TALK: DERIVED CATEGORIES AND SYSTEM THEORY?

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ABSTRACT. Buchsbaum and Grothendieck invented ABELian categories as the correct categorical generalization of module categories, i.e., as the most abstract way to do linear algebra. Derived categories are certain "completions" of ABELian categories which were later invented by Grothendieck and Verdier to prove the existence of certain operations and dualities (two of Grothendieck's six operations in algebraic geometry and the Verdier duality) which do not exist in the smaller ABELian category (of quasi-coherent sheaves). Nowadays derived categories popup everywhere in mathematics and one of their remarkable features is their ability to connect apparently remote fields of mathematics. The reason for this is that derived categories of very, very different ABELian categories might be equivalent. In the talk I will present several examples of these "tunnel effects". I will end with considering system theory where it would be nice to see the world on the other side of the tunnel.

1. FROM VECTORS TO COORDINATE VECTORS

The abstract notion of a vector space over a field k is a "coordinate-free" generalization of the "standard vector space" k^n. Some (engineering) students might find this a nightmare. But the story has a happy end. Once you introduce the notion of a basis you see that this generalization only helped you to get rid of "coordinates" which are the duals of a basis. Let us be more precise. Choose a basis $B = (B_1, \ldots, B_n)$ of the abstract n-dimensional vector space $\mathcal{V}$. If the vector space has an elaborated internal structure, e.g., being a space of functions on some complicated space, then the basis B will capture at least part of this internal complexity. Now take the dual basis $x = (x_1, \ldots, x_n)$ of B, i.e., the coordinate system with B as "axes" (remember, $x_i : \mathcal{V} \to k$ and $x_i(B_j) = \delta_{i,j}$). We define two maps, the **coordinate map**

$$\begin{aligned} \chi : \mathcal{V} &\to k^n \\ V &\mapsto (x_1(V), \ldots, x_n(V)) = x(V) \end{aligned}$$

and the **linear combination map**

$$\begin{aligned} \mathcal{V} &\leftarrow k^n : \nu \\ a \cdot B = \sum_{i=1}^{n} a_i \cdot B_i &\leftarrow\!\shortmid (a_1, \ldots, a_n) = a. \end{aligned}$$

We learned that $\chi : \mathcal{V} \rightleftarrows k^n : \nu$ are mutually inverse linear maps (this is in fact equivalent to x being the dual basis of B).

Example. Let $\mathcal{V}$ be the $\mathbb{R}$-vector space spanned by the basis $B = (\sin^2 x, \sin x \cos x, \cos^2 x)$.

$$\begin{array}{lcccl} \chi : & \mathcal{V} & \leftrightarrow & \mathbb{R}^3 & : \nu \\ \chi : & \sin 2x & \leftrightarrow & (0, 2, 0) & : \nu \\ \chi : & \cos 2x & \leftrightarrow & (-1, 0, 1) & : \nu \end{array}$$

So, the coordinate map $\chi = x(-)$ is able to send an internally complex vector space $\mathcal{V}$ to the standard one k^n, while the internal structure of $\mathcal{V}$ is concentrated in the inverse map ν; namely in the basis vectors $B_i \in \mathcal{V}$ which enter the definition of the linear combination map $\nu = - \cdot B$.

This sound like an elegant principle, and if an elegant principle appears once in mathematics it is certain that it will appear infinitely many times :-)

2. A tunnel effect: From sheaves on varieties to modules over rings

Here is another one which, at least at first sight, looks scary

$$\mathbf{R}\operatorname{Hom}(T,-) : D^b(X) \rightleftarrows D^b(\operatorname{End}(T)^{\mathrm{op}}) : -\otimes^{\mathbf{L}} T$$

and reads even scarier: The bounded derived category $D^b(X) = D^b(\mathfrak{Coh}\, X)$ of coherent sheaves on a Noetherian scheme X admitting a tilting sheaf $T = \bigoplus_i T_i$ is adjoint equivalent to the bounded derived category $D^b(\operatorname{End}(T)^{\mathrm{op}}) = D^b(\operatorname{End}(T)^{\mathrm{op}} - \mathrm{mod})$ of f.g. right modules over the endomorphism ring of the tilting object T. Wow!

What happened here?

Instead of ...	we are talking about
a single abstract vector space $\mathcal{V}$	$D^b(X)$, a whole (derived) category of coherent sheaves
the standard vector space k^n	$D^b(\operatorname{End}(T^{\mathrm{op}}))$, the whole (derived) category of f.g. modules
a basis $B = (B_1, \dots, B_n)$	$T = \bigoplus_i T_i$, a tilting sheaf
the coordinate map $\chi = x(-)$	$\mathbf{R}\operatorname{Hom}(T,-)$, the derived Hom-functor
the linear combination map $\nu = - \cdot B$	$-\otimes^{\mathbf{L}} T$, the derived tensor product functor

And in analogy with the above vector space situation, the functor $\mathbf{R}\operatorname{Hom}(T,-)$ does the Fourier analysis, while the (quasi-)inverse functor $-\otimes^{\mathbf{L}} T$ does the Fourier synthesis, i.e., the complex internal structure of $\mathfrak{Coh}\, X$ is now fully encoded in the derived tensor product functor $-\otimes^{\mathbf{L}} T$ defined in terms of a tilting sheaf $T = \bigoplus_i T_i \in \mathfrak{Coh}\, X$.

I will not formally define what an Abelian category $\mathcal{A}$ is: Roughly speaking, it is a category in which one can do linear algebra in the sense that we can add and subtract morphisms, perform direct sums of objects, define kernels and cokernels of morphisms, and where the fundamental homomorphism theorem is valid. We already know some examples: categories of vector spaces, categories of modules over unitial rings (like $\operatorname{End}(T)^{\mathrm{op}} - \mathrm{mod}$), categories of coherent sheaves over Noetherian schemes (like $\mathfrak{Coh}\, X$).

I also will not give a formal definition of the bounded derived category $D^b(\mathcal{A})$ of an Abelian category $\mathcal{A}$, but I will tell you how it may serve us. It is enough to know that it is a larger category containing not only objects of $\mathcal{A}$, but equivalence classes of complexes of objects in $\mathcal{A}$.

It often happens, that two very different Abelian categories $\mathcal{A}, \mathcal{B}$ become equivalent when we pass to their derived categories $D^b(\mathcal{A}) \simeq D^b(\mathcal{B})$. Draw picture. We already saw a general form of a nice example. The Abelian category $\mathfrak{Coh}\, \mathbb{P}^n$ of coherent sheaves on the projective space $X = \mathbb{P}^n$ is not equivalent to any module category for $n > 0$. But when we pass to the derived category $D^b(\mathbb{P}^n) = D^b(\mathfrak{Coh}\, \mathbb{P}^n)$ we get an equivalence to a derived category of modules, namely $D^b(\operatorname{End}(T)^{\mathrm{op}}) = D^b(\operatorname{End}(T)^{\mathrm{op}} - \mathrm{mod})$, where $T = \bigoplus_{i=0}^n T_i$ and $T_i = \Omega^i_{\mathbb{P}^n}(i)$, the ith exterior power of the twisted cotangent sheaf on $\mathbb{P}^n$. In fact, $D^b(\mathbb{P}^n)$ is generated by $\{\Omega^i_{\mathbb{P}^n}(i) \mid i = 0, \dots, n\}$ as a triangulated category [Bon89].

3. A simpler tunnel effect between two very different rings

I will show you a simpler example, which is in fact the crucial step in proving the above equivalence. This time it is about two graded module categories of two very different rings. Let V be an $n+1$ dimensional k vector space with basis $(e_0, \dots, e_n)$ and dual basis $(x_0, \dots, x_n)$. Consider the two graded rings

$$S = \operatorname{Sym} V^* = k[x_0, \dots, x_n] \quad \text{and} \quad E = \bigwedge V = k\langle e_0, \dots, e_n \rangle.$$

The two rings S and E are quite different and so their categories $\mathcal{A} = S - \mathrm{grmod}$ and $\mathcal{B} = E - \mathrm{grmod}$ of f.g. graded modules. For example, $\mathcal{B} = E - \mathrm{grmod}$ is a Frobenius category, i.e., the class of injectives and projectives coincide, while $\mathcal{A} = S - \mathrm{grmod}$ does not have enough injectives[1]. But

$$D^b(S - \mathrm{grmod}) \simeq D^b(E - \mathrm{grmod}).$$

The proof of this is not difficult. It simply about another way of expressing the structure of graded modules over S. This resembles the above Fourier analysis.

[1]Recall, we only consider f.g. modules.

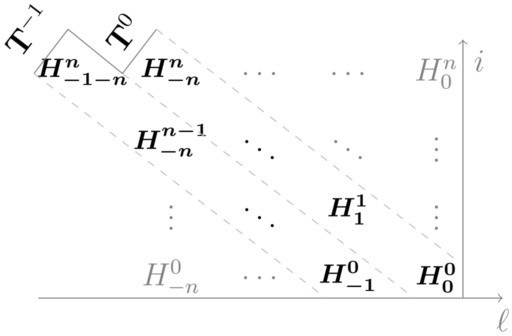
Example. Take $S := k[x_0, x_1]$ and
$$M := S_{\geq 1} = \langle x_0, x_1 \rangle_S$$
The indeterminates x_0 and x_1 induce maps between
$$M_1 = \langle x_0, x_1 \rangle_k$$
and
$$M_2 = \langle x_0^2, x_0 x_1, x_1^2 \rangle_k$$
given by
$$\mu_0^1 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \text{ and } \mu_1^1 := \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$
This yields a map a E with $\mu^1 = e_0\mu_0^1 + e_1\mu_1^1 : E^2 \to E^3$ between free modules over E. Taking gradings and covariance seriously, we must write $\mu^1 : \omega_E(-1)^2 \to \omega_E(-2)^3$, with $\omega_E = \mathrm{Hom}_k(E, k)$ a free (=injective) E-module of rank 1. Doing this for all pairs of subsequent layers M_i, M_{i+1} we obtain a complex over E:
$$\mathbf{E}(M) : 0 \to \omega_E(-1)^2 \xrightarrow[\mu^1]{\begin{pmatrix} e_0 & e_1 & 0 \\ 0 & e_0 & e_1 \end{pmatrix}} \omega_E(-2)^3 \xrightarrow[\mu^2]{\begin{pmatrix} e_0 & e_1 & 0 & 0 \\ 0 & e_0 & e_1 & 0 \\ 0 & 0 & e_0 & e_1 \end{pmatrix}} \omega_E(-3)^4 \to \cdots$$

In general we obtain the complex
$$\mathbf{E}(M) : \cdots \to \omega_E(-i) \otimes_k M_i \xrightarrow{\mu^i} \omega_E(-i-1) \otimes_k M_{i+1} \xrightarrow{\mu^{i+1}} \cdots,$$
where
$$\mu^i := \sum_{j=0}^n e_j \mu_j^i$$
and μ_j^i is the matrix representing the action of $x_j : M_i \to M_{i+1}$. $\mathbf{E}$ sends f.g. graded S-modules to left bounded linear f.g. free complexes over E which eventually become exact. In fact, $\mathbf{E}$ induces an equivalence of categories
$$\mathbf{E} : D^b(S - \mathrm{grmod}) \xrightarrow{\sim} D^b(E - \mathrm{grmod}).$$
This is called the BGG-correspondence (cf. [BGG78, EFS03]).

4. Back to the tunnel effect between a variety and a ring

Instead of explaining the Fourier synthesis for the derived tunnel between the two different rings S and E I will sketch the Fourier synthesis between the variety $X = \mathbb{P}^n$ and the endomorphism ring $\mathrm{End}(\bigoplus_{i=0}^n \Omega_{\mathbb{P}^n}^i(i))$ of the tilting object $T = \bigoplus_{i=0}^n \Omega_{\mathbb{P}^n}^i(i)$.

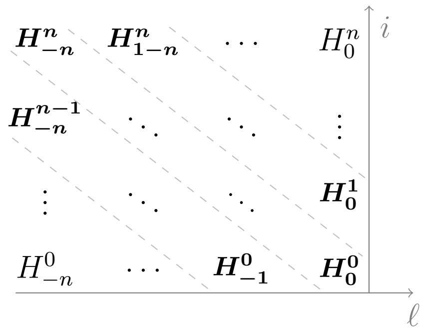
For the Fourier synthesis we need another complex, the so-called Tate resolution
$$\mathbf{T}(M) : \cdots \to \mathbf{T}^{-2}(M) \to \mathbf{T}^{-1}(M) \to \mathbf{T}^0(M) \to \mathbf{T}^1(M) \to \cdots.$$
Start with the exact E-complex $\mathbf{E}(M)_{\gg 0}$ and compute an infinite *minimal* free resolution $\mathbf{T}(M)$ to the left.
$$\mathbf{E}(M) :\ 0 \longrightarrow \omega_E(-1)^2 \xrightarrow{\begin{pmatrix} e_0 & e_1 & 0 \\ 0 & e_0 & e_1 \end{pmatrix}} \cdots$$
$$\mathbf{T}(M) : \cdots \to \omega_E(3)^2 \xrightarrow{\begin{pmatrix} e_0 \\ e_1 \end{pmatrix}} \omega_E(2)^1 \xrightarrow{\begin{pmatrix} e_0 \cdot e_1 \end{pmatrix}} \omega_E(0)^1 \xrightarrow{\begin{pmatrix} e_0 & e_1 \end{pmatrix}} \omega_E(-1)^2 \xrightarrow{\begin{pmatrix} e_0 & e_1 & 0 \\ 0 & e_0 & e_1 \end{pmatrix}} \cdots$$
The Tate resolution $\mathbf{T}(M)$, which only depends on the the sheafification $\mathcal{F} = \widetilde{M}$, encodes all sheaf cohomology groups $H_\ell^i = H^i(\mathcal{F}(\ell))$ as the socles of its modules $\mathbf{T}^i(M)$, more precisely, we have the decomposition
$$\mathbf{T}^i(M) = \mathbf{T}^i(\mathcal{F}) = \bigoplus_{j=0}^n \omega_E(j-i) \otimes_k H_{i-j}^j.$$



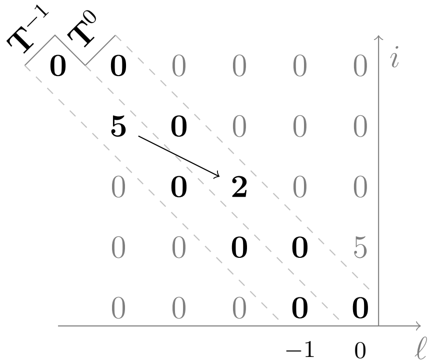
To recover $\mathcal{F} = \widetilde{M}$ from the TATE resolution $\mathbf{T}(\mathcal{F})$ replace $\omega_E(\ell)$ in $\mathbf{T}^i(\mathcal{F})$ by $\Omega^\ell_{\mathbb{P}^n}(\ell)$, the ℓth exterior power of the twisted cotangent sheaf. The morphisms between the $\omega_E(\ell)$'s can be naturally identified with morphisms between the $\Omega^\ell_{\mathbb{P}^n}(\ell)$'s, more precisely, $\operatorname{Hom}(\omega_E(i), \omega_E(j)) \cong \operatorname{Hom}(\Omega^i_{\mathbb{P}^n}(i), \Omega^j_{\mathbb{P}^n}(j))$. The resulting complex $\Omega(\mathbf{T}(\mathcal{F}))$ is (contrary to $\mathbf{T}(\mathcal{F})$) bounded and lives in cohomological degrees $-n, \dots, n$ (cf. [EFS03]). The complex

$$\Omega(\mathbf{T}(\mathcal{F})) = \mathbf{T}(\mathcal{F}) \otimes^{\mathbf{L}} \left(\bigoplus_{i=0}^{n} \Omega^i_{\mathbb{P}^n}(i) \right)$$

is isomorphic to $\mathcal{F} \cong H^0(\Omega(\mathbf{T}(\mathcal{F})))$ in $D^b(\mathbb{P}^n)$.



Now we see the usefulness of derived equivalences. The graded S-module M_{HM} (of twisted global sections) representing the so-called HORROCKS-MUMFORD bundle $\mathcal{E}_{\mathrm{HM}}$ has 19 generators with 35 relations. The derived equivalence now allows us to replace M_{HM} by a graded E-module having 2 generators with 5 relations: $\mathbf{T}^{-1}(\mathcal{E}_{\mathrm{HM}}) = \omega_E(4) \otimes_k H^3_{-4}$, $\mathbf{T}^0(\mathcal{E}_{\mathrm{HM}}) = \omega_E(2) \otimes_k H^2_{-2}$ with socles of dimension $\dim_k H^3_{-4} = \mathbf{5}$ and $\dim_k H^2_{-2} = \mathbf{2}$. The TATE morphism $\mathbf{T}^{-1}(\mathcal{E}_{\mathrm{HM}}) \to \mathbf{T}^0(\mathcal{E}_{\mathrm{HM}})$ is thus determined by a degree 2 map from H^3_{-4} to H^2_{-2}, i.e., by a k-linear map $\bigwedge^2 V^* \otimes H^3_{-4} \to H^2_{-2}$ which can be represented by the matrix



$$d^{-1}_{\mathrm{HM}} = \begin{pmatrix} e_1 \wedge e_4 & e_2 \wedge e_0 & e_3 \wedge e_1 & e_4 \wedge e_2 & e_0 \wedge e_3 \\ e_2 \wedge e_3 & e_3 \wedge e_4 & e_4 \wedge e_0 & e_0 \wedge e_1 & e_1 \wedge e_2 \end{pmatrix} \in E^{2 \times 5}.$$

5. SYSTEM THEORY

Linear systems over a ring of functional operators form an ABELian category $\mathcal{A}$. The derived category $D^b(\mathcal{A})$ has probably many other ABELian subcategories $\mathcal{B} \subset D^b(\mathcal{A})$ with a derived equivalence (or tunnel) $D^b(\mathcal{A}) \rightleftarrows D^b(\mathcal{B})$.

REFERENCES

[BGG78] I. N. Bernšteĭn, I. M. Gel′fand, and S. I. Gel′fand, *Algebraic vector bundles on* $\mathbf{P}^n$ *and problems of linear algebra*, Funktsional. Anal. i Prilozhen. **12** (1978), no. 3, 66–67. MR MR509387 (80c:14010a) 3

[Bon89] A. I. Bondal, *Representations of associative algebras and coherent sheaves*, Izv. Akad. Nauk SSSR Ser. Mat. **53** (1989), no. 1, 25–44. MR 992977 (90i:14017) 2

[EFS03] David Eisenbud, Gunnar Fløystad, and Frank-Olaf Schreyer, *Sheaf cohomology and free resolutions over exterior algebras*, Trans. Amer. Math. Soc. **355** (2003), no. 11, 4397–4426 (electronic). MR MR1990756 (2004f:14031) 3, 4