Computational Methods in Mathematical Systems Theory

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1 Scientific objective and/or technology of the collaboration

Over decades, a lot of effort has been dedicated with success to the modelling, the analysis and the control of linear dynamical systems evoluating with continuous or discrete time. However, over the last thirty years, for practical and theoretical reasons, different new classes of linear systems have been introduced in the literature of mathematical systems theory such as differential time-delay systems, partial differential systems, multidimensional discrete systems, hybrid systems, repetitive systems, convolutional codes... All these classes of systems are characterized by the fact that they are governed by new types of mathematical equations and need new techniques in order to analyze their structural properties and synthesize control laws.

With this growth of new types of linear systems and the specialization of the different fields, it is becoming more and more important to develop general concepts, techniques, algorithms and software which can handle a large number of classes of systems.

Recently, a collaboration has been developed in this challenging direction between the CAFÉ project at INRIA Sophia Antipolis (France) and the Lehrstuhl B für Mathematik, RWTH Aachen (Germany). In particular, in different publications in international journals and conferences, a long term program has been started in order to give some general answers. The main philosophy of this program can be described as follows: using the recent progress and implementations of Gröbner bases for non-commutative polynomial rings of functional operators (e.g., differential operators, time-delay/time-advance operators, shift operators, difference operators) in the effective algebra community and some ideas coming from algebraic analysis (D-module theory, homological algebra), we can now study effectively the structural properties of linear functional systems via some properties of associated modules. Important applications of these results to control theory have recently been shown.

In particular, F. Chyzak (ALGO project, INRIA Roquencourt) has developed two important packages called *Ore_algebra* and *Gröbner* in the well-known computer algebra system Maple (see `http://algo.inria.fr/chyzak/mgfun.html` for more details). The group of W. Plesken (Lehrstuhl B für Mathematik) has also developed and implemented Janet bases in a Maple package called *Janet* (see `http://wwwb.math.rwth-aachen.de/Janet`). These three packages allow us to work effectively in some different rings of functional operators called *Ore algebras*.

The scientific objective is to extend the existing framework of algebraic notions to study linear dynamical systems in the above mentioned generality and to use methods from computer algebra to find new knowledge about linear systems. A rather long-term objective is to find ways of handling non-linear systems.

The continuation and enlargement of the previous collaboration between INRIA Sophia Antipolis and Lehrstuhl B für Mathematik, RWTH Aachen, enables newcomers to enter this domain of research quickly. In particular, through this scientific exchange, the students at

Lehrstuhl B für Mathematik may obtain all the information from experts which they need in order to attain scientific progress.

2 State of collaboration / research

In the previous collaboration, the algebraic framework for the study of linear dynamical systems has been advanced and new symbolic software has been written to turn the new knowledge effective.

Methods from homological algebra are used to characterize structural properties of linear systems. Previous work in this domain of research has been collected in [1]. Moreover, recent progress in theoretical terms as well as for applications is described in the upcoming publications [4], [5].

These algebraic methods have been implemented in a new Maple package *OreModules* [2] which allows to decide controllability and parametrizability for the classes of linear systems which were enumerated in the previous section. Moreover, by means of *OreModules* one can check whether or not a linear multidimensional system is flat and compute effectively its parametrizations. The results can immediately be applied to solve motion planning or tracking problems. Many of its tools (computation of first integrals, construction of Brunovsky canonical forms, tools for linear quadratic optimal control problems) make *OreModules* a valuable software for both mathematicians and engineers. A first description of the package *OreModules* was given in [3].

The direct applicability of *OreModules* is demonstrated by a new library of examples [2]. All procedures of *OreModules* are explained on examples coming from control theory applications. This allows a wider distribution of the package. However, the library of examples is also of great educational value.

3 Complementarity of the teams

The projects CAFÉ and ALGO at INRIA Sophia Antipolis resp. INRIA Rocquencourt are known for their expertise in symbolic computation [7], [6].

D-module theory and effective algebraic analysis is studied by A. Quadrat in the CAFÉ project at INRIA Sophia Antipolis. His knowledge about systems of partial differential equations is fundamental for the future collaboration about computational methods in mathematical systems theory. Moreover, his research in algebraic and geometric methods in control theory provides the link between the part of symbolic computation and the engineering applications.

The research of M. Bronstein on symbolic integration methods is of direct use in the algebraic approach to linear dynamical systems. In the French-German collaboration, his symbolic software, e.g. *Bernina* [8], may advance *OreModules* and new effective methods for linear dynamical systems.

Characteristic sets offer an alternative computational approach to effective module theory. E. Hubert at the CAFÉ project does research on characteristic sets and implements effective methods in the Maple package *diffalg* [9].

The Maple library *Mgfun* [10] has recently been developed by the ALGO project at INRIA Rocquencourt, in particular by F. Chyzak, for the symbolic manipulation of a large class of special functions and combinatorial sequences. In particular, it offers the implementation of Gröbner bases for some classes of Ore algebras. The package *OreModules* is based on *Mgfun*.

In the algebraic study of linear systems, a module over a ring of functional operators is associated to each linear system. A main problem in this domain of research is to find bases of these modules. The members of Lehrstuhl B für Mathematik are engaged with the constructive aspects of this problem. In particular, A. Fabianska works on Quillen-Suslin's theorem, which ensures theoretically that such bases exist under suitable assumptions for linear systems with constant coefficients. Her implementation of a constructive version of Quillen-Suslin's theorem in Maple will be available soon.

Janet bases compete against Gröbner bases for a long time now, and they gain more and more ground. At Lehrstuhl B für Mathematik, implementations of Janet's algorithm [11] in Maple and C++ are developed and applied by D. Robertz. Janet bases are expected to excite new, rather efficient methods in symbolic computation.

Control problems in engineering applications provide great computational problems for symbolic methods. Lehrstuhl B für Mathematik is equipped with rather fast computers (AMD Opteron 64 bit processors, 16 GB memory) which allow to treat new realistic examples using the unifying algebraic approach.

At Lehrstuhl B für Mathematik, Dr. M. Barakat and D. Robertz started to design and implement a new software which incorporates abstract methods from homological algebra into the computer algebra system Maple. The resulting programs will handle modules in a way which is independent of the underlying ring. New impulses are to be expected, when this software is applied in the French-German collaboration.

Recently, new constructive results were also obtained by G. Hartjen in the theory of multidimensional systems [12].

Last but not least, the expertise of Lehrstuhl B für Mathematik in the domain of differential geometry [13] will open new horizons in the study of dynamical systems in this collaboration.

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