Computational Methods in Linear Control Theory

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1 Introduction

Over decades, lots of efforts have been dedicated with success to the *analysis* and *synthesis* of systems evoluating with (continuous or discrete) time. These systems are seen as dynamical systems (linear or non-linear, deterministic or stochastic) and are studied by means of different techniques adapted from differential geometry, functional analysis, algebra and probability.

However, over the last thirty years, from practical and theoretical reasons, different new classes of systems have been introduced such as differential time-delay systems, partial differential equations (PDE) with boundary control and observation, multidimensional systems, hybrid systems, repetitive systems, (multidimensional) convolutional codes... All these classes of systems are characterized by the fact that they are governed by new types of mathematical equations and need new techniques in order to analyze their structural properties and synthesize new control laws.

With this growth of new types of control systems, we are led to generalize certain well known results and techniques so that they can be used for more general classes of systems. However, it happens that sometimes we need to develop new concepts and tools because it is no more possible to extend the previous ones to a more general class of systems. When such a change of viewpoint appears, it usually gives a better understanding to what was previously known.

2 Scientific context of the project

The study of the *algebraic structural properties of linear systems* has clearly required the use of more and more powerful algebraic techniques. Let us illustrate this idea. In the seventies, the wish to study transfer matrices instead of transfer functions has led to the development of the polynomial approach [15] in which canonical forms of matrices of univariated polynomials were studied (e.g. Hermite and Smith forms, invariant factors, left/right/doubly coprimeness, diophantine equations...).

By the middle of the seventies, people became more and more interested in studying the canonical forms and primeness of polynomial matrices with two and three variables (i.e. with entries in $k[x_1, x_2, x_3]$). Indeed, generalizing linear systems defined by ordinary differential equations to differential time-delay systems ($\mathbb{R}[s, z]$), ordinary differential equations with parameters, 2-D and 3-D filters or circuits, systems over a ring ... [4, 5, 18, 35, 38, 39], they had to face the case of systems described by means of matrices with entries in a multivariate polynomial ring. All these new systems were called 2-D or 3-D systems and, more generally, *linear multidimensional systems*. One quickly realized that canonical forms such as Hermite, Smith and Popov forms did not exist anymore for matrices with entries in a polynomial ring with more than one variable.

Therefore, it is not very much surprising that, in the eighties, *Gröbner bases* were introduced in the study of linear multidimensional systems [6]. Let us notice that the – nowadays classical – paper of B. Buchberger on Gröbner bases was first published in the book of N. K. Bose [5] on linear multidimensional systems. Hence, it seems that the control theory community was one of the first mathematical communities to be interested in *symbolic methods* and *computer algebra*. A Gröbner basis reduces a system of polynomials to a certain *normal form* by respect to a certain ordering in the variables x_i [36]. It is an extension of the Gauss elimination algorithm to polynomial rings. However, Gröbner bases are just effective techniques for the computation of certain information on a matrix of polynomials: this theory does not introduce new general concepts for studying systems defined by means of polynomial matrices. Gröbner bases are now available in any symbolic software (e.g. the standard Maple).

In [16], R. E. Kalman has developed a *module theory* approach to 1-D systems, i.e. for linear time-invariant ordinary differential systems. More precisely, using the Laplace transform, he was led to study polynomial and rational matrices in one variable s. The introduction of module theory for multivariate polynomial rings in control theory was still missing. Indeed, module theory classifies the different *intrinsic properties* of linear systems. It gives a methodology for the study of algebraic properties of linear multidimensional systems. In the nineties, this step was done by U. Oberst when he introduced the idea of studying linear multidimensional systems (defined by matrices with entries in a commutative polynomial ring) by means of *algebraic analysis* and Gröbner bases [19]. In particular, he generalized and found intrinsic formulations of the different concepts of primeness previously defined in [4, 5, 38, 39]. Moreover, using algebraic analysis techniques, he was able to develop a perfect duality between his module theory approach to control systems and the behavioural one developed by J. C. Willems and his school for 1-D systems. Since the end of the nineties, a behavioural approach to linear multidimensional systems has been successfully developed in the literature. See [21, 40] for more details and references therein.

Algebraic analysis was developed in the seventies in order to study general systems of partial differential equations (PDE) with constant and variable coefficients using differential module theory and functional analysis [17, 20]. In the same years as [19], J. F. Pommaret was studing underdetermined systems of PDE coming from mathematical physics (e.g. elasticity, electromagnetism, hydrodynamics ...). He showed that this class of systems generalized classical linear multidimensional systems defined over commutative polynomial rings. See [23] for more details and references therein. Moreover, the work of M. Fliess on linear systems defined by ordinary differential equations with variable coefficients [10] also illustrated the need to pass from the polynomial point of view to the differential operators' one. Nowadays, this theory is called *partial differential control theory* [23]. Let us notice that this theory generalizes the theory of multidimensional systems but has not (yet) links with *infinite dimensional systems* [7] in which the control and observation appear on the boundary of the domain (see the last point of the next section).

Finally, *homological algebra* has been introduced in [28] for the study of partial differential control theory. This has led to generalizations of classical results on the analysis and synthesis of linear multidimensional systems. Moreover, using the formal theories of PDE (Spencer theory, Janet-Riquier theory, differential algebra, Gröbner bases for differential operators), the algebraic structural properties of such systems can effectively be checked using different algorithms developed in [28, 29, 30, 31, 32]. Hence, we are now in position to test effectively the algebraic properties of linear multidimensional systems (controllability, observability, flatness, poles and zeros, equivalences ...), compute different parametrizations of the systems (motion planning, tracking, π-polynomials) and propose certain feedback laws (poles placement, diophantine equations, Youla-Kucera parametrization of the controllers, optimal controllers). These different results can be applied at the same time to differential time-delay systems [12, 13, 32, 34], systems over rings [1, 35] and time-varying ordinary differential systems [10].

3 Aim of the project

Using the recent progress of Gröbner bases for differential operators, we are now in position to develop symbolic packages for the study of general linear multidimensional systems and partial control systems. Certain of the previous algorithms have recently been implemented by F. Chyzak and A. Quadrat using the library *Mgfun* of Maple (developed by the project ALGO at INRIA Rocquencourt). For instance, we are now able to check whether or not a linear multidimensional system is *flat* [11] and compute effectively its parametrizations (motion planning, tracking, π-polynomial).

Recently, J. F. Pommaret and A. Quadrat have visited my home institute at RWTH Aachen (Germany) during summer 2002. They lectured on linear multidimensional control systems and partial differential control theory. As I have improved an existing implementation of *Janet's algorithm* [14, 2, 3] in my diploma studies, I have also started to develop new routines computing the parametrizations of underdetermined systems of partial differential equations. Let us notice that a Janet basis gives more information on a linear system of PDE than a Gröbner basis. Moreover, our experience shows that C++ implementations in this area are faster than Maple implementations by a factor 1000.

This is why I believe that the development of a package using Janet bases in C++ could supersede what F. Chyzak and A. Quadrat have already done in Maple. A part of the project is to develop such a package under the supervision of A. Quadrat (CAFE project, INRIA Sophia Antipolis), using the expertise of the CAFE project on efficient symbolic computations (M. Bronstein). Such a package will allow anybody, unfamiliar with algebraic analysis and Gröbner/Janet bases, to check certain structural properties of linear multidimensional systems and to obtain certain control laws.

Secondly, it seems that we can easily extend partial differential control theory [23] to larger classes of systems including repetitive systems, codes, time-varying differential time-delay systems, finite difference or q-difference systems ... Indeed, recent work of F. Chyzak has shown that Gröbner bases still exist over a large class of non-commutative rings called *Ore algebra*. In particular, this class includes ordinary differential equations (polynomial or rational coefficients), partial differential equations (polynomial or rational coefficients), difference systems, q-difference systems ... Simple rules of commutation are needed to define an Ore algebra which allows us to mix different functional operators (partial differential operators, delay operators, difference or q-difference operators, delay systems over a finite field...). We believe that the results of [28] can easily be extended to this framework and be applied to certain classes of hybrid systems. Finally, the development of Janet bases for Ore algebras also seems to be an interesting new idea.

Thirdly, I want to understand which of the previous results can be extended to non-linear systems. I hope to learn some deeper differential algebra at INRIA Sophia Antipolis to treat some classes of non-linear systems. At a first step, I will learn the content of the papers [8, 9] on effective techniques for non-linear control systems defined by ordinary differential equations. Using the package *diffalg* of Maple developed by E. Hubert (CAFE project, INRIA Sophia Antipolis), the different algorithms of [8, 9] can nowadays easily and efficiently be developed on Maple 8. Using the fact that the classical Janet algorithm can be carried over to quasi-linear systems, it would be interesting to see if Janet bases can used instead of *characteristic sets* [8, 9].

To finish, the problem of parametrizing a non-linear system of ordinary/partial differential equations is called the *Monge problem*. A common group of the projects MIAOU (J.B. Pomet, L. Baratchard) and CAFE has been working on this problem. Hence, I could learn the state of art on this question attending this group.

Finally, I also want to use the opportunity to be at INRIA Sophia Antipolis to learn more on the module approach to linear infinite dimensional systems [7, 37] as it is developed in [30, 31, 32, 33] for robust stabilization problems. I also want to study the connections (recently developed by J. Tervo and A. Quadrat) between linear multidimensional systems and infinite dimensional systems using *Boutel de Monvel algebras* and *pseudo-differential operators*.

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