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Maintaining the positive definiteness of the matrices in reduced secant methods for equality constrained optimization

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We propose an algorithm for minimizing a function f on $\mathbb{R}^n$ in the presence of m equality constraints c that locally is a reduced secant method. The local method is globalized using a nondifferentiable augmented Lagrangian whose decrease is obtained by both a longitudinal search that decreases mainly f and a transversal search that decreases mainly $\|c\|$. Our main objective is to show that the longitudinal path can be designed to maintain the positive definiteness of the reduced matrices by means of the positivity of $\gamma_k^T \delta_k$, where γ_k is the change in the reduced gradient and δ_k is the reduced longitudinal displacement.

Key words: Augmented Lagrangian, constrained optimization, exact penalty function, global convergence, optimization algorithm, reduced secant method, superlinear convergence, Wolfe's step-size selection.

1. Introduction

We consider here the problem of minimizing a real-valued function f, defined on an open convex set ω in $\mathbb{R}^n$, subject to m $(m < n)$ nonlinear equality constraints c:

$$\min\{f(x): x \in \omega,\ c(x) = 0\}. \tag{1.1}$$

We shall suppose that the $m \times n$ Jacobian matrix of the constraints at a point x in ω, namely $A(x)$, is *surjective*, i.e., has full rank m. Then, if x_* is a local solution of (1.1), there exists a unique Lagrange multiplier λ_* in $\mathbb{R}^m$, such that the following first order optimality conditions are satisfied for $x = x_*$ and $\lambda = \lambda_*$ (e.g., see Fletcher (1981)):

$$c(x) = 0, \tag{1.2a}$$

$$\nabla f(x) + A(x)^T \lambda = 0. \tag{1.2b}$$

We have denoted by $\nabla f(x)$ the gradient vector of f at x.

Locally, the faster methods for solving (1.1) tackle the problem by trying to solve directly the system (1.2), whose solutions correspond to stationary points of the original problem. Two classes of local methods may be distinguished.

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The first class is formed of those algorithms whose step is an approximation of the Newton step for solving (1.2). Among them are the quasi-Newton methods, which may be introduced as follows. The Jacobian matrix of (1.2) at (x_*, λ_*) is given by

$$J_* := \begin{pmatrix} A(x_*) & 0 \\ L_* & A(x_*)^T \end{pmatrix},$$

where L_* is the Hessian according to x of the Lagrangian $l(x, \lambda) := f(x) + \lambda^T c(x)$ evaluated at (x_*, λ_*). If J_* is approximated by

$$J_k := \begin{pmatrix} A(x_k) & 0 \\ L_k & A(x_k)^T \end{pmatrix},$$

where L_k is a symmetric matrix of order n and if we denote by $\nabla_x l(x_k, \lambda_k)$ the gradient according to x of the Lagrangian, quasi-Newton methods consist in calculating approximations (x_k, λ_k) of (x_*, λ_*), by using the following scheme:

$$\begin{pmatrix} x_{k+1} \\ \lambda_{k+1} \end{pmatrix} = \begin{pmatrix} x_k \\ \lambda_k \end{pmatrix} - J_k^{-1} \begin{pmatrix} c(x_k) \\ \nabla_x l(x_k, \lambda_k) \end{pmatrix}.$$

Note that if L_k is positive definite, or only positive definite in Ker $A(x_k)$, the kernel of $A(x_k)$, i.e., $\xi^T L_k \xi > 0$ for all nonzero ξ in Ker $A(x_k)$, then J_k is non-singular and the previous iteration is well defined. This method is called the Successive Quadratic Programming (SQP) method because $x_{k+1} = x_k + d_k^{SQP}$, where d_k^{SQP} is obtained by solving successively in d the following quadratic programs:

$$\begin{aligned} &\min \quad \nabla f(x_k)^T d + \tfrac{1}{2} d^T L_k d, \\ &\text{s.t.} \quad d \in \mathbb{R}^n,\ c(x_k) + A(x_k) d = 0, \end{aligned} \tag{1.3}$$

and $\lambda_{k+1} = \lambda_{k+1}^{SQP}$, the associated multiplier. In this algorithm, L_k is updated at each iteration. This method has been extensively studied since the papers by Wilson (1963), Han (1976) and Powell (1978a,b,c) and we refer to Powell (1986) for a state of the art on the subject.

The second class of methods is based on the fact that the dimension of problem (1.1) is not n but $n-m$, which is the dimension of the manifold $M(x_*) := c^{-1}(0)$ on which f is minimized. Therefore, one may expect to find secant methods in which the updated matrices are of order $n-m$. This is certainly a realistic expectation if we force the iterates x_k to belong to the manifold $M(x_*)$, i.e., $c(x_k) = 0$ for all k. Indeed, in this case, c being a submersion, $M(x)$ is a submanifold of $\mathbb{R}^n$ (see, e.g., Leborgne (1982)) and there exists a smooth parametric representation ξ of $M(x_*)$ in a neighborhood V of x_*, i.e., a function $\xi : U \to M(x_*) \cap V$ such that $c(\xi(u)) = 0$ for all u in the open set U of $\mathbb{R}^{n-m}$. Therefore, working on the set U to minimize $f(\xi(u))$ will give the expected algorithm. But it is usually unrealistic to impose $c(x_k) = 0$ and fortunately, this is not necessary. As far as we know, the first studies on reduced secant methods (reduced because the order of the updated matrices is $n-m$ rather than n in the SQP method), without the feasibility condition ($c(x_k) = 0$),

are due to Gabay (1982b) and Coleman and Conn (1982a,b). Theoretically, the method proposed by Coleman and Conn seems better than the method studied by Gabay. Generally, the convergence of the latter is only *superlinear in two steps* (see Powell (1978c), Gabay (1982b), Byrd (1985) and Yuan (1985)), meaning that:

$$\|x_{k+1}-x_*\|/\|x_{k-1}-x_*\|\to 0, \tag{1.4}$$

while the convergence of the former has been proved to be *superlinear* (in one step) (see Byrd (1984) and Gilbert (1986 and 1989)), that is to say:

$$\|x_{k+1}-x_*\|/\|x_k-x_*\|\to 0. \tag{1.5}$$

This is a better rate of convergence than the rate of (1.4). Note that this rate of convergence can also be obtained by using the SQP method but with the necessity of updating a matrix of order n. Therefore, reduced secant methods become competitive and sometimes the only ones usable when the number m of constraints is large while the number $n-m$ of parameters remains reasonable.

In this work, we shall focus on some aspects of the method proposed by Coleman and Conn. But first, what is this method?

The presentation of the algorithm given below has been developed in Gilbert (1988 and 1989) and differs noticeably from the one given by Coleman and Conn (1982a). In particular, the formalism used here, which is due to Gabay (1982a), allows us to group together a set of methods that differ only by a choice of operators. The main idea is to build, at each point y in ω, a decomposition of $\mathbb{R}^n$ in two complementary subspaces and to do this smoothly in y. The first subspace is $\operatorname{Ker} A(y)$, the tangent space to $M(y) := c^{-1}(c(y))$ at y, for which a *basis* $Z(y)^-$ is *chosen*. This is to say that $Z(y)^-$ is an $n\times(n-m)$ matrix whose columns form a basis of $\operatorname{Ker} A(y)$. Therefore, for all y in ω, we have:

$$A(y)Z(y)^- = 0, \tag{1.6}$$

and $Z(y)^-$ is *injective*, i.e., $u\in\mathbb{R}^{n-m}$ and $Z(y)^-u=0$ imply $u=0$. The complementary subspace to $\operatorname{Ker} A(y)$ depends on the *choice* of a right inverse $A(y)^-$ of $A(y)$ and is given by $R(A(y)^-)$, the *range of* $A(y)^-$. Then, there exists a unique $(n-m)\times n$ matrix $Z(y)$ satisfying:

$$Z(y)Z(y)^- = I \quad \text{in } \mathbb{R}^{(n-m)\times(n-m)}, \tag{1.7}$$

$$Z(y)A(y)^- = 0 \quad \text{in } \mathbb{R}^{(n-m)\times m}. \tag{1.8}$$

One way to introduce the algorithm is to say that it tries to solve the system of optimality (1.2) by considering both equations separately and successively. Therefore, starting from an estimate (x_k, λ_k) of (x_*, λ_*), the next iterate (x_{k+1}, λ_{k+1}) is calculated in two steps:

$$y_k := x_k - R_k c(x_k) =: x_k + r_k, \tag{1.9}$$

$$x_{k+1} := y_k - Z(y_k)^- H_k g(y_k) =: y_k + t_k, \tag{1.10}$$

$$\lambda_{k+1} := -A(y_k)^{-\mathrm{T}}\nabla f(y_k) + A(y_k)^{-\mathrm{T}} L_k Z(y_k)^- H_k g(y_k). \tag{1.11}$$

In (1.9), r_k is the *restoration step* and R_k is the *restoration operator*, an $n \times m$ injective matrix, asymptotically close to $A(x_*)^-$. Here, we shall take $R_k := A(x_k)^-$, although $R_k := A(y_{k-1})^-$, which avoids the linearization of the constraints at x_k, is also possible without destroying superlinear convergence, but this is more difficult to analyze. In (1.10), t_k is the *tangent step* and H_k is a symmetric matrix of order $n-m$ that should be updated so as to remain positive definite (the main concern of this paper) and so as to have

$$(G_k - G_*)Z(y_k)t_k = o(\|t_k\|), \tag{1.12}$$

where $G_k := H_k^{-1}$ and G_* is the *reduced Hessian of the Lagrangian*. This is defined by

$$G_* := Z(x_*)^{-T} L_* Z(x_*)^-. \tag{1.13}$$

Condition (1.12) is a sufficient (but not necessary) condition of superlinear convergence for (x_k) (see Byrd (1984), Gilbert (1986 and 1989)). Usually, it is satisfied in practice (see Coleman and Conn (1984) and Gilbert (1988)). In (1.10), $g(y_k)$ is the *reduced gradient* of f at y_k and is defined by

$$g(y) := Z(y)^{-T}\nabla f(y).$$

Finally, in (1.11), L_k is an approximation of L_*.

Algorithm (1.9)-(1.10) differs from the one proposed by Coleman and Conn (1982a) because of their particular choice of operators. In their algorithm, $A(y)^-$ is the Moore-Penrose pseudo-inverse of $A(y)$, i.e., $A(y)^- := A(y)^T(A(y)A(y)^T)^{-1}$, and $Z(y)^-$ is an orthogonal basis of Ker $A(y)$. However, the local analysis of the method has shown that this particular choice of operators has no influence on the superlinear convergence of the method (see Hoyer (1986) and Gilbert (1989)). This means that it is possible to use this freedom of choice for other purposes.

Algorithm (1.9)-(1.11) calls for some comments. First, note that λ_k does not intervene in the calculation of x_{k+1} and λ_{k+1}. Therefore, from the superlinear convergence of the sequence (x_k, λ_k) (together), we can deduce the superlinear convergence of (x_k), while for (λ_k) we get

$$\|\lambda_{k+1} - \lambda_*\| / \|x_k - x_*\| \to 0. \tag{1.14}$$

We also see that the sequence (x_k) can be generated by (1.9)-(1.10) independently of the sequence (λ_k). We shall see, indeed, that the update of H_k does not require the knowledge of (λ_k). Therefore, algorithm (1.9)-(1.10) is a reduced method for (x_k), because the only matrix to update is H_k, which is of order $n-m$. On the other hand, the $n \times n$ matrix L_k intervenes in the calculation (1.11) of λ_{k+1}. We have said that, locally, this calculation is not necessary. However, in a global framework, some estimate of λ_* is useful. Therefore, we shall avoid the need of generating L_k by taking the following estimate:

$$\lambda(y) := -A(y)^{-T}\nabla f(y), \tag{1.15}$$

whose value at $y = x_*$ is λ_*.

The local algorithm (1.9)-(1.10) may be globalized by using a penalty function θ of the form:

$$\theta(x) := f(x) + \phi(c(x)), \tag{1.16}$$

where $\phi : \mathbb{R}^m \to \mathbb{R}_+$ satisfies $\phi(0) = 0$. If ϕ is convex, one may calculate $\theta'(x; t)$, the directional derivative of θ at x in a direction t. In particular, one finds: $\theta'(y_k; t_k) = \nabla f(y_k)^T t_k + \phi'(c(y_k); A(y_k)t_k)$ and because $t_k \in \operatorname{Ker} A(y_k)$, we have:

$$\theta'(y_k; t_k) = \nabla f(y_k)^T t_k = -g(y_k)^T H_k g(y_k) = g(y_k)^T Z(y_k) t_k. \tag{1.17}$$

This shows that it is useful to maintain H_k positive definite in order that t_k will be a descent direction of θ at y_k.

Our main objective in this paper is to design a globally and superlinearly convergent algorithm that locally reduces to the method (1.9)-(1.10) and that maintains the matrices H_k positive definite. To obtain this last property, we shall update the matrices H_k by the *inverse BFGS formula* (see, e.g., Dennis and Moré (1977)):

$$H_{k+1} = \left(I - \frac{\delta_k \gamma_k^T}{\gamma_k^T \delta_k}\right) H_k \left(I - \frac{\gamma_k \delta_k^T}{\gamma_k^T \delta_k}\right) + \frac{\delta_k \delta_k^T}{\gamma_k^T \delta_k}, \tag{1.18}$$

which we shall refer to by $H_{k+1} = \overline{\mathrm{BFGS}}(H_k, \gamma_k, \delta_k)$. In this formula, γ_k will be the change in the reduced gradient (a vector in $\mathbb{R}^{n-m}$) when making a certain displacement and δ_k is the corresponding "reduced" displacement (also a vector in $\mathbb{R}^{n-m}$). Then, it is well known that H_k transmits its positive definiteness to H_{k+1} if and only if

$$\gamma_k^T \delta_k > 0. \tag{1.19}$$

For us, this property will play a major part because we shall maintain the positive definiteness of H_k, precisely by trying to satisfy condition (1.19) *at each iteration of the algorithm.* In this paper, we shall not use other particular properties of the BFGS update. Therefore, other update formulae having the same property, such as those of the restricted Broyden's class, could be used.

This way of maintaining the positive definiteness of the generated matrices by realizing condition (1.19) is standard in unconstrained optimization. There, the vector γ_k is the change in the gradient of the objective function when the displacement $\delta_k = \rho_k d_k$ is done. Condition (1.19) is then tried to be satisfied by determining a suitable step size ρ_k in the direction d_k. Wolfe's step-size selection criteria are relevant for this determination. So, in order to realize (1.19) in our problem, we shall propose a generalization of Wolfe's criteria, which will require a particular design of the search path.

The fact that this strategy for maintaining H_k positive definite succeeds for reduced secant methods is rather comforting because it does not for SQP methods, in general. The reason for this is that, for these methods, the local analysis forces γ_k to be the change in the gradient of the Lagrangian for the displacement $\rho_k d_k^{\mathrm{SQP}}$, ρ_k is the step size. But, the Lagrangian is not necessarily bounded from below and may have a negative curvature in the direction d_k^{SQP}, even locally. Therefore, $\gamma_k^T \delta_k$ may be negative for any step size ρ_k (see Powell (1978b and 1984)).

The papers analyzing the update of matrices in reduced methods are due to Coleman and Conn (1984), Nocedal and Overton (1985) and Gilbert (1986). In the first paper the analysis is local and, as we shall see, (1.19) is automatically satisfied close to optimal points satisfying the usual second order optimality conditions, if the vectors γ_k and δ_k are correctly chosen (see below). The analysis of Nocedal and Overton, which concerns the algorithm studied by Gabay (1982b), is also local. Finally, the analysis in Gilbert (1986) is global but the reduced matrix is not updated if condition (1.19) is not satisfied. This does not prevent superlinear convergence from occurring because asymptotically (1.19) may be satisfied. However, even far from the solution it may be interesting to update the matrix in order to improve convergence. So one possibility would be to use Powell's (1978b) modification of γ_k (as for SQP methods), another one is proposed in this paper.

The paper is organized as follows. In Section 2, we specify the notation and state the hypotheses. In Section 3, we discuss the solutions adopted to realize condition (1.19) along the *longitudinal displacement* governed by the tangent step t_k. On the one hand, it is detailed how Wolfe's step-size selection procedure can be used to obtain (1.19) when the displacement is done on the manifold $M(y_k)$. On the other hand, a counter-example will show that a simple search along the direction t_k cannot assure Wolfe's criteria to be satisfied in general. However, a median solution can be obtained by extending to our problem an algorithm proposed by Lemaréchal (1981) to find a Wolfe point in unconstrained optimization. Here, a change in the direction of search is made each time an unfruitful attempt to realize (1.19) is done. Therefore, the longitudinal search path becomes piecewise linear, approximating roughly an "ideal" path on the manifold $M(y_k)$. The analysis in Section 3 is done using a penalty function having the general form (1.16). In Section 4, however, we insert the longitudinal search of Section 3 in a globally convergent algorithm by using tools that have been well developed in the specific literature since the paper by Han (1977). In particular, we motivate our choice of a nondifferentiable augmented Lagrangian

$$l_p(x, \mu) := f(x) + \mu^T c(x) + p\|c(x)\|, \tag{1.20}$$

where $\|\cdot\|$ is a norm on $\mathbb{R}^m$, as a merit function by the necessity to have a unit longitudinal step size asymptotically, being inspired in this respect by the work of Bonnans (1989). We shall also be more specific about the transversal displacement which consists of a simple linear search from x_k in the direction r_k, using Armijo's technique on the same penalty function (1.20). Finally, we give a theorem showing a global convergence result for the algorithm.

2. Hypotheses and notation

We shall suppose that ω is a convex open set of $\mathbb{R}^n$. The convexity of ω is not essential but it is assumed to discard technical problems when Taylor's theorem is

used. On the other hand, assuming ω open is essential because we do not consider here a problem with general constraints or inequality constraints. ω will be the set where f and c have nice properties. Usually, it will not be possible to take $\omega = \mathbb{R}^n$.

We suppose that on ω, f and c are sufficiently smooth, three times continuously differentiable will be enough, and that their derivatives are bounded, which can be satisfied if ω is bounded and small enough. Later, we shall suppose that the sequences (x_k) and (y_k) remain in ω, so, this supposes implicitly the boundedness of these sequences. We shall also suppose that c is a *submersion* on ω, i.e., that $A(y)$ is surjective for all y in ω. This is a rather strong hypothesis but a useful one because it allows to make the decomposition of $\mathbb{R}^n$ at each point y of ω in $\operatorname{Ker} A(y) = R(Z(y)^-)$ (see (1.6)) and $\operatorname{Ker} Z(y) = R(A(y)^-)$ (see (1.8)). Using (1.6), (1.7) and (1.8), we get

$$I = A(y)^- A(y) + Z(y)^- Z(y) \quad \text{in } \mathbb{R}^{n\times n}. \tag{2.1}$$

We shall also suppose that this decomposition is made in a smooth way. More precisely, the function

$$y \to (A(y)^-, Z(y)^-)$$

will be supposed twice continuously differentiable on ω and, as well as its derivatives, bounded on ω. Because $Z(y) = (0 \;\; I_{n-m})(A(y)^- Z(y)^-)^{-1}$, the function $y \to Z(y)$ will have the same property. This may also appear as a strong hypothesis if ω is large, but it can be satisfied in a neighbourhood of a solution x_* if some qualification hypothesis ($A(x_*)$ surjective) is satisfied. On this question, we refer to Byrd and Schnabel (1986).

We shall denote by x_* a solution of problem (1.1), i.e., a local minimizer satisfying the standard second order sufficient conditions of optimality (see Fletcher (1981), for instance). Therefore we shall suppose the existence of a Lagrange multiplier λ_* in $\mathbb{R}^m$ such that (1.2) is satisfied and such that the Hessian of the Lagrangian at (x_*, λ_*) is positive definite in the tangent space $\operatorname{Ker} A(x_*)$. In other words, G_* given in (1.13) will be supposed positive definite.

We shall denote by $\|\cdot\|$ any norm on $\mathbb{R}^n$ or $\mathbb{R}^m$ (and not necessarily the same norms on both spaces) and by $\|\cdot\|_D$ the dual norm for the Euclidean scalar product, i.e. $\|v\|_D := \sup\{v^T u : \|u\| \leq 1\}$. The norm $\|\cdot\|_2$ will denote the l_2-norm. Matrix norms will be supposed subordinated to the vector norms, i.e., $\|A\| := \sup\{\|Au\| : \|u\| \leq 1\}$. If $(u_k)_{k\geq 0}$ is a sequence of vectors and $(\alpha_k)_{k\geq 0}$ is a sequence of positive numbers, we shall write $u_k = O(\alpha_k)$ when $(\|u_k\|/\alpha_k)_{k\geq 0}$ is bounded and $u_k = o(\alpha_k)$ when $(\|u_k\|/\alpha_k)_{k\geq 0}$ converges to zero. The ith component of a vector u will be denoted by $u_{(i)}$.

3. The longitudinal displacement

In this section, we shall suppose that y_k is a point given in ω with $g(y_k) \neq 0$ (if $t_k = 0$, there will be no longitudinal displacement) and that H_k is positive definite.

Therefore, if θ is given by (1.16) with ϕ convex, we see by (1.17) that $\theta'(y_k; t_k)$ is negative.

In unconstrained minimization ($\min \psi(u)$), quasi-Newton methods locally try to approximate the Hessian of ψ at a solution u_*. Therefore, the change in the gradient of ψ between two successive iterates u_k and u_{k+1} gives some information on this Hessian. So, the vectors γ_k and δ_k used in the update formulae are usually taken as follows:

$$\gamma_k := \nabla\psi(u_{k+1}) - \nabla\psi(u_k),$$

$$\delta_k := u_{k+1} - u_k = \tau_k v_k,$$

where τ_k is some step size in a descent direction v_k of ψ at u_k. Wolfe's step-size selection procedure consists in finding a step size $\tau = \tau_k$ such that the following inequalities are satisfied:

$$\psi(u_k + \tau v_k) \leq \psi(u_k) + \alpha_1 \tau \nabla\psi(u_k)^{\mathrm{T}} v_k, \tag{3.1}$$

$$\nabla\psi(u_k + \tau v_k)^{\mathrm{T}} v_k \geq \alpha_2 \nabla\psi(u_k)^{\mathrm{T}} v_k, \tag{3.2}$$

where $0 < \alpha_1 < \alpha_2 < 1$. A step size τ satisfying both inequalities will be called *serious*. Condition (3.1) ensures a sufficient decrease of ψ, while condition (3.2) prevents the step size τ from being too small. A global convergence result can be obtained with these conditions, see Wolfe (1969). An important advantage of this way of selecting the step size in the framework of quasi-Newton methods is that condition (3.2) automatically ensures the positivity of $\gamma_k^{\mathrm{T}}\delta_k$ and, as a result, the positive definiteness of the updated matrices.

In reduced methods for constrained optimization, an approximation G_k of G_*, the projected Hessian of the Lagrangian, is updated. Here, it is the change in the reduced gradient that gives information on G_*, as suggested by the following formula (see Stoer (1984), Nocedal and Overton (1985)):

$$\nabla g(x_*) = \nabla(Z(x)^{-\mathrm{T}}(\nabla f(x) + A(x)^{\mathrm{T}}\lambda_*))(x_*) = Z(x_*)^{-\mathrm{T}} L_*, \tag{3.3}$$

where we used (1.6) and the second optimality condition in (1.2). Comparing (1.13) and (3.3), we see that G_* is a part of $\nabla g(x_*)$. This is essentially due to the unfeasibility of the method, because in this case, the successive iterates do not belong to a particular manifold of dimension $n-m$, and any function used to get information on G_* has to be defined on all $\mathbb{R}^n$. Hence, if, like g, this function takes its values in $\mathbb{R}^{n-m}$, its Jacobian is a matrix of dimension $(n-m)\times n$ and not of order $n-m$, like G_*. Therefore accurate information is obtained on G_* if, asymptotically, the change in the reduced gradient is given for a displacement along the tangent space $R(Z(x_*)^-)$. This is the basic idea of the update schemes for reduced secant methods, as those proposed independently by Nocedal and Overton (1985) and Gilbert (1986 and 1988). The scheme proposed by Coleman and Conn (1984) is related, but requires the use of a multiplier estimate. Following this strategy, when the unit step size is accepted (and it will be asymptotically), the matrix H_k will be updated by

the BFGS formula (1.18), with $\gamma_k = \gamma_k^1$ and $\delta_k = \delta_k^1$, where

$$\gamma_k^1 := g(x_{k+1}) - g(y_k),$$

$$\delta_k^1 := Z(y_k)t_k = Z(y_k)(x_{k+1} - y_k).$$

The step δ_k^1 is called the *reduced longitudinal displacement.* Note that another choice is possible but needs the use of an update criterion: see Nocedal and Overton (1985) for the algorithm of Gabay (1982b), and Gilbert (1986 and 1988) for algorithm (1.9)-(1.10).

Let us show now that the condition $(\gamma_k^1)^T\delta_k^1 > 0$ is usually satisfied asymptotically. Using (3.3) and supposing that the sequence (y_k) converges to a solution x_* of (1.1) with $t_k \to 0$, we have (we use $t_k = Z(y_k)^- \delta_k^1$):

$$\gamma_k^1 = Z(x_*)^{-T} L_* t_k + o(\|t_k\|) = G_* \delta_k^1 + o(\|\delta_k^1\|).$$

Hence, $(\gamma_k^1)^T\delta_k^1$ is positive for large k if G_* is positive definite. However, this condition (1.19) is not necessarily satisfied when y_k is far from x_*, even if a step size τ is introduced to scale the tangent step t_k:

$$\gamma_k^\tau := g(y_k + \tau t_k) - g(y_k),$$

$$\delta_k^\tau := \tau Z(y_k)t_k.$$

The following counter-example confirms this claim.

Suppose that $n = 2$, $m = 1$, $f(y) := y_{(2)}$, $c(y) := \frac{1}{2}(\|y\|_2^2 - 1)$ and take $\omega = \omega_\beta := \beta B \setminus \beta^{-1}\bar{B}$ with $\beta > 1$; B denotes the unit open ball. For these data, the unique solution of problem (1.1) is $y_{*(1)} = 0$ and $y_{*(2)} = -1$. We have $A(y) = y^T$. At any point in ω_β, we may use the following orthogonal decomposition of $\mathbb{R}^2$:

$$A(y)^- := y/\|y\|_2^2, \qquad Z(y)^- := \tilde{y},$$

where $\tilde{y}_{(1)} := y_{(2)}$ and $\tilde{y}_{(2)} := -y_{(1)}$. To these choices corresponds a unique matrix $Z(y)$ satisfying (1.7) and (1.8): it is given by $Z(y) = \tilde{y}^T/\|y\|_2^2$. The hypotheses of Section 2 are satisfied on ω_β for any $\beta > 1$. We have $g(y) = -y_{(1)}$, $t(y) = Hy_{(1)}\tilde{y}$ and $g(y + \tau t(y)) = -y_{(1)}(1 + \tau H y_{(2)})$. As $\delta^\tau = \tau H y_{(1)}$, if we suppose $y_{(1)}$ and H positive, the positivity of $(\gamma^\tau)^T\delta^\tau$ is equivalent to $g(y + \tau t(y)) > g(y)$, i.e., $-\tau H y_{(1)} y_{(2)} > 0$, which is never satisfied for any positive step size τ when $y_{(2)}$ is also positive.

On the other hand, Wolfe's conditions may be satisfied for a certain step size τ along a search path y_k^M starting from y_k and belonging to the manifold $M(y_k) := c^{-1}(c(y))$ (this submanifold of ω exists because c is a submersion). Let us consider, indeed, the search trajectory y_k^M, defined by the following ordinary differential equation (where the dot stands for a derivative according to τ):

$$\dot{y}_k^M(\tau) = Z(y_k^M(\tau))^- Z(y_k)t_k, \tag{3.4a}$$

$$y_k^M(0) = y_k. \tag{3.4b}$$

If we multiply to the left both sides of (3.4a) by $A(y_k^M(\tau))$, we see that $y_k^M(\tau) \in M(y_k)$. Let us introduce the function $\psi := \theta \circ y_k^M$, where θ is given in (1.16) with ϕ convex. We obtain, by a calculation similar to the one in (1.17):

$$\nabla\psi(\tau) = \theta'(y_k^M(\tau); Z(y_k^M(\tau))^- Z(y_k)t_k) = g(y_k^M(\tau))^T Z(y_k)t_k,$$

$$\nabla\psi(0) = \theta'(y_k; t_k) = g(y_k)^T Z(y_k)t_k.$$

Then, rewriting Wolfe's conditions (3.1)-(3.2) with this ψ, $u_k := 0 \in \mathbb{R}$ and $v_k := 1 \in \mathbb{R}$, we obtain:

$$\theta(y_k^M(\tau)) \leq \theta(y_k) + \alpha_1 \tau \theta'(y_k; t_k), \tag{3.5}$$

$$g(y_k^M(\tau))^T Z(y_k)t_k \geq \alpha_2 g(y_k)^T Z(y_k)t_k. \tag{3.6}$$

Considering the function $\tau \in \mathbb{R} \to \psi(\tau) \in \mathbb{R}$, it is clear that there exists a positive step size τ_k such that (3.5)-(3.6) are satisfied at $\tau = \tau_k$, if the solution of (3.4) exists for sufficiently large τ and if, for example, f is bounded from below on $M(y_k)$ (see the argument in Wolfe (1969)). Writing inequality (3.6) at $\tau = \tau_k$, we see that condition (1.19) is satisfied with $\gamma_k = \gamma_k^M$ and δ_k given by

$$\gamma_k^M := g(y_k^M(\tau_k)) - g(y_k),$$

$$\delta_k := \tau_k Z(y_k)t_k.$$

In view of the counter-example and the success of the path y_k^M, a possible direction of investigation could be to try to build an approximation of the path y_k^M, using an approximation scheme for the differential equation (3.4). But, on the one hand, it is very expensive to have an accurate approximation and, on the other hand, for any τ for which (3.6) would not be satisfied the question of the sharpness of the approximation would arise as a leitmotiv: as shown by the counter-example, the linear approximation $(y_k + \tau t_k)$ is sometimes inadequate, so, what about the current one? Fortunately, the situation can be sorted out by trying to satisfy both inequalities (3.5) and (3.6) in the following way.

Let us remark first that inequality (3.5) is satisfied for small positive τ along the linear path $y_k^0(\tau) := y_k + \tau t_k$ $(\tau > 0)$ instead of $y_k^M(\tau)$:

$$\theta(y_k^0(\tau)) \leq \theta(y_k) + \alpha_1 \tau \theta'(y_k; t_k). \tag{3.7}$$

Indeed, $\theta'(y_k; t_k)$ is negative and α_1 is less than 1. On the other hand, by continuity and because α_2 is less than 1, the inequality corresponding to (3.6),

$$g(y_k^0(\tau))^T Z(y_k)t_k \geq \alpha_2 g(y_k)^T Z(y_k)t_k, \tag{3.8}$$

is not satisfied for small positive τ along $y_k^0(\tau)$. Therefore, we may define $\tau_k^1 := \sup\{\tau' > 0: \forall \tau \in [0, \tau'[, y_k^0(\tau) \in \omega$, (3.7) is satisfied and (3.8) is not satisfied$\}$. If τ_k^1 is infinite, which is not possible if θ is bounded from below — see (3.7), or if $y_k^1 := y_k^0(\tau_k^1) \notin \omega$, we shall consider that the algorithm has failed. Otherwise, (3.7) is satisfied at y_k^1 (by continuity). Then, if (3.8) is satisfied at y_k^1 (which is the only possibility in the unconstrained case), τ_k^1 is a serious step size. Otherwise, this means,

by continuity, that (3.7) is not satisfied for $\tau > \tau_k^1$ in a neighborhood of τ_k^1. In this last case, $\theta(y_k^1) = \theta(y_k) + \alpha_1 \tau_k^1 \theta'(y_k; t_k)$ and the search to satisfy (3.7) and (3.8) may be pursued from y_k^1 in the direction $Z(y_k^1)^- Z(y_k) t_k$. To see this, it is enough to remark that, if $y_k^1(\tau)$ is defined by $y_k^1(\tau) := y_k^0(\tau)$ for $0 \leq \tau \leq \tau_k^1$ and $y_k^1(\tau) := y_k^1 + (\tau - \tau_k^1) Z(y_k^1)^- Z(y_k) t_k$ for $\tau > \tau_k^1$, the following inequality holds:

$$\theta(y_k^1(\tau)) \leq \theta(y_k) + \alpha_1 \tau \theta'(y_k; t_k),$$

for $\tau > \tau_k^1$ in a neighborhood of τ_k^1. This is true because, on the contrary, we would have a sequence of $\tau > \tau_k^1$, converging to τ_k^1, with

$$\theta(y_k^1(\tau)) - \theta(y_k^1) > \alpha_1 (\tau - \tau_k^1) \theta'(y_k; t_k).$$

Dividing by $(\tau - \tau_k^1)$ and taking the limit as τ tends to τ_k^1 would give:

$$(\theta \circ y_k^1)'(\tau_k^1; 1) \geq \alpha_1 \theta'(y_k; t_k) > \alpha_2 \theta'(y_k; t_k) = \alpha_2 g(y_k)^T Z(y_k) t_k.$$

But

$$\begin{aligned}(\theta \circ y_k^1)'(\tau_k^1; 1) &= \nabla f(y_k^1)^T Z(y_k^1)^- Z(y_k) t_k \\ &\quad + \phi'(c(y_k^1); A(y_k^1) Z(y_k^1)^- Z(y_k) t_k) \\ &= g(y_k^1)^T Z(y_k) t_k,\end{aligned}$$

because of (1.6). So, $g(y_k^1)^T Z(y_k) t_k > \alpha_2 g(y_k)^T Z(y_k) t_k$, which is a contradiction, because (3.8) was supposed to be violated at $y_k^1 = y_k^0(\tau_k^1)$. Now, we can continue and define $\tau_k^2 := \sup\{\tau' > \tau_k^1 : \forall \tau \in [\tau_k^1, \tau'[, y_k^1(\tau) \in \omega, \theta(y_k^1(\tau)) \leq \theta(y_k) + \alpha_1 \tau \theta'(y_k; t_k)$ and $g(y_k^1(\tau))^T Z(y_k) t_k < \alpha_2 g(y_k)^T Z(y_k) t_k\}$, $y_k^2 := y_k^1(\tau_k^2)$, and so on. Therefore, the search can be pursued along a piecewise linear path, as long as a serious step size is not met.

To obtain an implementable version of this algorithm, two questions, which constitute our program up to the end of this section, have to be clarified:

(i) the values τ_k^i of the step size at which the search is reoriented are not attainable by calculation and should be redefined.

(ii) the algorithm should be shown to terminate in a finite number of iterations.

The last question will be the subject of Theorem 3.3, while for the first question, we may refer to what is done in unconstrained optimization to find a serious step size in the sense of Wolfe. Indeed, in this case as well, if the step size corresponding to our τ_k^1 solves the problem, it is never calculated but only approximated and this is possible because there must exist a left neighborhood of τ_k^1 constituted of serious step sizes. For example, Lemaréchal (1981) has proposed an algorithm to find a serious step size in unconstrained optimization (see also, Fletcher (1980, Section 2.6), and Dennis and Schnabel (1983, Algorithm A 6.3.1 mod, p. 328)). Let us recall it here in terms of the function ψ introduced at the beginning of the section.

Line-search algorithm. (3.9)

```
1.   τ_L := 0; τ_U := ∞; choose (τ > 0);
2.   repeat
2.1.     if ((3.1) is not satisfied)
2.2.     then {τ_U := τ; τ := INTERPOL (τ_L, τ_U)}
2.3.     else {
2.4.         if ((3.2) is satisfied) then exit      /* τ is serious */
2.5.         else {
2.6.             τ_L := τ;
2.7.             if (τ_U = ∞) then τ := EXTRAPOL (τ_L)
2.8.             else τ := INTERPOL (τ_L, τ_U)
2.5.         }
2.3.     };
```

So, the algorithm tries to trap a step size like τ_k^1 in an interval $[\tau_L, \tau_U]$. The step size τ_U is said to be *too large* because it does not satisfy (3.1), hence some step size like τ_k^1 must exist in $[0, \tau_U]$. The step size τ_L is said to be *too small* because it is less than τ_U and satisfies (3.1) but not (3.2), hence some step size like τ_k^1 must exist in $[\tau_L, \tau_U]$. The algorithm uses two functions: INTERPOL gives a step size τ between the two finite values τ_L and τ_U and EXTRAPOL gives a step size τ greater than τ_L. Some conditions on these functions are required in order to ensure the global convergence of methods using such line-search algorithm.

We shall adapt this algorithm to our situation by modifying the direction of search each time a step size is recognized as too small. These step sizes will constitute our new τ_k^i's. Note that this change in the direction of search at a point y_k^i is free of charge because an inequality like (3.6) or (3.8) has to be tested at y_k^i and therefore the new basis $Z(y_k^i)^-$ is available and the new search direction $Z(y_k^i)^- Z(y_k)t_k$, as well. However, the fact to have to linearize the constraints and to compute the basis of the tangent space at each point y_k^i can be rather expensive for certain applications. It is therefore important to show that the unit step size is usually accepted asymptotically. This will be done in Section 4. Then, according to our experience with the line-search algorithm (3.9) in unconstrained optimization, it is not unrealistic to expect that the unit step size will be accepted at almost all iterations. Now, even if such is the case, the algorithm still requires two linearizations of the constraints per iteration (at y_k and x_{k+1}), which is more expensive than implementing an arc search from y_k to y_{k+1} on the penalty function using Armijo's rule as we would find in Gabay (1982b) or Gilbert (1989). This technique does not however guarantee the positivity of $(g(x_{k+1}) - g(y_k))^T Z(y_k)t_k$. Therefore, it is important to generalize the algorithm presented here to reduced secant methods that require only one linearization of the constraints per iteration.

Before stating our algorithm we need to define precisely the search path and to give the properties required for the interpolation and extrapolation functions.

Being given l positive numbers:

$$0=:\tau_k^0<\tau_k^1<\cdots<\tau_k^l, \tag{3.10}$$

we define by induction the points y_k^i and the piecewise linear trajectories $y_k^i(\tau)$, $\tau\geq 0$ for $0\leq i\leq l$. For $i=0$, we define

$$y_k^0:=y_k, \tag{3.11}$$

$$y_k^0(\tau):=y_k+\tau t_k=y_k+\tau Z(y_k)^- Z(y_k)t_k \quad \text{for } \tau\geq 0 \tag{3.12}$$

and for $1\leq i\leq l$, we define

$$y_k^i:=y_k^{i-1}(\tau_k^i), \tag{3.13}$$

$$y_k^i(\tau):=\begin{cases} y_k^{i-1}(\tau) & \text{for } 0\leq\tau\leq\tau_k^i,\\ y_k^i+(\tau-\tau_k^i)Z(y_k^i)^- Z(y_k)t_k & \text{for } \tau>\tau_k^i. \end{cases} \tag{3.14}$$

Therefore, if the dot stands for a right derivative, we have

$$\dot{y}_k^l(\tau_k^i)=Z(y_k^i)^- Z(y_k)t_k, \quad 0\leq i\leq l.$$

So, the path $y_k^l(\cdot)$ may be seen as an explicit Euler approximation of the solution y_k^M of (3.4) on $[0,\tau_k^l]$ for the discretization (3.10) in τ.

Lemma 3.1. *If $(\tau_k^l)_{l\geq 1}$ is an increasing sequence of positive numbers that converges to some $\tau_{U,k}$ and if $(y_k^l)_{l\geq 0}$, defined by (3.11)-(3.14), remains in ω, then $(y_k^l)_{l\geq 0}$ converges in $\mathbb{R}^n$.*

Proof. For $l\geq 1$, we have $y_k^l=y_k+S^l$, with

$$S^l:=\sum_{i=0}^{l-1}(\tau_k^{i+1}-\tau_k^i)Z(y_k^i)^- Z(y_k)t_k.$$

Because $y_k^i\in\omega$, $(Z(y_k^i)^-)_{i\geq 0}$ is bounded. Hence, for $l_2>l_1$, $\|S^{l_2}-S^{l_1}\|\leq C(\tau_k^{l_2}-\tau_k^{l_1})$, where C is a positive constant independent of l_1 and l_2. As $(\tau_k^l)_{l\geq 0}$ converges, $(S^l)_{l\geq 1}$ is a Cauchy sequence, hence converges in $\mathbb{R}^n$. And so does $(y_k^l)_{l\geq 0}$. □

Wolfe's criteria are then generalized as follows: find a non-negative integer l, $l+1$ positive numbers $(\tau_k^i)_{0\leq i\leq l}$ verifying (3.10) and a $\tau>\tau_k^l$ such that:

$$\theta(y_k^l(\tau))\leq\theta(y_k)+\alpha_1\tau\theta'(y_k;t_k), \tag{3.15}$$

$$g(y_k^l(\tau))^{\mathrm{T}}Z(y_k)t_k\geq\alpha_2 g(y_k)^{\mathrm{T}}Z(y_k)t_k, \tag{3.16}$$

where $0<\alpha_1<\alpha_2<1$ are given. See (1.17), for the value of $\theta'(y_k;t_k)$.

We shall need an *interpolation function* $J:\omega\times\mathbb{R}_+\to\mathbb{R}_+:=[0,\infty[$ such that:

$$(y,\tau)\to J(y,\tau) \text{ is continuous on } \omega\times\mathbb{R}_+, \tag{3.17}$$

$$\forall y\in\omega,\ \forall\tau>0:\quad J(y,\tau)\in\,]0,\tau[, \tag{3.18}$$

$$\forall y\in\omega,\ \forall\tau>0:\quad J_y^p(\tau)\to 0 \quad \text{as } p\to\infty, \tag{3.19}$$

where $J_y^p(\tau) := (J_y \circ \cdots (p \text{ times}) \cdots \circ J_y)(\tau)$ and $J_y(\tau) := J(y, \tau)$. This function will be used when we shall need a new trial step size between, say, τ_k^l and $\tau > \tau_k^l$. Then, we shall take $\tau' := \tau_k^l + J(y_k^l, \tau - \tau_k^l)$. By condition (3.18), $\tau' \in]\tau_k^l, \tau[$ (interpolation) and condition (3.19) guarantees that if the operation is repeated, the new step sizes will be closer and closer to τ_k^l. We can deduce from (3.17) and (3.18), that for $y \in \omega$ and $\tau \geq 0$, $J(y, \tau) = 0$ if and only if $\tau = 0$.

We shall also need an *extrapolation function* $E : \omega \to \mathbb{R}_+$ such that:

$$y \to E(y) \text{ is continuous on } \omega, \tag{3.20}$$

$$\forall y \in \omega: \quad E(y) > 0. \tag{3.21}$$

This function will be used when we shall need a new trial step size greater than, say, τ_k^l. Then, we shall take $\tau_k^l + E(y_k^l) > \tau_k^l$.

Lemma 3.2. *If $(y^l)_{l \geq 0}$ is a converging sequence in ω and $(\tau^l)_{l \geq 0}$ is a bounded sequence of positive numbers such that $J(y^l, \tau^l)$ converges to zero, then τ^l converges to zero.*

Proof. Let y in ω be the limit point of (y^l) and $(\tau^{l'})$ be a subsequence of (τ^l) that converges to some τ. Then, by (3.17), $J(y^{l'}, \tau^{l'}) \to J(y, \tau) = 0$, hence $\tau = 0$ and all the sequence $(\tau^l)_{l \geq 0}$ converges to zero. □

We can now state:

Longitudinal search algorithm. (3.22)

1. **if** $(t_k = 0)$ **then** exit;
2. $l := 0$; $\tau_k^0 := 0$; choose $(\tau > 0)$;
3. **repeat**
3.1. **if** $((y_k^l(\tau) \notin \omega)$ **or** ((3.15) is not satisfied))
3.2. **then** $\tau := \tau_k^l + J(y_k^l, \tau - \tau_k^l)$
3.3. **else** {
3.4. **if** ((3.16) is satisfied)
3.5. **then** $\{\tau_k := \tau$; $l_k := l$; exit$\}$ /* τ is serious */
3.6. **else** $\{l := l + 1$; $\tau_k^l := \tau$; $\tau := \tau_k^l + E(y_k^l)\}$
3.3. };

In statement 2, the choice $\tau = 1$ is recommended if the algorithm is used within the context of secant methods because in this case the unit step size is essential to obtain superlinear convergence. We have added in statement 3.1 another reason to decrease τ: the points y_k^l must belong to ω. Therefore a serious step size may not be found because ω is too small and the algorithm may loop in statement 3. This is one of two reasons for looping (see the other one in the theorem below).

Let us remark that the algorithm will not cycle between statement 3.1 and 3.2 because otherwise τ would decrease to τ_k^l by hypothesis (3.19); but, $y_k^l := y_k^l(\tau_k^l)$ is in ω by construction so $y_k^l(\tau)$ is also in ω for τ close to τ_k^l and, on the other hand, inequality (3.15) is satisfied for τ close to τ_k^l. Therefore the test 3.1 is always rejected after a finite number of loops 3.1-3.2. Consequently, if the algorithm loops in statement 3, a sequence $(y_k^l)_{l \geq 0}$ is built in ω.

We have seen that the search path $y_k^l(\cdot)$ is an Euler approximation of the solution of (3.4). From the way the discretization points τ_k^i are built by the algorithm above, we see that this approximation will be accurate if τ in statement 2 is chosen "small" and if $\forall y \in \omega$, $E(y)$ is "small". However, doing this is expensive and, in fact, useless because building a good approximation of y_k^M is not the aim of the algorithm.

We now give the main result of this section, which shows that, apart from some pathological situations, a serious step size is found in a finite number of iterations.

Theorem 3.3. *Let θ be the function defined on ω by* (1.16) *with ϕ convex and continuous on a neighborhood of $c(\omega)$. Let y_k be a point in ω such that $g(y_k) \neq 0$. Let H_k be a symmetric positive definite matrix of order $n-m$. Then, if the longitudinal search algorithm* (3.22) *with the definitions* (3.10)-(3.14) *and the hypotheses* (3.17)-(3.21) *is applied from y_k, one of the following situations occurs*:

(i) *the algorithm terminates in a finite number l_k of loops* 3.1-3.6, *with a point $x_{k+1} := y_k^{l_k}(\tau_k)$ satisfying both inequalities* (3.15) *and* (3.16) *with $l = l_k$ and $\tau = \tau_k$,*

(ii) *the algorithm builds a sequence $(y_k^l)_{l \geq 0}$ in ω and either $(\theta(y_k^l))_{l \geq 0}$ tends to $-\infty$ or $(y_k^l)_{l \geq 0}$ converges to a point on the boundary of the open set ω.*

Proof. Clearly, by the remark preceding the theorem, if a finite number l_k of points y_k^l are calculated, this means that a step size τ_k has been found in statement 3.5 and that $y_k^{l_k}(\tau_k)$ satisfies (3.15) and (3.16). So let us suppose the contrary, i.e., a sequence $(y_k^l)_{l \geq 0}$ is built in ω. Let us suppose also that the sequence $(\theta(y_k^l))_{l \geq 0}$ is bounded from below and that $(y_k^l)_{l \geq 0}$ does not converge to a point on the boundary of ω. We have to prove that these suppositions give a contradiction.

For all l, we have by construction:

$$\theta(y_k^l) \leq \theta(y_k) + \alpha_1 \tau_k^l \theta'(y_k; t_k), \tag{3.23}$$

$$g(y_k^l)^{\mathrm{T}} Z(y_k) t_k < \alpha_2 g(y_k)^{\mathrm{T}} Z(y_k) t_k. \tag{3.24}$$

Because $\theta'(y_k; t_k)$ is negative and $(\theta(y_k^l))_{l \geq 0}$ is bounded from below, (3.23) shows that $(\tau_k^l)_{l \geq 0}$ is bounded. As τ_k^l increases with l, the sequence converges to some $\tau_{\mathrm{U},k}$ and by Lemma 3.1, $(y_k^l)_{l \geq 0}$ converges to some $y_{\mathrm{U},k}$ in $\mathbb{R}^n$. According to the suppositions, $y_{\mathrm{U},k}$ is in ω.

Let us show that

$$\theta(y_{\mathrm{U},k}) = \theta(y_k) + \alpha_1 \tau_{\mathrm{U},k} \theta'(y_k; t_k) = \theta(y_k) + \alpha_1 \tau_{\mathrm{U},k} g(y_k)^{\mathrm{T}} Z(y_k) t_k. \tag{3.25}$$

In view of (3.23) and by continuity, it is enough to prove that the left-hand side of (3.25) is not less than the right-hand side. For this, let us remark that there exists an integer l^0 such that for $l \geq l^0$, we have for some $\tau_{\mathrm{U},k}^l > \tau_k^{l+1}$:

$$\tau_k^{l+1} = \tau_k^l + J(y_k^l, \tau_{\mathrm{U},k}^l - \tau_k^l). \tag{3.26}$$

Indeed, for sufficiently large l, τ_k^{l+1} is given by (3.26), i.e. by interpolation. Otherwise, we would have $\tau_k^{l+1} = \tau_k^l + E(y_k^l)$ for some subsequence of l's and at the limit on those l's, we would have, by (3.20), $E(y_{\mathrm{U},k}) = 0$, which is in contradiction with

hypothesis (3.21). Now, by construction, $\tau^l_{U,k} \leq \tau^l_k + E(y^l_k)$. So $(\tau^l_{U,k})_{l \geq l^0}$ is also bounded. Then, the limit in (3.26) and Lemma 3.2 show that $(\tau^l_{U,k})_{l \geq l^0}$ converges to $\tau_{U,k}$. As $y^l_k(\tau^l_{U,k}) = y^l_k + (\tau^l_{U,k} - \tau^l_k) Z(y^l_k)^- Z(y_k) t_k$ converges to $y_{U,k} \in \omega$, $y^l_{U,k}(\tau^l_{U,k}) \in \omega$ for large l. Therefore, (3.26) means that (3.15) is not satisfied at $y^l_k(\tau^l_{U,k})$, i.e.

$$\theta(y^l_k(\tau^l_{U,k})) > \theta(y_k) + \alpha_1 \tau^l_{U,k} \theta'(y_k; t_k).$$

The equality (3.25) is proved by taking the limit on l in this last inequality.

Taking the limit on l in (3.24) and using $0 < \alpha_1 < \alpha_2$, we see that there will be a contradiction (and therefore we shall have proved the theorem) if we show that

$$g(y_{U,k})^T Z(y_k) t_k \geq \alpha_1 g(y_k)^T Z(y_k) t_k. \tag{3.27}$$

For this, we build a sequence $(\eta^p)_{p \geq 0}$ of positive numbers converging to zero and a sequence $(z^p_k)_{p \geq 0}$ of points in $\mathbb{R}^n$ of the form

$$z^p_k := y_{U,k} + \eta^p Z(y_{U,k})^- Z(y_k) t_k, \tag{3.28}$$

such that:

$$z^p_k \in \omega \;\Rightarrow\; \theta(z^p_k) \geq \theta(y_k) + \alpha_1 (\tau_{U,k} + \eta^p) g(y_k)^T Z(y_k) t_k. \tag{3.29}$$

As $z^p_k \in \omega$ for large p, (3.29) and (3.25) give for large p:

$$\frac{\theta(z^p_k) - \theta(y_{U,k})}{\eta^p} \geq \alpha_1 g(y_k)^T Z(y_k) t_k.$$

Hence, (3.27) will follow by taking the limit on p in this inequality.

The sequences $(\eta^p)_{p \geq 0}$ and $(z^p_k)_{p \geq 0}$ are built by induction together with an increasing sequence of indices $(l^p)_{p \geq 0}$ such that for $p \geq 0$ and $l \geq l^p$, either $z^{p,l}_k := y^l_k(\tau^l_k + \eta^{p,l}) = y^l_k + \eta^{p,l} Z(y^l_k)^- Z(y_k) t_k \notin \omega$, where $\eta^{p,l} := J^p_{y^l_k}(E(y^l_k))$, or (3.15) is not satisfied at $z^{p,l}_k$, i.e. for $\tau = \tau^l_k + \eta^{p,l}$. In other words, for $p \geq 0$ and $l \geq l^p$:

$$z^{p,l}_k = \omega \;\Rightarrow\; \theta(z^{p,l}_k) > \theta(y_k) + \alpha_1 (\tau^l_k + \eta^{p,l}) g(y_k)^T Z(y_k) t_k. \tag{3.30}$$

We begin with η^0, z^0_k and l^0. We have already seen in getting (3.26) that we may find a positive index l^0 such that for $l \geq l^0$, τ^l_k is obtained by interpolation, which means that either $z^{0,l}_k = y^l_k(\tau^l_k + \eta^{0,l}) \notin \omega$ or (3.15) is not satisfied at $z^{0,l}_k$. This is precisely (3.30) for $p = 0$. Similarly, for $p \geq 1$, we may find an index $l^p \geq l^{p-1}$ (defined by induction) such that for all $l \geq l^p$, either $z^{p,l}_k \notin \omega$ or (3.15) is not satisfied at $z^{p,l}_k$. Indeed, otherwise, l^p being greater than l^{p-1}, we would have for a subsequence of l's:

$$\tau^{l+1}_k = \tau^l_k + J^p_{y^l_k}(E(y^l_k))$$

and the limit on l would give

$$(J_{y_{U,k}} \circ \cdots (p \text{ times}) \cdots \circ J_{y_{U,k}})(E(y_{U,k})) = 0.$$

This would imply $E(y_{U,k}) = 0$, in contradiction with (3.21). Therefore, for $l \geq l^p$, (3.30) is still satisfied. As $(z^{p,l}_k)_{l \geq 0}$ converges to z^p_k given by (3.28) with

$$\eta^p := J^p_{y_{U,k}}(E(y_{U,k})),$$

the limit in (3.30) gives (3.29). Moreover, $(\eta^p)_{p\geq 0}$ converges to zero because of property (3.19). □

Remarks 3.4. A consequence of Theorem 3.3 is that if θ is bounded below, $\omega = \mathbb{R}^n$ and the hypotheses of the theorem are satisfied, then the longitudinal search algorithm (3.22) will find a point $x_{k+1} := y_k^{l_k}(\tau_k)$ satisfying both inequalities (3.15) and (3.16). The situations in which the longitudinal search algorithm fails are similar to those posing problem to algorithm (3.9) for finding Wolfe points in unconstrained optimization. The first situation of Theorem 3.3(ii) corresponds to the case where a local minimum of an unbounded from below function f is searched. With the notation from the beginning of the section, the second situation of Theorem 3.3(ii) corresponds to the case where the objective function $\psi :]0, \infty[\to \mathbb{R}$ has positive derivative and negative curvature on, say $]0, 1[$. If $u_k \in]0, 1[$, algorithm (3.9) will search in $]0, u_k[$, where no Wolfe point can be found (Lemaréchal (1984)). The algorithm we propose cannot correctly face such situations, either.

Below, we shall need the following inequality.

Proposition 3.5. *With the hypotheses of Theorem* 3.3, *if starting from a point y_k in ω, the longitudinal search algorithm* (3.22) *gives a point x_{k+1} in ω and a step size τ_k, we have*

$$\|x_{k+1} - y_k\| \leq C\tau_k \|t_k\|, \tag{3.31}$$

where C is a positive constant that depends only on a bound of $Z(\cdot)$ and $Z(\cdot)^-$ on ω.

Proof. We have

$$x_{k+1} - y_k = \sum_{i=0}^{l_k - 1} (\tau_k^{i+1} - \tau_k^i) Z(y_k^i)^- Z(y_k) t_k,$$

from which (3.31) follows. □

4. The algorithm

In secant methods, it is commonly considered that a globalizing technique is successful if the unit step size is asymptotically accepted by the search algorithm, because then the superlinear convergence of the local method is not prevented from occurring. This means that the longitudinal search algorithm (3.22) should set $l_k = 0$ and $\tau_k = 1$ for all sufficiently large k. In fact, this depends on three factors: the search direction t_k, i.e. the matrix H_k, the penalty function θ and the constants α_1 and α_2 in (3.15)-(3.16).

Because G_k is updated to be a good approximation of the projected Hessian of the Lagrangian, which is a condition imposed by the local analysis, the point $(y_k + t_k)$ will be asymptotically a good approximation of a minimizer of the Lagrangian in the tangent plane $y_k + R(Z(y_k)^-)$. Note, indeed, that if $G_k = Z(y_k)^{-T} L(y_k, \lambda) Z(y_k)^-$, we have $t_k = \operatorname{argmin}\{l(y_k, \lambda) + \nabla_x l(y_k, \lambda)^T t + t^T L(y_k, \lambda) t/2 : t \in R(Z(y_k)^-)\}$, for any λ. Therefore, the unit step size has some chance of being accepted if θ is close to the Lagrangian. Finally, in addition to $0 < \alpha_1 < \alpha_2 < 1$, we need to impose the condition $\alpha_1 < \frac{1}{2}$, because then condition (3.15) accepts the minimum of a quadratic function.

We choose as penalty function the nondifferentiable augmented Lagrangian:

$$l_p(x, \mu) := f(x) + \mu^T c(x) + p\|c(x)\|, \tag{4.1}$$

where $\|\cdot\|$ is a norm on $\mathbb{R}^m$. This penalty function is exact, i.e., a solution x_* of problem (1.1) is a local minimizer (here strict) of $l_p(\cdot, \mu)$, if $p > \|\mu - \lambda_*\|_D$, where λ_* is the multiplier associated to x_* and $\|\cdot\|_D$ is the dual norm of $\|\cdot\|$ on $\mathbb{R}^m$. This result may be derived as a variant of a result of Han and Mangasarian (1979) by considering the problem $\min\{f(x) + \mu^T c(x) : x \in \omega, c(x) = 0\}$, which is equivalent to problem (1.1) or it may be directly obtained as in Bonnans (1989), where the penalty function (4.1) has been used in connection with the SQP method to obtain the admissibility of the unit step size.

Another possibility could have been to use the differentiable augmented Lagrangian, obtained by replacing $p\|c(x)\|$ by $(p/2)\|c(x)\|_2^2$ in (4.1), which is exact if $\mu = \lambda_*$ and if p is greater than some positive threshold (see Bertsekas (1982), for example). The advantage of l_p defined by (4.1) is that it can be exact with μ different from λ_* and that the threshold for p is easy to calculate. This is important, because, as we have said it, we shall need to make l_p close to the Lagrangian function. With the penalty function (4.1), this will be done simply by improving the estimate μ of λ_* as the iterates progress to a solution x_* and by decreasing p if necessary and if the requirement $p > \|\mu - \lambda_*\|_D$ allows it.

The path from y_k to x_{k+1}, given by Theorem 3.3, may be obtained by using $l_p(\cdot, \mu)$ as penalty function in algorithm (3.22). From (1.17), the directional derivative of l_p with respect to x in the direction t_k is given at (y_k, μ) by:

$$l_p'(y_k, \mu; t_k) = \nabla f(y_k)^T t_k = g(y_k)^T Z(y_k) t_k. \tag{4.2}$$

When H_k is non-singular, $g(y_k) = -G_k Z(y_k) t_k$, where $G_k := H_k^{-1}$. In this case, we shall also use:

$$\nabla f(y_k)^T t_k = g(y_k)^T Z(y_k) t_k = -t_k^T Z(y_k)^T G_k Z(y_k) t_k. \tag{4.3}$$

Now, it remains to give conditions for the feasibility of a linear search on l_p starting at x_k in the direction r_k. The directional derivative of l_p with respect to x in the direction $r_k := -A(x_k)^- c(x_k)$ is given at (x_k, μ) by:

$$l_p'(x_k, \mu; r_k) = (\lambda(x_k) - \mu)^T c(x_k) - p\|c(x_k)\|, \tag{4.4}$$

where we used the multiplier estimate $\lambda(x)$ given in (1.15). Remark that for the derivative of the last term of (4.1), noting $\nu(z) := \|z\|$, we have $(\nu \circ c)'(x_k; r_k) = \nu'(c(x_k); -c(x_k)) = -\nu(c(x_k))$, by definition of the directional derivative. Therefore, from (4.4), r_k is a descent direction of $l_p(\cdot, \mu)$ at x_k, if $p > \|\lambda(x_k) - \mu\|_D$. This shows that p will have to be adapted sometimes in order to preserve this inequality before doing the transversal step. We shall denote by p_k the value of the penalty parameter at iteration k. In the same way, we shall see that μ will have to be modified at some iterations and we shall denote by μ_k its value at iteration k. Therefore, a condition to satisfy at each iteration (from x_k to x_{k+1}) is:

$$p_k \geqslant \|\lambda(x_k) - \mu_k\|_D + p_{L,k}, \tag{4.5}$$

where $p_{L,k}$ is some positive number.

Let ρ_k denote the step size along the *transversal displacement* r_k:

$$y_k := x_k + \rho_k r_k. \tag{4.6}$$

We shall determine ρ_k by Armijo's (1966) procedure. Two parameters, α and β, are chosen in $]0, 1[$ and ρ_k is taken in the form

$$\rho_k := \beta^{b_k}, \tag{4.7}$$

where b_k is the smallest non-negative integer such that

$$x_k + \rho_k r_k \in \omega$$

and

$$l_{p_k}(x_k + \rho_k r_k, \mu_k) \leqslant l_{p_k}(x_k, \mu_k) + \alpha \rho_k l'_{p_k}(x_k, \mu_k; r_k). \tag{4.8}$$

If (4.5) is satisfied and x_k is in ω (an open set), such a b_k always exists.

We can now outline our reduced secant algorithm.

Algorithm RSA. (4.9)

1. Choose the constants: $0 < \alpha < 1$, $0 < \beta < 1$, $0 < \alpha_1 < \frac{1}{2}$, $\alpha_1 < a_2 < 1$, $0 < \varepsilon$, $1 < a_i$ $(i = 1, 2, 3)$.
2. Choose a point x_0 in ω and H_0, a symmetric positive definite matrix of order $n - m$.
3. Calculate $\lambda(x_0)$ by (1.15), choose $p_{L,0} > 0$, set $\mu_0 := \lambda(x_0)$ and $p_0 := S(p_{L,0})$ and set the indices $k := 0$ (iterations), $i := 0$ (adaptation of $p_{L,k}$), $j := 0$ (adaptation of p_k and μ_k).
4. Select a transversal step size ρ_k by Armijo's procedure (4.7)-(4.8) and set $y_k := x_k + \rho_k r_k$.
5. Execute the longitudinal search algorithm (3.22), starting with $\tau = 1$ and using the penalty function $l_{p_k}(\cdot, \mu_k)$ instead of $\theta(\cdot)$ in (3.15) to determine a step size τ_k and a point x_{k+1}, if possible.
6. Calculate $\varepsilon_k := \|g(y_k)\| + \|c(x_{k+1})\|$. If $\varepsilon_k < \varepsilon$ then stop.
7. Update H_k: $\gamma_k := g(x_{k+1}) - g(y_k)$, $\delta_k := \tau_k Z(y_k) t_k$, $H_{k+1} := \overline{\text{BFGS}}(H_k, \gamma_k, \delta_k)$.
8. Calculate $\lambda(x_{k+1})$ by (1.15).

9. Adapt $p_{L,k} \to p_{L,k+1}$.
10. Adapt $p_k \to p_{k+1}$ and $\mu_k \to \mu_{k+1}$.
11. Set $k := k+1$ and go to statement 4.

The algorithm calls for some comments. In statement 1, ε is a positive convergence threshold and is used in statement 6. The positive constants a_1, a_2 and a_3 will be used in the adaptation rules for $p_{L,k}$, p_k and μ_k (statements 9 and 10) given below. In statement 2, H_0 can be chosen as the identity matrix but this does not take into account the scaling of the problem. Therefore, a possible choice is to take $H_0 = I$ in the first longitudinal search (statement 5) and to calculate H_1 by updating $h_0 I$ rather than I, where $h_0 := \gamma_0^T \delta_0 / \gamma_0^T \gamma_0$. This term minimizes in h the condition number of $\overline{\text{BFGS}}(hI, \gamma_0, \delta_0)$, the inverse BFGS update of hI (see Oren and Spedicato (1976)), and seems to give good numerical results (see Shanno and Phua (1978)). In statement 3, $p_{L,0}$ should be taken large enough and the function $S:]0, \infty[\to]0, \infty[$ is supposed to satisfy the following properties:

$$S \text{ is nondecreasing on }]0, \infty[\quad \text{and} \quad S(a) \geq a, \ \forall a \in]0, \infty[, \tag{4.10}$$

$$\text{for } 0 < a \leq b < \infty, \quad S([a, b]) \text{ is finite,} \tag{4.11}$$

$$S(a) \to 0 \quad \text{as } a \to 0+. \tag{4.12}$$

This function S will be used again in the adaptation rules for $p_{L,k}$, p_k and μ_k and these properties will be useful to prove a global convergence result for the algorithm. For example, we may follow Bonnans (1989) and take $S(a) := \min\{10^q : a \leq 10^q, q \text{ integer}\}$. Statement 4 will always succeed because inequality (4.5) will be guaranteed by the adaptation rules for $p_{L,k}$, p_k and μ_k (statements 9 and 10) and because if statement 5 succeeds, the point x_k is in ω. Note that if $r_k = 0$, $\rho_k = 1$ is always accepted in statement 4! On the other hand, statement 5 may not succeed, as described in Theorem 3.3(ii) and commented in Remark 3.4. In order not to prevent superlinear convergence from occurring we suppose that the initial τ in the longitudinal search algorithm is chosen equal to 1. In statement 7, the inverse BFGS formula (1.18) is always well defined because, by construction, $\gamma_k^T \delta_k$ is positive.

Before stating the adaptation rules for $p_{L,k}$, p_k and μ_k in statements 9 and 10, we need to examine under what conditions the unit step sizes ρ_k and τ_k are accepted in both the transversal and longitudinal displacements. These are the contents of the following two propositions.

Proposition 4.1. *Suppose that Algorithm RSA* (4.9) *produces bounded sequences* (μ_k) *and* (p_k) *and a sequence* (x_k) *in* ω *that converges to a solution* x_* *of problem* (1.1). *Then, we have for large* k:

$$\begin{aligned} & l_{p_k}(x_k + r_k, \mu_k) - l_{p_k}(x_k, \mu_k) - \alpha l'_{p_k}(x_k, \mu_k; r_k) \\ & \quad = (1-\alpha) l'_{p_k}(x_k, \mu_k; r_k) + o(\|r_k\|). \end{aligned} \tag{4.13}$$

Proof. For large k, the segment $[x_k, x_k + r_k]$ is in ω. Then, f and c being twice differentiable on ω, Taylor's expansions give

$$f(x_k + r_k) = f(x_k) + \lambda(x_k)^{\mathrm{T}} c(x_k) + o(\|r_k\|)$$

and $c(x_k + r_k) = o(\|r_k\|)$. Consequently, (p_k) and (μ_k) being bounded, we get (4.13). □

Proposition 4.2. *Suppose that Algorithm RSA* (4.9) *produces a bounded sequence* (p_k), *a sequence* (x_k) *in* ω *that converges to a solution* x_* *of problem* (1.1), *a sequence* (μ_k) *that converges to the associated multiplier* λ_* *and a bounded sequence of non-singular reduced matrices* (H_k) *such that* (1.12) *is satisfied with* $G_k = H_k^{-1}$. *Then, we have for large* k:

$$\begin{aligned} &l_{p_k}(y_k + t_k, \mu_k) - l_{p_k}(y_k, \mu_k) - \alpha_1 l'_{p_k}(y_k, \mu_k; t_k) \\ &\quad \leq (\tfrac{1}{2} - \alpha_1) l'_{p_k}(y_k, \mu_k; t_k) + p_k O(\|t_k\|^2) + o(\|t_k\|^2), \end{aligned} \tag{4.14}$$

$$\alpha_2 g(y_k)^{\mathrm{T}} Z(y_k) t_k - g(y_k + t_k)^{\mathrm{T}} Z(y_k) t_k = \alpha_2 g(y_k)^{\mathrm{T}} Z(y_k) t_k + o(\|t_k\|^2). \tag{4.15}$$

Proof. As the sequence (y_k) converges also to x_* and (H_k) is bounded, the segment $[y_k, y_k + t_k]$ is in ω for large k. Then, f and c being C^2 on ω, Taylor's expansions give:

$$\begin{aligned} f(y_k + t_k) &= f(y_k) + \nabla f(y_k)^{\mathrm{T}} t_k + \tfrac{1}{2} f''(x_*) \cdot t_k^2 + o(\|t_k\|^2), \\ c(y_k + t_k) &= c(y_k) + \tfrac{1}{2} c''(x_*) \cdot t_k^2 + o(\|t_k\|^2). \end{aligned}$$

Hence, using $\mu_k \to \lambda_*$ and the boundedness of (p_k), we get

$$l_{p_k}(y_k + t_k, \mu_k) \leq l_{p_k}(y_k, \mu_k) + \nabla f(y_k)^{\mathrm{T}} t_k + \tfrac{1}{2} t_k^{\mathrm{T}} L_* t_k + p_k O(\|t_k\|^2) + o(\|t_k\|^2).$$

But, as $y \to Z(y)^-$ is continuous, $t_k = Z(y_k)^- Z(y_k) t_k = Z(x_*)^- Z(y_k) t_k + o(\|t_k\|)$. So, with (4.3), the last inequality becomes

$$\begin{aligned} &l_{p_k}(y_k + t_k, \mu_k) - l_{p_k}(y_k, \mu_k) - \alpha_1 \nabla f(y_k)^{\mathrm{T}} t_k \\ &\quad \leq (\tfrac{1}{2} - \alpha_1) \nabla f(y_k)^{\mathrm{T}} t_k - \tfrac{1}{2} t_k^{\mathrm{T}} Z(y_k)^{\mathrm{T}} (G_k - G_*) Z(y_k) t_k + p_k O(\|t_k\|^2) + o(\|t_k\|^2). \end{aligned}$$

From this inequality, (4.2) and (1.12), we deduce (4.14).

On the other hand, as $y \to Z(y)^-$ is C^1 on ω and f is C^2 on ω, g is C^1 on ω and, by a Taylor's expansion and (3.3), we get

$$g(y_k + t_k) = g(y_k) + Z(x_*)^{-\mathrm{T}} L_* t_k + o(\|t_k\|) = g(y_k) + G_* Z(y_k) t_k + o(\|t_k\|).$$

Hence, using again (4.3) and (1.12), we obtain

$$\begin{aligned} g(y_k + t_k)^{\mathrm{T}} Z(y_k) t_k &= g(y_k)^{\mathrm{T}} Z(y_k) t_k + t_k^{\mathrm{T}} Z(y_k)^{\mathrm{T}} G_* Z(y_k) t_k + o(\|t_k\|^2) \\ &= -t_k^{\mathrm{T}} Z(y_k)^{\mathrm{T}} (G_k - G_*) Z(y_k) t_k + o(\|t_k\|^2) \\ &= o(\|t_k\|^2), \end{aligned}$$

hence (4.15). □

Propositions 4.1 and 4.2 enable us to give conditions on the admissibility of the unit step sizes ρ_k and τ_k, i.e. on the non-positivity of the left-hand side of (4.13), (4.14) and (4.15). These conditions will guide us in the design of the adaptation rules for $p_{L,k}$, p_k and μ_k.

We have $r_k = O(\|c(x_k)\|)$ and inequality (4.5) implies that $l'_{p_k}(x_k, \mu_k; r_k) \leq -p_{L,k}\|c(x_k)\|$. Under this condition (4.5) and $\alpha < 1$, the estimate (4.13) becomes

$$l_{p_k}(x_k + r_k, \mu_k) - l_{p_k}(x_k, \mu_k) - \alpha l'_{p_k}(x_k, \mu_k; r_k) \leq -(1-\alpha)[p_{L,k} - \eta_k]\|c(x_k)\|,$$

where $\eta_k \to 0$. Therefore, the step size $\rho_k = 1$ will be accepted asymptotically if, in addition to the hypotheses of Proposition 4.1, $\alpha < 1$, (4.5) is satisfied at each iteration and $(p_{L,k})$ is bounded away from zero.

To exploit the estimates (4.14) and (4.15), let us remark that, under the hypotheses of Proposition 4.2, there exists constants C_1, C_2 and C_3 (independent of k), such that $\|H_k\|_2 \leq C_1$ and $\|t_k\| \leq C_2\|H_k g(y_k)\| \leq C_3\|Z(y_k)t_k\|_2$. Therefore, using (4.3) and the positive definiteness of G_k, we obtain

$$g(y_k)^T Z(y_k) t_k \leq -\|H_k\|_2^{-1}\|Z(y_k)t_k\|_2^2 \leq -C_1^{-1}C_3^{-2}\|t_k\|^2.$$

This shows that the left-hand side of (4.15) will be negative asymptotically if, in addition to the hypotheses of Proposition 4.2, $\alpha_2 > 0$ and H_k is positive definite. Whereas, the left-hand side of (4.14) will be negative asymptotically if, in addition to the hypotheses of Proposition 4.2, we have $\alpha_1 < \frac{1}{2}$, H_k positive definite and p_k sufficiently small. But, because p_k has to satisfy inequality (4.5), this means that $p_{L,k}$ must be small enough, although nonzero as we have just seen.

On the other hand, by modifying μ_k and p_k, we change the merit function at each iteration, which can prevent convergence. So, we have to proceed with caution, and, as in Bonnans (1989), we shall change μ_k and p_k only when the convergence progresses sufficiently. To test convergence, we shall need (ε_k is defined in statement 6 of Algorithm RSA):

$$\tilde{\varepsilon}_k := \min\{\varepsilon_i : 0 \leq i \leq k\}. \tag{4.16}$$

We can now make precise statements 9 and 10 of Algorithm RSA. Indices k, i and j are initialized in statement 3 of the algorithm. Index k is updated in statement 11, while indices i and j are updated in statements 9 and 10 respectively (see below).

Adaptation of $p_{L,k}$ (statement 9 of Algorithm RSA). (4.17)
if $((\tilde{\varepsilon}_k \leq \tilde{\varepsilon}_i / a_1)$ **and** $((l_k \neq 0)$ **or** $(\tau_k \neq 1)))$
then $\{i := k;\ p_{L,k+1} := p_{L,k}/a_2\}$ (4.18)
else $p_{L,k+1} := p_{L,k}$. (4.19)

Adaptation of p_k and μ_k (statement 10 of Algorithm RSA). (4.20)
if $(\tilde{\varepsilon}_k \leq \tilde{\varepsilon}_j / a_3)$
then $\{j := k;\ \mu_{k+1} := \lambda(x_{k+1});\ p_{k+1} := S(p_{L,k+1})\}$ (4.21)
else $\{\mu_{k+1} := \mu_k;\ p_{k+1} := \max(p_k, S(\|\lambda(x_{k+1}) - \mu_{k+1}\|_D + p_{L,k+1})\}$. (4.22)

We recall that the properties of function S have been given in (4.10)-(4.12). We see that, as long as $\tilde{\varepsilon}_k$ does not decrease sufficiently, $p_{L,k}$ and μ_k are not changed;

only p_k can be increased by (4.22) so as to satisfy (4.5). This procedure is recommended by Mayne and Polak (1982) for the globalization of SQP methods using the l_∞ exact penalty function. In our case, however, we have seen that it is necessary to have a penalty function close to the Lagrangian. Therefore, if $\tilde{\varepsilon}_k$ decreases sufficiently, we adapt μ_k and p_k in (4.21), and we decrease $p_{L,k}$ in (4.18) if, furthermore, the unit step size is not accepted.

The next theorem shows these rules work in case of convergence of (x_k).

Theorem 4.3. *Suppose that Algorithm RSA* (4.9) *with the adaptation rules* (4.17) *and* (4.20) *produces a sequence* (x_k) *in* ω *converging to a solution* x_* *of problem* (1.1) *and a bounded sequence of positive definite matrices* (H_k). *Suppose, as well, that condition* (1.12) *is satisfied with* $G_k := H_k^{-1}$. *Then, we have* $\rho_k = 1$, $l_k = 0$ *and* $\tau_k = 1$ *for k large enough.*

Proof. As $x_k \to x_*$, we have $c(x_k) \to 0$, hence $y_k \to x_*$, $g(y_k) \to 0$, $\varepsilon_k \to 0$ and, by (4.21), $\mu_k \to \lambda_*$.

We begin with the longitudinal displacement. Suppose that $l_k \neq 0$ or $\tau_k \neq 1$ infinitely often. Then, by (4.18), we would have $p_{L,k} \to 0$ and by (4.21), (4.22) and (4.12), $p_k \to 0$. Therefore, using Proposition 4.2 (and the comments after its proof), we see that the left-hand sides of (4.14) and (4.15) become negative for large k, and therefore $l_k = 0$ and $\tau_k = 1$ for large k: a contradiction.

Hence, $l_k = 0$ and $\tau_k = 1$ for large k and from (4.19) we see that $p_{L,k} = p_L > 0$, for large k. By using Proposition 4.1 (and the comments after the proof of Proposition 4.2), we see that the left-hand side of (4.13) becomes negative for large k, hence $\rho_k = 1$. □

We now prove a global convergence result for Algorithm RSA.

Theorem 4.4. *Suppose that Algorithm RSA* (4.9) *with the adaptation rules* (4.17) *and* (4.20) *produces sequences* (x_k) *and* (y_k) *in* ω *and a bounded sequence of positive definite matrices* (H_k) *with bounded inverses. Then, one of the following situations occurs*:

(i) $\lim \tilde{\varepsilon}_k = 0$,

(ii) $\mu_k = \mu$ *for large* k, (p_k) *is unbounded and* $\|\lambda(x_k)\|_D \to \infty$ *when* $k \to \infty$ *in* $\{k: p_k > p_{k-1}\}$,

(iii) $\mu_k = \mu$ *for large* k, $p_k = p$ *for large* k *and either* $l_p(x_k, \mu) \to -\infty$ *or for some subsequence* $\operatorname{dist}(x_k, \omega^c) \to 0$.

Remarks 4.5. Because the sequences (x_k) and (y_k) are generated by the algorithm, it is implicitly supposed that the longitudinal search algorithm succeeds at each iteration, which will occur if, for instance, l_p is bounded below and $\omega = \mathbb{R}^n$ (see Remarks 3.4). The situation (i) means that for a subsequence of indices k, $c(x_k) \to 0$ and $g(y_k) \to 0$. Therefore, if ω is bounded, there exists a subsequence of (x_k) that

converges to a solution of problem (1.1). The situation (ii) means that either $\{x_k: p_k > p_{k-1}\}$ is unbounded or has accumulation points $\bar{x}$ on the boundary of ω at which $\lambda(\bar{x})$ is not well defined by (1.15), for instance, because $A(\bar{x})$ has not full rank. In (iii), $\mathrm{dist}(x_k, \omega^c)$ is the distance from x_k to the complementary set of ω.

Proof of Theorem 4.4. Let us suppose that situation (i) does not occur. Then $\lim \tilde{\varepsilon}_k =: \tilde{\varepsilon}_\infty > 0$ and, by (4.19) and (4.22), $p_{L,k} = p_L$ for large k, $\mu_k = \mu$ for large k and (p_k) is a nondecreasing sequence for large k. The, either (p_k) is unbounded or (p_k) is bounded! In the first case, this means by (4.22) and (4.10) that $(\|\lambda(x_k)\|_D)$ is unbounded, and more precisely,

$$\|\lambda(x_k)\|_D \to +\infty \quad \text{for } k \to \infty \text{ in } \{k: p_k > p_{k-1}\},$$

which is the conclusion (ii) of the theorem.

On the other hand, if (p_k) is bounded, (4.22) and (4.11) show that p_k changes finitely often. So, $p_k = p$ for large k. We prove the rest of (iii) by contradiction, supposing that $l_p(x_k, \mu)$ is bounded from below and that (x_k) remains away from ω^c.

We have from Theorem 3.3, (4.8) and (4.5):

$$l_p(x_{k+1}, \mu) \leq l_p(y_k, \mu) \leq l_p(x_k, \mu).$$

Therefore, the sequences $(l_p(x_k, \mu))$ and $(l_p(y_k, \mu))$ converge to the same value.

According to the longitudinal displacement (i.e. to inequalities (3.15) and (3.16)), (4.2), the positive definiteness of H_k and the boundedness of (H_k^{-1}), we have

$$\begin{aligned} l_p(x_{k+1}, \mu) - l_p(y_k, \mu) &\leq \alpha_1 \tau_k l_p'(y_k, \mu; t_k) \\ &= -\alpha_1 \tau_k g(y_k)^T H_k g(y_k) \\ &\leq -\alpha_1 \tau_k \|H_k^{-1}\|_2^{-1} \|g(y_k)\|_2^2 \\ &\leq -\alpha_1 \tau_k C \|g(y_k)\|^2, \end{aligned} \tag{4.23}$$

$$(g(x_{k+1}) - g(y_k))^T Z(y_k) t_k \geq (1-\alpha_2) g(y_k)^T H_k g(y_k) \geq (1-\alpha_2) C \|g(y_k)\|^2. \tag{4.24}$$

Here and below, C denotes a positive constant independent of k. From (4.23) and the convergence of $(l_p(y_k, \mu))$ and $(l_p(x_k, \mu))$ to the same value, we deduce that $\tau_k \|g(y_k)\|^2 \to 0$. From (4.24), the boundedness of (H_k), the Lipschitz continuity of g (with the hypotheses of Section 2, g' is bounded on the convex set ω) and Proposition 3.5, we get

$$\|g(y_k)\|^2 \leq C \|x_{k+1} - y_k\| \, \|g(y_k)\| \leq C \tau_k \|g(y_k)\|^2.$$

Hence, $g(y_k) \to 0$.

Now, from the transversal search (i.e., from (4.8)), (4.4) and (4.5), we have

$$l_p(y_k, \mu) - l_p(x_k, \mu) \leq \alpha \rho_k [(\lambda(x_k) - \mu)^T c(x_k) - p \|c(x_k)\|] \leq -\alpha p_L \rho_k \|c(x_k)\|.$$

Hence, $\rho_k \|c(x_k)\| \to 0$. We are going to show that (ρ_k) is bounded away from zero. This will prove (iii) because then $c(x_k) \to 0$ and, with $g(y_k) \to 0$, we have $\varepsilon_k \to 0$,

which gives the expected contradiction with the initial assumption ($\tilde{\varepsilon}_k \geq \tilde{\varepsilon}_\infty > 0$). Again, we set about it by contradiction. So, let us suppose $\rho_k < 1$ and $\rho_k \to 0$ for k in a subsequence $\mathbb{K}$. We may consider that ρ_k/β is not accepted by the line search because Armijo's criterion (4.8) is not satisfied. Indeed, otherwise it would mean that for a subsequence $\mathbb{K}'$ of $\mathbb{K}$, $\tilde{x}_k := x_k + (\rho_k/\beta) r_k \notin \omega$ for $k \in \mathbb{K}'$. But $\rho_k \| c(x_k) \| \to 0$ implies that $(\rho_k/\beta) r_k \to 0$ ($(A(x_k)^-)$ is bounded) and therefore $(x_k)_{k \in \mathbb{K}'}$ would go closer and closer to a point $\tilde{x}_k$ not in ω, a situation that has been discarded by hypothesis. So, for k in $\mathbb{K}$, we have

$$l_p\left(x_k + \frac{\rho_k}{\beta} r_k, \mu\right) - l_p(x_k, \mu) > \alpha \frac{\rho_k}{\beta} [(\lambda(x_k) - \mu)^{\mathrm{T}} c(x_k) - p \| c(x_k) \|]. \quad (4.25)$$

Expanding $l_p(x_k + (\rho_k/\beta) r_k, \mu)$ about x_k gives

$$\begin{aligned} l_p\left(x_k + \frac{\rho_k}{\beta} r_k, \mu\right) &\leq f(x_k) + \frac{\rho_k}{\beta} \lambda(x_k)^{\mathrm{T}} c(x_k) + \rho_k^2 C \| r_k \|^2 \\ &\quad + \mu^{\mathrm{T}} c(x_k) - \frac{\rho_k}{\beta} \mu^{\mathrm{T}} c(x_k) + p\left(1 - \frac{\rho_k}{\beta}\right) \| c(x_k \| \\ &\leq l_p(x_k, \mu) + \frac{\rho_k}{\beta} [(\lambda(x_k) - \mu)^{\mathrm{T}} c(x_k) - p \| c(x_k) \|] \\ &\quad + \rho_k^2 C \| r_k \|^2. \end{aligned}$$

Therefore, with (4.25) and (4.5), this leads to

$$(1 - \alpha) \frac{\rho_k}{\beta} p_{\mathrm{L}} \| c(x_k) \| < C \rho_k^2 \| c(x_k) \|^2.$$

Therefore $\rho_k \| c(x_k) \|$ is positive for k in $\mathbb{K}$ and dividing by this factor in this inequality, we obtain a contradiction with the fact that $\rho_k \| c(x_k) \| \to 0$. □

Remark 4.6. The fact that, in situation (i) of the theorem, only a subsequence of (ε_k) converges to 0 (instead of all the sequence) does not come from the hypotheses of the theorem, which are rather strong. The boundedness of (H_k) and (H_k^{-1}) is usually enough to imply the convergence of all the sequence (ε_k) to zero, as this may be also observed in unconstrained optimization, see Wolfe (1969, Theorem 1). But it comes from the way the convergence is checked in statements 9 and 10 of the algorithm, using the *nonincreasing* sequence $(\tilde{\varepsilon}_k)_{k \geq 0}$ defined in (4.16). Indeed, suppose that, instead of being calculated by (4.16), $\tilde{\varepsilon}_k$ is given by

$$\tilde{\varepsilon}_k := \Phi(\{\varepsilon_i : 0 \leq i \leq k\}), \quad (4.26)$$

where the function Φ is such that, for any sequence $(\varepsilon_k)_{k \geq 0}$ of positive numbers, $\tilde{\varepsilon}_k$ is positive, $(\tilde{\varepsilon}_k)_{k \geq 0}$ is *nonincreasing* and

$$(\varepsilon_k \to 0) \Rightarrow (\tilde{\varepsilon}_k \to 0). \quad (4.27)$$

The function *min* in (4.16) is just an example of such Φ. Then, it can be shown that Theorems 4.3 and 4.4 remain true. Therefore, we would have a better convergence result in Theorem 4.4(i), if it were possible to find a function Φ such that we have also

$$(\tilde{\varepsilon}_k \to 0) \Rightarrow (\varepsilon_k \to 0). \tag{4.28}$$

Unfortunately, this is not possible. To see this, let us consider the following algorithm where ε_{k+1} is calculated by observing $\tilde{\varepsilon}_k$:

1. $l := 1$; $j := 1$; $\varepsilon_0 := 1$;
2. **for** $(k \geq 0)$ **do** {
2.1. $\tilde{\varepsilon}_k := \Phi\ (\{\varepsilon_i : 0 \leq i \leq k\})$;
2.2. **if** $(\tilde{\varepsilon}_k \geq 1/l)$
2.3. **then** $\{j := j+1;\ \varepsilon_{k+1} := 1/j\}$
2.4. **else** $\{l := l+1;\ \varepsilon_{k+1} := 1\}$
2. }

Let us show that if Φ satisfies (4.27), the sequences (ε_k) and $(\tilde{\varepsilon}_k)$ generated by this algorithm are such that (ε_k) does not converge to zero and $\tilde{\varepsilon}_k \to 0$. Therefore, Φ does not satisfy (4.28).

The sequence $(\tilde{\varepsilon}_k)$ converges to zero, because, otherwise, $(\tilde{\varepsilon}_k)$ being nonincreasing, there would exist a positive integer l_0 such that $\tilde{\varepsilon}_k \geq 1/l_0$, $\forall k \geq 0$, which would mean that statement 2.4 is executed at most l_0 times. Therefore, for large k, statement 2.3 would be executed and $\varepsilon_k \to 0$, which would be in contradiction with (4.27). Therefore, $\tilde{\varepsilon}_k \to 0$. But then, statement 2.4 is executed infinitely often, so (ε_k) does not converge to zero.

5. Conclusion

We have presented an algorithm for equality constrained optimization of the reduced type in which the projected Hessian of the Lagrangian is approximated by updating *at each iteration* a matrix H_k using the BFGS formula and two vectors: γ_k, the change in the reduced gradient, and δ_k, the corresponding reduced displacement.

The main purpose of the paper has been to show the possibility of obtaining the positivity of $\gamma_k^{\mathrm{T}}\delta_k$, which is essential to guarantee the positive definiteness of the reduced matrices H_k. This feature is due to a particular design of the longitudinal displacement, which minimizes the objective function f while roughly maintaining constant the value of the constraint function c. For this, we introduce a step size τ_k scaling the reduced displacement in $\mathbb{R}^{n-m}$, while the longitudinal displacement in $\mathbb{R}^n$ becomes piecewise linear. Wolfe's criteria are used to determine the step size τ_k.

The algorithm is made globally convergent by using a nondifferentiable augmented Lagrangian function. Another feature of the algorithm is to separate completely the longitudinal and transversal displacements: the step sizes of both displacements are determined by two different searches on the penalty function.

The technique used here to maintain the positive definiteness of the matrices H_k may be seen as a generalization to equality constrained optimization of Wolfe's step-size selection procedure in unconstrained optimization. It is well known that this technique cannot be used in the framework of quasi-Newton or SQP methods. As the technique works well in unconstrained optimization, this may be seen as an advantage of the reduced framework over the SQP methods. However, the algorithm proposed here always requires at least two (and exactly two, asymptotically) linearizations of the constraints for each superlinear step, which can be an important overcost in some applications. Therefore, the developed technique should be extended to those reduced methods that require only one linearization of the constraints per iteration.

As mentioned in the text, a weak point of the algorithm lies in the way the multipliers and the penalty parameters are adapted to improve the penalty function. Indeed, it requires from the algorithm to feel the closeness of a solution and therefore impoverishes the global convergence result (see Remark 4.6). We think that some progress might be obtained on this topic as well.

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