2-34-1 Quantum information and applications
Exercice sheet 1

Phase estimation

We want to solve the following problem.

Phase estimation
Input : a quantum unitary U acting on n qubits and an eigenvector $\|\psi\rangle$ of U with eigenvalue λ given as a quantum state. Goal : output λ.

Recall that an eigenvector $|\psi\rangle$ of U with eigenvalue λ means that $U(|\psi\rangle) = \lambda|\psi\rangle$. Because U is a unitary, $|\lambda| = 1$ so we can write $\lambda = e^{2\mathrm{i}\pi\phi}$ for some real number $\phi \in [0, 1)$ ($[0, 1[$ in French notation). We assume first that ϕ can be fully described with l bits of precision, *i.e.* there exists a natural number $C \in \mathbb{N}$ such that $\phi = \frac{C}{2^l}$.

We consider a quantum unitary Q satisfying

$$Q(|k\rangle|\psi\rangle) = |k\rangle U^k(|\psi\rangle).$$

for any $k \in \{0, \ldots, 2^l - 1\}$ and any state $|\psi\rangle$. We perform the following algorithm :

Algorithm for phase estimation

1. Start from $|0^l\rangle|\psi\rangle$ and apply F_{2^l} on the first register.
2. Apply Q on both registers.
3. Apply the inverse Fourier transform $F_{2^l}^{-1}$ on the first register and measure the first register which is C. Output $\frac{C}{2^l}$.

Exercice 1 : Inverse of the Fourier transform

Let G_{2^l} the quantum unitary operation acting on l qubits such that $\forall k \in \{0, \ldots, 2^l - 1\}$, we have $G_{2^l}(|k\rangle) = \frac{1}{\sqrt{2^l}} \sum_{j=0}^{2^l-1} \omega^{-jk}|j\rangle$. Show that G_{2^l} is the inverse of F_{2^l}.

Exercice 2 : Correctness of the algorithm

Write each step of the algorithm. Show that at the end of step 3, before the measurement, the quantum registers are in state $|C\rangle|\psi\rangle$.

Exercice 3 : Running time

Suppose U runs in time t. What is the running time of this algorithm ?

General case. If ϕ cannot be written with l bits of precision, we consider the closest approximation of ϕ of the form $\frac{C}{2^l}$. An error analysis (not detailed here) shows that the above procedure will find this C with probability at least $\frac{4}{\pi^2}$. By performing several iterations of this procedure, we can find the correct C, *i.e.* a good approximation of ϕ with a probability that exponentially converges to 1 in the number of iterations.

Fourier transform F_N for any N

We showed in the course how to perform the Fourier transform F_N when $N = 2^n$ for some $n \in \mathbb{N}$. Here, we show how to perform the Fourier transform for any N. F_N will act on a quantum register that can take N values from 0 to $N-1$ and

$$\forall k \in \{0, \dots, N-1\},\ F_N(|k\rangle) = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} \omega^{jk} |j\rangle.$$

where $\omega := e^{\frac{2i\pi}{N}}$. Let U_1 and U_2 two unitaries that do the following, $\forall k \in \{0, \dots, N-1\}$.

$$U_1(|k\rangle|0\rangle) = |k\rangle F_N(|k\rangle) \quad ; \quad U_2(F_N|k\rangle|0\rangle) = F_N(|k\rangle)|k\rangle.$$

Exercice 4 : Decomposing F_N

Using U_1, U_2 and basic quantum gates, show how to construct F_N.

Exercice 5 : Constructing U_1

Let S_N a quantum unitary such that $S_N(|0\rangle) = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} |j\rangle$. Since N is not a power of 2, S_N cannot be expressed as Hadamards but we can still easily construct such a unitary. Let also O_{mult} satisfying $O_{mult}(|k\rangle|j\rangle|0\rangle) = |k\rangle|j\rangle|kj \mod N\rangle$.

Algorithm for constructing U_1

1. Start from $|k\rangle|0\rangle|0\rangle$ and apply S_N on the second register.
2. Apply O_{mult} on the three registers.
3. Apply the unitary $|x\rangle \to \omega^x |x\rangle$ on the third register.
4. Apply O_{mult}^{-1} on the three registers.

Write the quantum state after each step. Show that this performs U_1, with the third register being an ancilla register.

Exercice 6 : Constructing U_2

Think of a way to construct a good approximation of U_2. Hint : consider the unitary O_{add} such that $O_{add}(|k\rangle) = |k+1 \mod N\rangle$. Show that $F_N(|k\rangle)$ is a eigenvector of O_{add} and use phase estimation.