

1 Higher-order dependency pairs

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Abstract. Arts and Giesl proved that the termination of a first-order rewrite system can be reduced to the study of its “dependency pairs”. We extend these results to rewrite systems on simply typed λ -terms by using Tait’s computability technique.

1.1 Introduction

Let \mathcal{F} be a set of function symbols, \mathcal{X} be a set of variables and \mathcal{R} be a set of rewrite rules over the set $\mathcal{T}(\mathcal{F}, \mathcal{X})$ of first-order terms. Let \mathcal{D} be the set of symbols occurring at the top of a rule left hand-side and $\mathcal{C} = \mathcal{F} \setminus \mathcal{D}$. The set $\mathcal{DP}(\mathcal{R})$ of *dependency pairs* of \mathcal{R} is the set of pairs (l, t) such that l is the left hand-side of a rule $l \rightarrow r \in \mathcal{R}$ and t is a subterm of r headed by some symbol $f \in \mathcal{D}$. The term t represents a potential recursive call. The chain relation is $\rightarrow_{\mathcal{C}} = \rightarrow_{\mathcal{R}_i}^* \rightarrow_{\mathcal{DP}_h}$, where $\rightarrow_{\mathcal{R}_i}^*$ is the reflexive and transitive closure of the restriction of $\rightarrow_{\mathcal{R}}$ to non-top positions and $\rightarrow_{\mathcal{DP}_h}$ is the restriction of $\rightarrow_{\mathcal{DP}}$ to top positions. Arts and Giesl prove in [1] that $\rightarrow_{\mathcal{R}}$ is strongly normalizing (SN) (or terminating, well-founded) iff the chain relation so is. Moreover, $\rightarrow_{\mathcal{C}}$ is terminating if there is a weak reduction ordering $>$ such that $\mathcal{R} \subseteq \geq$ and $\mathcal{DP}(\mathcal{R}) \subseteq >$ (only dependency pairs need to strictly decrease).

We would like to extend these results to higher-order rewriting. There are several approaches to higher-order rewriting. In Higher-order Rewrite Systems (HRSs) [7], terms and rules are simply typed λ -terms in β -normal η -long form, left hand-sides are patterns à la Miller and matching is modulo $\beta\eta$. An extension of dependency pairs for HRSs is studied in [10,9]. In Combinatory Reduction Systems (CRSs) [6], terms are λ -terms, rules are λ -terms with meta-variables, left hand-sides are patterns à la Miller and matching uses α -conversion and some variable occur-checks. The relation between the two kinds of rewriting is studied in [12]. It appears that the matching algorithms are similar and that, in HRSs, one does more β -reductions after having applied the matching substitution. But, in both cases, β -reduction is used at the meta-level for normalizing right hand-sides after the application of the matching substitution. So, a third more atomic approach is to have no meta-level β -reduction and add β -reduction at the object level. This is the approach that we consider in this paper.

So, we assume given a set \mathcal{R} of rewrite rules made of simply typed λ -terms and study the termination of $\rightarrow_{\beta} \cup \rightarrow_{\mathcal{R}}$ when using CRS-like matching. This

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clearly implies the termination of $\rightarrow_{\mathcal{R}}$ in the corresponding CRS or HRS. Another advantage of this approach is that we can rely on Tait's technique for proving termination [11,3]. This paper explores its use with dependency pairs. This is in contrast with [10,9].

In Tait's technique, to each type T , one associates a set $\llbracket T \rrbracket$ of terms of type T . Terms of $\llbracket T \rrbracket$ are said *computable*. Before giving some properties of computable terms, let us introduce a few definitions. The sets $\text{Pos}^+(T)$ and $\text{Pos}^-(T)$ of *positive and negative positions* in T are defined as follows:

- $\text{Pos}^+(B) = \{\varepsilon\}$ and $\text{Pos}^-(B) = \emptyset$ if B is a base type,
- $\text{Pos}^\delta(T \Rightarrow U) = 1 \cdot \text{Pos}^{-\delta}(T) \cup 2 \cdot \text{Pos}^\delta(U)$.

We use \mathbf{T} to denote a sequence of types T_1, \dots, T_n of length $|\mathbf{T}| = n$. The i -th argument of a function symbol $f : \mathbf{T} \Rightarrow B$ is *accessible* if B occurs only positively in T_i . Let $\text{Acc}(f)$ be the set of indexes of the accessible arguments of f . A base type B is *basic* if, for all $f : \mathbf{T} \Rightarrow B$ and $i \in \text{Acc}(f)$, T_i is a base type. After [3,4], given a relation R , *computability wrt R* can be defined so that the following properties are satisfied:

- (1) A computable term is strongly normalizable wrt $\rightarrow_{\beta} \cup R$.
- (2) A term of basic type is computable if it is SN wrt $\rightarrow_{\beta} \cup R$.
- (3) A term $v^{T \Rightarrow U}$ is computable if, for all t^T computable, vt is computable.
- (4) If t is computable then every reduct of t is computable.
- (5) A term $f\mathbf{t}$ is computable if all its reducts wrt $\rightarrow_{\beta} \cup R$ are computable.
- (6) If $f\mathbf{t}$ is computable then, for all $i \in \text{Acc}(f)$, t_i is computable.
- (7) If t contains no $f \in \mathcal{D}$ and σ is computable, then $t\sigma$ is computable.
- (8) Every term is computable whenever every $f \in \mathcal{D}$ is computable.

1.2 Admissible rules

An important property of the first-order case is that, given a term t , a substitution σ and a variable $x \in \mathcal{V}(t)$, $x\sigma$ is strongly normalizable whenever $t\sigma$ so is. This is not always true in the higher-order case. So, we need to introduce some restrictions on rules to keep this property.

Definition 1 (Admissible rules) A rule $f\mathbf{l} \rightarrow r$ is *admissible* if $\text{FV}(r) \subseteq \text{PCC}(\mathbf{l})$, where PCC is defined in Figure 1.1.

The Pattern Computability Closure (PCC) is called accessibility in [2]. It includes most usual higher-order patterns [8].

Lemma 2 If $f\mathbf{l} \rightarrow r$ is admissible, $\text{dom}(\sigma) \subseteq \text{FV}(\mathbf{l})$ and $\mathbf{l}\sigma$ is computable, then $\sigma|_{\text{FV}(r)}$ is computable.

Proof. We prove by induction that, for all $u \in \text{PCC}(\mathbf{t})$ and computable substitution θ such that $\text{dom}(\theta) \subseteq \text{FV}(u) \setminus \text{FV}(\mathbf{t})$, $u\theta$ is computable.

Fig. 1.1. Pattern Computability Closure [2]

$$\begin{array}{l}
 (\text{arg}) \quad t_i \in \text{PCC}(\mathbf{t}) \\
 (\text{acc}) \quad \frac{g\mathbf{u} \in \text{PCC}(\mathbf{t}) \quad i \in \text{Acc}(g)}{u_i \in \text{PCC}(\mathbf{t})} \\
 (\text{lam}) \quad \frac{\lambda y u \in \text{PCC}(\mathbf{t}) \quad y \notin \text{FV}(\mathbf{t})}{u \in \text{PCC}(\mathbf{t})} \\
 (\text{app-left}) \quad \frac{uy \in \text{PCC}(\mathbf{t}) \quad y \notin \text{FV}(\mathbf{t}) \cup \text{FV}(u)}{u \in \text{PCC}(\mathbf{t})} \\
 (\text{app-right}) \quad \frac{y^{U \Rightarrow T \Rightarrow U} u \in \text{PCC}(\mathbf{t}) \quad y \notin \text{FV}(\mathbf{t}) \cup \text{FV}(u)}{u \in \text{PCC}(\mathbf{t})}
 \end{array}$$

- (arg) Since $\text{dom}(\theta) = \emptyset$, $l_i \sigma \theta = l_i \sigma$ is computable by assumption.
- (acc) By induction hypothesis, $g\mathbf{u}\sigma$ is computable. Thus, by property (6), $u_i \sigma$ is computable.
- (lam) Let $\theta' = \theta|_{\text{dom}(\theta) \setminus \{y\}}$. Wlog, we can assume that $y \notin \text{codom}(\sigma\theta)$. Hence, $(\lambda y u)\sigma\theta' = \lambda y u\sigma\theta'$. Now, since $\text{dom}(\theta) \subseteq \text{FV}(u) \setminus \text{FV}(\mathbf{t})$, $\text{dom}(\theta') \subseteq \text{FV}(\lambda y u) \setminus \text{FV}(\mathbf{t})$. Thus, by induction hypothesis, $\lambda y u\sigma\theta'$ is computable. Since $y\theta$ is computable, by (3), $(\lambda y u\sigma\theta')y\theta$ is computable and, by (4), $u\sigma\theta'\{y \mapsto y\theta\}$ is computable. Finally, since $y \notin \text{dom}(\sigma\theta') \cup \text{codom}(\sigma\theta')$, $u\sigma\theta'\{y \mapsto y\theta\} = u\sigma\theta$.
- (app-left) Let $v : T_y$ computable and $\theta' = \theta \cup \{y \mapsto v\}$. Since $\text{dom}(\theta) \subseteq \text{FV}(u) \setminus \text{FV}(\mathbf{t})$ and $y \notin \text{FV}(\mathbf{t})$, $\text{dom}(\theta') = \text{dom}(\theta) \cup \{y\} \subseteq \text{FV}(uy) \setminus \text{FV}(\mathbf{t})$. Thus, by induction hypothesis, $(uy)\sigma\theta' = u\sigma\theta'v$ is computable. Since $y \notin \text{FV}(u)$, $u\sigma\theta' = u\sigma\theta$. Thus, $u\sigma\theta$ is computable.
- (app-right) Let $v = \lambda x^U \lambda y^T x$ and $\theta' = \theta \cup \{y \mapsto v\}$. By (3), v is computable. Since $\text{dom}(\theta) \subseteq \text{FV}(u) \setminus \text{FV}(\mathbf{t})$ and $y \notin \text{FV}(\mathbf{t})$, $\text{dom}(\theta') \subseteq \text{FV}(yu) \setminus \text{FV}(\mathbf{t})$. Thus, by induction hypothesis, $(yu)\sigma\theta' = v\sigma\theta'$ is computable. Since $y \notin \text{FV}(u)$, $u\sigma\theta' = u\sigma\theta$. Thus, by (4), $u\sigma\theta$ is computable. \square

1.3 Higher-order dependency pairs

In the following, we assume given a set \mathcal{R} of admissible rules. The sets $\text{FAP}(t)$ of *full application positions* of a term t and the *level* of a term t are defined as follows:

- $\text{FAP}(x) = \emptyset$ and $\text{level}(x) = 0$
- $\text{FAP}(\lambda x t) = 1 \cdot \text{FAP}(t)$ and $\text{level}(\lambda x t) = \text{level}(t)$

If $f \in \mathcal{D}$ then:

- $\text{level}(ft_1 \dots t_n) = 1 + \max\{\text{level}(t_i) \mid 1 \leq i \leq n\}$

– $\text{FAP}(ft_1 \dots t_n) = \{\varepsilon\} \cup \bigcup_{i=1}^n 1^{n-i} 2 \cdot \text{FAP}(t_i)$

If $t \neq ft_1 \dots t_n$ with $f \in \mathcal{D}$, then $\text{FAP}(tu) = 1 \cdot \text{FAP}(t) \cup 2 \cdot \text{FAP}(u)$ and $\text{level}(tu) = \max\{\text{level}(t), \text{level}(u)\}$.

Definition 3 (Dependency pairs) The set of *dependency pairs* is $\mathcal{DP} = \{l \rightarrow r|_p \mid l \rightarrow r \in \mathcal{R}, p \in \text{FAP}(r)\}$. The *chain relation* is $\rightarrow_{\mathcal{C}} = \rightarrow_{\mathcal{R}_i}^* \rightarrow_{\mathcal{DP}_h}$, where $\rightarrow_{\mathcal{R}_i}$ is the restriction of $\rightarrow_{\mathcal{R}}$ to non-top positions, and $\rightarrow_{\mathcal{DP}_h}$ is the restriction of $\rightarrow_{\mathcal{DP}}$ to top positions.

Since $\rightarrow_{\mathcal{C}} \subseteq \rightarrow_{\mathcal{R}}^+ \supseteq$, $\rightarrow_{\beta_{\mathcal{C}}}$ is terminating whenever $\rightarrow_{\beta_{\mathcal{R}}}$ so is.

Theorem 4 Assume that, for all $l \rightarrow r \in \mathcal{R}$ and $p \in \text{FAP}(r)$, $\text{FV}(r|_p) \subseteq \text{FV}(r)$ and $r|_p$ has the type of l (*). Then, $\rightarrow_{\beta_{\mathcal{R}}}$ is terminating if $\rightarrow_{\beta_{\mathcal{C}}}$ so is.

Proof. By (1), this is so if every term is computable wrt $\rightarrow_{\mathcal{R}}$. By (8), this is so if every $f^{T \Rightarrow B} \in \mathcal{D}$ is computable. By (3), this is so if, for all $\mathbf{t} : \mathbf{T}$ computable, $f\mathbf{t}$ is computable. We prove it by induction on $(f\mathbf{t}, \mathbf{t})$ with $(\rightarrow_{\mathcal{C}}, (\rightarrow_{\beta_{\mathcal{R}}})_{\text{lex}})_{\text{lex}}$ as well-founded ordering (H1). Indeed, by (1), \mathbf{t} are strongly normalizable wrt $\rightarrow_{\beta_{\mathcal{R}}}$. By (5), it suffices to prove that every reduct of $f\mathbf{t}$ is computable. If $\mathbf{t} \rightarrow_{\beta_{\mathcal{R}}} \mathbf{t}'$ then, by (H1), $f\mathbf{t}'$ is computable since, by (4), \mathbf{t}' are computable and $\rightarrow_{\beta_{\mathcal{C}}}(f\mathbf{t}') = \rightarrow_{\beta_{\mathcal{C}}}(f\mathbf{t})$. Now, assume that there is $f\mathbf{l} \rightarrow r \in \mathcal{R}$ and σ such that $\mathbf{t} = \mathbf{l}\sigma$. Since rules are admissible, by Lemma 2, $\sigma' = \sigma|_{\text{FV}(r)}$ is computable. We now prove that $r\sigma'$ is computable by induction on the level n of r (H2). Let p_1, \dots, p_k be the positions in r of the subterms of level $n - 1$; \mathbf{y}^i be the variables of $\text{FV}(r|_{p_i}) \setminus \text{FV}(r)$; x_1, \dots, x_k be distinct variables not occurring in r ; r' be the term obtained by replacing $r|_{p_i}$ by $x_i \mathbf{y}^i$ in r ; and $\theta = \{x_i \mapsto \lambda \mathbf{y}^i r|_{p_i} \sigma'\}$. We have $\text{level}(r') = 0$ and $r'\sigma'\theta \rightarrow_{\beta}^* r\sigma'$. If θ is computable then, by (7), $r'\sigma'\theta$ is computable and we are done. By (*), $\{\mathbf{y}^i\} = \emptyset$ and it suffices to prove that $r_{p_i} \sigma'$ is computable. For all $i \leq k$, $r|_{p_i}$ is of the form $g\mathbf{u}$ with $\text{level}(u_j) < n$. By (H2), $\mathbf{u}\sigma'$ are computable and, since $f\mathbf{t} \rightarrow_{\mathcal{C}} r|_{p_i} \sigma'$, by (H1), $x_i \theta$ is computable. \square

The condition on free variables is an important restriction since it is not satisfied by function calls with bound variables like in $(\text{lim } F) + x \rightarrow \text{lim } \lambda n(Fn + x)$.

Theorem 5 An higher-order reduction pair is two relations $(>, \geq)$ such that:

- $>$ is well-founded and stable by substitution,
- \geq is a reflexive and transitive rewrite relation containing \rightarrow_{β} ,
- $\geq \circ > \subseteq >$.

In the conditions of Theorem 4, $\rightarrow_{\beta_{\mathcal{C}}}$ is well-founded whenever $\mathcal{R} \subseteq \geq$ and $\mathcal{DP} \subseteq >$.

Proof. By (1), this is so if every term is computable wrt $\rightarrow_{\mathcal{C}}$. By (8), this is so if every $f^{T \Rightarrow B} \in \mathcal{D}$ is computable. By (3), this is so if, for all

$\mathbf{t} : \mathbf{T}$ computable, \mathbf{ft} is computable. We prove it by induction on $(\mathbf{ft}, \mathbf{t})$ with $(>, (\rightarrow_{\beta\mathcal{R}})_{\text{lex}})_{\text{lex}}$ as well-founded ordering (H1). Indeed, by (1) and Theorem 4, \mathbf{t} are strongly normalizable wrt $\rightarrow_{\beta\mathcal{R}}$. By (5), it suffices to prove that every reduct of \mathbf{ft} is computable. If $\mathbf{t} \rightarrow_{\beta\mathcal{R}} \mathbf{t}'$ then, by (H1), \mathbf{ft}' is computable since, by (4), \mathbf{t}' are computable and $>(\mathbf{ft}') \subseteq >(\mathbf{ft})$ since $\rightarrow_{\beta\mathcal{R}} \subseteq \geq$ and $\geq \circ > \subseteq >$. Now, assume that there is $\mathbf{ft} \rightarrow r \in \mathcal{DP}$ and σ such that $\mathbf{t} = \mathbf{l}\sigma$. Since rules are admissible, by Lemma 2, $\sigma' = \sigma|_{\text{FV}(r)}$ is computable. Since $\mathcal{DP} \subseteq >$ and $>$ is stable by substitution, $\mathbf{ft} > r\sigma'$. Thus, by (H1), $r\sigma'$ is computable. \square

An example of reduction pair can be given by using the higher-order recursive path ordering $>_{\text{horpo}}$ [5]. Take $\geq = (\rightarrow_{\beta} \cup >_{\text{horpo}})^+$ and $\geq = (\rightarrow_{\beta} \cup >_{\text{horpo}})^*$. The study of these two relations has to be done. However, $>_{\text{horpo}}$ does not take advantage of the fact that $>$ does not need to be monotonic. Such a relation is given by the weak higher-order recursive computability ordering $>_{\text{whorco}}$, whose monotonic closure strictly contains $>_{\text{horpo}}$ [4]. Moreover, $>_{\text{whorco}}$ is transitive, which is not the case of $>_{\text{horpo}}$. It would therefore be interesting to look for reduction pairs built from $>_{\text{whorco}}$.

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