A point on fixpoints in posets

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Let $(X, \leq)$ be a *non-empty strictly inductive poset*, that is, a non-empty partially ordered set such that every *non-empty chain* Y has a least upper bound $\mathrm{lub}(Y) \in X$, a chain being a subset of X totally ordered by $\leq$.

We are interested in sufficient conditions such that, given an element $a_0 \in X$ and a function $f : X \to X$, there is some ordinal k such that $a_{k+1} = a_k$, where a_k is the transfinite sequence of iterates of f starting from a_0 (implying that a_k is a fixpoint of f):

- $a_{k+1} = f(a_k)$
- $a_l = \mathrm{lub}\{a_k \mid k < l\}$ if l is a limit ordinal, *i.e.* $l = \mathrm{lub}(l)$

This note summarizes known results about this problem and provides a slight generalization of some of them.

1 Definitions

Let AC denote the axiom of choice[1] [Zer04].

Let $\mathbb{O}$ be the class of von Neumann ordinals [vN23].

Let $A = \{x \in X \mid \exists k \in \mathbb{O}, x = a_k\}$ be the set of all transfinite iterates of f from a_0.

Let N be the smallest subset of X containing a_0 and closed by f ($Z \subseteq X$ is closed by f if, for all $z \in Z$, $f(z) \in Z$) and non-empty lub's ($Z \subseteq X$ is closed by non-empty lub's if, for all $P \subseteq Z$, if $P \neq \emptyset$, then $\mathrm{lub}(P)$ exists and belongs to Z).

Given $Z \subseteq X$, let $\mathrm{glb}(Z)$ be, if it exists, the greatest lower bound of Z.

Let $\mathrm{PreFP}(f) = \{x \in X \mid x \leq f(x)\}$ be the set of *pre-fixpoints* of f, $\mathrm{PostFP}(f) = \{x \in X \mid x \geq f(x)\}$ be the set of *post-fixpoints* of f, and $\mathrm{FP}(f) = \mathrm{PreFP}(f) \cap \mathrm{PostFP}(f)$ be the set of fixpoints of f.

f is (resp. *strictly*) *monotone* (or *isotone* or *increasing*) (opposites: *anti-monotone*, *antitone* or *decreasing*) if $f(x) \leq f(y)$ whenever $x \leq y$ (resp. $f(x) < f(y)$ whenever $x < y$).

[1] Implicitly used by Cantor and others, but first explicitly mentioned (and rejected) by Peano in [Pea90].

f is (resp. *strictly*) *extensive* (or *inflationary* or *progressive*) (opposites: *reductive*, *deflationary* or *regressive*) if, for all x, $f(x) \geq x$ (resp. $f(x) > x$).

Note that: if X has a least element $\bot$, then it is a pre-fixpoint of f; if f is extensive, then *every* element of X is a pre-fixpoint of f; if f is monotone, then PreFP(f) and PostFP(f) are closed by f (pre-fixpoints are mapped to pre-fixpoints, and post-fixpoints to post-fixpoints).

2 Searching for the origins

We found useful references in [LNS82, Cam78, RR63, RR85, GKW12, Fel62]. The oldest result that we could find is:

In 1909, Hessenberg proved that, if X is a set of sets, $\leq$ is inclusion, and f is extensive, then f has a fixpoint [Hes09]. This is explained in [Fel62] (as mentioned in [Fel00][2] and [GKW12] p. 158). Unfortunately, these two papers are in German and I cannot get a more precise idea of their contents.

In 1922, Kuratowski proved that, if X is a set of sets, $\leq$ is inclusion, and f is extensive, then f has a fixpoint, namely lub(N) [Kur22]. He also proved that N is equal to A and that, if f is also monotone, then lub(N) is the smallest fixpoint of f that is greater than or equal to a_0.

In 1927, Knaster and Tarski proved that, if X is a set of sets, $\leq$ is inclusion and f is monotone, then f has a fixpoint [KT28].

In 1939, Tarski extended this result to arbitrary complete lattices (every subset, empty or not, has a lub and a glb) [Tar55]. In fact, he proved that FP(f) itself is a non-empty complete lattice. In 1951, Davis proved the converse, that is, a lattice is complete if every monotone map has a fixpoint [Dav55]. By the way, note that, in 1941, Frink proved that any complete lattice is compact in the interval topology [Fri42].

On the other hand, Tarski did not study whether the least fixpoint of f, proved to be glb(PostFP(f)), can be reached by transfinite iteration. However, he mentioned p. 305 that, if X is an ω-complete lattice (every countable subset has a lub), f is ω-continuous ($f(\text{lub}Y) = \text{lub}f(Y)$ for every countable $Y \subseteq X$)

[2] [Fel00]: "Returning to Hessenberg, his paper
Kettentheorie und Wohlordnung. Crelle 135 (1909) 81-133
can hardly be underestimated in its importance. Not that it was understood by his contemporaries. But Hessenberg, analyzing Zermelos second proof of the well ordering theorem, studied the general ways to construct well ordered subsets of ordered sets - with the one restriction that order always was inclusion and ordered sets were subfamilies of power sets. In the course of this, Hessenberg stated and proved the fixpoint theorem which thirty years later was rediscovered - for ordered sets now - by Nicolas Bourbaki. The amazing thing is that Hessenberg's proof is precisely the same as that given by Bourbaki ! (only that at one small point a simpler argument can be used due to the circumstance that Hessenberg's order is inclusion). For details, I refer to my article in Archiv d.Math. 13 (1962) 160-165 and to my book Naive Mengen und Abstrakte Zahlen from 1979 , p.200 ff ."

and a_0 is the least element $\perp$ of X, then $a_{\omega+1} = a_\omega$, a result sometimes attributed to Kleene because it is indeed used in his proof of the *first* recursion theorem with X being the set of partial functions in [Kle52] p. 348.

At the end of the 30's, Bourbaki[3] proved that, if X is a strictly inductive poset and f is extensive, then f has a fixpoint [Bou39, Bou49]. This is a generalization of Hessenberg and Kurakowski's results. This result was rediscovered or proved (because Bourbaki published a proof in 1949 only) at the end of the 40's or beginning of the 50's by many other people: [Kne50, Sze50, Wit50b, Wit50a, Vau52, Ina52].

Note that fixpoint theorems assuming that f is extensive easily follow from Zorn's maximal principle (equivalent to AC) saying that any non-empty inductive (= chain-complete) set of sets ordered by inclusion (every $\subseteq$-chain, including the empty one, has a lub) has a maximal element (for inclusion) [Zor35], or Tukey's maximal principle (equivalent to AC too) generalizing Zorn's one to arbitrary inductive posets (hence saying that any non-empty inductive poset has a maximal element) [Tuk40]. Indeed, A being a non-empty inductive poset (see Lemma 3 below), it has a maximal element a_k. Since $a_k \leq a_{k+1}$ and a_k is maximal, $a_{k+1} = a_k$. But the point of the previous authors was to prove Zermelo's result that AC implies the well order theorem (every set can be well ordered) [Zer04] without using ordinal theory (like Zermelo in the second proof of his theorem [Zer08]).

At the end of the 50's, by using Hartogs theorem (for any set A, there is an ordinal k that cannot be injected into A) [Har15], Rubin and Rubin proved that, if X is a strictly inductive set of sets, $\leq$ is inclusion and f is extensive, then $a_{k+1} = a_k$ for some k [RR63] (p. 18).

In 1957, Ward extended Tarski, Frink and Davis results to complete semi-lattices (every non-empty subset has a lub but not necessarily a glb) [War57]. Hence, a semi-lattice X is complete iff, for every $x \in X$, $x\downarrow = \{y \in X \mid y \leq x\}$ is compact in the interval topology; $\mathrm{FP}(f)$ is a complete semi-lattice if X is a complete semi-lattice and $f : X \to X$ is monotone; a semi-lattice is compact in the interval topology iff every monotone $f : X \to X$ has a fixed point.

In 1959, by refining Bourbaki's result, Abian and Brown extended Tarski's result to strictly inductive posets having a pre-fixpoint of f [AB61].

In 1962, using Hartogs theorem, Devidé proved that, in a complete lattice, if $f(x) = a_0 \vee g(x)$ with g monotone, then there is $k \in \mathbb{O}$ such that $a_{k+1} = a_k$ [Dev64] (note that f does not need to be extensive, although it is so on A).

Up to now, we have seen that all conditions for f to have a fixpoint are

[3] A result due to Chevalley after [Cam78].

requiring either f to be monotone or f to be extensive. So, one may wonder what relations are known between these two classes of functions, and whether one cannot devise a condition generalizing both.

As for a relation between monotony and extensivity, we have:

Lemma 1 ([Bou53] p. 35) On a well ordered poset $(X, \leq)$ (every non-empty subset has a least element), any strictly monotone function $f : X \to X$ is extensive.

Proof. Assume that f is not extensive. Then, the set $E = \{x \in X \mid f(x) < x\}$ is not empty. Let e be its least element. By definition, $f(e) < e$. By strict monotony, $f(f(e)) < f(e)$. Hence, $f(e) \in E$ and, $e \leq f(e)$ by definition of e. Contradiction. ■

As for a condition subsuming both notions, we have:

In 1969, Salinas extended the previous fixpoint theorems by requiring (P1) $a_0 \in \mathrm{PreFP}(f)$, *i.e.* $a_0 \leq f(a_0)$, and:

$$\text{(P2)} \quad f(x) \leq f(y) \text{ if } x \leq f(x) \leq y \text{ [Sal69].}$$

He provided two proofs, one using AC and Hartogs theorem, and another one not using AC but some notion of chain[4]. He also proved (using AC) that, if X is not strictly inductive but the set of upper bounds of every non-empty chain of X has a minimal element, f satisfies (P1) and (P2), and $\mathrm{glb}\{x, f(x)\}$ exists for all x, then f has a fixpoint.

In 1973, by using Bourbaki's theorem, Markowsky extended Tarski's results to inductive (= chain-complete) posets (every chain has a lub), that is, $\mathrm{FP}(f)$ is chain-complete if X is chain-complete, and proved the converse, that is, X is chain-complete if every monotone (or glb-preserving) map $f : X \to X$ has a *least* fixpoint [Mar76].

In 1974, Pasini extended Salinas' result by proving that $\mathrm{FP}(f)$ has a maximal element [Pas74].

In 1977, Cousot and Cousot studied the properties of $a = (a_k)_{k \in \mathbb{O}}$ when f is monotone and X a complete lattice (remarking however that their results extend to posets every chain of which has a lub and a glb) [CC79]. In particular, they proved that, in a complete lattice, A is bounded by every post-fixpoint bigger than or equal to a_0, hence that a can only converge to the *least* fixpoint of f bigger than or equal to a_0, and indeed converges to this fixpoint if f is monotone and $a_0 \leq f(a_0)$. They also extended Tarski's result by showing that $\mathrm{PreFP}(f)$ and $\mathrm{PostFP}(f)$ are non-empty complete lattices too.

[4]The notion of chain wrt a function f has been first introduced by Dedekind for defining the notion of infinite set [Ded88]. It has been used by Zermelo in his second proof that AC implies the well order theorem [Zer08], and studied in details by Hessenberg [Hes09].

3 Synthesis

In conclusion, the most direct argument not using AC why there must be some $k \in \mathbb{O}$ such that $a_{k+1} = a_k$ is the one of Rubin [RR63] based on Hartogs theorem [Har15]. We hereafter split this result in two parts by showing first that, by Hartogs theorem, f has a fixpoint if a is monotone, and second that, a is monotone if f satisfies Salinas conditions (P2).

Lemma 2 If a is monotone, then there is an ordinal k such that $a_{k+1} = a_k$.

Proof. By Hartogs theorem, there is an ordinal k that cannot be injected into A (the smallest such one is a cardinal). Therefore, $a|_k$, the restriction of a to k, is not an injection, that is, there are $l_1 < l_2 < k$ such that $a_{l_1} = a_{l_2}$. Since a is monotone, we have $a_{l_1+1} = a_{l_1}$. ■

Now, one can easily prove that a is monotone whenever a_0 is a pre-fixpoint of f and f satisfies Salinas condition (P2) above:

Lemma 3 a is monotone if $a_0 \leq f(a_0)$ and $f(x) \leq f(y)$ whenever $x \leq f(x) \leq y$, for all x and y in A.

Proof. We prove that $a_k \leq a_l$ whenever $k < l$ by induction on l (1).

- If l is a limit ordinal, then this is immediate.
- Otherwise, $l = m+1$ and $k \leq m$. If $k < m$ then, by induction hypothesis (1), we have $a_k \leq a_m$. We now prove that, for all $i \leq m$, $a_i \leq a_{i+1}$, by induction on i (2).
  - $i = 0$. $a_0 \leq a_1 = f(a_0)$ by assumption.
  - $i = j+1$. By induction hypothesis (2), $a_j \leq a_{j+1} = a_i$. Therefore, by (P2), $a_{j+1} = a_i \leq a_{i+1}$.
  - $i = \mathrm{lub}(i)$. Let $j < i$. By induction hypothesis (2), $a_j \leq a_{j+1}$. Since $j+1 < i \leq m$, by induction hypothesis (1), $a_{j+1} \leq a_i$. Hence, by (P2), $a_{j+1} \leq a_{i+1}$. Therefore, $a_j \leq a_{i+1}$ and $a_i = \mathrm{lub}\{a_j \mid j < i\} \leq a_{i+1}$. ■

Using a nice result of Abian and Brown [AB61] for *any* poset $(X, \leq)$ and *any* function f, we can go a little bit further and instead consider the condition:

(P2') $f(x) \leq f(y)$ if $x < f(x) \leq y$ and there is no z such that $x < z < f(x)$

Definition 1 ([AB61]) A set $C \subseteq X$ is an a_0-*chain* if:

- C is well ordered;
- C has a_0 as least element;
- C is closed by non-empty lub's;
- if $z \in C - \{\mathrm{lub}(C)\}$, then:

– $f(z) \in C$,
– $z < f(z)$,
– there is no $y \in C$ such that $z < y < f(z)$.

Let W be the set of elements $x \in X$ such that x is the lub of an a_0-chain.

In [Sal69], Salinas considered a similar (equal?) set called the set of admissible subsets of X.

Theorem 1 ([AB61]) For any poset $(X, \leq)$, $a_0 \in X$ and function $f : X \to X$:

- W is well ordered;
- W has a_0 as least element;
- if W has a lub ξ, then W is an a_0-chain with ξ as greatest element and $\xi \not< f(\xi)$.

Abian and Brown proved also that, for every $x \in W$, there is only one a_0-chain C such that $x = \text{lub}(C)$, namely $\{y \in W \mid y \leq x\}$.

Note also that W is not closed by f in general. However, they proved that, if $x \in W$ and $x \leq f(x)$, then $f(x) \in W$.

Theorem 2 In a non-empty strictly inductive poset $(X, \leq)$, if $a_0 \in \text{PreFP}(f)$ and $f : X \to X$ is monotone on W, then $\text{lub}(W)$ is a fixpoint of f.

Proof. We simply follow the proof of Abian and Brown and check that, indeed, the monotony of f is used only on elements of W.

Since X is strictly inductive, $\xi = \text{lub}(W)$ exists. By the previous theorem, W is an a_0-chain and $\xi \not< f(\xi)$. Since W has a_0 as least element, we have $a_0 \leq \xi$. Since $\xi \not< f(\xi)$, it suffices to check that $\xi \leq f(\xi)$. If $a_0 = \xi$, then this is immediate since, by assumption, $a_0 \leq f(a_0)$. Assume now that $a_0 < \xi$. Then, since W is an a_0-chain, we have $\xi \in W$ and $V = W - \{\xi\}$ not empty, thus $\theta = \text{lub}(V)$ exists and $\theta \leq \xi$. There are two cases:

- $\theta = \xi$. Let $x \in V$. Then, $x < \xi$, $x < f(x) \in W$ and, by monotony of f on W, $f(x) \leq f(\xi)$. Hence, for all $x \in V$, $x < f(\xi)$. Therefore, $\xi \leq f(\xi)$.
- $\theta < \xi = f(\theta)$. Then, by monotony of f on W, $\xi \leq f(\xi)$. ∎

Now, one can easily check that:

Lemma 4 f is monotone on W if f satisfies (P2').

We now provide precise statements for Abian and Brown's claim that $W = N$. (Salinas also proved in [Sal69] that his set of *admissible* subsets of X is N.)

Lemma 5 $W \subseteq N$.

Proof. If suffices to prove that W is included in every set Z containing a_0 and closed by f and non-empty lub's. We proceed by well-founded induction on $<$. Let $x \in W$. If $x = a_0$, then we are done. Assume now that $a_0 < x$. After Lemma 4 in [AB61], $x = \mathrm{lub}(C)$ with C the a_0-chain $\{y \in W \mid y \leq x\}$. We then proceed as in Theorem 2. Since C is an a_0-chain and $a_0 < x$, we have $D = C - \{x\}$ not empty, thus $\theta = \mathrm{lub}(D)$ exists and $\theta \leq x$. For all $y < x$, we have $y \in Z$ by induction hypothesis. Therefore, $\theta \in Z$ since Z is closed by non-empty lub's. If $\theta = x$, then we are done. Otherwise, $x = f(\theta) \in Z$ since Z is closed by f. ■

Lemma 6 $N \subseteq W$ if X is strictly inductive, $a_0 \in \mathrm{PreFP}(f)$ and f is monotone on W.

Proof. It suffices to show that W contains a_0 and is closed by f and non-empty lub's. By Theorem 1, we have $a_0 \in W$ and W an a_0-chain. Hence, W is closed by non-empty lub's. Let $\xi = \mathrm{lub}(W)$ and $x \in W$. If $x = \xi$, then $f(x) = x \in W$ since $\xi \in \mathrm{FP}(f)$ by Theorem 2. Otherwise, $x < \xi$ and $f(x) \in W$ since W is an a_0-chain. ■

For the sake of completeness, we also make precise Kuratowski's relations between A and N, when X is strictly inductive (for A to be well defined).

Lemma 7 ([Kur22]) $A \subseteq N$.

Proof. It suffices to prove that A is included in every set $Z \subseteq X$ containing a and closed by f and non-empty lub's, by transfinite induction. Let $a_k \in A$. If $k = 0$, then $a_k \in Z$ by assumption. If $k = j + 1$, then $a_k = f(a_j) \in Z$ since, by induction hypothesis, $a_j \in Z$ and Z is closed by f. Finally, if $k = \mathrm{lub}(k)$, then $a_k = \mathrm{lub}\{a_j \mid j < k\} \in Z$ since Z is closed by non-empty lub's and, for all $j < k$, $a_j \in Z$ by induction hypothesis. ■

Lemma 8 $N \subseteq A$ if a is monotone.

Proof. Since A contains a_0 and is closed by f, it suffices to prove that A is closed by non-empty lub's. Let Z be a non-empty subset of A. Then, there is a set K of ordinals such that $Z = \{x \mid \exists k \in K, x = a_k\}$. Since a is monotone, $\mathrm{lub}(Z)$ exists and equals a_k where $k = \mathrm{lub}(K)$ (every *set* of ordinals has a lub).■

Hence, we can conclude:

Theorem 3 If X is strictly inductive, $a_0 \in \mathrm{PreFP}(f)$ and f satisfies (P2'), then $N = W = A$. Therefore, N, W and A are a_0-chains and there is k such that $a_{k+1} = a_k = \mathrm{lub}(N)$ is the *least* fixpoint of f bigger than or equal to a_0.

Proof. Since f satisfies (P2'), f is monotone on W. Hence, $N = W$. Since $A \subseteq N$, f is monotone on A. Hence, a is monotone. Therefore, $N = A$. ■

Note that, if $\xi = \text{lub}(W)$, then f is both strictly monotone and strictly extensive on $W - \{\xi\}$.

We finish with some ultimate remarks.

In [Dev64], with X a complete lattice, Devidé takes $f(x) = a_0 \vee g(x)$ with g monotone. In this case, one can easily check that, if $a_0 \leq g(a_0)$, then f and g have the same set of fixpoints bigger than or equal to a_0. Moreover, the transfinite iterates of f and g are equal.

Now, consider $f(x) = x \vee g(x)$ with g monotone. Then, f is both monotone and extensive, and f and g are equal on $\text{PreFP}(g)$. Moreover, $\text{FP}(g) \subseteq \text{FP}(f) = \text{PostFP}(g)$ and, if X has at least two distinct elements, then $\text{FP}(f) \not\subseteq \text{FP}(g)$: if g is the constant function equal to the least element $\bot$ of X, then $\text{FP}(g) = \{\bot\}$ and $\text{FP}(f) = X \neq \{\bot\}$ since f is the identity and X has at least two elements. However, the *least* fixpoint of f is also the *least* fixpoint of g. Moreover, if $a_0 \leq g(a_0)$, then the transfinite iterates of f and g are equal.

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