3.9}$$

where the stepsize $\alpha_{k,i} > 0$ is determined along the direction

$$d_{k,i} = -Z_{k,i}^{-} H_k g_k - \hat{A}_{k,i}^{-} c_{k,i}. \tag{3.10}$$

This direction is obtained by evaluating the right hand side of (3.5) at a discretization point $x_{k,i}$ of the path p_k. The stepsize is chosen such that the following two conditions are satisfied for $\alpha = \alpha_{k,i}$:

$$x_{k,i} + \alpha d_{k,i} \in \Omega, \tag{3.11}$$

$$\Theta_{\mu_k,\sigma_{k,i}}(x_{k,i} + \alpha d_{k,i}) \le \Theta_{\mu_k,\sigma_{k,i}}(x_{k,i}) + \omega_1 \alpha \Theta'_{\mu_k,\sigma_{k,i}}(x_{k,i}; d_{k,i}). \tag{3.12}$$

Condition (3.12) imposes a sufficient decrease of the merit function and will be called the *Armijo condition*.

At each inner iteration i, the penalty parameter $\sigma_{k,i}$ may need to be adapted so that $d_{k,i}$ is a descent direction of $\Theta_{\mu_k,\sigma_{k,i}}$ at $x_{k,i}$. An adaptation rule will be given in Section 4.

Next the reduced Wolfe condition

$$g(x_{k,i+1})^\top \hat{Z}_k d_k \ge \omega_2 g_k^\top \hat{Z}_k d_k \tag{3.13}$$

is tested. If it holds, the PLS is completed and i_k is set to $i + 1$. Otherwise, the index i is increased by one and the search is pursued along a new direction $d_{k,i}$.

From the description of the algorithm, we have

$$x_{k+1} = x_{k,i_k} = x_k + \sum_{i=0}^{i_k-1} \alpha_{k,i} d_{k,i}.$$

It is interesting to compare the PLS algorithm with the rule consisting in skipping the update when $\gamma_k^\top \delta_k$ is not sufficiently positive. Indeed, the intermediate search directions $d_{k,i}$ are close to the SQP direction at $x_{k,i}$, with two differences however. First, the matrix M_k is kept unchanged as long as the reduced Wolfe condition is not satisfied, which is similar to the skipping rule strategy. On the other hand, the reduced gradient used in these directions is also kept unchanged. As we will see (Theorem 4.4), this gives the matrix a chance of being updated.

3.4. Computation of δ_k

The choice of δ_k is governed by the necessity to have $\delta_k \simeq x_{k+1} - x_k$, as required by the discussion in Section 3.1, and the desire to control precisely the positivity of $\gamma_k^\top \delta_k$ when

$A_k\delta_k = 0$. We have already observed that when $A_k\delta_k = 0$,

$$\gamma_k^\top \delta_k = (g_{k+1} - g_k)^\top Z_k \delta_k,$$

so that r_k cannot be used to get $\gamma_k^\top \delta_k > 0$.

Suppose that we choose $\delta_k = x_{k+1} - x_k$. Then, from the identity above, we have $\gamma_k^\top \delta_k = (g_{k+1} - g_k)^\top Z_k(x_{k+1} - x_k)$, and it is not clear how the reduced Wolfe condition (3.4) can ensure the positivity of $\gamma_k^\top \delta_k$, since $x_{k+1} - x_k$ is not parallel to d_k. For this reason, we prefer to take for δ_k the following approximation of $x_{k+1} - x_k$:

$$\begin{aligned}\delta_k &= -\alpha_k^{\mathrm{Z}} Z_k^- H_k g_k - \alpha_k^{\mathrm{A}} \hat{A}_k^- c_k \\ &\simeq \sum_{i=0}^{i_k-1} \alpha_{k,i}(-Z_{k,i}^- H_k g_k - \hat{A}_{k,i}^- c_{k,i}) \\ &= x_{k+1} - x_k.\end{aligned} \tag{3.14}$$

In (3.14), the *longitudinal stepsize* α_k^{Z} and the *transversal stepsize* α_k^{A} are defined by

$$\alpha_k^{\mathrm{Z}} = \sum_{i=0}^{i_k-1} \alpha_{k,i} \quad \text{and} \quad \alpha_k^{\mathrm{A}} = \sum_{i=0}^{i_k-1} \alpha_{k,i}\, e^{-\xi_{k,i}}, \tag{3.15}$$

with $\xi_{k,i} = \sum_{j=0}^{i-1} \alpha_{k,j}$. These definitions assume that the operators $Z_{k,i}^-$ and $\hat{A}_{k,i}^-$ remain close to Z_k^- and $\hat{A}_k^-$, respectively. Furthermore, the form of α_k^{A} aims at taking into account the fact that the value of c at $x_{k,i}$ is used in the search directions $d_{k,i}$, while it is c_k that is used in δ_k. It is based on the observation that along the path p_k defined by (3.5), we have $c(p_k(\xi)) = e^{-\xi} c_k$ (multiply both sides of (3.5) by $A_k(p_k(\xi))$ and integrate). After discretization: $c_{k,i} \simeq e^{-\xi_{k,i}} c_k$.

To check that our definition (3.14) of δ_k is appropriate, suppose that $A_k\delta_k = 0$. Then, we have $c_k = 0$, hence $\delta_k = \alpha_k^{\mathrm{Z}} Z_k^- \hat{Z}_k d_k$, and this allows us to write

$$\begin{aligned}\gamma_k^\top \delta_k &= (g_{k+1} - g_k)^\top Z_k \delta_k \\ &= \alpha_k^{\mathrm{Z}} (g_{k+1} - g_k)^\top \hat{Z}_k d_k \\ &> 0,\end{aligned}$$

by the reduced Wolfe condition (3.4). By a continuity argument, one can claim that $(g_{k+1} - g_k)^\top Z_k \delta_k$ is also positive when x_k is close to the constraint manifold, provided the stepsizes are determined by processes depending continuously on x_k and (3.4) is realized with strict inequality (in this case, the number of inner iterations in the PLS algorithm does not change in the neighborhood of a point on the constraint manifold). In the algorithm below, we shall not impose strict inequality in (3.4), because we believe that this continuity argument is not important in practice.

The conclusion of this discussion is that for any $k \geq 1$, one can find a (finite) $r_k \geq 0$ such that $\gamma_k^\top \delta_k > 0$, either because $A_k\delta_k \neq 0$ or because $A_k\delta_k = 0$ and $\gamma_k^\top \delta_k > 0$ by the reduced

Wolfe condition (3.4). In Section 6.1, we will specify a rule that can be used for updating the value of r_k and that has the property to minimize an estimate of the condition number of the updated matrices.

4. The piecewise line-search

In this section, we make more precise the PLS algorithm outlined in Section 3.3, show its well-posedness (Proposition 4.1), and prove its finite termination (Theorem 4.4).

4.1. Descent directions

A question we have not addressed so far is to know whether the ith inner search direction $d_{k,i}$, given by (3.10) and used in the PLS algorithm, is a descent direction of the merit function. The following proposition shows that this property holds when the penalty parameter $\sigma_{k,i}$ in the merit function is larger than a threshold that is easy to compute. For this, as in Proposition 3.2, the multiplier $\mu = \mu_k$ in the merit function is compared to the multiplier $\lambda_k^{\text{QP}}(x_{k,i})$ given by the quadratic program (2.9) at $x = x_{k,i}$. The multiplier μ_k is indexed by k because it will have to be modified at some iterations of the overall algorithm below.

Proposition 4.1. *Let $0 \le i < i_k$ be the index of an inner iteration of the PLS algorithm. Suppose that x_k is not a stationary point, that M_k is positive definite, and that*

$$\underline{\sigma}_k + \left\|\lambda_k^{\text{QP}}(x_{k,i}) - \mu_k\right\|_D \le \sigma_{k,i}, \tag{4.1}$$

where $\underline{\sigma}_k$ is a positive number. Then $d_{k,i}$ is a descent direction of $\Theta_{\mu_k,\sigma_{k,i}}$ at $x_{k,i}$, meaning that $\Theta'_{\mu_k,\sigma_{k,i}}(x_{k,i}; d_{k,i}) < 0$. For $i = 0$:

$$\Theta'_{\mu_k,\sigma_{k,0}}(x_{k,0}; d_{k,0}) \le -d_k^\top M_k d_k - \underline{\sigma}_k \|c_k\|_P, \tag{4.2}$$

while for $1 \le i < i_k$:

$$\Theta'_{\mu_k,\sigma_{k,i}}(x_{k,i}; d_{k,i}) < -\omega_2 g_k^\top H_k g_k - c_{k,i}^\top \hat{A}_{k,i}^{-\top} M_k \hat{A}_{k,i}^- c_{k,i} - \underline{\sigma}_k \|c_{k,i}\|_P. \tag{4.3}$$

Proof: For $i = 0$, the search direction is $d_{k,0} = d_k$ and the optimality conditions of (2.9) give

$$\nabla_x \ell\big(x_k, \lambda_k^{\text{QP}}(x_k)\big)^\top d_{k,0} = -d_k^\top M_k d_k.$$

For $i = 1, \ldots, i_k - 1$, we have by the optimality conditions of (2.9), formulae (2.13) and (3.10), identity (2.16), and the fact that the reduced Wolfe condition (3.13) is not satisfied at $x_{k,i}$:

$$\begin{aligned}
&\nabla_x l\big(x_{k,i}, \lambda_k^{\text{QP}}(x_{k,i})\big)^\top d_{k,i} \\
&\quad = -d_k^{\text{QP}}(x_{k,i})^\top M_k d_{k,i} \\
&\quad = -(-Z_{k,i}^- H_{k,i} g_{k,i} - \hat{A}_{k,i}^- c_{k,i})^\top M_k (-Z_{k,i}^- H_k g_k - \hat{A}_{k,i}^- c_{k,i}) \\
&\quad = -g_{k,i}^\top H_k g_k - c_{k,i}^\top \hat{A}_{k,i}^{-\top} M_k \hat{A}_{k,i}^- c_{k,i} \\
&\quad < -\omega_2 g_k^\top H_k g_k - c_{k,i}^\top \hat{A}_{k,i}^{-\top} M_k \hat{A}_{k,i}^- c_{k,i}.
\end{aligned}$$

Then, from the estimates above, (2.21), and (4.1), we see that (4.2) and (4.3) hold. Since x_k is not stationary, $d_{k,0} = d_k \neq 0$ and (4.2) shows that $d_{k,0}$ is a descent direction of $\Theta_{\mu_k,\sigma_{k,0}}$ at x_k. The strict inequality in (4.3) shows that for $i \geq 1$, $d_{k,i}$ is also a descent direction of $\Theta_{\mu_k,\sigma_{k,i}}$ at $x_{k,i}$. □

The preceding result suggests the following rule for updating the penalty parameter $\sigma_{k,i}$. Let us denote by $\sigma_k = \sigma_{k,0}$ the value of the penalty parameter at the beginning of the PLS. This value depends on the update of μ_k in the overall algorithm, which will be given in Section 5.1.

UPDATE RULE OF $\sigma_{k,i}$ $(1 \leq i < i_k)$:

if $\underline{\sigma}_k + \|\lambda_k^{\text{QP}}(x_{k,i}) - \mu_k\|_D \leq \sigma_{k,i-1}$ **then** $\sigma_{k,i} = \sigma_{k,i-1}$,
else $\sigma_{k,i} = \max(2\sigma_{k,i-1}, \underline{\sigma}_k + \|\lambda_k^{\text{QP}}(x_{k,i}) - \mu_k\|_D)$.

It follows that either

$$\sigma_{k,i} = \sigma_{k,i-1}, \tag{4.4}$$

or

$$\underline{\sigma}_k + \big\|\lambda_k^{\text{QP}}(x_{k,i}) - \mu_k\big\|_D > \sigma_{k,i-1} \quad \text{and} \quad \sigma_{k,i} \geq 2\sigma_{k,i-1}. \tag{4.5}$$

With this update rule, the search direction $d_{k,i}$ is a descent direction of $\Theta_{\mu_k,\sigma_{k,i}}$ at $x_{k,i}$. Then, by a standard argument, one can show that there is a stepsize α such that (3.11) and (3.12) hold. This shows that the PLS algorithm of Section 3.3 is well defined.

4.2. Finite termination

Before proving its finite termination, we give a precise description of the PLS algorithm. The algorithm starts at a point $x_k \in \Omega$ with a positive definite matrix M_k. It is assumed that the solution $(d_k, \lambda_k^{\text{QP}})$ of the quadratic program (1.5) is computed in the overall algorithm and that the penalty parameter σ_k satisfies the descent condition

$$\underline{\sigma}_k + \big\|\lambda_k^{\text{QP}} - \mu_k\big\|_D \leq \sigma_k,$$

for some $\underline{\sigma}_k > 0$ and a multiplier estimate μ_k given by the overall algorithm. It is also supposed that two constants ω_1 and ω_2 are given in $(0, 1)$ and a constant ρ is given in $(0, \frac{1}{2}]$.

PLS ALGORITHM:

0. Set $i = 0$, $x_{k,0} = x_k$, $d_{k,0} = d_k$, and $\sigma_{k,0} = \sigma_k$.
1. Find a stepsize $\alpha_{k,i}$ such that (3.11) and (3.12) hold for $\alpha = \alpha_{k,i}$. For this do the following:
   1.0. Set $j = 0$ and $\alpha_{k,i,0} = 1$.
   1.1. If (3.11) and (3.12) hold for $\alpha = \alpha_{k,i,j}$, set $\alpha_{k,i} = \alpha_{k,i,j}$, $x_{k,i+1} = x_{k,i} + \alpha_{k,i} d_{k,i}$, and go to Step 2.
   1.2. Choose $\alpha_{k,i,j+1} \in [\rho\alpha_{k,i,j}, (1-\rho)\alpha_{k,i,j}]$,
   1.3. Increase j by 1 and go to Step 1.1.
2. If the reduced Wolfe condition (3.13) holds, set $i_k = i + 1$, $x_{k+1} = x_{k,i_k}$, and terminate.
3. Otherwise, increase i by 1, compute the multiplier estimate $\lambda_k^{\text{QP}}(x_{k,i})$ as the multiplier of problem (2.9) with $x = x_{k,i}$, compute $d_{k,i}$ by (3.10), update $\sigma_{k,i}$ according to the rule given in Section 4.1, and go to 1.

The behavior of the PLS algorithm is analyzed in Theorem 4.4, the proof of which uses the two lemmas below. We recall that a real-valued function ϕ is *regular* at x (in the sense of Clarke [9, Definition 2.3.4]) if it has directional derivatives at x, and if for all h,

$$\phi'(x; h) = \limsup_{\substack{x' \to x \\ t \to 0+}} \frac{\phi(x' + th) - \phi(x')}{t}.$$

Lemma 4.2. *Suppose that f and c are continuously differentiable on Ω and let $x \in \Omega$. Suppose also that $x_k \to x$ in Ω, $d_k \to d$ in $\mathbb{R}^n$, and $\alpha_k \to 0$ in $\mathbb{R}_+$. Then, with the merit function $\Theta_{\mu,\sigma}$ defined by* (2.18), *we have*

$$\Theta'_{\mu,\sigma}(x; d) = \limsup_{k\to\infty} \frac{\Theta_{\mu,\sigma}(x_k + \alpha_k d_k) - \Theta_{\mu,\sigma}(x_k)}{\alpha_k}. \tag{4.6}$$

Proof: Since $\Theta_{\mu,\sigma}$ is Lipschitz continuous in a neighborhood of x and $d_k \to d$, one can readily substitute d_k by d in (4.6), so that it remains to prove that $\Theta_{\mu,\sigma}$ is regular at x. This is the case for $f(\cdot) + \mu^\top c(\cdot)$, since this function is continuously differentiable (use the corollary of Theorem 2.2.1 and Proposition 2.3.6 from [9]). For the regularity of the map $\|c(\cdot)\|_P$, use [9, Theorem 2.3.10] and the fact that the convexity of the norm implies its regularity [9, Theorem 2.3.6]. □

Lemma 4.3. *Suppose that M_k is positive definite, that $x \in \Omega \mapsto A(x)$ is continuous and bounded, and that the singular values of $A(x)$ are bounded away from zero on Ω. Then, $x \in \Omega \mapsto \hat{A}_k^-(x)$ is continuous and bounded.*

Proof: We have already seen that $u = -\hat{A}_k^-(x)c(x)$ is the solution of the quadratic subproblem (2.9), in which $\nabla f(x)$ is set to zero. Therefore there exists a multiplier λ such that

(u, λ) is solution of the corresponding first order optimality conditions:

$$\begin{aligned} M_k u + A(x)^\top \lambda &= 0 \\ c(x) + A(x)u &= 0. \end{aligned}$$

Canceling λ from these equations and observing that $c(x)$ is an arbitrary vector, the operator $\hat{A}_k^-(x)$ can be written as follows:

$$\hat{A}_k^-(x) = M_k^{-1} A(x)^\top \big(A(x) M_k^{-1} A(x)^\top \big)^{-1}. \tag{4.7}$$

Then, the continuity and boundedness of $\hat{A}_k^-$ follow from the hypotheses. □

Theorem 4.4. *Suppose that f and c are continuously differentiable on Ω and that $A(\cdot)$ and $Z^-(\cdot)$ are continuous and bounded on Ω. Suppose also that $A(\cdot)$ has its singular values bounded away from zero on Ω. If the PLS algorithm is applied from a point $x_k \in \Omega$ with a positive definite matrix M_k, then one of the following situations occurs.*

(i) *The number of iterations of the PLS algorithm is finite, in which case:*
  - (a) $x_{k+1} \in \Omega$,
  - (b) *at each inner iteration, the Armijo condition* (3.12) *holds*,
  - (c) *the reduced Wolfe condition* (3.4) *holds at* x_{k+1}.

(ii) *The algorithm builds a sequence $\{x_{k,i}\}_i$ in Ω and*
  - (a) *either $\{\sigma_{k,i}\}_i$ is unbounded, in which case $\{\lambda_k^{\text{QP}}(x_{k,i})\}_i$ is also unbounded*,
  - (b) *or $\sigma_{k,i} = \bar{\sigma}_k$ for large i, in which case either $\lim_{i\to\infty} \Theta_{\mu_k,\bar{\sigma}_k}(x_{k,i}) = -\infty$ or $\{x_{k,i}\}_i$ converges to a point on the boundary of Ω.*

Proof: Since $\Theta'_{\mu_k,\sigma_{k,i}}(x_{k,i}; d_{k,i}) < 0$ (Proposition 4.1), conditions (3.11) and (3.12) are satisfied for sufficiently small α, and thus the PLS algorithm does not cycle in Step 1. It is also clear that when the number of inner iterations is finite, the conclusions of situation (i) occur.

Note that if $g_k = 0$, then $\hat{Z}_k d_k = -H_k g_k = 0$. In this case, the reduced Wolfe condition (3.13) is trivially satisfied and the algorithm terminates at the first stepsize $\alpha_{k,0}$ satisfying conditions (3.11) and (3.12).

Suppose now that (i) does not occur, then $g_k \neq 0$ and a sequence $\{x_{k,i}\}_i$ is built, such that for $i \geq 1$:

$$x_{k,i} \in \Omega, \tag{4.8}$$

$$\Theta_{\mu_k,\sigma_{k,i}}(x_{k,i+1}) \leq \Theta_{\mu_k,\sigma_{k,i}}(x_{k,i}) + \omega_1 \alpha_{k,i} \Theta'_{\mu_k,\sigma_{k,i}}(x_{k,i}; d_{k,i}), \tag{4.9}$$

and

$$g(x_{k,i})^\top \hat{Z}_k d_k < \omega_2 g_k^\top \hat{Z}_k d_k. \tag{4.10}$$

Due to (4.4) and (4.5) it follows that either the sequence $\{\sigma_{k,i}\}_i$ is unbounded, which corresponds to conclusion (ii-a), or there exists an index $i_0 \geq 1$ such that $\sigma_{k,i} = \sigma_{k,i_0} = \bar{\sigma}_k$

for all $i \geq i_0$. It remains to show that in the latter case, the alternative in situation (ii-b) occurs. Up to the end of the proof, we simply denote by Θ the merit function $\Theta_{\mu_k,\bar{\sigma}_k}$. Let us prove situation (ii-b) by contradiction, assuming that the decreasing sequence $\{\Theta(x_{k,i})\}_i$ is bounded below and that $\{x_{k,i}\}$ does not converge to a point on the boundary of Ω.

Inequality (4.9) implies

$$\Theta(x_{k,i+1}) \leq \Theta(x_{k,i_0}) + \omega_1 \sum_{l=i_0}^{i} \alpha_{k,l} \Theta'(x_{k,l}; d_{k,l}).$$

But, by the positive definiteness of M_k and (4.3)

$$\Theta'(x_{k,i}; d_{k,i}) \leq -\omega_2 g_k^\top H_k g_k - \underline{\sigma}_k \|c_{k,i}\|_P. \tag{4.11}$$

The two latter inequalities, $\omega_1 > 0$, and our assumption on the fact that $\{\Theta(x_{k,i})\}_i$ is bounded below imply the convergence of the series

$$\sum_{i\geq 0} \alpha_{k,i} < \infty \quad \text{and} \quad \sum_{i\geq 0} \alpha_{k,i} \|c_{k,i}\|_P < \infty. \tag{4.12}$$

In particular, $\alpha_{k,i} \to 0$.

By definition of $x_{k,i}$ we have

$$x_{k,i+1} = x_k + \sum_{l=0}^{i} \left(-\alpha_{k,l} Z_{k,l}^{-\top} H_k g_k - \alpha_{k,l} \hat{A}_{k,l}^{-} c_{k,l}\right).$$

Since $Z^-(\cdot)$ and $\hat{A}_k^-(\cdot)$ are bounded on Ω (by hypothesis and Lemma 4.3), the convergence of the series in (4.12) implies that the series defining $x_{k,i+1}$ above is absolutely convergent. It follows that the sequence $\{x_{k,i}\}_i$ converges to a point $\bar{x}_k$. By our assumptions, $\bar{x}_k$ must be in Ω. Therefore, using the continuity of Z^-, c, and $\hat{A}_k^-$ (Lemma 4.3), we have when $i \to \infty$:

$$d_{k,i} \to \bar{d}_k = -Z^-(\bar{x}_k) H_k g_k - \hat{A}_k^-(\bar{x}_k) c(\bar{x}_k).$$

In Step 1.0, the PLS algorithm takes $\alpha_{k,i,0} = 1$ and we have $\alpha_{k,i} \to 0$. Therefore, for all large i, there must exists some index j_i such that $\alpha_{k,i} \in [\rho\alpha_{k,i,j_i}, (1-\rho)\alpha_{k,i,j_i}]$. This means that either (3.11) or (3.12) is not verified for $\alpha = \alpha_{k,i,j_i}$. But for i large, condition (3.11) holds for $\alpha = \alpha_{k,i,j_i}$, because $x_{k,i} \to \bar{x}_k \in \Omega$, $\{d_{k,i}\}_i$ is bounded, and $\alpha_{k,i,j_i} \to 0$. Therefore, for all large i, it is the Armijo condition (3.12) that is not satisfied for $\alpha = \alpha_{k,i,j_i}$. This can be written

$$\omega_1 \Theta'(x_{k,i}; d_{k,i}) < \frac{\Theta(x_{k,i} + \alpha_{k,i,j_i} d_{k,i}) - \Theta(x_{k,i})}{\alpha_{k,i,j_i}}.$$

When $i \to \infty$, the form of $\Theta'(x_{k,i}; d_{k,i})$ (see for example (2.20)) shows that the left hand side of the inequality above converges to $\omega_1 \Theta'(\bar{x}_k; \bar{d}_k)$. For the right hand side, we use

Lemma 4.2, so that by taking the $\limsup_{i\to\infty}$ in the inequality, we obtain $\omega_1 \Theta'(\bar{x}_k; \bar{d}_k) \le \Theta'(\bar{x}_k; \bar{d}_k)$. Since $\omega_1 < 1$, this implies that $\Theta'(\bar{x}_k; \bar{d}_k) \ge 0$.

On the other hand, taking the limit in (4.11) when $i \to \infty$ and recalling that $g_k \neq 0$, we obtain

$$\Theta'(\bar{x}_k; \bar{d}_k) \le -\omega_2 g_k^\top H_k g_k - \underline{\sigma}_k \|c(\bar{x}_k)\|_P < 0,$$

a contradiction that concludes the proof. □

5. Convergence results

In this section, we give a global convergence result, assuming the boundedness of the generated matrices M_k and their inverses. Despite this strong assumption, we believe that such a result is useful in that it shows that the different facets of the algorithm introduced in the previous sections can fit together.

The results given in this section deal with the behavior of the sequence of iterates x_k, so that it is implicitly assumed that a sequence $\{x_k\}$ is actually generated and therefore that the PLS algorithm has finite termination each time it is invoked.

Recall that the value of the penalty parameter σ_k must be updated such that the descent condition

$$\underline{\sigma}_k + \left\|\lambda_k^{\text{QP}} - \mu_k\right\|_D \le \sigma_k \tag{5.1}$$

is satisfied. This corresponds to (4.1) with $i = 0$. In (5.1), $\underline{\sigma}_k$ is a positive number that is adapted at some iterations.

5.1. Admissibility of the unit stepsize

Admissibility of the unit stepsize by Armijo's condition on the penalty function Θ_{μ_k,σ_k} is studied by Bonnans [3], but in a form that is not appropriate for our algorithm. Proposition 5.1 gives a version suitable for us. Conditions for the admissibility of the unit stepsize by the reduced Wolfe inequality are given in Proposition 5.2. These results are obtained by expanding f and c about the current iterate x_k. They are useful for determining how and when the multiplier μ_k and the penalty parameter σ_k have to be adapted.

Proposition 5.1 requires that the multiplier estimate λ_k^{QP} be used in the descent condition (5.1). It also requires that μ_k be sufficiently close to the optimal multiplier and that the penalty parameter σ_k be sufficiently small. In other words, near the solution, the merit function has to be sufficiently close to the Lagrangian function. This implies that, in the overall algorithm, μ_k will have to be reset to λ_k^{QP} and σ_k will have to be decreased at some iterations.

Proposition 5.1. *Suppose the f and c are twice continuously differentiable in a convex neighborhood of a local solution x_* satisfying the second order sufficient condition of optimality* (1.3)–(1.4). *Suppose also that $x_k \to x_*$, $d_k \to 0$, $\omega_1 < 1/2$, the descent* *condition* (5.1) *holds*, *and*

$$d_k^\top (M_k - L_*^r) d_k \geq o(\|d_k\|^2), \tag{5.2}$$

in which r is a nonnegative scalar such that L_*^r is positive definite. Then, there exists a constant $\varepsilon > 0$ such that when

$$\|\mu_k - \lambda_*\| \leq \varepsilon \quad \text{and} \quad 0 \leq \sigma_k \leq \varepsilon,$$

and when k is sufficiently large, *the unit stepsize is accepted by the Armijo inequality*:

$$\Theta_{\mu_k,\sigma_k}(x_k + d_k) \leq \Theta_{\mu_k,\sigma_k}(x_k) + \omega_1 \Theta'_{\mu_k,\sigma_k}(x_k; d_k).$$

Proof: Since $d_k \to 0$, a second order expansion of $f(x_k + d_k)$ about x_k gives with (1.6):

$$\begin{aligned} f(x_k + d_k) &= f_k + \nabla f_k^\top d_k + \frac{1}{2} d_k^\top \nabla^2 f(x_*) d_k + o(\|d_k\|^2) \\ &= f_k - d_k^\top M_k d_k + \left(\lambda_k^{\text{QP}}\right)^\top c_k + \frac{1}{2} d_k^\top \nabla^2 f(x_*) d_k + o(\|d_k\|^2). \end{aligned}$$

Similarly, for any component $c_{(i)}$ of c, we have with (1.6)

$$c_{(i)}(x_k + d_k) = \frac{1}{2} d_k^\top \nabla^2 c_{(i)}(x_*) d_k + o(\|d_k\|^2).$$

Combining these two estimates and using (1.6), (2.21), and the hypotheses on μ_k and σ_k, we get

$$\begin{aligned} &\Theta_{\mu_k,\sigma_k}(x_k + d_k) - \Theta_{\mu_k,\sigma_k}(x_k) - \omega_1 \Theta'_{\mu_k,\sigma_k}(x_k; d_k) \\ &\quad = -d_k^\top M_k d_k + \left(\lambda_k^{\text{QP}} - \mu_k\right)^\top c_k - \sigma_k \|c_k\|_P + \frac{1}{2} d_k^\top L_* d_k - \omega_1 \Theta'_{\mu_k,\sigma_k}(x_k; d_k) \\ &\qquad + (\|\mu_k - \lambda_*\| + \sigma_k)\, O(\|d_k\|^2) + o(\|d_k\|^2) \\ &\quad \leq (1 - \omega_1) \Theta'_{\mu_k,\sigma_k}(x_k; d_k) + \frac{1}{2} d_k^\top L_*^r d_k - \frac{r}{2} \|c_k\|_2^2 + \varepsilon O(\|d_k\|^2) + o(\|d_k\|^2). \end{aligned}$$

Now, splitting $(1 - \omega_1)$ in $(\frac{1}{2} - \omega_1) + \frac{1}{2}$, using the fact that $\Theta'_{\mu_k,\sigma_k}(x_k; d_k) \leq -d_k^\top M_k d_k$ (see (4.2)), the nonnegativity of r, $\omega_1 < 1/2$, the positive definiteness of L_*^r (which with (5.2) implies that $d_k^\top M_k d_k \geq C' \|d_k\|^2$, for some constant $C' > 0$), and (5.2), we obtain

$$\begin{aligned} &\Theta_{\mu_k,\sigma_k}(x_k + d_k) - \Theta_{\mu_k,\sigma_k}(x_k) - \omega_1 \Theta'_{\mu_k,\sigma_k}(x_k; d_k) \\ &\quad \leq \left(\frac{1}{2} - \omega_1\right) \left(-d_k^\top M_k d_k\right) - \frac{1}{2} d_k^\top \left(M_k - L_*^r\right) d_k + \varepsilon O(\|d_k\|^2) + o(\|d_k\|^2) \\ &\quad \leq -C \|d_k\|^2 + \varepsilon O(\|d_k\|^2) + o(\|d_k\|^2), \end{aligned}$$

for some constant $C > 0$. Since the right hand side is negative when k is large and ε is sufficiently small, the proposition is proved. □

Proposition 5.1 suggests a way of updating the parameters μ_k, σ_k, and $\underline{\sigma}_k$: μ_k should be close to λ_k^{QP} and σ_k should be kept small. The latter condition may require decreasing $\underline{\sigma}_k$.

In order to ensure convergence, we allow μ_k to change only when the iterates make sufficient progress towards a local solution. This is measured by the following quantity:

$$\varepsilon_k = \min(\|g_k\| + \|c_k\|_P, \varepsilon_{k-1}), \quad k \geq 0 \tag{5.3}$$

($\epsilon_{-1} = \|g_0\| + \|c_0\|_P$). It follows that the sequence $\{\varepsilon_k\}$ is nonincreasing and that $\lim_{k\to\infty} \varepsilon_k = 0$ if and only if $\liminf_{k\to\infty} \|g_k\| + \|c_k\|_P = 0$.

Now, suppose that a new iterate x_{k+1} has been computed by the PLS algorithm. Recall that $\alpha_{k,0}$ is the first stepsize along the direction d_k at which the Armijo condition (3.12) is satisfied. Our update rule for $\underline{\sigma}_k$ uses two constants $a_1 > 1$ and $a_2 > 1$. Let k' be the index of the last iteration at which $\underline{\sigma}_k$ has been updated. Initially k' is set to 0 in the overall algorithm.

UPDATE RULE OF $\underline{\sigma}_k$:

if $\varepsilon_{k+1} \leq \varepsilon_{k'}/a_1$ and $\alpha_{k,0} \neq 1$
then $k' = k+1$ and $\underline{\sigma}_{k+1} = \underline{\sigma}_k/a_2$
else $\underline{\sigma}_{k+1} = \underline{\sigma}_k$.

In other words, $\underline{\sigma}_k$ is decreased when the iterates progress to a local solution, although the unit stepsize is not accepted along the direction d_k.

Recall that σ_{k,i_k-1} used in the update rule below is the value of the penalty parameter at the end of the PLS. Let $a_3 > 1$ be another constant. The rule below updates an index k'', initially set to 0 in the overall algorithm. It is the index of the last iteration at which μ_k has been set to λ_k^{QP}.

UPDATE RULE OF μ_k AND σ_k:

if $\varepsilon_{k+1} \leq \varepsilon_{k''}/a_3$ **then** $k'' = k+1$, $\mu_{k+1} = \lambda_{k+1}^{\text{QP}}$, and $\sigma_{k+1} = \underline{\sigma}_{k+1}$,
else $\mu_{k+1} = \mu_k$ and set σ_{k+1} according to:

if $\underline{\sigma}_{k+1} + \|\lambda_{k+1}^{\text{QP}} - \mu_{k+1}\|_D \leq \sigma_{k,i_k-1}$ **then** $\sigma_{k+1} = \sigma_{k,i_k-1}$,
else $\sigma_{k+1} = \max(2\sigma_{k,i_k-1}, \underline{\sigma}_{k+1} + \|\lambda_{k+1}^{\text{QP}} - \mu_{k+1}\|_D)$.

Note that, when μ_k is unchanged, the parameter σ_k is updated by the same rule as $\sigma_{k,i}$ in Section 4.1. It follows that as long as μ_k is kept constant, the sequence

$$\sigma_{k-1,i_{k-1}-1}, \quad \sigma_{k,0}(=\sigma_k), \quad \sigma_{k,1}, \ldots, \sigma_{k,i_k-1}, \quad \sigma_{k+1,0}(=\sigma_{k+1}), \quad \sigma_{k+1,1}, \ldots$$

is nondecreasing and satisfies (4.4) or (4.5).

Let us consider now, the admissibility of the unit stepsize for the reduced Wolfe condition.

Proposition 5.2. *Let $x_* \in \Omega$ satisfying the optimality condition* (1.3). *Suppose that $A(\cdot)$ and $Z^-(\cdot)$ are continuous at x_* and that g is continuously differentiable on a convex neighborhood of x_*. Suppose also that $x_k \to x_*$, $d_k \to 0$, $\omega_2 > 0$, $\{M_k\}$ and $\{M_k^{-1}\}$ are bounded*, $c_k = O(\|g_k\|)$, *and*

$$\left(Z_k^{-\top} M_k - Z_*^{-\top} L_*\right) d_k = o(\|d_k\|). \tag{5.4}$$

Then, *when k is sufficiently large*, *the unit stepsize is accepted by the reduced Wolfe condition*:

$$g(x_k + d_k)^\top \hat{Z}_k d_k \geq \omega_2 g(x_k)^\top \hat{Z}_k d_k.$$

Proof: By the continuity of $Z^-(\cdot)$ and the boundedness of $\{M_k\}$, the matrices $H_k^{-1} = Z_k^{-\top} M_k Z_k^-$ form a bounded sequence, so that the eigenvalues of H_k are bounded away from zero when $k \to \infty$. Similarly, the boundedness of $\{M_k^{-1}\}$ and the fact that there is a constant $C > 0$ such that $\|Z_k^- u\| \geq C\|u\|$ imply that $\{H_k\}$ is bounded. Now, using formula (4.7), the continuity of $A(\cdot)$ and the boundedness of $\{M_k\}$ and $\{M_k^{-1}\}$, we deduce that $\{\hat{A}_k^-\}$ is bounded.

By the boundedness of $\{Z_k^-\}$, $\{H_k\}$, $\{\hat{A}_k^-\}$, and the estimate $c_k = O(\|g_k\|)$, we have that $d_k = -Z_k^- H_k g_k - \hat{A}_k^- c_k = O(\|g_k\|)$. Also, by (1.6), we have $Z_k^{-\top} M_k d_k = -g_k$. Then, using (2.6) and next (5.4), we obtain

$$\begin{aligned} g(x_k + d_k) &= g_k + \nabla g(x_*)^\top d_k + o(\|d_k\|) \\ &= -\left(Z_k^{-\top} M_k - Z_*^{-\top} L_*\right) d_k + o(\|d_k\|) \\ &= o(\|g_k\|). \end{aligned}$$

Since $\hat{Z}_k d_k = -H_k g_k = O(\|g_k\|)$, we finally have

$$g(x_k + d_k)^\top \hat{Z}_k d_k - \omega_2 g_k^\top \hat{Z}_k d_k = \omega_2 g_k^\top H_k g_k + o(\|g_k\|^2),$$

which is positive for k large, by the boundedness of $\{H_k^{-1}\}$ and $\omega_2 > 0$. □

Proposition 5.2 is important for designing the overall algorithm in an efficient manner, because it suggests that a criterion should be used to decide when the PLS algorithm should be launched. Suppose indeed that, contrary to what is required by the hypotheses of this proposition $c_k \neq O(\|g_k\|)$, say for $k \in \mathcal{K}$. If the PLS algorithm is launched at all iterations $k \in \mathcal{K}$, it will try to satisfy the reduced Wolfe condition (3.4), although it is not guaranteed that it will succeed with a unit stepsize, even asymptotically. Therefore, the overall algorithm may fail to converge superlinearly. On the other hand, $\{\|g_k\|/\|c_k\|\}_{k\in\mathcal{K}}$ tends to zero, which means that x_k approaches x_* approximately tangently to the reduced gradient manifold $\{g = 0\}$. In this case, the displacement d_k has a small longitudinal component, so that the information that one can draw on the tangent part of the Hessian of the Lagrangian is likely to be imprecise. In this case, updating M_k by adjusting the parameter r_k only looks perfectly adequate. It is also feasible, since $c_k \neq 0$ for $k \in \mathcal{K}$. These observations suggest not launching a PLS when $c_k \neq O(\|g_k\|)$. This leads to the following criterion ($K > 0$ is a constant).

PLS CRITERION:

if $\|c_k\| \leq K\|g_k\|$ **then** call the PLS algorithm,
else perform only the first inner iteration of the PLS algorithm.

It follows that whenever $\|c_k\| > K\|g_k\|$, the next iterate x_{k+1} is set to $x_{k,1}$, which only satisfies condition (3.11) and the Armijo inequality (3.12), but not necessarily the reduced Wolfe condition (3.4).

Note finally that the conditions (5.2) and (5.4) on the updated matrix M_k, used in Propositions 5.1 and 5.2 are both satisfied when $(M_k - L_*^r)d_k = o(\|d_k\|)$, which is a reasonable condition to expect from the quasi-Newton theory.

5.2. Global convergence

We are now in position to give a complete description of our BFGS version of the SQP algorithm.

OVERALL ALGORITHM

0. Choose three constants $a_i > 1$ ($i = 1, 2, 3$) for the update of μ_k, σ_k, and $\underline{\sigma}_k$; a constant $K > 1$ for the PLS criterion; and constants $\omega_1 \in (0, \frac{1}{2})$, $\omega_2 \in (0, 1)$, and $\rho \in (0, \frac{1}{2}]$ for the PLS algorithm.
   Choose a starting point $x_0 \in \Omega$ and an initial symmetric positive definite matrix $M_0 \in \mathbb{R}^{n\times n}$.
   Set $k = k' = k'' = 0$ (the indices k' and k'' are reset by the update rules of μ_k, σ_k, and $\underline{\sigma}_k$).
   Solve the SQP subproblem (1.5) (with $k = 0$) giving $(d_0, \lambda_0^{\text{QP}})$.
   Choose $\underline{\sigma}_0 > 0$, set $\mu_0 = \lambda_0^{\text{QP}}$ and $\sigma_0 = \underline{\sigma}_0$.
1. **if** $\|c_k\| \leq K\|g_k\|$ **then** call the PLS algorithm,
   else perform only the first inner iteration of the PLS algorithm.
   This gives a new iterate x_{k+1}.
2. Compute γ_k and δ_k by formula (3.2) and (3.14), where $r_k \geq 0$ is set such that $\gamma_k^\top \delta_k > 0$, and update M_{k+1} by the BFGS formula (1.7).
3. Solve the SQP subproblem giving $(d_{k+1}, \lambda_{k+1}^{\text{QP}})$.
4. Update μ_{k+1}, σ_{k+1}, and $\underline{\sigma}_{k+1}$ by the rules given in Section 5.1.
5. Increase k by 1 and go to 1.

A possible way of determining r_k in Step 2 is described in Section 6.1. Our global convergence result below does not require that we specify the update rule of r_k, since its value intervenes in the update of the matrices M_k only and that $\{M_k\}$ and $\{M_k^{-1}\}$ are supposed bounded.

Below, we denote by $\text{dist}(x, \Omega^c)$ the Euclidean distance between a point x and the complementary set of Ω.

Theorem 5.3. *Suppose that Ω is convex, that f and c are differentiable on Ω with Lipschitz continuous derivatives, and that $Z^-(\cdot)$ is bounded on Ω. If the overall algorithm above generates a sequence $\{x_k\}$, using a bounded sequence of matrices $\{M_k\}$ with bounded inverses, then one of the following situations occurs.*

(i) *The algorithm converges in the sense that*

$$\liminf_{k\to\infty}(\|g_k\| + \|c_k\|_P) = 0.$$

(ii) *There exists k_0 such that for all $k \geq k_0$, $\mu_k = \mu$, and either*

- (a) *the set $\{\sigma_{k,i} : k \geq 0, 0 \leq i < i_k\}$ is unbounded, implying that the set $\{\lambda_k^{\text{QP}}(x_{k,i}) : k \geq 0, 0 \leq i < i_k\}$ is also unbounded, or*
- (b) *there exists $k_1 \geq k_0$ such that $\sigma_{k,i} = \sigma$, for all $k \geq k_1$ and $0 \leq i < i_k$, in which case $\Theta_{\mu,\sigma}(x_k) \to -\infty$ or $\liminf_{k\to\infty} \text{dist}(x_k, \Omega^c) = 0$.*

Proof: Suppose that conclusion (i) does not occur. By the definition (5.3) of ε_k, this is equivalent to $\lim_{k\to\infty} \varepsilon_k > 0$. The update rules of μ_k, σ_k and $\underline{\sigma}_k$, given in Section 5.1, and (4.4) and (4.5) imply that there exists an index k_0, such that for all $k \geq k_0$, $\mu_k = \mu$, $\underline{\sigma}_k = \underline{\sigma}$ and the sequence $\sigma_{k,0} (= \sigma_k)$, $\sigma_{k,1}, \ldots, \sigma_{k,i_k-1}$, $\sigma_{k+1,0} (= \sigma_{k+1})$, $\sigma_{k+1,1}, \ldots,$ is nondecreasing. This sequence is either unbounded, in which case conclusion (ii-a) follows, or there exists $k_1 \geq k_0$ such that $\sigma_{k,i} = \sigma$ for all $k \geq k_1$ and $0 \leq i < i_k$.

It remains to prove that in the latter case the alternative given in (ii-b) holds. This is done by contradiction, assuming that $\Theta_{\mu,\sigma}(x_k)$ is bounded from below and that $\liminf_{k\to\infty} \text{dist}(x_k, \Omega^c) > 0$.

Since the Armijo inequality (3.12) holds at each inner iteration of the PLS algorithm (conclusion (i-b) of Theorem 4.4), we have

$$\Theta_{\mu,\sigma}(x_{k+1}) \leq \Theta_{\mu,\sigma}(x_k) + \omega_1 \sum_{i=0}^{i_k-1} \alpha_{k,i} \Theta'_{\mu,\sigma}(x_{k,i}; d_{k,i}).$$

Note that this inequality holds even if the PLS criterion is not satisfied (Step 1 of the overall algorithm). Recall that we have denoted $x_{k,0} = x_k$ and $d_{k,0} = d_k$. Using the fact that $d_{k,i}$ is a descent direction of $\Theta_{\mu,\sigma}$ at $x_{k,i}$, the previous inequality gives

$$\Theta_{\mu,\sigma}(x_{k+1}) \leq \Theta_{\mu,\sigma}(x_k) + \omega_1 \alpha_{k,0} \Theta'_{\mu,\sigma}(x_k; d_k).$$

With (4.2), this inequality implies

$$0 \leq \omega_1 \alpha_{k,0} \big(d_k^\top M_k d_k + \underline{\sigma} \|c_k\|_P\big) \leq \Theta_{\mu,\sigma}(x_k) - \Theta_{\mu,\sigma}(x_{k+1}).$$

Adding over k and using the boundedness assumption on $\Theta_{\mu,\sigma}(x_k)$, we deduce the convergence of the series

$$\sum_{k\geq 0}\alpha_{k,0}d_k^\top M_k d_k < \infty \quad \text{and} \quad \sum_{k\geq 0}\alpha_{k,0}\|c_k\|_P < \infty. \tag{5.5}$$

If $\liminf \alpha_{k,0} > 0$, then the convergence of these series and the boundedness of $\{M_k^{-1}\}$ would imply that $d_k \to 0$ and $c_k \to 0$, and since $g_k = -Z_k^{-\top}M_k d_k$ (see (2.16)) we would have $(\|g_k\|+\|c_k\|_P) \to 0$, in contradiction with our initial assumption. Thus, a subsequence $\{\alpha_{k,0}\}_{k\in\mathcal{K}}$ converges to 0. This means that either condition (3.12) or the Armijo condition (3.12) is not accepted for some $\alpha = \alpha_{k,0,j_k} \leq \rho^{-1}\alpha_{k,0}$ (Step 1 of the PLS algorithm). This can be written

$$x_k + \alpha_{k,0,j_k}d_k \notin \Omega \tag{5.6}$$

or

$$\Theta_{\mu,\sigma}(x_k + \alpha_{k,0,j_k}d_k) > \Theta_{\mu,\sigma}(x_k) + \omega_1\alpha_{k,0,j_k}\Theta'_{\mu,\sigma}(x_k; d_k). \tag{5.7}$$

Let us show that (5.6) does not hold for large k. Using the convergence of the first series in (5.5), the boundedness of $\{M_k^{-1}\}$, and $\alpha_{k,0} \leq 1$, we have $\alpha_{k,0}\|d_k\| \to 0$. But for $k \in \mathcal{K}$, $\alpha_{k,0,j_k} \leq \rho^{-1}\alpha_{k,0}$, so that $\alpha_{k,0,j_k}\|d_k\| \to 0$, which with (5.6) implies that $\liminf \operatorname{dist}(x_k, \Omega^c) \to 0$, in contradiction with our assumptions.

Now, we show that (5.7) leads to a contradiction, which will prove the theorem. Expanding f and c at the first order and using the Lipschitz continuity of ∇f and ∇c, we obtain the following estimates, when $\alpha \in (0, 1]$ and $x_k + \alpha d_k \in \Omega$:

$$\begin{aligned} f(x_k + \alpha d_k) &\leq f_k + \alpha\nabla f_k^\top d_k + C\alpha^2\|d_k\|^2, \\ \mu^\top c(x_k + \alpha d_k) &\leq (1-\alpha)\mu^\top c_k + C\alpha^2\|d_k\|^2, \end{aligned}$$

and

$$\|c(x_k + \alpha d_k)\|_P \leq (1-\alpha)\|c_k\|_P + C\alpha^2\|d_k\|^2,$$

where C denotes a constant independent of k. Then,

$$\begin{aligned} \Theta_{\mu,\sigma}(x_k + \alpha d_k) &= f(x_k + \alpha d_k) + \mu^\top c(x_k + \alpha d_k) + \sigma\|c(x_k + \alpha d_k)\|_P \\ &\leq f_k + \alpha\nabla f_k^\top d_k + (1-\alpha)\mu^\top c_k + (1-\alpha)\sigma\|c_k\|_P + C\alpha^2\|d_k\|^2 \\ &\leq \Theta_{\mu,\sigma}(x_k) + \alpha\Theta'_{\mu,\sigma}(x_k; d_k) + C\alpha^2\|d_k\|^2. \end{aligned}$$

The last inequality and (5.7) imply

$$0 < (1-\omega_1)\Theta'_{\mu,\sigma}(x_k; d_k) + C\alpha_{k,0,j_k}\|d_k\|^2.$$

Finally, using the inequality $\Theta'_{\mu,\sigma}(x_k; d_k) \leq -d_k^\top M_k d_k - \underline{\sigma}\|c_k\|_P$, the boundedness of $\{M_k^{-1}\}$, and next $\alpha_{k,0,j_k} \to 0$, when $k \to \infty$ in $\mathcal{K}$, we obtain a contradiction

$$0 < -(1-\omega_1)\underline{\sigma}\|c_k\|_P - C\|d_k\|^2 \leq 0.$$

This contradiction concludes the proof. □

Note that in outcome (i) of Theorem 5.3, we only have a subsequence of the iterates x_k, for which $\|g_k\| + \|c_k\|_P \to 0$. This is the price that the algorithm pays to allow for a decrease of the penalty parameter σ_k. A local analysis of the algorithm could show that all the sequence (c_k, g_k) actually tends to zero.

We conclude this section by a result specifying the conditions under which the unit stepsize is accepted asymptotically in the overall algorithm.

Proposition 5.4. *Let $x_* \in \Omega$ satisfying the optimality condition* (1.3) *and* (1.4). *Suppose that $A(\cdot)$ and $Z^-(\cdot)$ are continuous at x_* and that f and c are twice continuously differentiable and g is continuously differentiable on a convex neighborhood of x_*. If the overall algorithm generates a sequence $\{x_k\}$ converging to x_*, such that $d_k \to 0$, $\{M_k\}$ and $\{M_k^{-1}\}$ are bounded, and $(M_k - L_*^r)d_k = o(\|d_k\|)$ for some $r \geq 0$ such that L_*^r is positive definite. Then, for sufficiently large k, there is only one inner iteration in the PLS algorithm* (*i.e.*, $i_k = 1$) *and the unit stepsize is accepted by the Armijo inequality* (*i.e.*, $\alpha_{k,0} = 1$).

Proof: Since x_k converges to a local solution of the problem, then $g_k \to 0$ and $c_k \to 0$. Thus $\varepsilon_k \to 0$ and $\lambda_k^{\mathrm{QP}} \to \lambda_*$ (use (1.6) and $M_k d_k \to 0$).

Note first that $i_k = 1$ for large k, either because the PLS criterion does not hold or because of Proposition 5.2.

Suppose now that $\alpha_{k,0} \neq 1$ for a subsequence of iterates. Then, the update rule of $\underline{\sigma}_k$ implies the convergence $\underline{\sigma}_k \to 0$ (it is here that the index k' is useful). In the same way, the update rule of μ_k and σ_k implies $\mu_k \to \lambda_*$ and $\sigma_k \to 0$ (usefulness of the index k''). It follows from Proposition 5.1 that $\alpha_{k,0} = 1$ for large k, contradicting our initial assumption. □

This result shows that, under reasonable assumptions, the PLS will finally act like a standard Armijo line-search close to a solution. As a result, this technique is essentially useful away from a solution, where nonconvexity can be encountered.

6. Implementation issues

In this section, we discuss some issues related to the implementation of the algorithm described in the previous sections.

6.1. *Computation of r_k*

Formula (2.10) shows that only the part $Z^-(x)^\top M_k$ of M_k plays a role in the determination of the SQP direction and that this direction is well defined provided $Z^-(x)^\top M_k Z^-(x)$ is nonsingular. In this case, because of (2.1), adding a positive multiple of $A_k^\top A_k$ to M_k does not modify the SQP direction. In our case, r_k is aimed at forcing M_{k+1} to approach the Hessian of the augmented Lagrangian $L_* + r_k A_*^\top A_*$. But since the matrix M_{k+1} depends nonlinearly on r_k, via the vector γ_k and the BFGS formula (1.7), the value of r_k affects d_{k+1}. This discussion suggests, however, that this value could be set from considerations based only on the matrix update.

In the algorithm below, we choose r_k in order to minimize an estimate of the condition number of the matrix M_{k+1}. This estimate is the function

$$\omega(M) = \frac{\operatorname{tr} M}{\det M^{1/n}}$$

introduced by Dennis and Wolkowicz [13]. Interestingly enough, minimizing $\omega(M)$ on a subset $\mathcal{S}$ of the set of positive definite matrices is equivalent to minimizing in $(\zeta, M) \in (0, +\infty) \times \mathcal{S}$ the function $\psi(\zeta M)$, where $\psi(M) = \operatorname{tr}(M) - \ln\det(M)$ is the condition number estimate introduced by Byrd and Nocedal [6]. In both cases, one tries to find the matrix $M \in \mathcal{S}$, in a certain sense the closest to the set $\{\zeta I : \zeta > 0\}$ of positive definite matrices with unit condition number.

The next proposition analyzes the problem of minimizing $\omega(M_{k+1})$ with respect to r_k.

Proposition 6.1. *Let η, δ, and $\tilde{\gamma}$ be vectors in $\mathbb{R}^n$ such that $\eta^\top\delta > 0$ and let $\gamma(r) = \tilde{\gamma} + r\eta$, where r is a scalar parameter belonging to the interval $\mathcal{R} = \{r : \gamma(r)^\top\delta > 0\} = (-\tilde{\gamma}^\top\delta/\eta^\top\delta, +\infty)$. Let M be a positive definite matrix and let $M(r)$ be the matrix obtained by the BFGS formula, using $\gamma(r)$ and δ:*

$$M(r) = M - \frac{M\delta\delta^\top M}{\delta^\top M\delta} + \frac{\gamma(r)\gamma(r)^\top}{\gamma(r)^\top\delta}.$$

Then, the function $r \mapsto \omega(M(r))$ is uniquely minimized on $\mathcal{R}$ by

$$\bar{r} = \frac{\bar{t} - \tilde{\gamma}^\top\delta}{\eta^\top\delta}, \tag{6.1}$$

where

$$\bar{t} = a + (a^2 + b)^{\frac{1}{2}}, \quad a = \frac{c_1}{2(n-1)c_2}, \quad b = \frac{n+1}{n-1}\frac{c_0}{c_2}, \quad c_0 = \left\| \gamma\left(-\frac{\tilde{\gamma}^\top\delta}{\eta^\top\delta}\right)\right\|^2,$$
$$c_1 = \operatorname{tr} M - \frac{\|M\delta\|^2}{\delta^\top M\delta} + 2\left(\frac{\tilde{\gamma}^\top\eta}{\eta^\top\delta} - \|\eta\|^2\frac{\tilde{\gamma}^\top\delta}{(\eta^\top\delta)^2}\right), \quad c_2 = \frac{\|\eta\|^2}{(\eta^\top\delta)^2}.$$

Proof: Let us show that $r \mapsto \omega(M(r))$ is pseudoconvex on $\mathcal{R}$, which in particular implies that any stationary point is a global minimizer (see [26]). By using

$$\operatorname{tr} M(r) = \operatorname{tr} M - \frac{\|M\delta\|^2}{\delta^\top M\delta} + \frac{\|\gamma(r)\|^2}{\gamma(r)^\top\delta}$$

and (see [28])

$$\det M(r) = \frac{\gamma(r)^\top \delta}{\delta^\top M \delta} \det M,$$

we have $\operatorname{tr} M(r) = \varphi(\gamma(r)^\top \delta)$, where $\varphi(t) = \frac{c_0}{t} + c_1 + c_2 t$ with the c_i given in the statement of the proposition. Since c_0 is nonnegative, φ is convex on $(0, +\infty)$ and, because $r \mapsto \gamma(r)^\top \delta$ is affine, $\operatorname{tr} M(\cdot)$ is convex on $\mathcal{R}$. On the other hand, $M(r)$ is positive definite for $r \in \mathcal{R}$ and $r \mapsto \det M(r)$ is affine. Hence the function $\omega(M(\cdot))$ is the quotient of a positive convex function and a positive concave function and thus is pseudoconvex on $\mathcal{R}$ (see [26, p. 148]).

Now by using $\omega(M(r)) = (\frac{\delta^\top M \delta}{\det M})^{\frac{1}{n}} \tilde{\omega}(\gamma(r)^\top \delta)$, with $\tilde{\omega}(t) = t^{-\frac{1}{n}} \varphi(t)$, a straightforward calculation shows that $\bar{t}$ is the unique stationary point of $\tilde{\omega}(\cdot)$ on $(0, +\infty)$, hence $\bar{r}$ is the unique global minimum of $\omega(M(\cdot))$ on $\mathcal{R}$. □

As we said above, setting $\eta = A_k^\top A_k \delta_k$, $\delta = \delta_k$, and $\tilde{\gamma} = \tilde{\gamma}_k = Z_k^\top (g_{k+1} - g_k) - A_k^\top (\lambda_{k+1} - \lambda_k)$ in Proposition 6.1, we take in our algorithm $r_k = \bar{r}$ given by formula (6.1). This value is not defined when $\eta_k^\top \delta_k = \|A_k \delta_k\|^2 = 0$. In this case, the discussion in Section 3.4 has shown that setting $r_k = 0$ is appropriate because the PLS alone ensures the positivity of the scalar product $\gamma_k^\top \delta_k$. From a numerical point of view, the difficulty is now to decide when $\|A_k \delta_k\|$ is sufficiently small so that r_k can be set to zero in formula (3.2), while preserving the positivity of $\gamma_k^\top \delta_k$.

To determine how small $\|A_k \delta_k\|$ must be, we compare $\tilde{\gamma}_k^\top \delta_k^{\mathrm{z}}$ to $\tilde{\gamma}_k^\top \delta_k^{\mathrm{A}}$, where δ_k^{z} and δ_k^{A} are the longitudinal and transversal components of δ_k, respectively:

$$\delta_k^{\mathrm{z}} = -\alpha_k^{\mathrm{z}} Z_k^- H_k g_k \quad \text{and} \quad \delta_k^{\mathrm{A}} = -\alpha_k^{\mathrm{A}} \hat{A}_k^- c_k.$$

Suppose that the reduced Wolfe condition (3.4) holds. Then $\tilde{\gamma}_k^\top \delta_k^{\mathrm{z}} = \alpha_k^{\mathrm{z}} (g_{k+1} - g_k)^\top \hat{Z}_k d_k$ is positive and when

$$\tilde{\gamma}_k^\top \delta_k^{\mathrm{A}} \geq -\beta \tilde{\gamma}_k^\top \delta_k^{\mathrm{z}}, \tag{6.2}$$

for some constant $\beta \in (0, 1)$, we have $\tilde{\gamma}_k^\top \delta_k = \tilde{\gamma}_k^\top \delta_k^{\mathrm{z}} + \tilde{\gamma}_k^\top \delta_k^{\mathrm{A}} \geq (1 - \beta) \tilde{\gamma}_k^\top \delta_k^{\mathrm{z}} > 0$, so that r_k can be set to zero. Therefore, we adopt the following rule for the update of r_k.

COMPUTATION OF r_k:

if (3.4) and (6.2) hold **then** $r_k = 0$,
else $r_k = \max(0, \bar{r}_k)$, where $\bar{r}_k$ is given by (6.1), in which $\eta = A_k^\top A_k \delta_k$, $\delta = \delta_k$, and $\tilde{\gamma} = Z_k^\top (g_{k+1} - g_k) - A_k^\top (\lambda_{k+1} - \lambda_k)$.

In our numerical experiment, we set $\beta = 0.1$ in (6.2).

Since $\tilde{\gamma}_k$ aims at approximating $L_* \delta_k$ (see (3.1)), (6.2) may be seen as a way of comparing the curvatures of the Lagrangian along δ_k^{A} and along δ_k^{z}. The rule above for computing r_k can be read as follows: if the curvature of the Lagrangian in the transversal direction is positive or not too negative (with respect to the longitudinal curvature), set r_k to zero, otherwise use formula (6.1) to set r_k to a positive value. One can also say that (6.2) is a way of measuring the smallness of $\|A_k\delta_k\|$, since when this quantity vanishes, $\delta_k^{\wedge} = 0$ and (6.2) readily holds.

6.2. *Using a QR factorization of $A^\top$*

In our experiment, the matrices $A^-(x)$ and $Z^-(x)$ described in Section 2 are obtained from a QR factorization of $A(x)^\top$:

$$A(x)^\top = (\, Y^-(x) \quad Z^-(x)\,) \begin{pmatrix} R(x) \\ 0 \end{pmatrix} = Y^-(x)R(x),$$

where $Y^-(x)$ and $Z^-(x)$ are respectively $n \times m$ and $n \times (n-m)$ matrices, such that $(Y^-(x) \quad Z^-(x))$ is orthogonal, and $R(x)$ is an order m upper triangular matrix. Clearly, the columns of $Z^-(x)$ span the null space of $A(x)$, as desired. We choose as right inverse of $A(x)$ the Moore-Penrose pseudo-inverse $A(x)^\top(A(x)A(x)^\top)^{-1}$, which can be computed by

$$A^-(x) = Y^-(x)R(x)^{-\top}.$$

With this choice for Z^- and A^-, it follows that $Z(x) = Z^-(x)^\top$.

6.3. *Weighted augmentation*

Byrd, Tapia and Zhang [8, p. 216] emphasize that, due to the augmentation term $A_k^\top A_k$ in the vector γ_k, badly scaled constraints may have some negative effects on the next updated matrix. These authors prefer using a weighted augmentation term $A_k^\top W_k A_k$, where W_k is an order m weighting matrix. They suggest using $W_k = (A_k A_k^\top)^{-1}$, because with the notation of Section 6.2,

$$A_k^\top W_k A_k = A_k^\top(A_k A_k^\top)^{-1} A_k = Y_k^- Y_k^{-\top}$$

is a well conditioned matrix.

It is clear that the same technique can be adopted in our algorithm. Therefore, in our experiment we replace the augmentation term $A_k^\top A_k \delta_k$ by $Y_k^- Y_k^{-\top}\delta_k$.

6.4. *Scaling the tangential direction*

It is shown in [20] that the number of inner iterations in the PLS algorithm can be reduced by using a scaling factor $\tau_{k,i} > 0$ on the longitudinal component of the inner search direction $d_{k,i}$. This leads to redefining the direction $d_{k,i}$ as

$$d_{k,i} = -\tau_{k,i} Z_{k,i}^- H_k g_k - \hat{A}_{k,i}^- c_{k,i}.$$

For $i = 0$, we set $\tau_{k,0} = 1$, so that when a unit stepsize is accepted ($i_k = 1$ and $\alpha_{k,0} = 1$), a plain SQP step is taken and superlinear convergence of the overall algorithm may occur.

With this change, the vector δ_k is still given by (3.14), but the longitudinal stepsize α_k^{Z} has to be computed by

$$\alpha_k^{\mathrm{Z}} = \sum_{i=0}^{i_k-1} \alpha_{k,i}\tau_{k,i}.$$

It is not difficult to extend the finite termination result of Section 4.2 (Theorem 4.4), when $d_{k,i}$ is given as above, provided $\tau_{k,i}$ is maintained in a fixed interval: $0 < \underline{\tau}_k \le \tau_{k,i} \le \bar{\tau}_k$ for all $i \ge 0$. The global convergence result (Theorem 5.3) is not affected by the scaling factors $\tau_{k,i}$, since $\tau_{k,0} = 1$ and only the progress obtained by first inner iteration of the PLS is used in the convergence proof.

6.5. *Speeded-up PLS technique*

There is another way of speeding up the PLS algorithm that is useful in practice to reduce the number of inner iterations (this is discussed in [20]). It consists in resetting the right hand side of the Wolfe inequality (3.13) to $\omega_2 g_{k,i}^\top \hat{Z}_k d_k$, when the current iterate $x_{k,i}$ makes $g_{k,i}^\top \hat{Z}_k d_k$ more negative than at all the previous inner iterations. Specifically, it consists in replacing the reduced Wolfe condition (3.4) by the less demanding inequality

$$g_{k+1}^\top \hat{Z}_k d_k \ge \omega_2 \min_{0\le i<i_k} g_{k,i}^\top \hat{Z}_k d_k. \tag{6.3}$$

Since this inequality is more rapidly satisfied than (3.4), the finite termination result (Theorem 4.4) is still valid. Also, the global convergence theorem still holds, since the Wolfe condition (3.4) plays no role in its proof.

Let us denote by l_k the greatest index i realizing the minimum in the right hand side of (6.3). When this condition is used in place of (3.4) and $l_k \ne 0$, the vectors γ_k and δ_k have to be modified. Aiming at having $\delta_k \simeq x_{k+1} - x_{k,l_k}$, the same reasoning as in Sections 3.1 and 3.4 shows that it is appropriate to set (we take into account the suggestions given in Sections 6.3 and 6.4):

$$\gamma_k = Z_{k,l_k}^\top \big(g_{k+1} - g_{k,l_k}\big) - A_{k,l_k}^\top \big(\lambda_{k+1} - \lambda_{k,l_k}\big) + r_k Y_{k,l_k}^- Y_{k,l_k}^{-\top} \delta_k$$
$$\delta_k = -\alpha_{k,l_k}^{\mathrm{Z}} Z_{k,l_k}^- H_k g_k - \alpha_{k,l_k}^{\mathrm{A}} \hat{A}_{k,l_k}^- c_{k,l_k},$$

where $\alpha_{k,l_k}^{\mathrm{Z}}$ and $\alpha_{k,l_k}^{\mathrm{A}}$ are defined by

$$\alpha_{k,l_k}^{\mathrm{Z}} = \sum_{i=l_k}^{i_k-1} \alpha_{k,i}\tau_{k,i} \quad \text{and} \quad \alpha_{k,l_k}^{\mathrm{A}} = \sum_{i=l_k}^{i_k-1} \alpha_{k,i} e^{-\xi_{k,i}},$$

with $\xi_{k,i} = \sum_{j=l_k}^{i-1} \alpha_{k,j}$. Note that when $l_k = 0$ and $\tau_{k,i} = 1$, we recover the preceding formulae (3.2), (3.14), and (3.15).

Observe finally that when $A_{k,l_k}\delta_k = 0$, then $c_{k,l_k} = 0$ and we have

$$\begin{aligned}\gamma_k^\top\delta_k &= (g_{k+1} - g_{k,l_k})^\top Z_{k,l_k}\delta_k\\ &= \alpha^{\mathrm{Z}}_{k,l_k}(g_{k+1} - g_{k,l_k})^\top \hat{Z}_k d_k\\ &> 0,\end{aligned}$$

by (6.3). Hence, one can always have $\gamma_k^\top\delta_k > 0$ by adjusting the value of r_k.

7. Numerical experiment

Our experiments were performed on a Power Macintosh 6100/66 in Matlab (release 4.2c.1) with a machine epsilon of about 2×10^{-16}. We have taken the same list of test problems as in the paper of Byrd, Tapia, and Zhang [8]. In each case, the standard starting points (those given in [25, 33]) were used, except for problems 12, 316–322, 336 and 338, for which we have used $x_0 = 10^{-4}(1, \dots, 1)$, since the Jacobian matrix is rank deficient at the standard initial point $x_0 = 0$, which is not supported by our theory.

Three updating methods have been tested:

- `Powell`: Powell's corrections, in which, at each iteration, γ_k is set to γ_k^{P} given by formula (1.10), with θ calculated as described after this formula ($\eta = 0.1$);
- `BTZ`: the Byrd, Tapia, and Zhang approach, in which γ_k is given by formula (1.11), the rules (1.12) and (1.13), with $\nu_{\mathrm{BTZ}} = \beta_{\mathrm{BTZ}} = 0.01$, and the weighted augmentation described in Section 6.3;
- `PLS`: the algorithm presented in this paper.

All the methods use the same merit function (2.18), with $\|\cdot\|_P = \|\cdot\|_1$, and the same technique to update the parameters μ, σ, and $\underline{\sigma}$. The constants, used by the update rules (see Section 5.1), are set as follows: $a_1 = 1.0001$, $a_2 = 10$, $a_3 = 1.0001$, and $\underline{\sigma}_0 = \|\lambda_0\|_\infty/10$. The constant used in the Armijo inequality is set to $\omega_1 = 0.01$. For `Powell` and `BTZ` algorithms, each stepsize α_k is found by a backtracking line-search along d_k, as in Step 1 of the PLS algorithm. In Step 1.2 of the PLS algorithm, quadratic or cubic interpolation formulae are used and we set $\rho = 0.1$. The constant for the reduced Wolfe condition is $\omega_2 = 0.9$ and the one used in the PLS criterion is $K = 100\|c_0\|/\|g_0\|$. Initially, $M_0 = I$ and the pre-update scaling $\gamma_k^\top\delta_k/\|\delta_k\|^2 I$ is done before the first update by the BFGS formula. The stopping criterion for all the methods is

$$\|g_k\|_2 + \|c_k\|_2 \le \varepsilon_{tol}(\|g_0\|_2 + \|c_0\|_2), \quad \text{with } \varepsilon_{tol} = 10^{-7}.$$

The results are presented in Table 1. The columns in this table are labeled as follows: `P` is the problem number given in [25, 33], `n` is the number of variables, `m` is the number of constraints, `ng` is the number of gradient calculations and constraint linearizations (the main computation cost), `nf` is the number of function evaluations, `cr` is the number of

Table 1. Results.

		Powell			BTZ			PLS			
P	n:m	ng/nf	cr	κ_2	ng/nf	cr	κ_2	ng/nf	cr	κ_2	ii
7	2:1	14/15	2	1.	18/19	2	1.	10/10	1	1.	1
12	2:1	22/22	1	0.	28/34	1	0.	20/20	2	0.	2
43	4:2	11/12	1	1.	11/12	—	1.	9/10	—	0.	—
61	3:2	**	17	16.	42/53	5	1.	18/18	7	1.	2
63	3:2	12/13	7	10.	9/9	2	2.	8/8	5	0.	—
78	5:3	9/10	1	2.	9/10	1	1.	8/9	1	1.	—
100	7:2	14/17	—	2.	14/17	—	2.	12/15	—	2.	—
104	8:4	27/28	—	3.	27/28	—	3.	25/26	—	3.	1
316	2:1	33/34	2	0.	34/42	4	0.	32/36	2	0.	3
318	2:1	32/45	15	0.	*	183	0.	25/25	9	0.	1
319	2:1	***	20	Inf	40/66	10	1.	25/26	12	1.	1
320	2:1	***	38	Inf	32/40	3	1.	22/22	12	1.	2
321	2:1	***	157	Inf	31/37	3	2.	29/35	16	1.	4
322	2:1	38/47	22	4.	28/33	3	4.	23/25	10	3.	3
335	3:2	24/32	5	8.	25/31	1	6.	23/29	7	2.	6
336	3:2	31/32	1	3.	34/60	4	5.	29/36	2	4.	6
338	3:2	35/37	24	11.	128/334	7	9.	31/40	10	5.	7
355	4:1	35/60	3	6.	24/34	2	2.	18/21	—	2.	2
373	9:6	*	7	20.	27/29	1	14.	16/18	—	9.	—
10	2:1	11/11	—	1.	11/11	—	1.	11/12	—	1.	2
11	2:1	8/9	—	1.	8/9	—	1.	10/13	—	1.	4
26	3:1	25/26	—	4.	25/26	—	4.	25/26	1	5.	—
40	4:3	7/8	2	5.	9/9	1	8.	7/7	1	0.	1
46	5:2	36/40	—	6.	36/40	—	6.	36/40	—	6.	—
47	5:3	29/33	7	6.	26/29	2	7.	28/30	—	4.	—
60	3:1	9/10	—	1.	9/10	—	1.	10/11	—	1.	—
66	3:2	8/8	—	1.	8/8	—	1.	8/8	—	2.	—
71	4:3	6/6	—	1.	6/6	—	1.	6/6	—	2.	1
72	4:2	21/21	4	1.	21/21	—	1.	26/26	—	1.	7
77	5:2	19/20	—	1.	19/20	—	1.	19/21	2	1.	2
79	5:3	12/13	1	2.	12/13	—	2.	12/13	—	2.	—
80	5:3	7/7	—	1.	7/7	—	1.	7/7	1	1.	1
81	5:3	10/11	3	9.	9/9	2	9.	11/11	3	2.	2
93	6:2	24/26	1	3.	26/28	—	3.	27/29	—	3.	1
317	2:1	29/33	10	3.	*	183	0.	30/37	7	0.	5

(*Continued on next page.*)

Table 1. (*Continued*).

		Powell			BTZ			PLS			
P	n:m	ng/nf	cr	κ_2	ng/nf	cr	κ_2	ng/nf	cr	κ_2	ii
6	2:1	11/15	2	3.	9/10	1	2.	12/12	1	1.	2
27	3:1	19/21	4	3.	18/20	—	3.	29/36	1	3.	4
29	3:1	13/17	1	4.	9/9	1	1.	12/12	—	1.	1
39	4:2	13/13	—	2.	13/13	—	2.	20/26	—	2.	4
56	7:4	**	5	13.	10/11	2	2.	13/15	2	2.	—
65	3:1	13/17	1	2.	11/11	1	1.	15/23	—	1.	4
216	2:1	11/14	—	2.	11/14	—	2.	17/26	2	0.	3
219	4:2	15/16	—	2.	15/16	—	2.	25/33	4	3.	4
375	10:9	10/11	2	4.	99/268	7	3.	15/19	6	2.	4
106	8:6	**	7	12.	*	2	11.	*	2	0.	197

corrections of γ_k (Powell's corrections in algorithm `Powell` or $r_k > 0$ in `BTZ` and `PLS`), κ_2 gives the logarithm in base 10 of the ℓ_2 condition number of the matrix M_k at the last iteration, and `ii` is the total number of inner iterations in the PLS. A symbol '*' in the table indicates that the algorithm failed to find the solution in less than 201 linearizations. Other failures are of two kinds: either the number of backtrackings in the line-search exceeds 10 (symbol '**') or the matrix M_k is so badly conditioned that d_k is not a descent direction (symbol '***').

One can make the following comments on the results of these experiments. First, we do not pretend that this numerical experiment reflects the average behavior of the tested algorithms. It was mainly done to see whether our algorithm could be implemented and to compare it with other techniques on a small set of problems. Furthermore, the dimensions of the problems are very small, which prevents us from drawing firm conclusions. With this in mind, one can however quote the following points.

1. The four parts of the table are sorted according to the efficiency of `PLS` with respect to the two other techniques. We take the counter `ng` as a measure of efficiency, because it contributes the most to the overall computational cost. The first part of the table gathers the cases where `PLS` has a counter `ng` better (strictly lower) than the other algorithms, sometimes dramatically (see Problems 61, 318, 319, 320, 355, and 373). For a large subset of the problems (second part of the table), the efficiency of `PLS` is more or less the same as the best of the other two algorithms: the difference with the best counter `ng` does not exceed 25%. Finally, we observe that `PLS` does not always improve the efficiency of the SQP algorithm (third part of the table). The last part of the table contains the single problem on which all the algorithms fail.
2. In all the cases, except the cycling observed in problem 106, on which all the algorithms fail, the total number of inner iterations used by `PLS` is always small.
3. The technique for determining the augmentation parameter r_k in `PLS` by minimizing an estimate of the condition number of the updated matrix (Section 6.1) makes its condition

number smaller in PLS than in the other algorithms. This technique could also be used for algorithm BTZ. In order to make a clear comparison between BTZ and PLS we also ran the former using formula (6.1) of Proposition 6.1 for the computation of r_k. More precisely, we set $r_k = \max(\bar{r}_k, r_k^{\text{BTZ}})$, where $\bar{r}_k$ is the value given by (6.1) and r_k^{BTZ} is the one given by formula (1.12). Curiously, we did not observe an improvement on the counters ng and nf on average, though the condition number of the matrices M_k at the last iteration were better than those given in Table 1.

4. Finally, regarding the usual counters (ng and nf), there is no clear winner. PLS is the most efficient on 42% of the test problems (first part of the table), one of the most efficient on 36% of the test problems (second part of the table) and not the most efficient on the other problems. On the other hand, PLS is more robust in that it fails less often than Powell or BTZ.

8. Conclusion

This paper proposes a technique for maintaining the positive definiteness of the updated matrices in the quasi-Newton version of the SQP algorithm for equality constrained optimization problems. The overall algorithm generates approximations of the Hessian of the augmented Lagrangian, whose positive definiteness is obtained via the realization of a reduced Wolfe condition and an adequate setting of the augmentation parameter. The globalization is obtained by means of a nondifferentiable augmented Lagrangian function as merit function. Finite termination of the search algorithm, global convergence (assuming boundedness of the generated matrices and their inverses) and asymptotic admissibility of the unit stepsize are proved.

What is new in the proposed approach is the use of a piecewise line-search algorithm, mimicking what is done for unconstrained problems. To tell it vaguely, this algorithm ensures the positive definiteness of the matrices in the space tangent to the constraints. We believe that this is a conceptual improvement on the algorithm proposed by Byrd, Tapia, and Zhang [8], with which our method has similarities.

The numerical experiment indicates that the proposed algorithm can be more robust than existing approaches, although there is not a uniform improvement on the usual counters.

Finally, not all the facets of the algorithm have been studied. It is hoped that a precise analysis of the possibility to have superlinear convergence of the method (such as in the papers [8] and [19]) may improve the algorithm. We plan to undertake this study.

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