Institut Polytechnique de Paris
Master Program in Optimization

Advanced Continuous Optimization

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— May 26, 2021 —

Lectures: November 16, 2020 → January 11, 2020
Examination: January 18, 2020

Keep a close eye to
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Outline

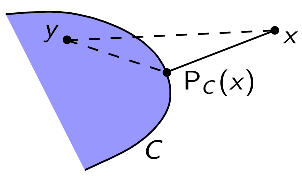
Background
Convex analysis (notation)

- $\bar{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$.
- $\mathbb{R}_+ := \{t \in \mathbb{R} : t \geqslant 0\}$ and $\mathbb{R}_{++} := \{t \in \mathbb{R} : t > 0\}$.
- B, $\bar{B}$: open and closed unit balls centered at the origin; for $r \geqslant 0$: $B(x, r) = x + rB$ and $\bar{B}(x, r) = x + r\bar{B}$.
- $\mathbb{E}$, $\mathbb{F}$, $\mathbb{G}$ usually denote Euclidean vector spaces.

Background
Convex analysis (projection)

- *Definition*. For a *nonempty*, *closed*, *convex* set C in a *Euclidean* space $\mathbb{E}$, with scalar product $\langle \cdot, \cdot \rangle$ and associated norm $\| \cdot \|$, the problem

$$\inf_{y \in C} \|y - x\|$$



has a *unique* solution, called the projection of x on C and denoted $\mathrm{P}_C(x)$.

- *Characterization*. For $x \in \mathbb{E}$ and $\bar{x} \in C$, there hold

$$\begin{aligned} \bar{x} = \mathrm{P}_C(x) \quad &\iff \quad \langle y - \bar{x}, \bar{x} - x \rangle \geqslant 0, \quad \forall y \in C, \\ &\iff \quad \langle y - \bar{x}, y - x \rangle \geqslant 0, \quad \forall y \in C, \\ &\iff \quad \langle y - x, \bar{x} - x \rangle \geqslant \|\bar{x} - x\|^2, \quad \forall y \in C. \end{aligned}$$

Background
Convex analysis (relative interior I)

- The affine hull of $P \subseteq \mathbb{E}$ is the smallest affine space containing P:

$$\operatorname{aff} P := \bigcap \{A : A \text{ is an affine space containing } P\}.$$

- The relative interior of $P \subseteq \mathbb{E}$ is its interior in aff P:

$$\operatorname{ri} P := \{x \in P : \exists\, r > 0 \text{ such that } [B(x,r) \cap \operatorname{aff} P] \subseteq P\}.$$

In finite dimension, the following holds

$$C \text{ convex and nonempty} \quad \Longrightarrow \quad \begin{cases} \operatorname{ri} C \neq \varnothing, \\ \operatorname{aff} C = \operatorname{aff}(\operatorname{ri} C). \end{cases}$$

Background
Convex analysis (relative interior II)

Proposition (relative interior criterion)

Let C be a nonempty convex set and $x \in \mathbb{E}$. Then

$$x \in \operatorname{ri} C \text{ and } y \in \overline{C} \quad \Longrightarrow \quad [x, y) \subseteq \operatorname{ri} C.$$

$$x \in \operatorname{ri} C \quad \Longleftrightarrow \quad \forall\, x_0 \in C \text{ (or aff } C),\ \exists\, t > 1 : \quad (1-t)x_0 + tx \in C.$$

Let C be a nonempty convex set. Then

- $\operatorname{ri} C$ is convex,
- $\overline{C}$ is convex and $\operatorname{aff} C = \operatorname{aff} \overline{C}$,
- $\overline{\operatorname{ri} C} = \overline{C}$ and $\operatorname{ri} \overline{C} = \operatorname{ri} C$ (i.e., the last operation prevails).

A point $x \in C$ is said absorbing if $\forall\, d \in \mathbb{E}$, $\exists\, t > 0$ such that $x + td \in C$.

- $x \in \operatorname{int} C \iff x$ is absorbing.

Background

Convex analysis (dual cone and Farkas lemma)

- The (*positive*) *dual cone* of a set $P \subseteq \mathbb{E}$ is defined by

$$P^+ := \{d \in \mathbb{E} : \langle d, x \rangle \geqslant 0, \ \forall x \in P\}.$$

The negative dual cone of a set P is $P^- := -P^+$.

Lemma (Farkas, generalized)

Let $\mathbb{E}$ and $\mathbb{F}$ be two Euclidean spaces, $A : \mathbb{E} \to \mathbb{F}$ a linear map, and K a nonempty convex cone of $\mathbb{E}$. Then

$$\overline{A(K)} = \{y \in \mathbb{F} : A^* y \in K^+\}^+.$$

- $A^* : \mathbb{F} \to \mathbb{E}$ is defined by: $\forall (x, y) \in \mathbb{E} \times \mathbb{F}$, $\langle A^* y, x \rangle = \langle y, Ax \rangle$.
- One cannot get rid of the closure on $A(K)$ in general.
- If K is polyhedral, then $A(K)$ is polyhedral, hence closed.
- For $K = \mathbb{E}$, one recovers $\mathcal{R}(A) = \mathcal{N}(A^*)^\perp$.

Background

Convex analysis (tangent and normal cones)

- Tangent cone
  - Let C be a convex set of $\mathbb{E}$ and $x \in C$.
  - The cone of feasible directions for C at x is $\mathrm{T}_x^f C := \mathbb{R}_+(C - x)$.
  - The tangent cone to C at x is the closure of the previous one

$$\mathrm{T}_x C \equiv \mathrm{T}_C(x) = \overline{\mathbb{R}_+(C - x)}.$$

- Normal cone
  - Let C be a convex set of $\mathbb{E}$ and $x \in C$.
  - The normal cone to C at x is

$$\mathrm{N}_x C \equiv \mathrm{N}_C(x) = \{d \in \mathbb{E} : \langle x' - x, d \rangle \leqslant 0, \ \forall x' \in C\}.$$

  - There hold

$$\mathrm{N}_x C = (\mathrm{T}_x C)^- \qquad \text{and} \qquad \mathrm{T}_x C = (\mathrm{N}_x C)^-.$$

Background

Convex analysis (asymptotic cone I)

Let $\mathbb{E}$ be a vector space of finite dimension and C be a nonempty closed convex set of $\mathbb{E}$.

- The asymptotic cone of C is

$$C^\infty := \{d \in \mathbb{E} : C + \mathbb{R}_+ d \subseteq C\} = \{d \in \mathbb{E} : C + d \subseteq C\}.$$

- Properties
  - C^∞ is closed.
  - For any $x \in C$:

$$\begin{aligned} C^\infty &= \{d \in \mathbb{E} : x + \mathbb{R}_+ d \subseteq C\} = \bigcap_{t>0} \frac{C - x}{t} \\ &= \left\{ d \in \mathbb{E} : \exists \{x_k\} \subseteq C,\ \exists \{t_k\} \to \infty \text{ such that } \frac{x_k}{t_k} \to d \right\}. \end{aligned}$$

  - Boundedness by calculation:

$$C \text{ is bounded} \quad \iff \quad C^\infty = \{0\}.$$

Background

Convex analysis (asymptotic cone II)

- Two calculation rules (there are many more)
  - If $K \neq \varnothing$, then K is a closed convex cone $\iff K^\infty = K$.
  - For an arbitrary collection $\{C_i\}_{i \in I}$ of closed convex sets C_i with nonempty intersection:

$$(\cap_{i \in I} C_i)^\infty = \cap_{i \in I} C_i^\infty.$$

- Example
  - Let $A : \mathbb{E} \to \mathbb{F}$ and $B : \mathbb{E} \to \mathbb{G}$ be linear maps, $a \in \mathbb{F}$, $b \in \mathbb{G}$, K be a nonempty closed convex cone of $\mathbb{G}$, and

$$P := \{x \in \mathbb{E} : Ax = a,\ Bx \in b + K\} \neq \varnothing.$$

Then

$$P^\infty = \{d \in \mathbb{E} : Ad = 0,\ Bd \in K\}.$$

Background

Convex analysis (strict separation of convex sets)

The sets S_1 and S_2 in a Euclidean vector space $\mathbb{E}$ are said to be strictly separable if there exists a vector $\xi \in \mathbb{E}$ (necessarily nonzero) such that

$$\sup_{x_1 \in S_1} \langle \xi, x_1 \rangle < \inf_{x_2 \in S_2} \langle \xi, x_2 \rangle .$$

Proposition (strict separation of convex sets)

One can strictly separate two disjoint nonempty closed convex sets C_1 and $C_2 \subseteq \mathbb{E}$ in any of the following situations

1. $C_1 - C_2$ *is closed,*
2. $C_1^\infty \cap C_2^\infty = \{0\}$,
3. C_1 *or* C_2 *is compact,*
4. C_1 *and* C_2 *are polyhedral.*

Background

Convex analysis (convex polyhedron)

A convex polyhedron in $\mathbb{E}$ is a set of the form

$$P = \{x \in \mathbb{E} : Ax \leqslant b\},$$

where $A : \mathbb{E} \to \mathbb{R}^m$ is a linear map and $b \in \mathbb{R}^m$. It is a closed set. For $x \in P$, define

$$I(x) := \{i \in [1:m] : (Ax - b)_i = 0\}.$$

- $(\{x_k\} \to x) \implies I(x_k) \subseteq I(x)$ for large k.
- If $T : \mathbb{E} \to \mathbb{F}$ is linear, then $T(P)$ is a convex polyhedron.
- If P_1 and P_2 are polyhedra, then $P_1 + P_2$ is a polyhedron.
- $\mathrm{T}_x P = \mathrm{T}_x^f P = \{d \in \mathbb{E} : (Ad)_{I(x)} \leqslant 0\}$.
  $I(x_1) \subseteq I(x_2) \implies \mathrm{T}_{x_1} P \supseteq \mathrm{T}_{x_2} P$.
- $\mathrm{N}_x P = \mathrm{cone}\{A^* e_i : i \in I(x)\}$ (A^*: adjoint of A for the scalar product of $\mathbb{E}$).
  $I(x_1) \subseteq I(x_2) \implies \mathrm{N}_{x_1} P \subseteq \mathrm{N}_{x_2} P$.

Background

Convex analysis (asymptotic function I)

The domain and the epigraph of a function $f : \mathbb{E} \to \mathbb{R} \cup \{+\infty\}$ are the sets

$$\operatorname{dom} f := \{x \in \mathbb{E} : f(x) < +\infty\} \quad \text{and} \quad \operatorname{epi} f := \{(x, \alpha) \in \mathbb{E} \times \mathbb{R} : f(x) \leqslant \alpha\}.$$

Let $\overline{\operatorname{Conv}}(\mathbb{E})$ be the set of closed (i.e., epi f is closed) proper (i.e., epi $f \neq \varnothing$) convex (i.e., epi f is convex) functions.

Proposition (asymptotic function f^∞)

If $f \in \overline{\operatorname{Conv}}(\mathbb{E})$, *then*

1. $(\operatorname{epi} f)^\infty$ *is the epigraph of a function* $f^\infty : \mathbb{E} \to \mathbb{R} \cup \{+\infty\}$,
2. *for all* $x \in \operatorname{dom} f$ *and all* $d \in \mathbb{E}$

$$f^\infty(d) = \lim_{t \to \infty} \frac{f(x + td) - f(x)}{t} = \lim_{t \to \infty} \frac{f(x + td)}{t},$$

3. $\operatorname{dom} f^\infty \subseteq (\operatorname{dom} f)^\infty$,
4. $f^\infty \in \overline{\operatorname{Conv}}(\mathbb{E})$.

Background

Convex analysis (asymptotic function II)

The sublevel set of $f : \mathbb{E} \to \mathbb{R} \cup \{+\infty\}$ of level $\nu \in \mathbb{R}$ is the set

$$L_\nu(f) := \{x \in \mathbb{E} : f(x) \leqslant \nu\}.$$

Proposition (existence of a bounded set of minimizers)

If $f \in \overline{\operatorname{Conv}}(\mathbb{E})$, *then*

1. $\forall \nu \in \mathbb{R}$ *such that* $L_\nu(f) \neq \varnothing$, *the following holds*

$$\left[L_\nu(f)\right]^\infty = \{d \in \mathbb{E} : f^\infty(d) \leqslant 0\},$$

2. *the following properties are equivalent:*
   - i. $\exists \nu \in \mathbb{R}$: $L_\nu(f)$ *is nonvoid and bounded,*
   - ii. $\forall \nu \in \mathbb{R}$: $L_\nu(f)$ *is bounded,*
   - iii. $\operatorname{Arg\,min} f$ *is nonvoid and bounded,*
   - iv. $\forall d \in \mathbb{E} \setminus \{0\}$: $f^\infty(d) > 0$.

Background

Convex analysis (subdifferential)

A subgradient at $x \in \mathbb{E}$ of $f \in \text{Conv}(\mathbb{E})$ is a vector $x^* \in \mathbb{E}$ such that

$$f(y) \geqslant f(x) + \langle x^*, y - x \rangle, \quad \forall\, y \in \mathbb{E}.$$

The subdifferential $\partial f(x)$ of f at x is the set of its subgradients at x. f is said to be subdifferentiable at x if $\partial f(x) \neq \varnothing$.

Proposition (characterization of subgradients)

For $f \in \overline{\text{Conv}}(\mathbb{E})$, $x \in \text{dom}\, f$, $x^* \in \mathbb{E}$, here are equivalent propreties

1. $x^* \in \partial f(x)$,
2. $f'(x; d) \geqslant \langle x^*, d \rangle, \quad \forall\, d \in \mathbb{E}$,
3. $x \in \text{Arg min}_{y \in \mathbb{E}} \big(f(y) - \langle x^*, y \rangle\big) = \text{Arg max}_{y \in \mathbb{E}} \big(\langle x^*, y \rangle - f(y)\big)$.

Background

Nonsmooth analysis (multifunction I)

- A multifunction T (or set-valued mapping) between two sets $\mathbb{E}$ and $\mathbb{F}$ is a function from $\mathbb{E}$ to $\mathcal{P}(\mathbb{F})$, the set of the subsets of $\mathbb{F}$. Notation:

$$T : \mathbb{E} \multimap \mathbb{F} : x \mapsto T(x) \subseteq \mathbb{F}.$$

  Same concept as a binary relation (i.e., the data of a part of $\mathbb{E} \times \mathbb{F}$).

- The graph, the domain, the range of $T : \mathbb{E} \multimap \mathbb{F}$ are defined by

$$\begin{aligned} \mathcal{G}(T) &:= \{(x, y) \in \mathbb{E} \times \mathbb{F} : y \in T(x)\}, \\ \mathcal{D}(T) &:= \{x \in \mathbb{E} : (x, y) \in \mathcal{G}(T) \text{ for some } y \in \mathbb{F}\} = \pi_{\mathbb{E}} \mathcal{G}(T), \\ \mathcal{R}(T) &:= \{y \in \mathbb{F} : (x, y) \in \mathcal{G}(T) \text{ for some } x \in \mathbb{E}\} = \pi_{\mathbb{F}} \mathcal{G}(T). \end{aligned}$$

- The image of a part $P \subseteq \mathbb{E}$ by T is

$$T(P) := \bigcup_{x \in P} T(x).$$

Background
Nonsmooth analysis (multifunction II)

- The inverse of a multifunction $T : \mathbb{E} \multimap \mathbb{F}$ (it always exists!) is the multifunction $T^{-1} : \mathbb{F} \multimap \mathbb{E}$ defined by

$$T^{-1}(y) := \{x \in \mathbb{E} : y \in T(x)\}.$$

Hence

$$y \in T(x) \quad \Longleftrightarrow \quad x \in T^{-1}(y).$$

- When $\mathbb{E}$, $\mathbb{F}$ are topological/metric spaces, a multifunction $T : \mathbb{E} \multimap \mathbb{F}$ is said to be
  - closed at $x \in \mathbb{E}$ if $y \in T(x)$ when $(x_k, y_k) \in \mathcal{G}(T)$ converges to (x, y),
  - closed if $\mathcal{G}(T)$ is closed in $\mathbb{E} \times \mathbb{F}$ (i.e., T is closed at any $x \in \mathbb{E}$),
  - upper semi-continuous at $x \in \mathbb{E}$ if $\forall\, \varepsilon > 0$, $\exists\, \delta > 0$, $\forall\, x' \in x + \delta B$, one has $T(x') \subseteq T(x) + \varepsilon B$ (in this definition, B may be the open or closed ball at any place).

Background
Nonsmooth analysis (multifunction III)

- When $\mathbb{E}$, $\mathbb{F}$ are vector spaces, a multifunction $T : \mathbb{E} \multimap \mathbb{F}$ is said to be convex if $\mathcal{G}(T)$ is convex in $\mathbb{E} \times \mathbb{F}$. This is equivalent to saying that $\forall\, (x_0, x_1) \in \mathbb{E}^2$ and $\forall\, t \in [0, 1]$:

$$T((1-t)x_0 + tx_1) \supseteq (1-t)T(x_0) + tT(x_1).$$

Note that

$$T \text{ convex and } C \text{ convex in } \mathbb{E} \quad \Longrightarrow \quad T(C) \text{ convex in } \mathbb{F}.$$

Background

Nonsmooth analysis (Lipschitz continuity)

- Let $\mathbb{E}$ and $\mathbb{F}$ be two normed spaces and $F : \mathbb{E} \to \mathbb{F}$ be a function.
- F is Lipschitz on a set $U \subseteq \mathbb{E}$ if

$$\exists\, L \geqslant 0, \quad \forall\, (x, x') \in U^2 : \quad \|F(x) - F(x')\| \leqslant L\|x - x'\|.$$

- F is Lipschitz near $x \in \mathbb{E}$ if it is Lipschitz on some neighborhood of x.
- F is locally Lipschitz on an open set $\Omega \subseteq \mathbb{E}$ if it is Lipschitz near any point of Ω.

Background

Optimization (a generic problem)

One considers the generic optimization problem

$$(P_X) \quad \begin{cases} \min\ f(x) \\ x \in X. \end{cases}$$

where

- $f : \mathbb{E} \to \mathbb{R}$ ($\mathbb{E}$ is a Euclidean vector space),
- X is a set of $\mathbb{E}$ (possibly nonconvex).

Définitions:

- solution or (global) minimum $x_* \in X$ if $\forall\, x \in X$, $f(x_*) \leqslant f(x)$,
- local minimum $x_* \in X$ if $\exists V \in \mathcal{N}(x_*)$, $\forall\, x \in X \cap V$, $f(x_*) \leqslant f(x)$,
- strict local/global minimum x_* if $f(x_*) < f(x)$ above when $x \neq x_*$.

Background

Optimization (tangent cone to a nonconvex set)

- A direction $d \in \mathbb{E}$ is tangent to $X \subseteq \mathbb{E}$ at $x \in X$ (in the sense of Bouligand) if

$$\exists \{x_k\} \subseteq X, \quad \exists \{t_k\} \downarrow 0 : \quad \frac{x_k - x}{t_k} \to d.$$

- The tangent cone to X at x (in the sense of Bouligand) is the set of tangent directions. It is denoted by

$$\mathrm{T}_x X \qquad \text{or} \qquad \mathrm{T}_X(x).$$

- Properties Let $x \in X$.

$$\mathrm{T}_x X \text{ is closed.}$$

$$X \text{ is convex} \quad \Longrightarrow \quad \mathrm{T}_x X \text{ is convex and } \mathrm{T}_x X = \overline{\mathbb{R}_+(X - x)}.$$

Background

Optimization (Peano-Kantorovich NC1)

Theorem (Peano-Kantorovich NC1)

If x_* is a local minimizer of (P_X) and f is differentiable at x_*, then

$$\nabla f(x_*) \in (\mathrm{T}_{x_*} X)^+.$$

- The gradient of f at x is denoted by $\nabla f(x) \in \mathbb{E}$ and is defined from the derivative $f'(x)$ by

$$\forall\, d \in \mathbb{E} : \quad \langle \nabla f(x_*), d \rangle = f'(x) \cdot d.$$

- Recall that a convex function $f : \mathbb{E} \to \mathbb{R} \cup \{+\infty\}$ has directional derivatives $f'(x; d) \in \overline{\mathbb{R}}$ for all $x \in \operatorname{dom} f$ and all $d \in \mathbb{E}$.

Proposition (NSC1 for a convex problem)

Suppose that X is convex, f is convex on X, and $x_* \in X$. Then x_* is a global solution to (P_X) if and only if

$$\forall x \in X : \quad f'(x_*; x - x_*) \geqslant 0.$$

Proof. Straightforward, using the convexity inequality

$$\forall\, x \in X : \quad f(x) \geqslant f(x_*) + f'(x_*; x - x_*).$$

Let $\mathbb{E}$, $\mathbb{F}$ be Euclidean vector spaces. The equality constrained problem is

$$(P_E) \quad \left\{ \begin{array}{l} \inf_x \; f(x) \\ c(x) = 0, \end{array} \right.$$

where $f : \mathbb{E} \to \mathbb{R}$, $c : \mathbb{E} \to \mathbb{F}$ are smooth (possibly non convex) functions.

- The feasible set is denoted by

$$X_E := \{x \in \mathbb{E} : c(x) = 0\}.$$

- (P_E) is said to be convex if f is convex and X_E is convex.

Theorem (NC1 for (P_E), Lagrange, XVIIIth)

If x_* is a local minimum of (P_E), if f and c are differentiable at x_*, and if c is qualified for representing X_E at x_* in the sense (2) *below, then there exists a multiplier $\lambda_* \in \mathbb{F}$ such that*

$$\nabla_x \ell(x_*, \lambda_*) = 0, \qquad (1a)$$

$$c(x_*) = 0. \qquad (1b)$$

Some explanations.

- The constraint c is qualified for representing X_E at $x \in X_E$ if

$$\mathrm{T}_x X_E = \mathrm{T}'_x X_E := \mathcal{N}(c'(x)). \qquad (2)$$

  Qualification holds of $c'(x)$ is surjective (sufficient condition of CQ).
- The Lagrangian of (P_E) is the function

$$\ell : (x, \lambda) \in \mathbb{E} \times \mathbb{F} \mapsto \ell(x, \lambda) = f(x) + \langle \lambda, c(x) \rangle .$$

Theorem (NC2 for (P_E))

If x_* is a local minimum of (P_E), if f and c are twice differentiable at x_*, and if (1a) *holds for some $\lambda_* \in \mathbb{F}$, then*

$$\forall\, d \in \mathrm{T}_{x_*} X_E : \quad \left\langle \nabla^2_{xx} \ell(x_*, \lambda_*) d, d \right\rangle \geqslant 0. \qquad (3)$$

- Inequality in (3) is not necessarily true for $d \in \mathcal{N}(c'(x_*)) \setminus \mathrm{T}_{x_*} X_E$.
- $\nabla^2_{xx} \ell(x_*, \lambda_*)$ is not necessarily positive semi-definite (even if qualification holds).

Theorem (SC2 for (P_E))

If f and c are twice differentiable at x_*, if (1) *holds for some $\lambda_* \in \mathbb{F}$, and if*

$$\forall\, d \in \mathrm{T}_{x_*} X_E \setminus \{0\} : \quad d^{\mathsf{T}} \nabla^2_{xx} \ell(x_*, \lambda_*) d > 0, \qquad (4)$$

then x_* is a strict local minimum of (P_E).

- (4) stronger (hence conclusion holds) if inequality holds $\forall\, d \in \mathcal{N}(c'(x_*)) \setminus \{0\}$.

A generic form of the nonlinear optimization problem:

$$(P_{EI}) \quad \begin{cases} \inf_x \ f(x) \\ c_E(x) = 0 \\ c_I(x) \leqslant 0, \end{cases}$$

where $f : \mathbb{E} \to \mathbb{R}$, E and I form a partition of $[1:m]$, $c_E : \mathbb{E} \to \mathbb{R}^{m_E}$, and $c_I : \mathbb{E} \to \mathbb{R}^{m_I}$ are smooth (possibly non convex) functions.

- The feasible set is denoted by

$$X_{EI} := \{x \in \mathbb{E} : c_E(x) = 0,\ c_I(x) \leqslant 0\}.$$

- We say that an inequality constraint is active at $x \in X_{EI}$ if $c_i(x) = 0$.
- The set of indices of active inequality constraints is denoted by

$$I^0(x) := \{i \in I : c_i(x) = 0\}.$$

- (P_{EI}) is said to be convex if f is convex and X_{EI} is convex.

Theorem (NC1 for (P_{EI}), Karush-Kuhn-Tucker (KKT))

If x_* is a local minimum of (P_{EI}), if f and $c = (c_E, c_I)$ are differentiable at x_*, and if c is qualified for representing X_{EI} at x_* in the sense (6) below, then there exists a multiplier $\lambda_* \in \mathbb{R}^m$ such that

$$\nabla_x \ell(x_*, \lambda_*) = 0, \quad (5a)$$

$$c_E(x_*) = 0, \quad (5b)$$

$$0 \leqslant (\lambda_*)_I \perp c_I(x_*) \leqslant 0. \quad (5c)$$

Some explanations.

- The Lagrangian of (P_{EI}) is the function

$$\ell : (x, \lambda) \in \mathbb{E} \times \mathbb{R}^m \mapsto \ell(x, \lambda) = f(x) + \lambda^{\mathsf{T}} c(x).$$

- The complementarity condition (5c) means

$$(\lambda_*)_I \geqslant 0, \quad (\lambda_*)_I^{\mathsf{T}} c_I(x_*) = 0, \quad \text{and} \quad c_I(x_*) \leqslant 0.$$

- The tangent cone $\mathrm{T}_x X_{EI}$ is always contained in the linearizing cone

$$\mathrm{T}'_x X_{EI} := \{d \in \mathbb{E} : c'_E(x)\cdot d = 0,\ c'_{I^0(x)}(x)\cdot d \leqslant 0\}.$$

- It is said that the constraint c is qualified for representing X_{EI} at x if

$$\mathrm{T}_x X_{EI} = \mathrm{T}'_x X_{EI}. \tag{6}$$

- Sufficient conditions of qualification: continuity and/or differentiability and one of the following
  - (CQ-A) $c_{E\cup I^0(x)}$ is affine near x (Affinity),
  - (CQ-S) c_E is *affine*, $c_{I^0(x)}$ is componentwise convex, $\exists\,\hat{x} \in X_{EI}$ such that $c_{I^0(x)}(\hat{x}) < 0$ (Slater),
  - (CQ-LI) $\sum_{i\in E\cup I^0(x)} \alpha_i \nabla c_i(x) = 0 \Longrightarrow \alpha = 0$ (Linear Independence),
  - (CQ-MF) $\sum_{i\in E\cup I^0(x)} \alpha_i \nabla c_i(x) = 0$ and $\alpha_{I^0(x)} \geqslant 0 \Longrightarrow \alpha = 0$ (Mangasarian-Fromovitz).

Proposition (other forms of (CQ-MF))

Suppose that $c_{E\cup I^0(x)}$ is differentiable at $x \in X_{EI}$. Then the following properties are equivalent:

1. *(CQ-MF) holds at x,*
2. $\forall\, v \in \mathbb{R}^m$, $\exists\, d \in \mathbb{E}$: $c'_E(x)\cdot d = v_E$ *and* $c'_{I^0(x)}(x)\cdot d \leqslant v_{I^0(x)}$,
3. $c'_E(x)$ *is surjective and* $\exists\, d \in \mathbb{E}$: $c'_E(x)\cdot d = 0$ *and* $c'_{I^0(x)}(x)\cdot d < 0$.

Background

Optimization (problem (P_{EI}) – SC1 for convex problem)

Theorem (SC1 for convex (P_{EI}))

If
- *f is a convex function and X_{EI} is a convex set,*
- *f and c are differentiable at $x_* \in X_{EI}$,*
- *there is a $\lambda_* \in \mathbb{F}$ such that (x_*, λ_*) satisfies (5),*

then x_* is a global minimum of (P_{EI}).

No need of constraint qualification.

The goal of the first part of this course is to extend the previous NC1 and SC1 to a more general problem and to derive second order optimality conditions.

Background

Optimization (linear optimization duality)

- Let $c \in \mathbb{E}$ (a Euclidean vector space), $A : \mathbb{E} \to \mathbb{R}^m$ and $B : \mathbb{E} \to \mathbb{R}^p$ linear, $a \in \mathbb{R}^m$, and $b \in \mathbb{R}^p$.
- A linear optimization problem (P_L) and its dual (D_L) read

$$(P_L) \quad \begin{cases} \inf_{x\in\mathbb{E}} \ \langle c, x\rangle \\ Ax = a \\ Bx \leqslant b \end{cases} \quad \text{and} \quad (D_L) \quad \begin{cases} \sup_{(y,s)\in\mathbb{R}^m\times\mathbb{R}^p} \ a^{\mathsf{T}}y - b^{\mathsf{T}}s \\ A^*y - B^*s = c \\ s \geqslant 0. \end{cases}$$

- **Properties**

$$(P_L) \text{ has a solution} \iff \operatorname{val}(P_L) \in \mathbb{R},$$

$$\operatorname{val}(D_L) \leqslant \operatorname{val}(P_L) \qquad \text{[named weak duality]},$$

$$(P_L),\ (D_L) \text{ feasible} \iff \operatorname{Sol}(P_L) \neq \varnothing \iff \operatorname{Sol}(D_L) \neq \varnothing. \qquad (7)$$

When (7) holds [named strong duality], $\operatorname{val}(D_L) = \operatorname{val}(P_L)$.

Background

Algorithmics (speeds of convergence)

Let $\mathbb{E}$ be a normed space and $\{x_k\} \subseteq \mathbb{E}$ be a sequence converging to $\bar{x}$.

- $\{x_k\}$ is said to converge linearly, if $\exists\, r \in [0,1)$ and $K \in \mathbb{N}$ such that $\forall\, k \geqslant K$, one has $\|x_{k+1} - \bar{x}\| \leqslant r\|x_k - \bar{x}\|$.
  - Depends on the norm of $\mathbb{E}$.
- $\{x_k\}$ is said to converge superlinearly, if $x_{k+1} - \bar{x} = o(\|x_k - \bar{x}\|)$.
  - Independent of the norm of $\mathbb{E}$.
  - Faster than linear convergence.
  - Typical of the quasi-Newton methods.
- $\{x_k\}$ is said to converge quadratically, if $x_{k+1} - \bar{x} = O(\|x_k - \bar{x}\|^2)$.
  - Independent of the norm of $\mathbb{E}$.
  - Faster than superlinear convergence.
  - Typical of Newton's method.

Lemma

If $\{x_k\} \to x_*$ superlinearly, then $\{x_{k+1} - x_k\} \sim \{x_k - x_*\}$.

Background

Algorithmics (Dennis & Moré criterion for superlinear convergence)

Let $F : \mathbb{E} \to \mathbb{F}$ and consider the nonlinear system to solve in $x \in \mathbb{E}$:

$$F(x) = 0.$$

A quasi-Newton algorithm locally generates a sequence $\{x_k\}$ by the recurrence

$$F(x_k) + M_k(x_{k+1} - x_k) = 0, \tag{8}$$

where $M_k \in \mathcal{L}(\mathbb{E}, \mathbb{F})$ is an approximation of $F'(x_k)$, generated by the algorithm.

Proposition (Dennis & Moré criterion for superlinear convergence)

If
- *F is differentiable at a zero x_* of F,*
- *$F'(x_*)$ is nonsingular,*
- *$\{x_k\}$ generated by (8) converges to x_*,*

then the convergence is superlinear if and only if

$$\Big(M_k - F'(x_*)\Big)(x_{k+1} - x_k) = o(\|x_{k+1} - x_k\|).$$

Optimality conditions

First order optimality conditions for (P_G) (the problem I)

- We consider the problem

$$(P_G) \quad \begin{cases} \min f(x) \\ c(x) \in G, \end{cases}$$

  where
  - $f : \mathbb{E} \to \mathbb{R}$ ($\mathbb{E}$ is a Euclidean vector space),
  - $c : \mathbb{E} \to \mathbb{F}$ ($\mathbb{F}$ is another Euclidean vector space),
  - G is nonempty closed convex set in $\mathbb{F}$.
- The feasible set is denoted by

$$X_G := \{x \in \mathbb{E} : c(x) \in G\} = c^{-1}(G).$$

- (P_G) is said to be convex if f is convex and X_G is convex.

$$T : \mathbb{E} \multimap \mathbb{F} : x \mapsto c(x) - G \text{ is convex} \quad \Longrightarrow \quad X_G \text{ is convex.}$$

Optimality conditions

First order optimality conditions for (P_G) (the problem II)

Some examples of optimization problems that can be written in the form (P_G).

- The nonlinear optimization problem

$$(P_{EI}) \quad \begin{cases} \inf_{x \in \mathbb{R}} \ f(x) \\ c_E(x) = 0 \text{ and } c_I(x) \leqslant 0, \end{cases}$$

  where $f : \mathbb{E} \to \mathbb{R}$, E and I form a partition of $[1:m]$, $c_E : \mathbb{E} \to \mathbb{R}^{m_E}$, and $c_I : \mathbb{E} \to \mathbb{R}^{m_I}$ are smooth (possibly non convex) functions.
- The linear semidefinite optimization problem

$$(P_{\text{SDO}}) \quad \begin{cases} \inf_{X \in \mathcal{S}^n} \ \langle C, X \rangle \\ A(X) = b \text{ and } X \succcurlyeq 0, \end{cases}$$

  where $C \in \mathcal{S}^n$, $A : \mathcal{S}^n \to \mathbb{R}^m$ is linear, and $b \in \mathbb{R}^m$.
- The composite optimization problem

$$\inf_{x \in \mathbb{E}} (g \circ f)(x),$$

  where $f : \mathbb{E} \to \mathbb{F}$ and $g : \mathbb{F} \to \mathbb{R}$.

Proposition (tangent and linearizing cones)

If c is differentiable at $x \in X_G$, then

$$\mathrm{T}_x X_G \subseteq \mathrm{T}'_x X_G := \{d \in \mathbb{E} : c'(x) \cdot d \in \mathrm{T}_{c(x)} G\}.$$

- $\mathrm{T}'_x X_G$ is called the linearizing cone to X at x.
- The equality $\mathrm{T}_x X_G = \mathrm{T}'_x X_G$ is not guaranteed.
- The constraint function c is said to be qualified for representing X_G at x if

$$\mathrm{T}_x X_G = \mathrm{T}'_x X_G, \tag{9a}$$

$$c'(x)^*[(\mathrm{T}_{c(x)} G)^+] \text{ is closed.} \tag{9b}$$

Theorem (NC1 for (P_G))

If x_* is a local minimum of (P_G), if f and c are differentiable at x_*, and if c is qualified for representing X_G at x_* in the sense (9a)-(9b), then

1. *there exists a multiplier $\lambda_* \in \mathbb{F}$ such that*

$$\nabla f(x_*) + c'(x_*)^* \lambda_* = 0, \tag{10a}$$

$$\lambda_* \in \mathrm{N}_{c(x_*)} G. \tag{10b}$$

2. *if, furthermore, $G \equiv K$ is a convex cone, then (10b) can be written*

$$K^- \ni \lambda_* \perp c(x_*) \in K \qquad \text{or} \qquad c(x_*) \in \mathrm{N}_{\lambda_*} K^-. \tag{10c}$$

One recognizes in (10a) the gradient of the Lagrangian wrt x

$$\ell : \mathbb{E} \times \mathbb{F} \to \mathbb{R} : x \mapsto \ell(x, \lambda) = f(x) + \langle \lambda, c(x) \rangle$$

and in (10c) the complementarity conditions.

Theorem (SC1 for convex (P_G))

If
- *f is a convex function and X_G is a convex set,*
- *f and c are differentiable at $x_* \in X_G$,*
- *there is a $\lambda_* \in \mathbb{F}$ such that (x_*, λ_*) satisfies (10a)-(10b),*

then x_* is a global minimum of (P_G).

The goal of the next slides is to highlight and analyze a condition (Robinson's condition) that

1. provides an *error bound* for the feasible set $y + X_G$ (y small),
2. claims the *stability* of the feasible set X_G,
3. ensures that c is *qualified* for representing X_G.

We say that Robinson's condition holds at $x \in X_G$ if

$$\text{(CQ-R)} \qquad \boxed{0 \in \operatorname{int}\Big(c(x) + c'(x) \cdot \mathbb{E} - G\Big).} \qquad (11)$$

We will see below that

- it is useful since it
  - provides an error bound for small perturbations $y + X_G$ of X_G,
  - claims the stability of the feasible set X_G with respect to small perturbations,
  - shows that the constraint function c is qualified for representing X_G at x, in the sense (9a)-(9b),
- it generalizes to (P_G) the Mangasarian-Fromovitz constraint qualification (CQ-MF) for (P_{EI}).

Theorem ((CQ-R) and metric regularity)

If c is continuously differentiable near $x_0 \in X_G$, then the following properties are equivalent:

1. *(CQ-R) holds at $x = x_0$,*
2. *there exists a constant $\mu \geqslant 0$, such that $\forall\,(x, y)$ near $(x_0, 0)$:*

$$\operatorname{dist}(x, c^{-1}(y + G)) \leqslant \mu \operatorname{dist}(c(x), y + G). \tag{12}$$

Condition 2 is named *metric regularity* since it is equivalent to that property (see its definition below) for the multifunction

$$T : \mathbb{E} \multimap \mathbb{F} : x \mapsto c(x) - G.$$

An error bound is an estimation of the distance to a set by a quantity easier to compute.

Corollary (Robinson's error bound)

If c is continuously differentiable near $x_0 \in X_G$ and if (CQ-R) holds at $x = x_0$, then there exists a constant $\mu \geqslant 0$, such that

$$\forall\, x \text{ near } x_0 : \quad \operatorname{dist}(x, X_G) \leqslant \mu \operatorname{dist}(c(x), G). \tag{13}$$

- $\operatorname{dist}(x, X_G)$ is often difficult to evaluate in $\mathbb{E}$,
- $\operatorname{dist}(c(x), G)$ is often easier to evaluate in $\mathbb{F}$ (it is the case if $c(x)$ is easy to evaluate and G is simple),
- useful in *theory* (e.g., for proving that (CQ-R) implies constraint qualification), in *algorithmics* (e.g., for proving [speed of] converge) or in *practice* (for estimating $\operatorname{dist}(x, X_G)$).

Optimality conditions

First order optimality conditions for (P_G) (stability wrt small perturbations)

The estimate (12) readily implies the following stability result.

Corollary (stability wrt small perturbations)

If c is continuously differentiable near some $x_0 \in X_G$ and if (CQ-R) holds at x_0, then, for all small $y \in \mathbb{F}$:

$$\{x \in \mathbb{E} : c(x) \in y + G\} \neq \varnothing.$$

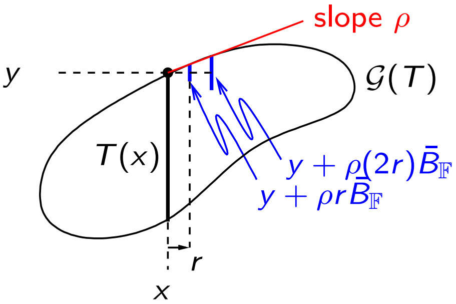
Optimality conditions

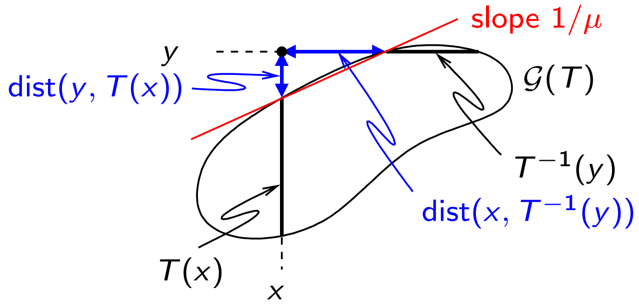
First order optimality conditions for (P_G) (open multifunction theorem I) [26, 23, 5]

Let $T : \mathbb{E} \multimap \mathbb{F}$ be a multifunction and $\bar{B}_{\mathbb{E}}$ and $\bar{B}_{\mathbb{F}}$ be the closed balls of $\mathbb{E}$ and $\mathbb{F}$.

- T is open at $(x_0, y_0) \in \mathcal{G}(T)$ with ratio $\rho > 0$ if there exist a neighborhood W of (x_0, y_0) and a radius $r_{\max} > 0$ such that for all $(x, y) \in W \cap \mathcal{G}(T)$ and $r \in [0, r_{\max}]$: $y + \rho\, r\, \bar{B}_{\mathbb{F}} \subseteq T(x + r\, \bar{B}_{\mathbb{E}})$.
- T is metric regular at $(x_0, y_0) \in \mathcal{G}(T)$ with modulus $\mu > 0$ if for all (x, y) near (x_0, y_0): $\operatorname{dist}(x, T^{-1}(y)) \leqslant \mu \operatorname{dist}(y, T(x))$.

Difference: $(x, y) \in \mathcal{G}(T)$ for the openness, but not for the metric regularity.





Both estimate the change in $T(x)$ with x (but these are not infinitesimal notions), either from inside $G(T)$ (openness) or outside (metric regularity).

Extention of the *open mapping theorem* for linear (continuous) maps to (nonlinear) convex multifunctions.

Theorem (open multifunction theorem, finite dimension)

If $T : \mathbb{E} \multimap \mathbb{F}$ is convex and $(x_0, y_0) \in \mathcal{G}(T)$, then the following properties are equivalent:

1. $y_0 \in \operatorname{int} \mathcal{R}(T)$,
2. *for all* $r > 0$, $y_0 \in \operatorname{int} T(x + r\bar{B}_{\mathbb{E}})$,
3. T *is open at* (x_0, y_0) *with rate* $\rho > 0$,
4. T *is metric regular at* (x_0, y_0) *with modulus* $\mu > 0$.

One can take $\mu = 1/\rho$ *in point 4 if* ρ *is given by point 3.*

Theorem (metric regularity diffusion)

Suppose that

- $c : \mathbb{E} \to \mathbb{F}$ *a continuous function,* $G \neq \varnothing$ *a closed convex set of* $\mathbb{F}$,
- $T : \mathbb{E} \multimap \mathbb{F} : x \mapsto c(x) - G$ *is* μ*-metric regular at* $(x_0, y_0) \in \mathcal{G}(T)$,
- $\delta : \mathbb{E} \to \mathbb{F}$ *is Lipschitz near* x_0 *with modulus* $L < 1/\mu$,
- $\tilde{T} : \mathbb{E} \multimap \mathbb{F} : x \mapsto c(x) + \delta(x) - G$.

Then $\tilde{T}$ *is also metric regular at* $(x_0, y_0 + \delta(x_0)) \in \mathcal{G}(\tilde{T})$ *with modulus* $\mu/(1 - L\mu)$: *for all* (x, y) *near* $(x_0, y_0 + \delta(x_0))$, *the following holds*

$$\operatorname{dist}(x, \tilde{T}^{-1}(y)) \leqslant \frac{\mu}{1 - L\mu} \operatorname{dist}(y, \tilde{T}(x)). \tag{14}$$

No need of convexity.

Optimality conditions

First order optimality conditions for (P_G) (qualification with (CQ-R) I)

Proposition (other forms of (CQ-R))

Suppose that c is differentiable at $x \in X_G$. Then the following properties are equivalent

1. $0 \in \operatorname{int}(c(x) + c'(x)\mathbb{E} - G)$ *[this is (CQ-R)],*
2. $c'(x)\mathbb{E} - \mathrm{T}^f_{c(x)}\, G = \mathbb{F}$,
3. $c'(x)\mathbb{E} - \mathrm{T}_{c(x)}\, G = \mathbb{F}$,
4. $\overline{c'(x)\mathbb{E} - \mathrm{T}_{c(x)}\, G} = \mathbb{F}$.

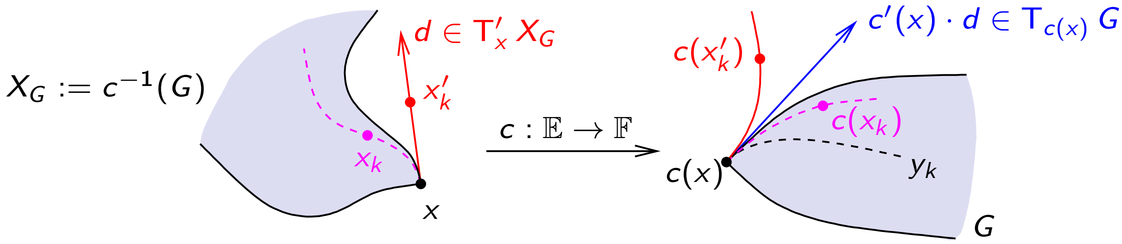
Optimality conditions

First order optimality conditions for (P_G) (qualification with (CQ-R) II)

Proposition (qualification with (CQ-R))

Suppose that c is continuously differentiable near $x \in X_G$ and that (CQ-R) holds at x. Then c is qualified for representing X_G at x.

The figure below shows how (CQ-R) is used to create an appropriate sequence $\{x_k\}$ in $X_G := c^{-1}(G)$ to get qualification (the figure requires some oral explanations, though …).



Proposition (Gauvin's boundedness property)

Suppose that f and c are differentiable at $x_* \in X_G$ and that the set Λ_* of multipliers $\lambda_* \in \mathbb{F}$ satisfying (10a)-(10b) *is nonempty. Then*

1. $\Lambda_*^\infty = [c'(x_*)\mathbb{E} - \mathrm{T}_{c(x_*)}\, G]^+$,
2. *Λ_* is bounded if and only if (CQ-R) holds.*

The property was originally established for problem (P_{EI}) and (CQ-MF) [10].

Let x_* be a local solution to (P_{EI}), λ_* be an associated optimal multiplier, and $L_* := \nabla^2_{xx}\ell(x_*, \lambda_*)$.

In view of the second order optimality conditions of the equality constrained optimization problem (P_E), it is tempting to claim that

$$\forall\, d \in \mathrm{T}_{x_*} X_{EI} : \quad \langle L_* d, d\rangle \geqslant 0.$$

But this is not guaranteed!

- The good cone is not $\mathrm{T}_{x_*} X_{EI}$ but the critical cone C_* (to be defined).
- The multiplier λ_* must be chosen, depending on $d \in C_*$.

- Critical cone at $x \in X_{EI}$ (it is a part of $\mathrm{T}'_x X_{EI}$)

$$C(x) := \{d \in \mathbb{E} : c'_E(x) \cdot d = 0,\ c'_{I^0(x)}(x) \cdot d \leqslant 0,\ f'(x) \cdot d \leqslant 0\}.$$

- Short notation for index sets

$$\begin{aligned} I_*^0 &:= \{i \in I : c_i(x_*) = 0\} := I^0(x_*), \\ I_*^{0+} &:= \{i \in I : c_i(x_*) = 0,\ (\lambda_*)_i > 0\}, \\ I_*^{00} &:= \{i \in I : c_i(x_*) = 0,\ (\lambda_*)_i = 0\}. \end{aligned}$$

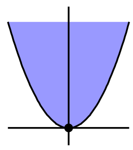
- Other forms of the critical cone at a stationary pair (x_*, λ_*):

$$\begin{aligned} C_* &= \{d \in \mathbb{E} : c'_E(x_*) \cdot d = 0,\ c'_{I_*^0}(x_*) \cdot d \leqslant 0,\ f'(x_*) \cdot d = 0\}, \\ &= \{d \in \mathbb{E} : c'_{E \cup I_*^{0+}}(x_*) \cdot d = 0,\ c'_{I_*^{00}}(x_*) \cdot d \leqslant 0\}. \end{aligned}$$

Three instructive examples

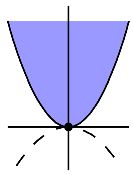
1. Strong NC2 (not always true): $\forall\, \lambda_* \in \Lambda_*$, $\forall\, d \in C_*$: $\langle L_* d, d\rangle \geqslant 0$.

$$\begin{cases} \min\ x_2 \\ x_2 \geqslant x_1^2, \end{cases}$$



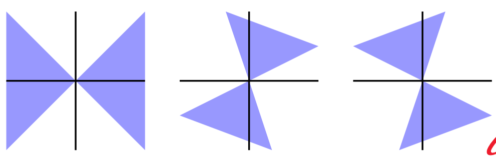
2. Semi-strong NC2 (not always true): $\exists\, \lambda_* \in \Lambda_*$, $\forall\, d \in C_*$: $\langle L_* d, d\rangle \geqslant 0$.

$$\begin{cases} \min\ x_2 \\ x_2 \geqslant x_1^2 \\ x_2 \geqslant -\frac{1}{2}x_1^2. \end{cases}$$



3. Weak NC2 (always true): $\forall\, d \in C_*$, $\exists\, \lambda_* \in \Lambda_*$: $\langle L_* d, d\rangle \geqslant 0$.

$$\begin{cases} \min\ x_3 \\ x_3 \geqslant (x_1 + x_2)(x_1 - x_2) \\ x_3 \geqslant (x_2 + 3x_1)(2x_2 - x_1) \\ x_3 \geqslant (2x_2 + x_1)(x_2 - 3x_1). \end{cases}$$



Optimality conditions

Second order optimality conditions (NC2)

Notation at a stationary point x_*:

- $\Lambda_* := \{\lambda_* \in \mathbb{R}^m : (x_*, \lambda_*)$ satisfies the KKT system (5)$\}$.
- $L_* := \nabla^2_{xx}\ell(x_*, \lambda_*)$ for some specified $\lambda_* \in \Lambda_*$.

Theorem (NC2 for (P_{EI}))

Suppose that

- x_* *is a local minimum of* (P_{EI}),
- f *and* c_E *are* C^2 *near* x_*, $c_{I^0_*}$ *is twice differentiable at* x_*, $c_{I\setminus I^0_*}$ *is continuous at* x_*,
- *(CQ-MF) holds at* x_*,

then $\forall\, d \in C_*$, $\exists\, \lambda_* \in \Lambda_*$ *such that* $\langle L_* d, d\rangle \geqslant 0$.

These conditions are named weak second order necessary conditions and also read

$$\forall\, d \in C_* : \quad \max_{\lambda_* \in \Lambda_*} \langle L_* d, d\rangle \geqslant 0. \tag{15}$$

Optimality conditions

Second order optimality conditions (SC2 I)

Theorem (SC2 for (P_{EI}))

Suppose that

- f *and* $c_{E\cup I^0_*}$ *are twice differentiable at* x_*,
- (x_*, λ_*) *verifies the KKT conditions* (5),
- *the following equivalent conditions hold*

$$\forall\, d \in C_* \setminus \{0\}, \quad \exists\, \lambda_* \in \Lambda_* : \quad \langle L_* d, d\rangle > 0, \tag{16a}$$

$$\exists\, \bar\gamma > 0, \quad \forall\, d \in C_*, \quad \exists\, \lambda_* \in \Lambda_* : \quad \langle L_* d, d\rangle \geqslant \bar\gamma\, \|d\|^2, \tag{16b}$$

then $\forall\, \gamma \in [0, \bar\gamma)$, $\exists$ *a neighborhood* V *of* x_*, $\forall\, x \in (V \setminus \{x_*\}) \cap X_{EI}$:

$$f(x) > f(x_*) + \frac{\gamma}{2}\, \|x - x_*\|^2. \tag{17}$$

In particular, x_* *is a strict local minimum of* (P_{EI}).

Optimality conditions

Second order optimality conditions (SC2 II)

- Condition (17) is called the quadratic growth property.
- No need of a constraint qualification assumption.
- The property

$$\exists\, \lambda_* \in \Lambda_* : \quad \forall\, d \in C_* \setminus \{0\}, \quad \langle L_* d, d\rangle > 0$$

is stronger than (16) and is called the semi-strong SC2.

- The even stronger property

$$\forall\, \lambda_* \in \Lambda_* : \quad \forall\, d \in C_* \setminus \{0\}, \quad \langle L_* d, d\rangle > 0$$

is called the strong SC2.

Perturbation analysis

Stability result for (P_G) with a polyhedral cone G I [22]

- Let $\mathbb{P}$ be a vector space. For $p \in \mathbb{P}$, consider the perturbed problem

$$(P_K^p) \quad \begin{cases} \min_x \; f(x,p) \\ c(x,p) \in K, \end{cases}$$

where $f : \mathbb{E} \times \mathbb{P} \to \mathbb{R}$ smooth, $c : \mathbb{E} \times \mathbb{P} \to \mathbb{F}$ smooth, $K \subseteq \mathbb{F}$ convex polyhedral cone.

- Optimality system at $x \in \mathbb{E}$: $\exists\, \lambda \in \mathbb{F}$ such that

$$\begin{cases} \nabla_x f(x,p) + c'_x(x,p)^* \lambda = 0 \\ K^- \ni \lambda \perp c(x,p) \in K. \end{cases} \tag{18}$$

- The multiplier multifunction $\Lambda : \mathbb{E} \times \mathbb{P} \multimap \mathbb{F}$ is defined at $(x,p) \in \mathbb{E} \times \mathbb{P}$ by

$$\Lambda(x,p) := \{\lambda \in \mathbb{F} : (x,p,\lambda) \text{ satisfies } (18)\}.$$

- The stationary multifunction $\Sigma : \mathbb{P} \multimap \mathbb{E}$ is defined at $p \in \mathbb{P}$ by

$$\Sigma(p) := \{x \in \mathbb{E} : \Lambda(x,p) \neq \varnothing\}.$$

Assume the framework defined above.

Proposition (stability of (P_K))

If
- $f(\cdot, p_0)$ and $c(\cdot, p_0)$ are C^2 near $x_0 \in \mathbb{E}$ for some $p_0 \in \mathbb{P}$,
- f'_x, c and c'_x are Lipschitz continuous near (x_0, p_0),
- $0 \in \operatorname{int}\big(c(x_0, p_0) + c'_x(x_0, p_0)\mathbb{E} - K\big)$,
- $x_0 \in \Sigma(p_0)$,
- $\exists\, \lambda_0 \in \Lambda(x_0, p_0)$ such that strong SC2 holds for $(P_K^{p_0})$,

then $\exists\, L \geqslant 0$, such that $\forall\, p$ near p_0:

1. $\Sigma(p) \neq \varnothing$,
2. $\forall\, x \in \Sigma(p)$ near x_0, $\forall\, \lambda \in \Lambda(x, p)$:

$$\operatorname{dist}\big((x, \lambda), \{x_0\} \times \Lambda(x_0, p_0)\big) \leqslant L\, \|p - p_0\|.$$

Two classes of linearization methods that are used to solve systems with nonsmoothness.

1. Methods that capture much of the local behavior of the system.

   Features
   - Expensive iteration (nonlinear), fast convergence, easy to globalize.

   Examples
   - The Josephy-Newton algorithm for functional inclusions.
   - The SQP algorithm for (P_{EI}).

2. Methods that use a single piece of the local behavior of the system.

   Features
   - Cheap iteration (linear), fast convergence, difficult to globalize.

   Example
   - The semismooth Newton algorithm for nonsmooth system of equations.

Linearization methods

Josephy-Newton algorithm for functional inclusions (functional inclusion I)

Let $\mathbb{E}$ and $\mathbb{F}$ be Euclidean vector spaces of the same dimension, $F : \mathbb{E} \to \mathbb{F}$ be a smooth function, and $N : \mathbb{E} \multimap \mathbb{F}$ be a multifunction.

We consider the functional inclusion problem

$$(P_{FI}) \qquad \boxed{F(x) + N(x) \ni 0.} \tag{19}$$

Interested in algorithmic issues (not theoretical ones, like existence of solution).

Examples

1. The variational problem if $N = \mathrm{N}_X$ the normal cone to $X \subseteq \mathbb{E}$ at x ($= \varnothing$ if $x \notin X$):
$$(P_V) \qquad F(x) + \mathrm{N}_X(x) \ni 0. \tag{20}$$
2. The variational inequality problem is (P_V) with $X = C$ (a closed convex set):
$$(P_{VI}) \qquad \begin{cases} x \in C \\ \langle F(x), y - x \rangle \geqslant 0, \quad \forall y \in C. \end{cases} \tag{21}$$
3. The complementarity problem is (P_{FI}) with $N = N_{K^+} \circ G$ (closed convex cone $K \subseteq \mathbb{F}$, $G : \mathbb{E} \to \mathbb{F}$)
$$(P_{CP}) \qquad K^+ \ni G(x) \perp F(x) \in K. \tag{22}$$

Linearization methods

Josephy-Newton algorithm for inclusions (functional inclusion II)

Examples (continued)

4. The Peano-Kantorovitch NC1 of problem $\min\{f(x) : x \in X\}$ reads
$$\nabla f(x) + \mathrm{N}_X(x) \ni 0.$$
5. The first order optimality conditions for (P_G), when $G \equiv K$ is a closed convex *cone*, can be written
$$\tilde{K}^- \ni \tilde{x} \perp \tilde{F}(\tilde{x}) \in \tilde{K},$$
with the variable $\tilde{x} := (x, \lambda) \in \mathbb{E} \times \mathbb{F}$,
$$\tilde{F}(\tilde{x}) := \begin{pmatrix} \nabla f(x) + c'(x)^* \lambda \\ -c(x) \end{pmatrix}, \quad \text{and} \quad \tilde{K} := \{0_{\mathbb{E}}\} \times (-K). \tag{23}$$
6. If N is the constant multifunction $x \multimap \{0_{\mathbb{R}}^{m_E}\} \times \mathbb{R}_+^{m_I} \subseteq \mathbb{R}^m \equiv \mathbb{F}$ where E and I make a partition of $[1:m]$, (P_{FI}) becomes the system of equalitites and inequalitites
$$F_E(x) = 0 \qquad \text{and} \qquad F_I(x) \leqslant 0.$$

Linearization methods

Josephy-Newton algorithm for inclusions (the JN algorithm) [15, 16]

Algorithm (Josephy-Newton algorithm for solving (P_{FI}))

Given x_k, compute x_{k+1} as a solution to the problem in x:

$$F(x_k) + M_k(x - x_k) + N(x) \ni 0, \tag{24}$$

where $M_k = F'(x_k)$ or an approximation to it.

Remarks

- Only F is linearized, not N (is the reason for the chosen structure of (P_{FI})).
- (24) captures more information from (P_{FI}) than a "simple" linearization.
- (24) is often a nonlinear problem, hence yielding an expensive iteration.
- Makes sense computationally if N is sufficiently simple.

Linearization methods

Josephy-Newton algorithm for inclusions (semi-stability) [3]

A solution x_* to (P_{FI}) is said semi-stable if $\exists\, \sigma_1 > 0$ and $\exists\, \sigma_2 > 0$ such that

$$\left.\begin{array}{r}(x,p) \in \mathbb{E} \times \mathbb{F} \\ F(x) + N(x) \ni p \\ \|x - x_*\| \leqslant \sigma_1\end{array}\right\} \quad \Longrightarrow \quad \|x - x_*\| \leqslant \sigma_2 \|p\|.$$

Remarks

- The perturbed inclusion $F(x) + N(x) \ni p$ is not required to have a solution.
- Semistability implies that x_* is the unique solution to (P_{FI}) on $\bar{B}(x_*, \sigma_1)$.
- If $N \equiv \{0\}$, then semi-stability of x_* $\Leftrightarrow$ injectivity of $F'(x_*)$ (hence, nonsingularity if $\dim \mathbb{E} = \dim \mathbb{F}$).

Proposition (semi-stability characterization for polyhedral VI)

Consider problem (P_{VI}) *with a convex* polyhedron C *and* F *being* C^1 *near a solution* x_*. *Then the following properties are equivalent:*

1. x_* *is semi-stable,*
2. x_* *is an isolated solution to*
$$F(x_*) + F'(x_*)(x - x_*) + \mathrm{N}_C(x) \ni 0,$$
3. *one has* $\langle F'(x_*)(x - x_*), x - x_* \rangle > 0$ *when* $x \in C \setminus \{x_*\}$ *satisfies*
$$\langle F(x_*), x - x_* \rangle = 0 \quad \text{and} \quad F(x_*) + F'(x_*)(x - x_*) + \mathrm{N}_C(x_*) \ni 0,$$
4. x_* *is the unique solution to*
$$\mathrm{N}_C(x) \subseteq \mathrm{N}_C(x_*),\ \langle F(x_*), x - x_* \rangle = 0,\ \mathbb{R}_+ F(x_*) + F'(x_*)(x - x_*) + \mathrm{N}_C(x) \ni 0.$$

Proposition (speed of convergence of quasi-Newton methods)

Suppose that F *is* C^1 *near a* semi-stable *solution* x_* *to* (P_{FI}). *Let* $\{x_k\}$ *be a sequence generated by algorithm* (24), *converging to* x_*.

1. *If* $(M_k - F'(x_*))(x_{k+1} - x_k) = o(\|x_{k+1} - x_k\|)$, *then* $\{x_k\}$ *converges superlinearly.*
2. *If* $(M_k - F'(x_*))(x_{k+1} - x_k) = O(\|x_{k+1} - x_k\|^2)$ *and* F *is* $C^{1,1}$ *near* x_*, *then* $\{x_k\}$ *converges quadratically.*

Corollary (speed of convergence of Newton's method)

Suppose that F *is* C^1 *near a* semi-stable *solution* x_* *to* (P_{FI}). *Let* $\{x_k\}$ *be a sequence generated by algorithm* (24) *with* $M_k = F'(x_k)$, *converging to* x_*. *Then*

1. $\{x_k\}$ *converges superlinearly,*
2. *if, furthermore,* F *is* $C^{1,1}$ *near* x_*, *then* $\{x_k\}$ *converges quadratically.*

Linearization methods

Josephy-Newton algorithm for inclusions (local convergence) [3]

A solution x_* to (P_{FI}) is said hemi-stable if $\forall\,\alpha > 0$, $\exists\,\beta > 0$, $\forall\,x_0 \in \bar{B}(x_*, \beta)$, the "linearized" inclusion in x

$$F(x_0) + F'(x_0)(x - x_0) + N(x) \ni 0$$

has a solution in $\bar{B}(x_*, \alpha)$.

Theorem (local convergence of JN)

Suppose that F is C^1 near a semi-stable and hemi-stable solution x_* to (P_{FI}). Then $\exists\,\varepsilon > 0$ such that if $x_1 \in \bar{B}(x_*, \varepsilon)$, then

1. *the JN algorithm (24) with $M_k = F'(x_k)$ can generate $\{x_k\} \subseteq \bar{B}(x_*, \varepsilon)$,*
2. *any sequence $\{x_k\}$ generated in $\bar{B}(x_*, \varepsilon)$ by the JN algorithm converges superlinearly to x_* (quadratically if F is $C^{1,1}$).*

Linearization methods

The SQP algorithm (overview)

Recall the equality and inequality constrained problem

$$(P_{EI}) \quad \begin{cases} \inf_x \; f(x) \\ c_E(x) = 0 \\ c_I(x) \leqslant 0. \end{cases}$$

Three popular methods to solve (P_{EI})

- Augmented Lagrangian methods: a dual method that generates $\{(\lambda_k, r_k)\} \subseteq \mathbb{R}^m \times \mathbb{R}_{++}$ by
  - $\inf\limits_{x,\; s\geqslant 0} \left(f(x) + (\lambda_k)_E^{\mathsf{T}} c_E(x) + \frac{r_k}{2}\|c_E(x)\|_2^2 + (\lambda_k)_I^{\mathsf{T}}(c_I(x){+}s) + \frac{r_k}{2}\|c_I(x){+}s\|_2^2 \right)$,
  - $\lambda_{k+1} := (\lambda_k + r_k c(x_k))^{\#}$ and $r_{k+1} = ?$ (heuristics for nonlinear problems).
- SQP methods: it is a linearization method on the KKT system (see below).
- Interior point methods, which can be viewed as a penalization method solving (approximately) a sequence of problems (25) below with $\mu \downarrow 0$, thanks to the SQP algorithm:

$$\begin{cases} \min_{(x,s)} \; f(x) - \mu \sum_{i\in I} \log s_i \\ c_E(x) = 0, \quad c_I(x) + s = 0. \end{cases} \tag{25}$$

Linearization methods

The SQP algorithm (definition - KKT is a nonlinear complementarity problem)

- Similarly to Newton's method in unconstrained optimization, the SQP algorithm is conceptually interested in solutions of the first order optimality (KKT) system in (x, λ) of (P_{EI}):

$$\nabla_x \ell(x, \lambda) = 0, \tag{26a}$$
$$c_E(x) = 0, \tag{26b}$$
$$0 \leqslant \lambda_I \perp c_I(x) \leqslant 0. \tag{26c}$$

- This system in (x, λ) can be written like (P_{FI}) or (P_{CP}), namely

$$F(x, \lambda) + \mathrm{N}_{K^+}(x, \lambda) \ni 0 \qquad \text{or} \qquad K^+ \ni (x, \lambda) \perp F(x, \lambda) \in K, \tag{27}$$

with the data

$$F(x, \lambda) = \begin{pmatrix} \nabla_x \ell(x, \lambda) \\ -c(x) \end{pmatrix} \qquad \text{and} \qquad K = \{0_{\mathbb{E}}\} \times (\{0_{\mathbb{R}^{m_E}}\} \times \mathbb{R}^{m_I}_+). \tag{28}$$

Hence, $K^+ = \mathbb{E} \times \left(\mathbb{R}^{m_E} \times \mathbb{R}^{m_I}_+\right)$.

Linearization methods

The SQP algorithm (definition - the SQP algorithm viewed as a JN method) [22]

- The SQP algorithm (SQP for *Sequential Quadratic Programming*) for solving (P_{EI}) is the JN algorithm (24) on the functional inclusion (27)-(28):

$$\nabla_x \ell(x_k, \lambda_k) + M_k(x - x_k) + c'(x_k)^*(\lambda - \lambda_k) = 0, \tag{29a}$$
$$c_E(x_k) + c_E'(x_k) \cdot (x - x_k) = 0, \tag{29b}$$
$$0 \leqslant \lambda_I \perp \left(c_I(x_k) + c_I'(x_k) \cdot (x - x_k)\right) \leqslant 0, \tag{29c}$$

where $M_k = L_k := \nabla^2_{xx} \ell(x_k, \lambda_k)$ or an approximation to it ($M_k \not\simeq \nabla^2 f(x_k)$!).

- (29) is formed of the KKT conditions of the osculating quadratic problem

$$(\text{OQP}) \qquad \begin{cases} \min_x \ \langle \nabla f(x_k), x - x_k \rangle + \frac{1}{2} \langle M_k(x - x_k), x - x_k \rangle \\ c_E(x_k) + c_E'(x_k) \cdot (x - x_k) = 0, \\ c_I(x_k) + c_I'(x_k) \cdot (x - x_k) \leqslant 0. \end{cases} \tag{30}$$

The primal-dual solution (x_{k+1}, λ_{k+1}) to the OQP is the new iterate.

Algorithm (local SQP)

From (x_k, λ_k) *to* (x_{k+1}, λ_{k+1})*:*

1. *If* (26) *holds at* $(x, \lambda) = (x_k, \lambda_k)$*, stop.*
2. *Compute a primal-dual stationary point* (x_{k+1}, λ_{k+1}) *of the OQP* (30)*.*

Remarks

- The OQP's are still hard to solve (not just a linear system, expensive iteration):
  - If $M_k = L_k \not\succcurlyeq 0$, the OQP is NP-hard.
  - One of the good reasons for taking $M_k \simeq L_k$ with $M_k \succ 0$, updated by a quasi-Newton method; the OQP is then polynomial, but still difficult.
- Other (non local) difficulties to overcome:
  - What if the linearized constraints are incompatible?
  - What if the OQP is unbounded?
- Nothing is done for forcing the convergence from remote starting points.

Proposition (semi-stability and SC2)

If x_* *is a local minimum of* (P_{EI}) *and* λ_* *is an associated mutiplier, then the following properties are equivalent:*

1. (x_*, λ_*) *is semi-stable,*
2. $\Lambda_* = \{\lambda_*\}$ *and* x_* *satisfies* $SC2$*.*

At a local solution, semi-stability implies hemi-stability:

Proposition (SC for hemi-stability)

If
- x_* *is a local minimum of* (P_{EI})*,*
- (x_*, λ_*) *satisfies the KKT conditions,*
- (x_*, λ_*) *is semi-stable,*

then (x_*, λ_*) *is hemi-stable.*

Theorem (local convergence of SQP)

If
- *f and c are $C^{2,1}$ near a local minimizer x_* of (P_{EI}),*
- *there is a unique multiplier λ_* associated with x_*,*
- *SC2 is satified,*

then there exists a neighborhood V of (x_*, λ_*) such that if the first iterate $(x_1, \lambda_1) \in V$, then

1. *the SQP algorithm can generate $\{(x_k, \lambda_k)\}$ in V,*
2. *any sequence $\{(x_k, \lambda_k)\}$ generated in V by the SQP algorithm converges quadratically to (x_*, λ_*).*

- Consider the nonsmooth penalty function $\Theta_\sigma : \mathbb{E} \to \mathbb{R}$ associated with (P_{EI}):
$$\Theta_\sigma(x) := f(x) + \sigma \|c(x)^\#\|,$$
where for $v \in \mathbb{R}^m$, $v^\# \in \mathbb{R}^m$ is defined by: $(v^\#)_i = v_i$ when $i \in E$ and $(v^\#)_i = v_i^+ = \max(0, v_i)$ when $i \in I$.
- The dual norm of $\|\cdot\|$ is defined by
$$\|u\|_D := \sup_{\|v\| \leqslant 1} u^\top v.$$

Proposition (exact penalty property)

If
- *f and c are C^2 near a local minimizer x_* of (P_{EI}),*
- *the set Λ_* of associated optimal multipliers is nonempty,*
- *weak SC2 holds,*
- *$\sigma \geqslant \sup\{\|\lambda_*\|_D : \lambda_* \in \Lambda_*\}$ and $\sigma > \|\hat{\lambda}_*\|_D$ for some $\hat{\lambda}_* \in \Lambda_*$,*

then x_* is a strict local minimum of Θ_σ.

Recall the osculating quadratic problem at (x_k, λ_k): $M_k \simeq L(x_k, \lambda_k)$ and

$$(\text{OQP})_k \qquad \begin{cases} \min_d \ \langle \nabla f(x_k), d \rangle + \frac{1}{2} \langle M_k d, d \rangle \\ (c(x_k) + c'(x_k)d)^\# = 0. \end{cases} \tag{31}$$

We often make the following assumption (true for ℓ_p norms)

$$\| \cdot^\# \| : v \in \mathbb{R}^m \mapsto \|v^\#\| \text{ is convex.}$$

Proposition (descent direction)

If
- $(d_k, \lambda_k^{\mathrm{QP}})$ *is a stationary pair of* (31),
- $\| \cdot^\# \|$ *is convex,*

then

1. $\Theta'_\sigma(x_k; d_k) \leqslant -\langle M_k d_k, d_k \rangle + (\lambda_k^{\mathrm{QP}})^\mathsf{T} c(x_k) - \sigma \|c(x_k)^\#\|$,
2. $\Theta'_\sigma(x_k; d_k) < 0$, *if* $\sigma \geqslant \|\lambda_k^{\mathrm{QP}}\|_D$, $M_k \succ 0$, *and* x_k *is not stationary.*

The SQP algorithm with linesearch forces $M_k \succ 0$ and minimizes the changing nondifferentiable merit function Θ_{σ_k} along the SQP directions d_k.

Algorithm (global SQP)

Given $(x_k, \lambda_k, M_k) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathcal{S}^n_{++}$, *compute* $(x_{k+1}, \lambda_{k+1}, M_{k+1})$ *by*

- *solve* (31) *to get a PD solution* $(d_k, \lambda_k^{\mathrm{QP}})$ *(if any),*
- *impose* $\sigma_k \geqslant \|\lambda_k^{\mathrm{QP}}\|_D + \bar{\sigma}$ *keeping* σ_k *constant if* $\{\lambda_k^{\mathrm{QP}}\}$ *is bounded,*
- *linesearch:* $\alpha_k > 0$ *such that* $\Theta_{\sigma_k}(x_k + \alpha_k d_k) \leqslant \Theta_{\sigma_k}(x_k) + \omega \alpha_k \Delta_k$,
- $x_{k+1} := x_k + \alpha_k d_k$ *and* $\lambda_{k+1} := \lambda_k + \alpha_k(\lambda_k^{\mathrm{QP}} - \lambda_k)$,
- *update* $M_k \curvearrowright M_{k+1}$.

We have used $\Delta_k := \langle \nabla f(x_k), d_k \rangle - \sigma_k \|c(x_k)^\#\|$ as a negative over-estimate of $\Theta'_{\sigma_k}(x_k; d_k)$.

Theorem (global convergence of SQP)

If
- f *and* c *are* $C^{1,1}$,
- $\|\cdot^{\#}\|$ *is convex,*
- $\{M_k\}$ *and* $\{M_k^{-1}\}$ *are* $\succ 0$ *and bounded,*
- (31) *has a PD solution* $(d_k, \lambda_k^{\rm QP})$ *for all* $k \geqslant 1$,
- $\{\lambda_k^{\rm QP}\}$ *is bounded,*
- $\Theta_{\sigma_k}(x_k)$ *is bounded below,*

then the KKT conditions are satisfied asymptotically, meaning that $\nabla_x \ell(x_k, \lambda_k^{\rm QP}) \to 0$, $c(x_k)^{\#} \to 0$, $(\lambda_k^{\rm QP})_I \geqslant 0$, *and* $(\lambda_k^{\rm QP})_I^{\mathsf T} c_I(x_k) \to 0$.

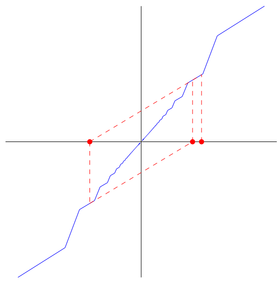
- Let $\mathbb{E}$ and $\mathbb{F}$ be (finite dimensional) normed spaces and $\Omega \subseteq \mathbb{E}$ be open.
- We consider the problem of fiding a zero of a nonsmooth function $F : \Omega \to \mathbb{F}$:
$$F(x) = 0. \tag{32}$$
- **Examples**
  1. The CP $(0 \leqslant x \perp \Phi(x) \geqslant 0)$ can be represented by (32):
  $$\Psi(x, \Phi(x)) = 0,$$
  where $\Psi(u, v) = \{\psi(u_i, v_i)\}_i$ and ψ is a C-function, meaning that $\psi(a, b) = 0$ iff $a \geqslant 0$, $b \geqslant 0$, and $ab = 0$. Examples (F = Fischer):
  $$\psi_{\min}(a, b) = \min(a, b) \quad \text{and} \quad \psi_F(a, b) = \sqrt{a^2 + b^2} - (a + b).$$
  2. The VI problem (find $x \in C$ s.t. $\langle \Phi(x), y - x\rangle \geqslant 0$ for all $y \in C$) can be written llike (32):
  $$\mathrm{P}_C(x - \Phi(x)) - x = 0.$$

Linearization methods

The semismooth Newton method (motivation)

- Suppose that $F : \mathbb{E} \to \mathbb{F}$ is Lipschitz continuous near $x_* \in \Omega$.
- Then $\partial_C F(x) \neq \varnothing$ for x near x_* and one could think of the Newton-like algorithm generating $\{x_k\}$ by
$$x_{k+1} = x_k - J_k^{-1} F(x_k),$$
provided some nonsingular J_k can be found in $\partial_C F(x_k)$.
- Actually, this algorithm may not converge locally as in the example below [18]:

$$F : \mathbb{R} \to \mathbb{R} \text{ is Lipschitz},$$
$$F(0) = 0,$$
$$\partial_C F(0) = [\tfrac{1}{2}, 2] \not\ni 0,$$
$$\lim_{k\to\infty} \frac{F(x_k) - F(0) - F'(x_k)(x_k - 0)}{|x_k - 0|} \geqslant 1.$$



- Semismoothness is an assumption on F that prevents such a cycling (arbitrary close to the zero).

Linearization methods

The semismooth Newton method (generalized differentiability I) [6]

For a function $F : \mathbb{E} \to \mathbb{F}$ ($\mathbb{E}$, $\mathbb{F}$ are finite dimensional vector spaces), denote

$$\mathcal{D}_F := \{x \in \mathbb{E} : F \text{ is Fréchet-differentiable at } x\}.$$

Theorem (Rademacher, 1919)

If F is Lipschitz near any point of an open set $\Omega \subseteq \mathbb{E}$, then the Lebesgue measure of $\Omega \setminus \mathcal{D}_F$ is zero; in particular $\mathcal{D}_F$ is dense in Ω, i.e., $\Omega \subseteq \overline{\mathcal{D}_F}$.

- The B-differential (B honoring Bouligand) of F at x is the set
$$\partial_B F(x) := \{J \in \mathcal{L}(\mathbb{E}, \mathbb{F}) : \exists\, \{x_k\} \subseteq \mathcal{D}_F \text{ such that } x_k \to x,\ F'(x_k) \to J\}.$$
- The C-differential (C for Clarke) of F at x is the set
$$\partial_C F(x) := \operatorname{co} \partial_B F(x).$$

Proposition (compactness and upper semi-continuity)

If F is L-Lipschitz near x, then

1. $\partial_C F(x)$ *is nonempty compact* ($\subseteq L\bar{B}$) *and convex,*
2. $\partial_C F$ *is upper semi-continuous at* x.

$\partial_C F(x)$ is said to be nonsingular if any $J \in \partial_C F(x)$ is nonsingular.

Proposition (nonsingularity diffusion)

If F is Lipschitz near x and $\partial_C F(x)$ is nonsingular, then there are constants $C > 0$ and $\delta > 0$ such that

$$\sup_{\substack{x' \in \bar{B}(x,\delta) \\ J \in \partial_C F(x')}} \max\left(\|J\|, \|J^{-1}\|\right) \leqslant C.$$

Let $\mathbb{E}$, $\mathbb{F}$, ... be normed spaces of finite dimension and Ω be open in $\mathbb{E}$.

- $F : \Omega \to \mathbb{F}$ is semismooth at $x \in \Omega$ if

  (S_1) F is Lipschitz near x,
  (S_2) $F'(x; h)$ exists for all $h \in \mathbb{E}$,
  (S_3) for $h \to 0$ in $\mathbb{E}$, the following holds

$$\sup_{J \in \partial_C F(x+h)} \frac{\|F(x+h) - F(x) - Jh\|}{\|h\|} \to 0.$$

- $F : \Omega \to \mathbb{F}$ is strongly semismooth at $x \in \Omega$ if it is semismooth at x with (S_3) strengthened by

  (S_4) for h small in $\mathbb{E}$, the following holds

$$\sup_{J \in \partial_C F(x+h)} \|F(x+h) - F(x) - Jh\| = O(\|h\|^2).$$

Linearization methods

The semismooth Newton method (semismoothness - properties I) [9, 14]

Proposition (differentiable function)

If $F : \Omega \to \mathbb{F}$ is C^1 ($C^{1,1}$) near $x \in \Omega$, then F is (strongly) semismooth at x.

Proposition (convex function)

If $f : \Omega \to \mathbb{R}$ is convex in a convex neighborhood of $x \in \Omega$, then f is semismooth at x.

$F : \Omega \to \mathbb{F}$ is said to be piecewise semismooth at $x \in \Omega$ if there exist a neighborhood V of x and functions $F_i : V \to \mathbb{F}$, with $i \in I$ (I finite), which are semismooth at x, such that • F is continuous on V and, • for all $y \in V$, $F(y) = F_i(y)$ for some $i \in I$.

Proposition (piecewise semismooth function)

If $F : \Omega \to \mathbb{F}$ is piecewise semismooth at $x \in \Omega$, then F is semismooth at x.

When the pieces are affine, F is said to be piecewise affine at x.

Proposition (piecewise affine function)

If $F : \Omega \to \mathbb{F}$ is piecewise affine at $x \in \Omega$, then F is strongly semismooth at x.

Linearization methods

The semismooth Newton method (semismoothness - properties II) [9, 14]

Proposition (componentwise semismooth function)

Let $F_1 : \Omega \to \mathbb{F}_1$, $F_2 : \Omega \to \mathbb{F}_2$, and $x \in \Omega$. Then $(F_1, F_2) : \Omega \to \mathbb{F}_1 \times \mathbb{F}_2$ is (strongly) semismooth at $x \in \Omega$ if and only if F_1 and F_2 are (strongly) semismooth at x.

Proposition (composition of functions)

If $F : \Omega \to \mathbb{F}$ is (strongly) semismooth at $x \in \Omega$, V is a neighborhood of $F(x)$, and $G : V \to \mathbb{G}$ is (strongly) semismooth at $F(x)$, then $G \circ F$ is (strongly) semismooth at x.

Proposition (calculus)

If $F_1 : \Omega \to \mathbb{F}$ and $F_2 : \Omega \to \mathbb{F}$ are (strongly) semismooth at $x \in \Omega$, then the following functions are (strongly) semismooth at x (for the last two, $\mathbb{F} = \mathbb{R}^m$):

$$F_1 + F_2, \quad \langle F_1, F_2 \rangle, \quad \max(F_1, F_2), \quad \text{and} \quad \min(F_1, F_2).$$

Examples

- The ℓ_p norm, for $1 \leqslant p \leqslant \infty$, is *strongly* semismooth.
- The min C-function $\psi_{\min} : (a, b) \in \mathbb{R}^2 \mapsto \min(a, b)$ is *strongly* semismooth.
- The Fischer C-function $\psi_F : (a, b) \in \mathbb{R}^2 \mapsto \sqrt{a^2 + b^2} - (a + b)$ is *strongly* semismooth.
- The projector P_K on the convex set $K := \{x \in \mathbb{E} : c(x) \leqslant 0\}$ is semismooth at x, provided
  - $c : \mathbb{E} \to \mathbb{R}^m$ is C^2 and componentwise convex,
  - the constant rank constraint qualification (CQ-CR) holds at $\mathrm{P}_K(x)$ [(CQ-LI) is certainly fine].

Algorithm (semismooth Newton for equations)

Given $x_k \in \mathbb{E}$, compute $x_{k+1} \in \mathbb{E}$ as a solution to

$$F(x_k) + J_k(x - x_k) = 0,$$

for some nonsingular $J_k \in \partial_C F(x_k)$ (if any).

Remarks

- To work well the algorithm needs smoothness and regularity assumptions.
- There is a single linear system to solve per iteration (i.e., cheap iteration).

Theorem (local convergence of the semismooth Newton method)

If
- $F(x_*) = 0$,
- F *is semismooth at* x_*,
- $\partial_C F(x_*)$ *is nonsingular,*

then

1. *there is a neighborhood* V *of* x_* *such that the semismooth Newton algorithm starting at* $x_1 \in V$ *is well defined and generates a sequence in* V*, converging to* x_* *superlinearly,*
2. *if* F *is strongly semismooth, then the convergence is quadratic.*

Notation and first properties

- $\mathcal{S}^n$ is the Euclidean space of symmetric $n \times n$ real matrices, equipped with the scalar product

$$\langle \cdot, \cdot \rangle : (A, B) \in (\mathcal{S}^n)^2 \mapsto \langle A, B \rangle = \operatorname{tr}(AB) = \textstyle\sum_{ij} A_{ij} B_{ij} \in \mathbb{R}.$$

- $\mathcal{S}^n_+$ is the cone of $\mathcal{S}^n$ made of the positive semidefinite matrices:

$$\begin{aligned} A \succcurlyeq 0 \quad &\overset{\text{def}}{\Longleftrightarrow} \quad A \in \mathcal{S}^n_+ \quad \Longleftrightarrow \quad \lambda(A) \subseteq [0, +\infty), \\ A \succcurlyeq 0 \quad &\Longleftrightarrow \quad \forall\, B \succcurlyeq 0 : \ \langle A, B \rangle \geqslant 0, \\ \text{if } A \succcurlyeq 0 \text{ and } B \succcurlyeq 0, \text{ then} \quad &\langle A, B \rangle = 0 \quad \Longleftrightarrow \quad AB = 0, \\ \mathrm{T}_A \mathcal{S}^n_+ &= \{D \in \mathcal{S}^n : v^\mathsf{T} D v \geqslant 0, \ \text{for all } v \in \mathcal{N}(A)\}. \end{aligned} \tag{33}$$

By (33), $\mathcal{S}^n_+$ is self-dual, meaning that $(\mathcal{S}^n_+)^+ = \mathcal{S}^n_+$.

- $\mathcal{S}^n_{++}$ is the cone of $\mathcal{S}^n$ made of the positive definite matrices:

$$\begin{aligned} A \succ 0 \quad &\overset{\text{def}}{\Longleftrightarrow} \quad A \in \mathcal{S}^n_{++} \quad \Longleftrightarrow \quad \lambda(A) \subseteq (0, +\infty), \\ A \succ 0 \quad &\Longleftrightarrow \quad \forall\, B \in \mathcal{S}^n_+ \setminus \{0\} : \ \langle A, B \rangle > 0, \\ A \succcurlyeq 0 \text{ and } [v^\mathsf{T} M v > 0, \ \forall\, v \in \mathcal{N}(A) \setminus \{0\}] \quad &\Longrightarrow \quad M + rA \succ 0 \text{ for large} \end{aligned}$$

Semidefinite optimization

Problem definition (primal and dual problems)

The primal and (Lagrange) dual of the SDO problem read

$$(P)\quad \begin{cases} \inf\ \langle C, X\rangle \\ \mathcal{A}(X) = b \\ X \succcurlyeq 0 \end{cases} \qquad \text{and} \qquad (D)\quad \begin{cases} \sup\ \langle b, y\rangle \\ \mathcal{A}^*(y) + S = C \\ S \succcurlyeq 0, \end{cases} \tag{34}$$

where

- $C \in \mathcal{S}^n$ and $b \in \mathbb{R}^m$,
- $\mathcal{A} : \mathcal{S}^n \to \mathbb{R}^m$ is linear ($\mathcal{A}^* : \mathbb{R}^m \to \mathcal{S}^n$ is its adjoint).

Notation

- Feasible sets: $\mathcal{F}_P := \{X \in \mathcal{S}^n_+ : \mathcal{A}(X) = b\}$, $\mathcal{F}_D := \{(y, S) \in \mathbb{R}^m \times \mathcal{S}^n_+ : \mathcal{A}^*(y) + S = C\}$, and $\mathcal{F} := \mathcal{F}_D \times \mathcal{F}_D$.
- Strictly feasible sets: $\mathcal{F}^s_P := \{X \in \mathcal{S}^n_{++} : \mathcal{A}(X) = b\}$, $\mathcal{F}^s_D := \{(y, S) \in \mathbb{R}^m \times \mathcal{S}^n_{++} : \mathcal{A}^*(y) + S = C\}$, and $\mathcal{F}^s := \mathcal{F}^s_D \times \mathcal{F}^s_D$.
- Optimal values: $\mathrm{val}(P)$ and $\mathrm{val}(D)$.
- Solution sets: $\mathrm{Sol}(P)$ and $\mathrm{Sol}(D)$.

Semidefinite optimization

Problem definition (Lagrange dualization consequences)

The Lagrangian of problem (P) is the function $\ell : \mathcal{S}^n \times \mathbb{R}^m \times \mathcal{S}^n \to \mathbb{R}$ defined at $(X, y, S) \in \mathcal{S}^n \times \mathbb{R}^m \times \mathcal{S}^n$ by

$$\ell(X, y, S) = \langle C, X\rangle - \langle y, A(X) - b\rangle - \langle S, X\rangle .$$

Proposition (consequences of the Lagrangian dualization)

1. $\mathrm{val}(D) \leqslant \mathrm{val}(P)$.
2. $(X, y, S) \in \mathcal{F} \implies \langle C, X\rangle - \langle b, y\rangle = \langle X, S\rangle \geqslant 0$.
3. $(X, y, S) \in \mathcal{F}$, $\langle X, S\rangle = 0$
   $\iff X \in \mathrm{Sol}(P)$, $(y, S) \in \mathrm{Sol}(D)$, $\mathrm{val}(D) = \mathrm{val}(P)$,
   $\iff (X, (y, S))$ *is a saddle-point of* ℓ *on* $\mathcal{S}^n \times (\mathbb{R}^m \times \mathcal{S}^n_+)$.

Remarks

- One says that there is a duality gap if $\mathrm{val}(D) < \mathrm{val}(P)$.
- $X \in \mathrm{Sol}(P)$, $(y, S) \in \mathrm{Sol}(D) \quad \not\Rightarrow \quad \mathrm{val}(D) = \mathrm{val}(P)$.

Semidefinite optimization
Problem definition (examples of SDO formulations)

The Schur complement of $A \succ 0$ in

$$K := \begin{pmatrix} A & B \\ B^\mathsf{T} & C \end{pmatrix}$$

is $(A\,|\,K) := C - B^\mathsf{T} A^{-1} B$. The following holds

$$K \succ 0 \quad \Longleftrightarrow \quad \begin{cases} A \succ 0 \\ (A\,|\,K) \succ 0. \end{cases}$$

Examples of SDO modelling

1. Linear optimization.
2. Convex quadratic optimization.
3. Global minimization of polynomials (relaxation if # variables $\geqslant 2$).
4. Rank relaxation of a QCQP.
5. Many more ...

Semidefinite optimization
Existence of solution

- Strong duality of linear optimization no longer holds (since the linear image of a closed convex cone is not necessarily closed).
- Here are conditions for having nonempty compact sets of solutions.

Proposition (compact sets of solutions)

1. $\mathcal{F}_P \times \mathcal{F}_D^s \neq \varnothing \quad \Longrightarrow \quad \mathrm{Sol}(P) \neq \varnothing$ *and compact.*
2. $\mathcal{F}_P^s \times \mathcal{F}_D \neq \varnothing \quad \Longrightarrow \quad \mathrm{Sol}(D) \cap (\mathcal{R}(\mathcal{A}) \times \mathcal{S}^n) \neq \varnothing$ *and compact.*
3. $\mathcal{F}^s \neq \varnothing \quad \Longrightarrow \quad \mathrm{Sol}(P)$ *and* $\mathrm{Sol}(D) \cap (\mathcal{R}(\mathcal{A}) \times \mathcal{S}^n) \neq \varnothing$ *and compact.*

In these cases, there is no duality gap: $\mathrm{val}(D) = \mathrm{val}(P)$.

- The sufficient condition in 1 is (CQ-R) for (D).
  The sufficient conditions in 2 are (CQ-R) for (P).

Proposition (optimality conditions)

Suppose that $\mathcal{F}^s \neq \varnothing$, then

$$(X,(y,S)) \in \text{Sol}(P) \times \text{Sol}(D) \quad \Longleftrightarrow \quad \left\{ \begin{array}{ll} \mathcal{A}^*(y) + S = C, & S \succcurlyeq 0, \\ \mathcal{A}(X) = b, & X \succcurlyeq 0, \\ \langle X, S \rangle = 0. & \end{array} \right. \tag{35}$$

- There are good reasons to generate iterates well inside $\mathcal{F}^s_P$. This is obtained analytically (not geometrically) by an interior penalization:

$$(P) \quad \curvearrowright \quad (P_\mu) \quad \left\{ \begin{array}{l} \inf_X \ \langle C, X \rangle + \mu \, \text{ld}(X) \\ A(X) = b, \end{array} \right.$$

$$(D) \quad \curvearrowright \quad (D_\mu) \quad \left\{ \begin{array}{l} \sup_{(y,S)} \ \langle b, y \rangle - \mu \, \text{ld}(S) \\ A^*(y) + S = C, \end{array} \right.$$

  where $\text{ld} : \mathcal{S}^n \to \overline{\mathbb{R}}$ is the strictly convex and closed function defined at X by

$$\text{ld}(X) := \left\{ \begin{array}{ll} -\log\det(X) & \text{if } X \succ 0 \\ +\infty & \text{otherwise.} \end{array} \right.$$

- Three properties of ld (with $X \succ 0$ and H, $K \in \mathcal{S}^n$):

$$\text{ld}'(X) \cdot H = -\left\langle X^{-1}, H \right\rangle,$$
$$\text{ld}''(X) \cdot (H,K) = \left\langle X^{-1} H X^{-1}, K \right\rangle,$$
$$\text{ld}^\infty = \mathcal{I}_{\mathcal{S}^n_+}.$$

The central path is the smooth curve $\mathscr{C} : \mu \in \mathbb{R}_{++} \mapsto$ the unique solution to

$$(O_\mu) \quad \begin{cases} A^*(y) + S = C, & S \succ 0, \\ A(X) = b, & X \succ 0, \\ XS = \mu I. \end{cases} \tag{36}$$

Proposition (existence and smoothness of the central path)

Suppose that $\mathcal{F}^s \neq \varnothing$ and $\mu > 0$. Then,

1. *the system (O_μ) has a solution (X_μ, y_μ, S_μ), unique in $\mathcal{S}^n_{++} \times \mathcal{R}(\mathcal{A}) \times \mathcal{S}^n_{++}$,*
2. *the map $\mu \in \mathbb{R}_{++} \mapsto (X_\mu, y_\mu, S_\mu) \in \mathcal{S}^n_{++} \times \mathcal{R}(\mathcal{A}) \times \mathcal{S}^n_{++}$ is C^∞.*

A primal-dual path-following interior-point algorithm generates iterates $z_k := (X_k, y_k, S_k) \in \mathcal{F}^s$ in a neighborhood $V(\theta)$ of the central path $\mathscr{C}$ ($\theta \in (0,1)$ is a parameter that determines the size of the neighborhood). Each iteration proceeds along a Newton direction aiming a moving point on $\mathscr{C}$, whose central parameter if $\sigma\bar{\mu}(z)$ where $\sigma \in (0,1)$ and $\bar{\mu}(z) := \langle X, S \rangle / n$.

Algorithm (primal-dual path-following IP)

From one iterate z to the next one z_+.

1. *Let d be the Newton direction on a symmetrized version of $(O_{\sigma\bar{\mu}(z)})$.*
2. *Determine a large stepsize $\alpha > 0$ such that $z + \alpha d \in V(\theta)$.*
3. $z_+ := z + \alpha d$.

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