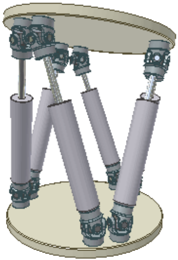
Parametric polynomial systems and linkages

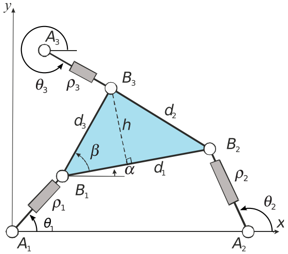
Guillaume Moroz

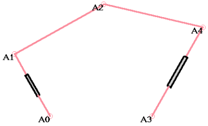
Inria Nancy - Grand Est

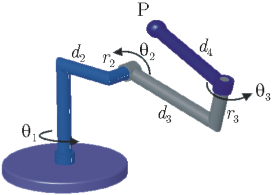
Supelec, February 18, 2015

Linkages



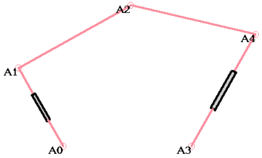

y
A_3
θ_3
ρ_3
B_3
d_2
d_3
h
β
B_2
α
d_1
ρ_2
B_1
ρ_1
θ_2
θ_1
x
A_1
A_2


A2
A4
A1
A0
A3


P
d_2
r_2
θ_2
d_4
θ_3
d_3
r_3
θ_1

Modeling

Parallel PR-PRR



- Actuator variables
  - r_1, r_2
- Pose variables
  - x, y
- Passive variables
  - θ_1, θ_2

- Equations

$$(F)\begin{cases} x = cos(\frac{2\pi}{3})r_1 + cos(\theta_1) \\ x = 1 + cos(\frac{\pi}{3})r_2 + cos(\theta_2) \\ y = sin(\frac{2\pi}{3})r_1 + sin(\theta_1) \\ y = 1 + sin(\frac{\pi}{3})r_2 + sin(\theta_2) \end{cases}$$

Parametric system

$$S \quad : \quad \left\{ \begin{array}{rl} f_1(\underline{T}, \underline{X}) & = 0 \\ & \vdots \\ f_k(\underline{T}, \underline{X}) & = 0 \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{rl} g_1(\underline{T}, \underline{X}) & \neq 0 \\ & \vdots \\ g_r(\underline{T}, \underline{X}) & \neq 0 \end{array} \right.$$

$$f_i, g_j \in \mathbb{Q}[\underbrace{T_1, \cdots, T_s}_{parameters}, \underbrace{X_1, \cdots, X_n}_{unknowns}]$$

- Parametric system S
- Solutions: $\mathcal{C} \subset \mathbb{C}^s \times \mathbb{C}^n$

Parametric system

$$S_{\underline{t_0}} : \left\{ \begin{array}{rl} f_1(\underline{t_0}, \underline{X}) &= 0 \\ &\vdots \\ f_k(\underline{t_0}, \underline{X}) &= 0 \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{rl} g_1(\underline{t_0}, \underline{X}) &\neq 0 \\ &\vdots \\ g_r(\underline{t_0}, \underline{X}) &\neq 0 \end{array} \right.$$

$$f_i, g_j \in \mathbb{Q}[\underbrace{T_1, \cdots, T_s}_{parameters}, \underbrace{X_1, \cdots, X_n}_{unknowns}]$$

- Parametric system S
- Solutions: $\mathcal{C} \subset \mathbb{C}^s \times \mathbb{C}^n$
- For almost all $\underline{t_0} \in \mathbb{C}^s$: $S_{\underline{t_0}}$ has finitely many complex solutions.

Parametric system

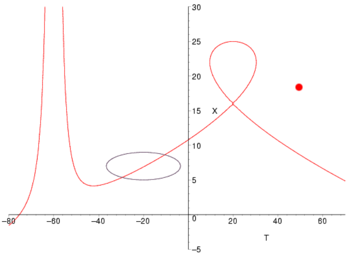
$$S^{\mathbb{R}} : \left\{ \begin{array}{l} f_1(\underline{T}, \underline{X}) = 0 \\ \vdots \\ f_k(\underline{T}, \underline{X}) = 0 \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{l} g_1(\underline{T}, \underline{X}) > 0 \\ \vdots \\ g_r(\underline{T}, \underline{X}) > 0 \end{array} \right.$$

$$f_i, g_j \in \mathbb{Q}[\underbrace{T_1, \cdots, T_s}_{parameters}, \underbrace{X_1, \cdots, X_n}_{unknowns}]$$

- Parametric system S
- Solutions: $\mathcal{C} \subset \mathbb{C}^s \times \mathbb{C}^n$
- For almost all $\underline{t_0} \in \mathbb{C}^s$: $S_{\underline{t_0}}$ has finitely many complex solutions.

In the applications we are interested in $\mathcal{C}_{\mathbb{R}} \subset \mathbb{R}^s \times \mathbb{R}^n$

Parametric system



- Parametric system S
- Solutions: $\mathcal{C} \subset \mathbb{C}^s \times \mathbb{C}^n$
- For almost all $\underline{t_0} \in \mathbb{C}^s$: $S_{\underline{t_0}}$ has finitely many complex solutions.

In the applications we are interested in $\mathcal{C}_{\mathbb{R}} \subset \mathbb{R}^s \times \mathbb{R}^n$

Applications and general problem

Robotics: Parallel robots [McAree, Daniel, Wenger, Chablat, ...]

Vision: Camera calibration [Gao, Tang, Yang, ...]

Academic: Haas systems [Dickenstein, Rojas, Rusek, Shih]

General problem: classification of the parameters' space

- Number of solutions of $S_{\underline{t_0}}$ depends on $\underline{t_0}$

$\Longrightarrow$ Classification of the parameters

State of the art (non exhaustive)

- Collins (1970): Cylindrical Algebraic Decomposition
  - Implementations (QEPCAD, Redlog, Mathematica, ...) , Efficient in practice for less than 3 variables
  - Worst case doubly exponential in the number of variables
- Weispfenning (1992): Comprehensive Gröbner bases
  - Implementations (Singular, Maple, Risa/Asir, ...)
  - Time complexity not well understood
- Grigoriev, Vorobjov (1999): Maps of vector of multiplicities
  - Time complexity analysis
  - Difficult to implement efficiently
- Lazard, Rouillier (2004): Minimal discriminant variety
  - Computed with Gröbner bases and CAD
  - Relatively efficient in practice and in theory under some assumptions
  - General case: combinatorial factors spoiled practical efficiency

Discriminant variety and classification

Discriminant variety

$$S \; : \quad \left\{ \begin{array}{rl} f_1(\underline{T}, \underline{X}) & = 0 \\ \vdots & \\ f_k(\underline{T}, \underline{X}) & = 0 \end{array} \right. \qquad \left\{ \begin{array}{rl} g_1(\underline{T}, \underline{X}) & \neq 0 \\ \vdots & \\ g_r(\underline{T}, \underline{X}) & \neq 0 \end{array} \right.$$

$$f_i, g_j \in \mathbb{Q}[\underbrace{T_1, \cdots, T_s}_{parameters}, \underbrace{X_1, \cdots, X_n}_{unknowns}]$$

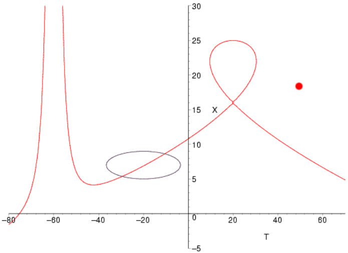
- $\pi : \; \mathcal{C} = V(S) \rightarrow \mathbb{C}^s$ canonical projection
  $(\underline{t}, \underline{x}) \mapsto \underline{t}$

Definition: covering map

Given a connected open set $U \subset \mathbb{C}^s$, we say that (π, U) is a covering map if:

- $\pi^{-1}(U) = \mathcal{C}_1 \cup \cdots \cup \mathcal{C}_m$
- $\pi_{|\mathcal{C}_i} : \mathcal{C}_i \rightarrow U$ is a diffeomorpism
- $\mathcal{C}_i \cup \mathcal{C}_j = \emptyset$

Discriminant variety



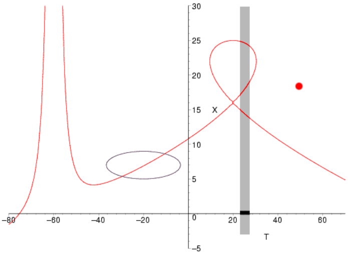
- $\pi : \mathcal{C} = V(S) \rightarrow \mathbb{C}^s$ canonical projection
  $(\underline{t}, \underline{x}) \mapsto \underline{t}$

Definition: covering map

Given a connected open set $U \subset \mathbb{C}^s$, we say that (π, U) is a covering map if:

- $\pi^{-1}(U) = \mathcal{C}_1 \cup \cdots \cup \mathcal{C}_m$
- $\pi_{|\mathcal{C}_i} : \mathcal{C}_i \rightarrow U$ is a diffeomorpism
- $\mathcal{C}_i \cup \mathcal{C}_j = \emptyset$

Discriminant variety



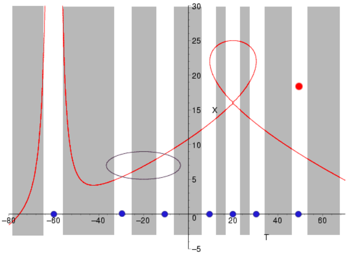
- $\pi : \mathcal{C} = V(S) \rightarrow \mathbb{C}^s$ canonical projection
  $(\underline{t}, \underline{x}) \mapsto \underline{t}$

Definition: covering map

Given a connected open set $U \subset \mathbb{C}^s$, we say that (π, U) is a covering map if:

- $\pi^{-1}(U) = \mathcal{C}_1 \cup \cdots \cup \mathcal{C}_m$
- $\pi_{|\mathcal{C}_i} : \mathcal{C}_i \rightarrow U$ is a diffeomorpism
- $\mathcal{C}_i \cup \mathcal{C}_j = \emptyset$

Discriminant variety

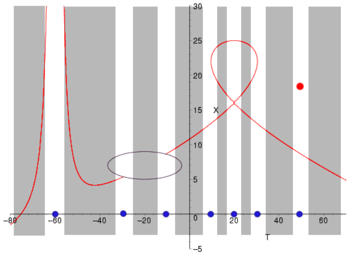


Definition: Discriminant variety

$D(\mathcal{C}) \subset \mathbb{C}^s$ s.t. for all connected open set $U \subset \mathbb{C}^s \setminus D(\mathcal{C})$

(π, U) is a covering map

Discriminant variety



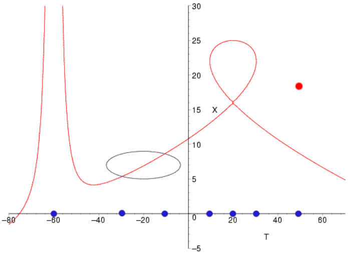
Property of the complex discriminant variety in the real

For all connected open set $U \subset \mathbb{R}^s \setminus D(\mathcal{C})$

$(\pi_{\mathbb{R}}, U)$ is a covering map

Number of real roots of $S_p^{\mathbb{R}}$ constant for all $p \in U$

Discriminant variety



Definition: Minimal discriminant variety

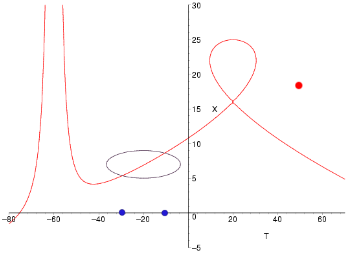
The intersection of all the discriminant varieties of S.

$$D_{min}(\mathcal{C}) = V(D_1(\underline{T}), \ldots, D_m(\underline{T}))$$
$$D_i \in \mathbb{Q}[T_1, \ldots, T_s]$$

Discriminant variety

- Different components:

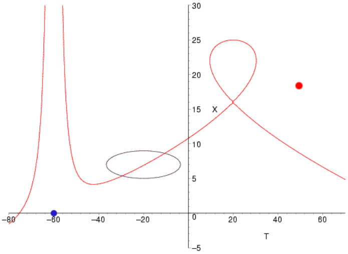
$$D_{min}(\mathcal{C}) = \begin{cases} D_{ineq}(\mathcal{C}): & \text{projection of } \overline{\mathcal{C}} \cap \cup_i V\left(g_i(\underline{T}, \underline{X})\right) \\ D_{\infty}(\mathcal{C}): & \text{divergence of the solutions} \\ D_{mult}(\mathcal{C}): & \text{projection of the multiple solutions} \\ D_{sd}(\mathcal{C}): & \text{components of dimension} < s \end{cases}$$



Discriminant variety

- Different components:

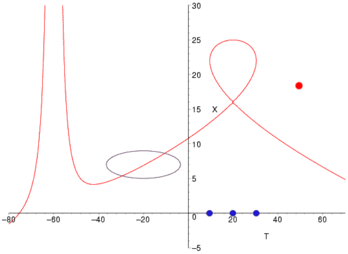
$$D_{min}(\mathcal{C}) = \begin{cases} D_{ineq}(\mathcal{C}): & \text{projection of } \overline{\mathcal{C}} \cap \cup_i V(g_i(\underline{T}, \underline{X})) \\ D_{\infty}(\mathcal{C}): & \text{divergence of the solutions} \\ D_{mult}(\mathcal{C}): & \text{projection of the multiple solutions} \\ D_{sd}(\mathcal{C}): & \text{components of dimension} < s \end{cases}$$



Discriminant variety

- Different components:

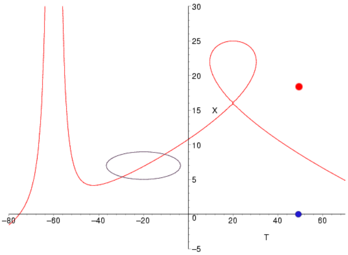
$$D_{min}(\mathcal{C}) = \begin{cases} D_{ineq}(\mathcal{C}): & \text{projection of } \overline{\mathcal{C}} \cap \cup_i V(g_i(\underline{T}, \underline{X})) \\ D_{\infty}(\mathcal{C}): & \text{divergence of the solutions} \\ D_{mult}(\mathcal{C}): & \text{projection of the multiple solutions} \\ D_{sd}(\mathcal{C}): & \text{components of dimension} < s \end{cases}$$



Discriminant variety

- Different components:

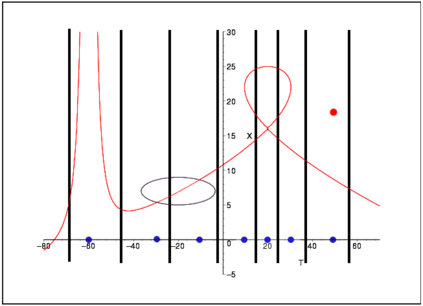
$$D_{min}(\mathcal{C}) = \begin{cases} D_{ineq}(\mathcal{C}): & \text{projection of } \overline{\mathcal{C}} \cap \cup_i V(g_i(\underline{T}, \underline{X})) \\ D_{\infty}(\mathcal{C}): & \text{divergence of the solutions} \\ D_{mult}(\mathcal{C}): & \text{projection of the multiple solutions} \\ D_{sd}(\mathcal{C}): & \text{components of dimension} < s \end{cases}$$



Discriminant variety

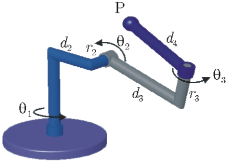
- Different components:

$$D_{min}(\mathcal{C}) = \begin{cases} D_{ineq}(\mathcal{C}): & \text{projection of } \overline{\mathcal{C}} \cap \cup_i V(g_i(\underline{T}, \underline{X})) \\ D_{\infty}(\mathcal{C}): & \text{divergence of the solutions} \\ D_{mult}(\mathcal{C}): & \text{projection of the multiple solutions} \\ D_{sd}(\mathcal{C}): & \text{components of dimension} < s \end{cases}$$



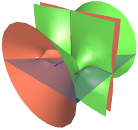
Describing the real roots with the discriminant variety

Input



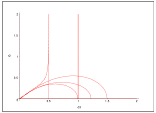
$\rightarrow$

$$S : \begin{cases} f_1(T, X) = 0 \\ \vdots \\ f_k(T, X) = 0 \end{cases} \quad \begin{cases} g_1(T, X) > 0 \\ \vdots \\ g_r(T, X) > 0 \end{cases}$$



Parametric system

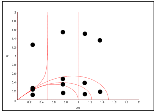
$D_{min}(\mathcal{C}) : D_1(T) = 0, .., D_m(T) = 0$



Discriminant variety

$\rightarrow$

$\{p_0, \ldots, p_k\}$



Witness points

$\rightarrow$

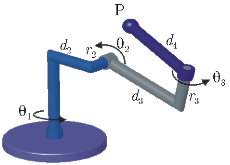
p_0	p_1	$\cdots$	p_k
3	0	$\cdots$	4

Number of real roots

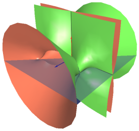
Output

Describing the real roots with the discriminant variety

Input

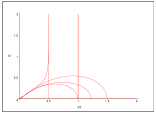


$$S : \begin{cases} f_1(T, X) = 0 \\ \vdots \\ f_k(T, X) = 0 \end{cases} \quad \begin{cases} g_1(T, X) > 0 \\ \vdots \\ g_r(T, X) > 0 \end{cases}$$



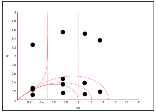
Parametric system

$$D_{min}(\mathcal{C}) : D_1(T) = 0, .., D_m(T) = 0$$



Discriminant variety

$\{p_0, \ldots, p_k\}$



Witness points

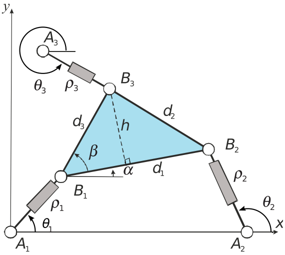
p_0	p_1	$\cdots$	p_k
3	0	$\cdots$	4

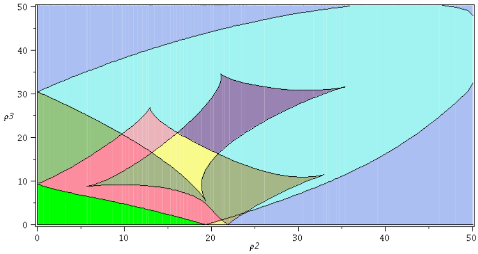
Number of real roots

Output

Example

- 3-RPR: a 9-bar linkage
  - Parallel robot
  - r_1 fixed
  - Parameter space Q: r_2, r_3
  - Workspace W: $B_{1x}, B_{1y}, \alpha_x, \alpha_y$
  - Constraint equations:
    $f_1 = f_2 = f_3 = f_4 = 0$
- Discriminant variety and partition of Q





Cuspidal points

System (S):

$$I : \begin{cases} f_1 = 0 \\ f_2 = 0 \\ f_3 = 0 \\ f_4 = 0 \end{cases}$$

$$\mathcal{J}(I) : j_0 := \det(d\vec{f_1}, d\vec{f_2}, d\vec{f_3}, d\vec{f_4}) = 0$$

$$\mathcal{J}(I + \mathcal{J}(I)) : \begin{cases} j_0 := \det(d\vec{f_1}, d\vec{f_2}, d\vec{f_3}, d\vec{f_4}) = 0 \\ j_1 := \det(d\vec{f_1}, d\vec{f_2}, d\vec{f_3}, d\vec{j_0}) = 0 \\ j_2 := \det(d\vec{f_1}, d\vec{f_2}, d\vec{j_0}, d\vec{f_4}) = 0 \\ j_3 := \det(d\vec{f_1}, d\vec{j_0}, d\vec{f_3}, d\vec{f_4}) = 0 \\ j_4 := \det(d\vec{j_0}, d\vec{f_2}, d\vec{f_3}, d\vec{f_4}) = 0 \end{cases}$$

Cuspidal points

System (S):

$$I : \begin{cases} f_1 = 0 \\ f_2 = 0 \\ f_3 = 0 \\ f_4 = 0 \end{cases}$$

$$\mathcal{J}(I) : j_0 := \det(d\vec{f_1}, d\vec{f_2}, d\vec{f_3}, d\vec{f_4}) = 0$$

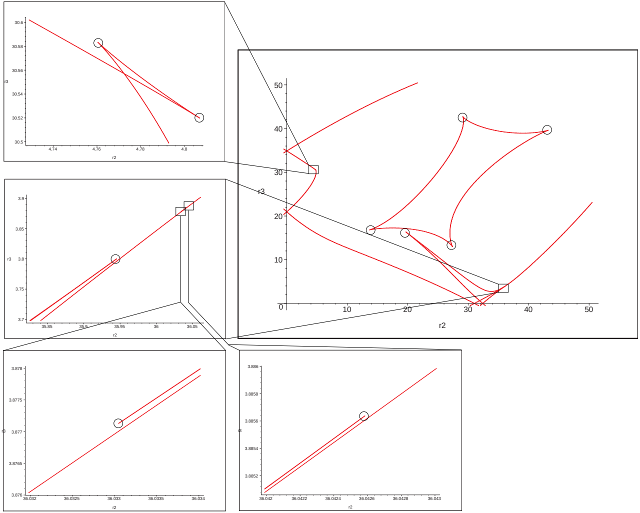
$$\mathcal{J}(I + \mathcal{J}(I)) : \begin{cases} j_0 := \det(d\vec{f_1}, d\vec{f_2}, d\vec{f_3}, d\vec{f_4}) = 0 \\ j_1 := \det(d\vec{f_1}, d\vec{f_2}, d\vec{f_3}, d\vec{j_0}) = 0 \\ j_2 := \det(d\vec{f_1}, d\vec{f_2}, d\vec{j_0}, d\vec{f_4}) = 0 \\ j_3 := \det(d\vec{f_1}, d\vec{j_0}, d\vec{f_3}, d\vec{f_4}) = 0 \\ j_4 := \det(d\vec{j_0}, d\vec{f_2}, d\vec{f_3}, d\vec{f_4}) = 0 \end{cases}$$

- Curve in $\mathbb{C}^7$ (determinantal ideal)
- Description:
  - r_1 : parameter
  - $r_2, r_3, t_x, t_y, u_x, u_y$: unknowns
  - $N : x \mapsto \#\{\text{real solutions of } (S) \text{ for } r_1 = x\}$

Classification of cuspidal configurations

N		0		2		4		2
r_1	0.000	]——[	0.148	]——[	1.655	]——[	1.660	]——[
N		4		6		8		6
r_1	2.261	]——[	2.975	]——[	9.18678	]——[	9.18686*	]——[
N		8		6		8		6
r_1	9.257662	]——[	9.257677*	]——[	10.9056649	]——[	10.9056683*	]——[
N		8		6		8		6
r_1	14.579115749	]——[	14.579115757*	]——[	20.555	]——[	20.562	]——[
N		8		10		8		6
r_1	26.786	]——[	28.094	]——[	28.107	]——[	28.257	]——[
N		8		6			4	
r_1	30.740	]——[	30.779	]——[	30.946	]————————		

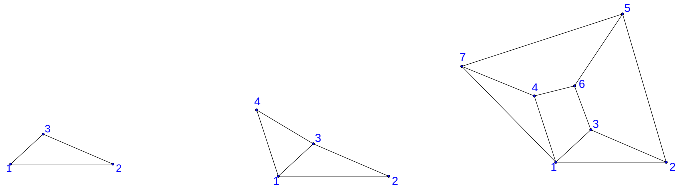
10 cuspidal points



Parameter space for $r_1 = 28.10$

11-bar linkage

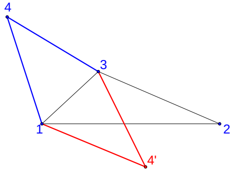
Planar rigid linkage



3-bar   5-bar   11-bar

Constraints

- Fixed length bars: c_{ij}
- Free revolute joints
- Zero degree of freedom



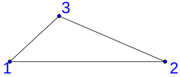
- Several assembly modes
- Number depends on c_{ij}
- Max number of assembly modes?

Properties of minimally rigid linkages

- Construction steps

Henneberg steps: H_1 and H_2

- 3-bar rigid linkage

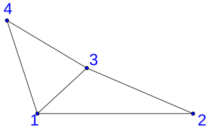


Properties of Rigid Linkages

- Construction steps

Henneberg steps: H_1 and H_2

- 5-bar rigid linkage

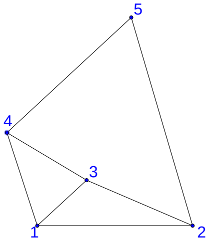


Properties of Rigid Linkages

- Construction steps

Henneberg steps: H_1 and H_2

- 7-bar rigid linkage

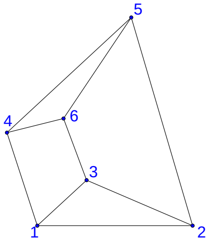


Properties of Rigid Linkages

- Construction steps

Henneberg steps: H_1 and H_2

- 9-bar rigid linkage

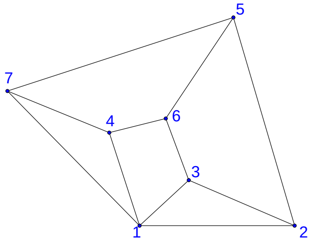


Properties of Rigid Linkages

- Construction steps

Henneberg steps: H_1 and H_2

- 11-bar rigid linkage



Properties known before [Emiris and M. 11]

Maximal number of assembly modes

bars	3	5	7	9	11	13	15	17
upper	2	4	8	24	64	128	512	2048
lower	2	4	8	24	48	96	288	576

Theorem

A linkage is minimally rigid $\Leftrightarrow$ It is constructed with H_1 and H_2

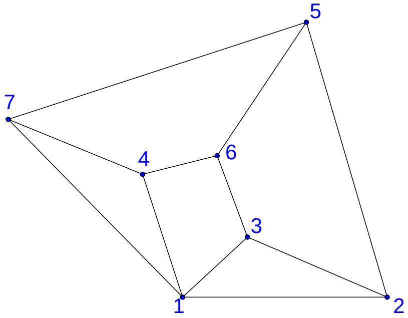
Corollary

$$\#Links = 2\#Joints - 3$$

Outline

1. Upper Bound
   - Algebraic Modeling
   - Mixed Volume
2. Lower Bound
   - Adaptive Sampling

Algebraic Modeling I



- c_{ij}: 10 parameters
- x_i, y_i: 14 variables

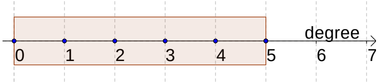
$$\begin{cases} x_1 = 0, y_1 = 0 \\ x_2 = 1, y_2 = 0 \end{cases}$$

$$\begin{cases} {x_3}^2 + {y_3}^2 &= c_{13} \\ (x_3 - 1)^2 + {y_3}^2 &= c_{23} \\ (x_5 - 1)^2 + {y_5}^2 &= c_{25} \\ (x_6 - x_3)^2 + (y_6 - y_3)^2 &= c_{36} \\ {x_4}^2 + {y_4}^2 &= c_{14} \end{cases} \qquad \begin{cases} {x_7}^2 + {y_7}^2 &= c_{17} \\ (x_6 - x_4)^2 + (y_6 - y_4)^2 &= c_{46} \\ (x_5 - x_6)^2 + (y_5 - y_6)^2 &= c_{56} \\ (x_7 - x_5)^2 + (y_7 - y_5)^2 &= c_{57} \\ (x_4 - x_7)^2 + (y_4 - y_7)^2 &= c_{47} \end{cases}$$

Number of solutions

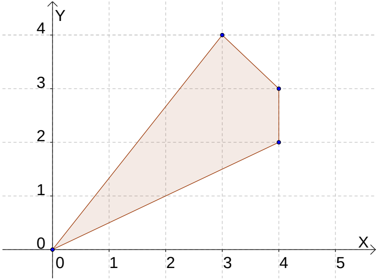
- Mixed Volume: $n!Volume(Support)$

1 variable



$$1 - X + 3X^2 - X^3 + 6X^4 - 5X^5 = 0$$

2 variables



$$\begin{cases} 1 - X^4Y^2 + 7X^4Y^3 - 4X^3Y^4 = 0 \\ 8 + 6X^4Y^2 - 5X^4Y^3 - X^3Y^4 = 0 \end{cases}$$

- Our system: 2^{10}

Algebraic Modeling II

- Distance matrix

$$
\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \end{array}
\begin{bmatrix}
 & v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 \\
0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & c_{12} & c_{13} & c_{14} & x_{15} & x_{16} & c_{17} \\
1 & c_{12} & 0 & c_{23} & x_{24} & c_{25} & x_{26} & x_{27} \\
1 & c_{13} & c_{23} & 0 & x_{34} & x_{35} & c_{36} & x_{37} \\
1 & c_{14} & x_{24} & x_{34} & 0 & x_{45} & c_{46} & c_{47} \\
1 & x_{15} & c_{25} & x_{35} & x_{45} & 0 & c_{56} & c_{57} \\
1 & x_{16} & x_{26} & c_{36} & c_{46} & c_{56} & 0 & x_{67} \\
1 & c_{17} & x_{27} & x_{37} & c_{47} & c_{57} & x_{67} & 0
\end{bmatrix}
$$

Theorem

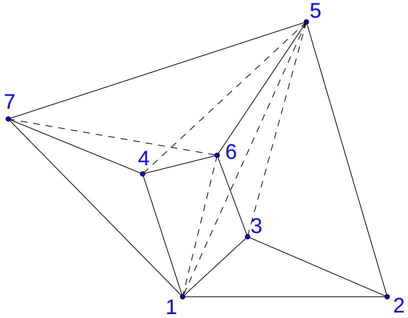
The distance matrix has rank 4.

Corollary

All the 5x5 minors vanish.

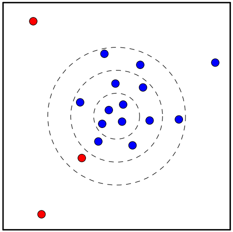
Algebraic Modeling II

$$\begin{cases} D(0,4,5,6,7)(c_{46}, c_{47}, c_{56}, c_{57}, x_{45}, x_{67}) & = & 0 \\ D(0,1,4,6,7)(c_{14}, c_{17}, c_{46}, c_{47}, x_{16}, x_{67}) & = & 0 \\ D(0,1,4,5,7)(c_{14}, c_{17}, c_{47}, c_{57}, x_{15}, x_{45}) & = & 0 \\ D(0,1,2,3,5)(c_{12}, c_{13}, c_{25}, c_{23}, x_{15}, x_{35}) & = & 0 \\ D(0,1,3,5,6)(c_{13}, c_{36}, c_{56}, x_{15}, x_{16}, x_{35}) & = & 0 \end{cases}$$

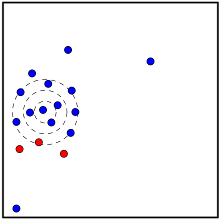


- Upper Bound
  - Mixed volume: 56
- Lower Bound?

- Uniform sampling
  - No linkage found with 56 assembly modes
- Adaptive sampling
  - Simulated annealing
  - Cross-Entropy Method



⇒



Step k

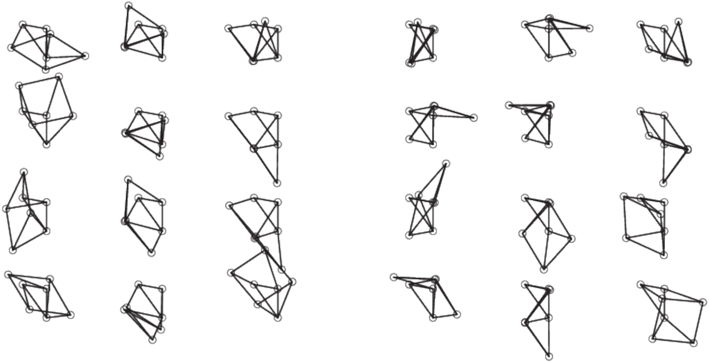
Step k+1

Results

- Random simulations for different sampling methods

Uniform	Simulated annealing	Cross-entropy
44 (572)	52 (17)	52 (199)
42 (196)	54 (247)	54 (132)
48 (27)	48 (362)	52 (186)
44 (200)	52 (14)	54 (130)
42 (200)	54 (547)	56 (497)
44 (424)	54 (315)	56 (328)
46 (48)	56 (425)	56 (454)
42 (170)	50 (585)	54 (375)
42 (18)	54 (26)	56 (552)
46 (366)	52 (474)	56 (355)

Results



Conclusion

- 9-bar linkage
  - Discriminant variety can be computed:
    - on the equation constraints
    - on the cuspidal equation constraints
  - Classification of the parameter space
- 11-bar linkage
  - No complete classification of the parameter space
  - Distance matrices and mixed volume:
    - at most 56 assembly modes
  - simulated annealing and cross entropy method:
    - a 11-bar linkage with exactly 56 assembly modes
- n vertices linkage
  - State-of-the-art: $\Omega(2.89^n)$ and $O(4^n)$ possible embeddings
  - New lower bound: $\Omega(2.3^n)$

Merci !