HORPO with Computability Closure : A Reconstruction

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Outline

RPO [Dershowitz 79]

build a WF ordering on terms from a WF ordering on symbols

ingredients:

(i) a WF ordering $>_{\mathcal{F}}$ on symbols

(ii) a status $\mathrm{stat}_f \in \{\mathrm{lex}, \mathrm{mul}\}$ for each symbol $f \in \mathcal{F}$

$s > t$ if:

1. $s = f(\vec{s})$ and either:
   (a) $(\exists i)\ s_i \geq t$
   (b) $t = g(\vec{t})$, $f =_{\mathcal{F}} g$, $\vec{s} >_{\mathrm{stat}_f} \vec{t}$ and $(\forall j)\ s > t_j$
   (c) $t = g(\vec{t})$, $f >_{\mathcal{F}} g$ and $(\forall j)\ s > t_j$

HORPO [Jouannaud-Rubio 99]

additional ingredient: a congruence $=_{\mathcal{T}_\mathcal{S}}$ on types

$s > t$ if $type(s) =_{\mathcal{T}_\mathcal{S}} type(t)$ and either:

1. $s = f(\vec{s})$ and either:
   (a) $(\exists i)\ s_i \geq t$
   (b) $t = g(\vec{t})$, $f =_\mathcal{F} g$, $\vec{s} >_{\mathrm{stat}_f} \vec{t}$ and $(\forall j)\ s > t_j$ or $(\exists i)\ s_i \geq t_j$
   (c) $t = g(\vec{t})$, $f >_\mathcal{F} g$ or $g = @$, and $(\forall j)\ s > t_j$ or $(\exists i)\ s_i \geq t_j$
2. $s = @(s_1, s_2)$ and:
   (c) $t = @(t_1, t_2)$ and $\vec{s} >_{\mathrm{mul}} \vec{t}$
3. $s = \lambda x.u$ and:
   (c) $t = \lambda y.v$, $type(x) =_{\mathcal{T}_\mathcal{S}} type(y)$ and $u^z_x > v^z_y$ (z fresh)

Well-foundedness proof

uses a model based on Tait-Girard's notion of reducibility:

- terms are interpreted by themselves
- types are interpreted by reducibility candidates
- typing is interpreted by set membership

a set S of terms is a *reducibility candidate* if:

1. $S \subseteq \mathrm{SN}(>)$
2. $>(S) \subseteq S$
3. if $t \neq \lambda x.u$ and $>(t) \subseteq S$, then $t \in S$

Interpretation of types

$>$ is WF if the interpretation $\_ \mapsto [\![\_]\!]$ is valid, that is:

1. for all type T, $[\![T]\!]$ is a reducibility candidate
2. every term is *computable*: if $t : T$ then $t \in [\![T]\!]$

in particular, the following interpretation is valid:

- $[\![\mathbb{B}]\!] = \mathrm{SN}$
- $[\![U \rightarrow V]\!] = \{t \in \mathcal{T} \mid \forall u \in [\![U]\!], tu \in [\![V]\!]\}$

Outline

HORPO with Computability Closure [Jouannaud-Rubio 99]

this model provides a way for improving HORPO

given a function $\mathrm{CC} : \mathcal{T} \to \mathcal{P}(\mathcal{T})$ such that
every $t \in \mathrm{CC}(f(\vec{s}))$ is computable whenever $\vec{s}$ are computable

$s > t$ if $type(s) =_{\mathcal{T}_\mathcal{S}} type(t)$ and either:

1. $s = f(\vec{s})$ and either:
   (a) $(\exists i)\ s_i \geq t$
   (b) $t = g(\vec{t})$, $f =_\mathcal{F} g$, $\vec{s} >_{\mathrm{stat}_f} \vec{t}$ and $(\forall j)\ s > t_j$ or $t_j \in \mathrm{CC}(s)$
   (c) $t = g(\vec{t})$, $f >_\mathcal{F} g$ or $g = @$, and $(\forall j)\ s > t_j$ or $t_j \in \mathrm{CC}(s)$

...

Computability Closure

inductive formulation of the general schema [Jouannaud-Okada 91]

taking $\succ = \rightarrow_\beta$, $t \in \mathrm{CC}^{\succ}(s, X)$ if either:

1. $s = f(\vec{s})$ and either:
   (a) $(\exists i)$ $s_i = t$ or $s_i \rhd t$, $type(t) \in \mathcal{S}$ and $\mathrm{FV}(t) \subseteq \mathrm{FV}(s_i)$
   (b) $t = g(\vec{t})$, $f =_{\mathcal{F}} g$, $\vec{s}$ $(\succ \cup \rhd)_{\mathrm{stat}_f}$ $\vec{t}$ and $(\forall j)$ $t_j \in \mathrm{CC}(s, X)$
   (c) $t = g(\vec{t})$, $f >_{\mathcal{F}} g$ or $g = @$, and $(\forall j)$ $t_j \in \mathrm{CC}(s, X)$
4. (a) $s = f(\vec{s})$ and $t \in X$
   (b) $s = f(\vec{s})$, $t = \lambda y.w$ and $w^z_y \in \mathrm{CC}(s, X \cup \{z\})$ (z fresh)
5. $s = f(\vec{s})$, $t \prec u$ and $u \in \mathrm{CC}(s, X)$

can we better integrate HORPO and the computability closure ?
[LPAR'06]

Computability closure ordering [Blanqui 06]

let $\mathrm{CR}(R)$ be the smallest rewrite relation containing the pairs (t, u) such that $u \in \mathrm{CC}^{\rightarrow_\beta \cup \rightarrow_R}(t, \emptyset)$ and $type(t) =_{\mathcal{T}_\mathcal{S}} type(u)$

CR is monotone and its fixpoint $\succ$ is WF and contains HORPO

$C[s] \succ C[t]$ if $type(s) =_{\mathcal{T}_\mathcal{S}} type(t)$ and $s \succ_h^\emptyset t$; $s \succ_h^X t$ if either:

1. $s = f(\vec{s})$ and either:
   (a) $(\exists i)\ s_i = t$ or $s_i \rhd t$, $type(t) \in \mathcal{S}$ and $\mathrm{FV}(t) \subseteq \mathrm{FV}(s_i)$
   (b) $t = g(\vec{t})$, $f =_\mathcal{F} g$, $\vec{s}\ (\succ \cup \rhd)_{\mathrm{stat}_f}\ \vec{t}$ and $(\forall j)\ s \succ_h^X t_j$
   (c) $t = g(\vec{t})$, $f >_\mathcal{F} g$ or $g = @$, and $(\forall j)\ s \succ_h^X t_j$
4. (a) $s = f(\vec{s})$ and $t \in X$
   (b) $s = f(\vec{s})$, $t = \lambda y.w$ and $s \succ_h^{X \cup \{z\}} w_y^z$ (z fresh)
5. $s = f(\vec{s})$, $s \succ_h^X u$ and $u \succ t$

unfortunately, this ordering is not decidable

Outline

Limitation of HORPO

functions defined by pattern-matching on strictly positive types

example: $0 : \mathbb{O}, \quad s : \mathbb{O} \rightarrow \mathbb{O}, \quad lim : (\mathbb{N} \rightarrow \mathbb{O}) \rightarrow \mathbb{O},$
$rec : \mathbb{O} \times \alpha \times (\mathbb{O} \rightarrow \alpha \rightarrow \alpha) \times ((\mathbb{N} \rightarrow \mathbb{O}) \rightarrow (\mathbb{N} \rightarrow \alpha) \rightarrow \alpha) \rightarrow \alpha$

$rec(lim(F), U, V, W) \rightarrow @(W, F, \lambda n.rec(@(F, n), U, V, W))$

we need to have $lim(F) > @(F, n)$

Interpretation of types [Mendler 87]

because it is (strictly) positive, $\mathbb{O}$ can be interpreted as the fixpoint of a monotone function on the lattice of reducibility candidates

$$\phi(S) = \{t \in \mathrm{SN} \mid t >^* su \Rightarrow u \in S, t >^* \mathit{lim}(F) \Rightarrow F \in [\![\mathbb{N}]\!] \to S\}$$

this provides a WF ordering on $[\![\mathbb{O}]\!]$:

$$t \sqsupset u \text{ if } o(t) > o(u)$$

where $o(t)$ is the smallest ordinal $\mathfrak{a}$ such that $t \in \phi^{\mathfrak{a}}(\bot)$

in particular, for all $n \in [\![\mathbb{N}]\!]$, $\mathit{lim}(F) \sqsupset @(F, n)$

Accessibility

a computable term may have non-computable subterms

$s \rhd_{\mathrm{acc}} t$ if

1. $s = f(\vec{s})$ and
   (a) $(\exists i)\ s_i \unrhd_{\mathrm{acc}} t$ and $type(s)$ occurs only positively in $type(t)$

- $\mathrm{Pos}^{+}(\mathbb{B}) = \{\varepsilon\}$
- $\mathrm{Pos}^{-}(\mathbb{B}) = \emptyset$
- $\mathrm{Pos}^{\delta}(U \to V) = 1.\mathrm{Pos}^{-\delta}(U) \cup 2.\mathrm{Pos}^{\delta}(V)$

if $s \rhd_{\mathrm{acc}} t$ and s is computable, then t is computable

Outline

New HORPO $>^{\emptyset}_{\mathcal{T}_\mathcal{S}}$

$s >^X_{\mathcal{T}_\mathcal{S}} t$ if $type(s) =_{\mathcal{T}_\mathcal{S}} type(t)$ and $s >^X t$

$s >^X t$ if either:

1. $s = f(\vec{s})$ and either:
   (a) $(\exists i)\ s_i \trianglerighteq_{\mathrm{acc}} \geq^X_{\mathcal{T}_\mathcal{S}} t$
   (b) $t = g(\vec{t})$, $f =_{\mathcal{F}} g$, $\vec{s}\ (>^{\emptyset}_{\mathcal{T}_\mathcal{S}} \cup \triangleright_{\mathrm{acc}} \geq^X_{\mathcal{T}_\mathcal{S}})_{\mathrm{stat}_f}\ \vec{t}$ and $(\forall j)\ s >^X t_j$
   (c) $t = g(\vec{t})$, $f >_{\mathcal{F}} g$ or $g = @$, and $(\forall j)\ s >^X t_j$

where $s \triangleright_{\mathrm{acc}} \geq^X_{\mathcal{T}_\mathcal{S}} t$ if $(\exists w)(\exists \vec{x} \in X)\ s \triangleright_{\mathrm{acc}} w$ and $@(w, \vec{x}) \geq_{\mathcal{T}_\mathcal{S}} t$

for instance: $lim(F) \triangleright_{\mathrm{acc}} \geq^{\{n\}}_{\mathcal{T}_\mathcal{S}} @(F, n)$

New HORPO $>^{\emptyset}_{\mathcal{T}_{\mathcal{S}}}$

2. $s = @(s_1, s_2)$ and either:
   (a) $(\exists i)\ s_i \trianglerighteq_{\mathrm{acc}} \geq^{X}_{\mathcal{T}_{\mathcal{S}}} t$
   (b) $t = @(t_1, t_2)$ and $\vec{s}\ (>^{X}_{\mathcal{T}_{\mathcal{S}}})_{\mathrm{mul}}\ \vec{t}$
   (c) $s_1 = \lambda x.w$ and $w^{s_2}_{x} \geq^{X} t$
3. $s = \lambda x.u$ and either:
   (a) $u^{z}_{x} \geq^{X}_{\mathcal{T}_{\mathcal{S}}} t$ (z fresh)
   (b) $t = \lambda y.v$, $type(x) =_{\mathcal{T}_{\mathcal{S}}} type(y)$ and $u^{z}_{x} >^{X} v^{z}_{y}$ (z fresh)
   (c) $u = @(v, x)$, $x \notin \mathrm{FV}(v)$ and $v \geq^{X} t$
4. (a) $x \notin \mathcal{X}$ and $t \in X$
   (b) $x \notin \mathcal{X}$, $s \neq \lambda x.u$, $t = \lambda y.w$ and $s >^{X \cup \{z\}} w^{z}_{y}$ (z fresh)

Conclusion

the new HORPO:

- more comparable terms
- less type checking
- better handling of bound variables
- handling of strictly positive types
- nice integration of the computability closure

future work:

- extension to the calculus of constructions