Notes on the theory of cardinals

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These notes gather basic results on cardinal theory and, in particular, the cofinality of an ordinal and regular cardinals. More material can for instance be found in [2, 3].

First note that:

Lemma 1 If α and β are two ordinals, $\alpha < \beta$ and β is a limit ordinal, then $\alpha + 1 < \beta$.

Proof. Since $\alpha < \beta$, we have $\alpha + 1 \leq \beta$. If $\alpha + 1 \geq \beta$, then $\beta = \alpha + 1$. But β is a limit ordinal. Therefore, $\alpha + 1 < \beta$ (ordinals are totally ordered). ∎

1 Monotone and extensive functions on a poset

Definition 2 (Extensive function) A function $f : X \to X$ is *extensive* if, for all $x \in X$, $x \leq f(x)$.

Lemma 3 On a well-ordered set, a strictly monotone function is extensive.

Proof. Assume that $S = \{x \in X \mid x > f(x)\} \neq \emptyset$. Then, let a be the least element of S (X is well ordered). Hence, $a > f(a)$. By strict monotony, $f(a) > f(f(a))$. Therefore, $f(a) \in S$ and $a \leq f(a)$. Contradiction. ∎

Lemma 4 If f is a monotone injection from α to β, then $\alpha \leq \beta$.

Proof. Since α is well ordered, f is extensive. Hence, for all $x < \alpha$, $x \leq f(x) < \beta$. If $\beta < \alpha$, then $\beta < \beta$. Contradiction. ∎

2 Order type

We write $x \simeq y$ if x and y are two isomorphic posets, that is, when there is a monotone bijection f from x to y such that f^{-1} is monotone too.

Definition 5 (Order type) The *order type* of a well-ordered set X, $o(X)$, is the smallest ordinal isomorphic to X.

Lemma 6 If α is an ordinal and $X \subseteq \alpha$, then $o(X) \leq \alpha$.

Proof. Since X is a set of ordinals, X is well ordered. Therefore, $o(X)$ is well defined. Since $o(X) \simeq X$ and $X \subseteq \alpha$, there is a monotone injection from $o(X)$ to α. Therefore, $o(X) \leq \alpha$. ■

Lemma 7 If $f : X \to Y$ is a strictly monotone function between two well-ordered sets, then $o(X) \leq o(Y)$.

Proof. Let $g : Y \to o(Y)$ be a monotone bijection. We have $X \simeq \mathrm{Im}(f) \simeq g[\mathrm{Im}(f)] = \{g(y) \in o(Y) \mid y \in \mathrm{Im}(f)\}$. Thus, $o(X) = o(g[\mathrm{Im}(f)]) \leq o(Y)$. ■

3 Cofinal and unbounded subsets of a poset

Definition 8 (Cofinal and unbounded subsets) A subset X of an ordered set Y is *cofinal* (resp. *unbounded*) if, for all $y \in Y$, there is $x \in X$ such that $y \leq x$ (resp. $y < x$). A function $f : X \to Y$ is cofinal (resp. unbounded) if its image $\mathrm{Im}(f)$ is cofinal (resp. unbounded).

Note that every extensive function is cofinal.

Note that an unbounded subset is cofinal but a cofinal subset does not need to be unbounded.

Lemma 9 If α is an ordinal and X is an unbounded subset of α, then α is a limit ordinal and $\sup X = \alpha$.

Proof. Assume that $\alpha = \beta + 1$ for some β. Then, $\beta \in \alpha$. Since X is unbounded, there is $x \in X$ such that $\beta < x$. But, since $x \in X$ and $X \subseteq \alpha$, $x \leq \alpha = \beta + 1$. Contradiction.

We now prove that $\sup X = \alpha$. Since $X \subseteq \alpha$, $X \leq \alpha$. So, $\sup X \leq \alpha$. Assume now that $\sup X < \alpha$. Since X is unbounded, there is $x \in X$ such that $\sup X < x$. But, since $x \in X$, $x \leq \sup X$. Contradiction. ■

Lemma 10 α is a limit ordinal iff every cofinal subset of α is unbounded.

Proof.

- $\Rightarrow$ Let α be a limit ordinal and X be a cofinal subset of α. Assume that X is bounded, that is, there is $\beta < \alpha$ such that $X \leq \beta$. Since α is a limit ordinal, $\beta + 1 < \alpha$. Since X is cofinal, there is $x \in X$ such that $\beta + 1 \leq x$. But $x \leq \beta$. Contradiction.

- $\Leftarrow$ Assume that α is not a limit ordinal. Then $\alpha = \beta + 1$ for some β and $\{\beta\}$ is a bounded cofinal subset of α. Contradiction. ■

4 Cofinality of an ordinal

For the cofinality of an ordinal, I found the following definitions:

Definition 11 (Cofinality) Let $\mathrm{cf}_{\mathrm{c}}(\alpha)$ be the smallest ordinal β such that there is a cofinal function $f : \beta \to \alpha$.

If α is a limit ordinal, let $\mathrm{cf}_{\mathrm{m}}(\alpha)$ be the smallest ordinal β such that there is a strictly monotone function $f : \beta \to \alpha$ such that $\sup \mathrm{Im}(f) = \alpha$.

Let $\mathrm{cf}_{\mathrm{o}}(\alpha)$ be the smallest order type of a cofinal subset of α.

$\mathrm{cf}_{\mathrm{c}}(\alpha)$ is well defined since the identity function is cofinal.

$\mathrm{cf}_{\mathrm{m}}(\alpha)$ is well defined since the identity function is strictly monotone and satisfies $\sup \mathrm{Im}(\mathrm{id}_\alpha) = \alpha$ since α is a limit ordinal.

Note that $\mathrm{cf}_{\mathrm{m}}(\alpha)$ is defined on limit ordinals only. Indeed, if $\alpha = \alpha' + 1$ for some α' then, for any $f : \beta \to \alpha$, $\sup \mathrm{Im}(f) \leq \alpha' < \alpha$.

$\mathrm{cf}_{\mathrm{o}}(\alpha)$ is well defined since α is cofinal in α.

It follows that:

Lemma 12 For $x \in \{\mathsf{c}, \mathsf{m}, \mathsf{o}\}$, $\mathrm{cf}_x(\alpha) \leq \alpha$.

Note that $\mathrm{cf}_{\mathrm{c}}(\alpha + 1) = \mathrm{cf}_{\mathrm{o}}(\alpha + 1) = 1$.

We are now going to see that these definitions are however all equivalent on limit ordinals:

Lemma 13 1. $\mathrm{cf}_{\mathrm{c}}(\alpha) \leq \mathrm{cf}_{\mathrm{m}}(\alpha)$.

2. If α is a limit ordinal, then $\mathrm{cf}_{\mathrm{m}}(\alpha) \leq \mathrm{cf}_{\mathrm{c}}(\alpha)$.

3. $\mathrm{cf}_{\mathrm{c}}(\alpha) \leq \mathrm{cf}_{\mathrm{o}}(\alpha)$.

4. If α is a limit ordinal, then $\mathrm{cf}_{\mathrm{o}}(\alpha) \leq \mathrm{cf}_{\mathrm{c}}(\alpha)$.

Proof.

1. Since every strictly monotone function on an ordinal is extensive and thus cofinal.

2. Let $\beta = \mathrm{cf}_{\mathrm{c}}(\alpha)$ and $f : \beta \to \alpha$ cofinal. By wellfounded recursion, there is g such that, for all $x < \beta$, $g(x) = \max(f(x), S_g(x) + 1)$, where $S_g(0) = 0$ and, for all $x > 0$, $S_g(x) = \sup_{y<x} g(y)$. If $y < x$, then $g(y) < g(x)$. So, g is strictly monotone. Now, for all x, $f(x) \leq g(x)$. Hence, $S_f(x) \leq S_g(x)$. Since f is cofinal and α is a limit ordinal, f is unbounded and $S_f(\beta) = \alpha$. So, $\alpha \leq S_g(\beta)$. Let now γ be the smallest ordinal x such that $\alpha \leq S_g(x)$. We have $\gamma \leq \beta$. Moreover, for all $x < \gamma$, $S_g(x) < \alpha$. Since α is a limit ordinal, $S_g(x) + 1 < \alpha$. And since $f(x) < \alpha$, we have $g(x) < \alpha$. Therefore, $S_g(\gamma) = \alpha$ and $\mathrm{cf}_{\mathrm{m}}(\alpha) \leq \gamma \leq \beta = \mathrm{cf}_{\mathrm{c}}(\alpha)$.

3. $\mathrm{cf_o}(\alpha) = o(X)$ where X is a cofinal subset of α. Let $f : o(X) \to X$ be an isomorphism between $o(X)$ and X, and $g : o(X) \to \alpha$ be the function such that $g(x) = f(x)$. Then, g is cofinal since $\mathrm{Im}(g) = \mathrm{Im}(f) = X$ and X is cofinal. Therefore, $\mathrm{cf_c}(\alpha) \le \mathrm{cf_o}(\alpha)$.

4. Let $\beta = \mathrm{cf_c}(\alpha)$ and $f : \beta \to \alpha$ be cofinal. We have seen in (2) that, since α is a limit ordinal, there are $\gamma \le \beta$ and $g : \gamma \to \alpha$ strictly monotone and cofinal. Thus, $\mathrm{Im}(g)$ is cofinal and $o(\mathrm{Im}(g)) = \gamma$. Therefore, $\mathrm{cf_o}(\alpha) \le \beta$. ■

In the following, when α is a limit ordinal, we write $\mathrm{cf}(\alpha)$ to denote any one of these definitions.

5 Initial ordinals

We write $x \sim y$ if x and y are two equipotent sets, that is, when there is a bijection f from x to y.

Definition 14 (Initial ordinal) An ordinal α is *initial* if it is equipotent to no smaller ordinals.

Lemma 15 α is initial iff, for all $\beta < \alpha$, α cannot be injected into β.

Proof. The $\Leftarrow$ part is immediate. Assume now that there is $\beta < \alpha$ and an injection $f : \alpha \to \beta$. Then, $\alpha \sim \mathrm{Im}(f) \simeq o(\mathrm{Im}(f))$ and $o(\mathrm{Im}(f)) < \alpha$ since $\mathrm{Im}(f) \subseteq \beta$ and $\beta < \alpha$. ■

Lemma 16 An infinite initial ordinal is a limit ordinal.

Proof. If α is infinite, then $\alpha + 1 \sim \alpha$. Take $f : \alpha + 1 \to \alpha$ such that $f(\alpha) = 0$; for all $\beta < \omega$, $f(\beta) = \beta + 1$; and for all $\beta \in [\omega, \alpha[$, $f(\beta) = \beta$. ■

Lemma 17 $\mathrm{cf_c}(\alpha)$ is initial.

Proof. By definition of $\beta = \mathrm{cf_c}(\alpha)$, there is a cofinal function $f : \beta \to \alpha$. Assume that β is not initial, that is, there is $\gamma < \beta$ and a bijection $g : \gamma \to \beta$. Now, let $y \in \alpha$. Since f is cofinal, there is $x \in \beta$ such that $y \le f(x)$. But, $f(x) = (f \circ g)(g^{-1}(x))$. Therefore, $f \circ g : \gamma \to \alpha$ is cofinal. Contradiction. ■

6 Cardinal of a set

Definition 18 (Cardinal) The *cardinal of a set* X, written $|X|$, is the smallest ordinal equipotent to X (requires the axiom of choice if X is not equipped with a particular well order). An ordinal α is a cardinal if there is some set X such that $\alpha = |X|$.

Lemma 19 $|X| = \alpha$ iff $\alpha \sim X$ and there is no injection from α to $\beta < \alpha$.

Proof.

- $\Rightarrow$ Assume that there is an injection f from α to $\beta < \alpha$. Then, $\alpha \sim \mathrm{Im}(f) \simeq o(\mathrm{Im}(f))$ and $o(\mathrm{Im}(f)) < \alpha$ since $\mathrm{Im}(f) \subseteq \beta < \alpha$.
- $\Leftarrow$ If there is no injection from α to $\beta < \alpha$, then there is no bijection from α to $\beta < \alpha$. ■

Lemma 20 For every ordinal α, $|\alpha| \leq \alpha$.

Proof. Since $\alpha \sim \alpha$. ■

Lemma 21 α is a cardinal iff α is initial iff $|\alpha| = \alpha$.

Proof.

- $1 \Rightarrow 2$ Assume that $\alpha = |X|$ for some X. If α is not initial, then there is $\beta < \alpha$ such that $\beta \sim \alpha$. Since $\alpha = |X|$ and $|X| \sim X$, there is therefore $\beta < |X|$ such that $\beta \sim X$. Contradiction.
- $2 \Rightarrow 3$ Since α is initial, $|\alpha| \geq \alpha$. But, since $|\alpha| \leq \alpha$, we have $|\alpha| = \alpha$.
- $3 \Rightarrow 1$ Immediate. ■

Hence, initial and cardinal are synonyms.

Lemma 22 1. If $f : X \to Y$ is injective, then $|X| \leq |Y|$.

2. If $f : X \to Y$ is surjective, then $|Y| \leq |X|$.

Proof.

1. Since $X \sim |X|$ and $Y \sim |Y|$, there is an injection from $|X|$ to $|Y|$. Therefore, $|X| \leq |Y|$.
2. Let R be the equivalence relation on X such that xRx' iff $f(x) = f(x')$, and $\gamma : X/R \to X$ be a choice function, that is, $\gamma(x) \in x$. The function $f/R : X/R \to Y$ mapping the class of x to $f(x)$ is injective. It is also surjective since f is surjective. The function γ is injective too. Therefore, the function $\gamma \circ (f/R)^{-1}$ is an injection from Y to X. ■

Lemma 23 1. If $\alpha \leq \beta$, then $|\alpha| \leq |\beta|$.

2. If $|\alpha| < |\beta|$, then $\alpha < |\beta|$.

Proof.

1. Since $\alpha \leq \beta$, there is an injection from α to β.
2. If $\alpha \geq |\beta|$, then $|\alpha| \geq ||\beta|| = |\beta|$. ■

Lemma 24 $\mathrm{cf}(\alpha) \leq |\alpha|$.

Proof. Since $\mathrm{cf}(\alpha) \leq \alpha$ and $\mathrm{cf}(\alpha)$ is initial, $\mathrm{cf}(\alpha) = |\,\mathrm{cf}(\alpha)| \leq |\alpha|$. ■

Lemma 25 If λ is an infinite cardinal and $(\kappa_\alpha)_{\alpha<\lambda}$ is a family of non-zero cardinals, then $\sum_{\alpha<\lambda} \kappa_\alpha = \max(\lambda, \sup_{\alpha<\lambda} \kappa_\alpha)$.

Proof. Let $S = \sum_{\alpha<\lambda} \kappa_\alpha$ and $\kappa = \sup_{\alpha<\lambda} \kappa_\alpha$. Since for all $\alpha < \lambda$, $\kappa_\alpha \leq \kappa$, we have $S \leq \sum_{\alpha<\lambda} \kappa \leq \lambda\kappa = \max(\lambda, \kappa)$. Now, $\lambda = \sum_{\alpha<\lambda} 1 \leq S$ since, for all $\alpha < \lambda$, $\kappa_\alpha \neq 0$. And since for all $\alpha < \lambda$, $\kappa_\alpha \leq S$, we have $\kappa \leq S$. ■

7 Hartogs ordinal

Definition 26 (Hartogs ordinal) Given a set X, let $h(X)$ be the smallest ordinal that cannot be injected into X.

The proof of the existence of $h(X)$ is due to Hartogs [1]. Clearly:

Lemma 27 $h(X)$ is initial.

Lemma 28 If α is initial, then $h(\alpha)$ is the least initial ordinal greater than α.

Proof. First, $\alpha < h(\alpha)$. Otherwise, there is an injection from $h(\alpha)$ to α. Now, assume that β is an initial ordinal such that $\alpha < \beta < h(\alpha)$. Since $\beta < h(\alpha)$, there is an injection from β to α. But, since $\alpha < \beta$ and β is initial, there is no injection from β to α. Contradiction. ■

Definition 29 Let $\mathcal{C}$ be the class of ordinals defined by wellfounded recursion as follows:

- $w_0 = \omega$
- $w_{\alpha+1} = h(\omega_\alpha)$
- $w_\lambda = \sup_{\alpha<\lambda} \omega_\alpha$ if $\lambda = \sup \lambda$.

Lemma 30 1. ω is strictly monotone and extensive.

2. $\mathcal{C}$ is the class of all infinite initial ordinals.

Proof.

1. First, one can easily check that ω is monotone and that, for all α, $\omega_\alpha < \omega_{\alpha+1}$. Assume now that $\alpha < \beta$ and $\omega_\alpha = \omega_\beta$. If $\beta = \gamma + 1$, then $\alpha \leq \gamma$ and $\omega_\alpha \leq \omega_\gamma < \omega_{\gamma+1} = \omega_\alpha$. Contradiction. Assume now that β is a limit ordinal. Then, $\alpha + 1 < \beta$ and $\omega_\alpha < \omega_{\alpha+1} \leq \omega_\beta = \omega_\alpha$. Contradiction.

2. We first prove that every element of $\mathcal{C}$ is initial. $\omega_0 = \omega$ is initial. For all α, $\omega_{\alpha+1}$ is initial. Let now λ be a limit ordinal and assume that, for all $\alpha < \lambda$, ω_α is initial. If ω_λ is not initial, then there is $\beta < \omega_\lambda$ such that $\beta \sim \omega_\lambda$. Since $\omega_\lambda = \sup_{\alpha<\lambda} \omega_\alpha$, there is $\alpha < \lambda$ such that $\beta < \omega_\alpha$. Hence, $\omega_\lambda \sim \beta < \omega_\alpha \leq \omega_\lambda$. Contradiction.

   It remains to prove that every infinite initial ordinal belongs to $\mathcal{C}$. Since ω is extensive, for all α, $\alpha \leq \omega_\alpha$. We now prove that, for all α, for all infinite initial ordinal $\beta < \omega_\alpha$, there is $\gamma < \beta$ such that $\beta = \omega_\gamma$, by induction on α. If $\alpha = 0$, this is immediate since there is no infinite ordinal smaller than ω_0. Assume now that $\alpha = \alpha' + 1$. Then, $\omega_\alpha = h(\omega_{\alpha'})$ and $\beta \leq \omega_{\alpha'}$. If $\beta < \omega_{\alpha'}$ then we can conclude by induction hypothesis. Assume finally that $\alpha = \sup \alpha$. Then, there is $x < \alpha$ such that $\beta < \omega_x$ and we can conclude by induction hypothesis. ■

Hence, every cardinal is equal to some ω_α for some α, and we can study cardinals by studying $\mathcal{C}$.

A cardinal of the form $\omega_{\alpha+1}$ is called a *successor cardinal*. A cardinal of the form ω_λ with $\lambda = \sup \lambda$ is called a *limit cardinal*.

8 Regular ordinals

Definition 31 (Regular ordinal) A infinite cardinal κ is *regular* if $\mathrm{cf}(\kappa) = \kappa$, and *singular* otherwise.

Lemma 32 Let κ be a regular cardinal and X a subset of κ. If X is unbounded, then $|X| = \kappa$. Equivalently, if $|X| < \kappa$, then X is bounded.

Proof. If X is unbounded, then X is cofinal. So, $\mathrm{cf}(\kappa) \leq o(X)$. Now, since $X \subseteq \kappa$, $o(X) \leq \kappa$. Therefore, $\kappa = \mathrm{cf}(\kappa) \leq |o(X)| = |X| \leq \kappa$. ■

Lemma 33 κ is singular iff there are a set I and a family of infinite cardinals $(\kappa_i)_{i\in I}$ such that $\kappa = \sum_{i\in I} \kappa_i$, $|I| < \kappa$ and, for all $i \in I$, $\kappa_i < \kappa$.

Proof.

- $\Rightarrow$ Assume that κ is singular and let $\lambda = \mathrm{cf}(\kappa)$. Then, there is a strictly monotone and extensive function $f : \lambda \to \kappa$ such that $\sup_{x<\lambda} f(x) = \kappa$, that is, $\kappa = \bigcup_{x<\lambda} f(x)$. Now, $\kappa = \biguplus_{x<\lambda} g(x)$ where $g(x) = f(x) - \{y \in f(x) | y < x\}$. Therefore, $\kappa = \sum_{x<\lambda} |g(x)|$ with $\lambda < \kappa$ and, for all $x < \lambda$, $|g(x)| \leq |f(x)| < \kappa$.

- $\Leftarrow$ By definition, there is a bijection f from $|I|$ to I. Hence, $\kappa = \sum_{\alpha<|I|} \kappa_{f(\alpha)} = \max(|I|, \nu)$ where $\nu = \sup_{\alpha<|I|} \kappa_{f(\alpha)} = \sup\{\kappa_i | i \in I\}$. Since $|I| < \kappa$, $\kappa = \nu$. Hence, $X = \{k_{f(\alpha)} | \alpha < |I|\}$ is cofinal and $\mathrm{cf}(\kappa) \leq o(X) = |I| < \kappa$. ■

Lemma 34 $h(\kappa)$ is regular.

Proof. If $h(\kappa)$ is singular, then there are a set I and a family $(\kappa_i)_{i\in I}$ of infinite cardinals such that $h(\kappa) = \sum_{i\in I} \kappa_i$, $|I| < h(\kappa)$ and, for all $i \in I$, $\kappa_i < h(\kappa)$. Since $h(\kappa)$ is the smallest cardinal greater than κ, $|I| \leq \kappa$ and, for all $i \in I$, $\kappa_i \leq \kappa$. Therefore, $h(\kappa) = \max(|I|, \sup_{i\in I} \kappa_i) \leq \kappa$. Contradiction. ■

Hence, every successor cardinal is regular. What about limit cardinals? There are arbitrary large limit cardinals that are singular. For instance, for all ordinal α, $\sup\{\omega_{\alpha+i} | i < \omega\}$ is a singular limit cardinal greater than ω_α and thus greater than α. So, is there any uncountable regular limit cardinal? Such a cardinal must be a fixpoint of ω, but this is not enough since, for all cardinal κ_0, $\sup\{\kappa_i | i < \omega\}$, where $\kappa_{i+1} = \omega_{\kappa_i}$, is singular. In fact, an uncountable regular limit cardinal is called *weakly accessible*: the existence of such a cardinal is not provable in ZFC. An uncountable limit cardinal κ is called *strongly accessible* if it is regular and, for all $\lambda < \kappa$, $2^\lambda < \kappa$. (Under the Generalized Continuum Hypothesis saying that $h(\kappa) = 2^\kappa$, the two notions are equivalent.)

References

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[3] W. Weiss. An introduction to set theory, 2008. Lecture notes.