Phase estimation

We want to solve the following problem.

Input : a quantum unitary $U$ acting on $n$ qubits and an eigenvector $|\psi\rangle$ of $U$ with eigenvalue $\lambda$ given as a quantum state.

Goal : output $\lambda$.

Recall that an eigenvector $|\psi\rangle$ of $U$ with eigenvalue $\lambda$ means that $U(|\psi\rangle) = \lambda |\psi\rangle$. Because $U$ is a unitary, $|\lambda| = 1$ so we can write $\lambda = e^{2\pi i \phi}$ for some real number $\phi \in [0, 1)$ ([0, 1[ in French notation). We assume first that $\phi$ can be fully described with $l$ bits of precision, i.e. there exists a natural number $C \in \mathbb{N}$ such that $\phi = \frac{C}{2^l}$.

We consider a quantum unitary $Q$ satisfying

$$Q(|k\rangle |\psi\rangle) = |k\rangle U^k(|\psi\rangle).$$

for any $k \in \{0, \ldots, 2^l - 1\}$ and any state $|\psi\rangle$. We perform the following algorithm :

\begin{itemize}
  \item 1. Start from $|0^l\rangle |\psi\rangle$ and apply $F_{2^l}$ on the first register.
  \item 2. Apply $Q$ on both registers.
  \item 3. Apply the inverse Fourier transform $F_{2^l}^{-1}$ on the first register and measure the first register which is $C$. Output $\frac{C}{2^l}$.
\end{itemize}

Exercice 1 : Inverse of the Fourier transform

Let $G_{2^l}$ the quantum unitary operation acting on $l$ qubits such that $\forall k \in \{0, \ldots, 2^l - 1\}$, we have $G_{2^l}(|k\rangle) = \frac{1}{\sqrt{2^l}} \sum_{j=0}^{2^l-1} \omega^{-jk} |j\rangle$. Show that $G_{2^l}$ is the inverse of $F_{2^l}$.

Exercice 2 : Correctness of the algorithm

Write each step of the algorithm. Show that at the end of step 3, before the measurement, the quantum registers are in state $|C\rangle |\psi\rangle$. 


Exercice 3 : Running time
Suppose $U$ runs in time $t$. What is the running time of this algorithm?

**General case.** If $\phi$ cannot be written with $l$ bits of precision, we consider the closest approximation of $\phi$ of the form $\frac{C}{2^l}$. An error analysis (not detailed here) shows that the above procedure will find this $C$ with probability at least $\frac{4}{\pi^2}$. By performing several iterations of this procedure, we can find the correct $C$, i.e. a good approximation of $\phi$ with a probability that exponentially converges to 1 in the number of iterations.

**Fourier transform $F_N$ for any $N$**
We showed in the course how to perform the Fourier transform $F_N$ when $N = 2^n$ for some $n \in \mathbb{N}$. Here, we show how to perform the Fourier transform for any $N$. $F_N$ will act on a quantum register that can take $N$ values from 0 to $N-1$ and

$$\forall k \in \{0, \ldots, N-1\}, \quad F_N(|k\rangle) = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} \omega^{jk} |j\rangle,$$

where $\omega := e^{2\pi i / N}$. Let $U_1$ and $U_2$ two unitaries that do the following, $\forall k \in \{0, \ldots, N-1\}$.

$$U_1(|k\rangle|0\rangle) = |k\rangle F_N(|k\rangle) \quad U_2(F_N|k\rangle|0\rangle) = F_N(|k\rangle)|k\rangle.$$

**Exercice 4 : Decomposing $F_N$**
Using $U_1, U_2$ and basic quantum gates, show how to construct $F_N$.

**Exercice 5 : Constructing $U_1$**
Let $S_N$ a quantum unitary such that $S_N(|0\rangle) = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} |j\rangle$. Since $N$ is not a power of 2, $S_N$ cannot be expressed as Hadamards but we can still easily construct such a unitary. Let also $O_{\text{mult}}$ satisfying $O_{\text{mult}}(|k\rangle|j\rangle|0\rangle) = |k\rangle|j\rangle|kj \mod N\rangle$.

<table>
<thead>
<tr>
<th>Algorithm for constructing $U_1$</th>
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<tbody>
<tr>
<td>1. Start from $</td>
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<tr>
<td>2. Apply $O_{\text{mult}}$ on the three registers.</td>
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<tr>
<td>3. Apply the unitary $</td>
</tr>
<tr>
<td>4. Apply $O_{\text{mult}}^{-1}$ on the three registers.</td>
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Write the quantum state after each step. Show that this performs $U_1$, with the third register being an ancilla register.

**Exercice 6 : Constructing $U_2$**
Think of a way to construct a good approximation of $U_2$. Hint : consider the unitary $O_{\text{add}}$ such that $O_{\text{add}}(|k\rangle) = |k + 1 \mod N\rangle$. Show that $F_N(|k\rangle)$ is a eigenvector of $O_{\text{add}}$ and use phase estimation.