On the Monge problem for multidimensional linear systems

This talk is dedicated to Manuel and his family

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Introduction

Parametrizability problem

• Let D be a **ring of functional operators** and $\mathcal{F}$ a **functional space** which satisfies:

$$\forall\, P_1,\, P_2 \in D,\ \forall\, y_1,\, y_2 \in \mathcal{F}:\ \ P_1\, y_1 + P_2\, y_2 \in \mathcal{F}.$$

Let us consider $R \in D^{q \times p}$ and:

$$\ker_{\mathcal{F}}(R.) \triangleq \{\eta \in \mathcal{F}^p \,|\, R\, \eta = 0\}.$$

• **Question**: When does $Q \in D^{p \times m}$ exist s.t.:

$$\ker_{\mathcal{F}}(R.) = \mathrm{im}_{\mathcal{F}}(Q.) \triangleq Q\, \mathcal{F}^m?$$

$\Rightarrow Q$ is called a **parametrization** of $\ker_{\mathcal{F}}(R.)$.

• **Example**: $D = \mathbb{R}(t)\left[\frac{d}{dt}\right],\ \mathcal{F} = C^\infty(\mathbb{R})$,

$$R = \left(\frac{d^2}{dt^2} + \alpha(t)\,\frac{d}{dt} + 1,\ -\frac{d}{dt} - \alpha(t)\right) \in D^{1 \times 2}.$$

$$\ddot{y}(t) + \alpha(t)\,\dot{y}(t) + y(t) - \dot{u}(t) - \alpha(t)\,u(t) = 0$$

$$\Leftrightarrow \begin{cases} y(t) = \dot{\xi}(t) + \alpha(t)\,\xi(t), \\ u(t) = \ddot{\xi}(t) + \alpha(t)\,\dot{\xi}(t) + (\dot{\alpha}(t) + 1)\,\xi(t). \end{cases}$$

• **Example**: $D = \mathbb{R}\left[\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right], \mathcal{F} = C^\infty(\mathbb{R}^3)$,

$$\begin{array}{rcl} \mathrm{div}\vec{A} = 0 & \Leftrightarrow & \exists\, \vec{B} \in \mathcal{F}^3:\ \vec{A} = \mathrm{curl}\, \vec{B}, \\ \mathrm{curl}\, \vec{B} = \vec{0} & \Leftrightarrow & \exists\, f \in \mathcal{F}:\ \vec{B} = \mathrm{grad}\, f. \end{array}$$

State of art

• **Theorem**: If $\mathcal{F}$ is an **injective cogenerator** left D-module, then we have the equivalences:

1. $\mathcal{B} = \ker_{\mathcal{F}}(R.)$ is **parametrizable**.

2. $\exists\, Q \in D^{p \times m}: \; \ker_{\mathcal{F}}(R.) = Q\,\mathcal{F}^m$.

3. $\exists\, Q \in D^{p \times m}: \; \ker_D(.Q) = D^{1 \times p}\, R$,

   where $\ker_D(.Q) = \{\lambda \in D^{1 \times p} \,|\, \lambda\, Q = 0\}$.

4. The left D-module $M = D^{1 \times p}/(D^{1 \times q}\, R)$ is **torsion-free**, namely:

$$t(M) = \{m \in M \,|\, \exists\, 0 \neq P \in D : \; P\, m = 0\} = 0.$$

(Pillai-Shankar/Pommaret-Quadrat 98)

• **Example**: $D = \mathbb{R}\left[\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right], \mathcal{F} = C^{\infty}(\mathbb{R}^n)$.

• **Effective algorithms** are available in the **Maple package** OREMODULES for computing $t(M)$ and Q.

`http://wwwb.math.rwth-aachen.de/OreModules.`

Autonomous elements

• **Definition**: If M is a left D-module, then **a torsion element of** M is any element belonging to:

$$t(M) = \{m \in M \,|\, \exists\, 0 \neq P \in D : \; P\,m = 0\}.$$

• **Definition**: **An autonomous element** of the behaviour $\mathcal{B} = \ker_{\mathcal{F}}(R.)$ is an element of the form

$$\Psi = \sum_{i=1}^{p} P_i\,\eta_i, \quad \eta = (\eta_1, \ldots, \eta_p)^T \in \mathcal{B}, \quad P_i \in D,$$

which satisfies $\exists\, 0 \neq P \in D : \; P\,\Psi = 0$.

• **Example**: Non-controllable element:

$$\left\{ \begin{array}{l} \dot{x}_1 = x_2 + u, \\ \dot{x}_2 = x_1 + u, \end{array} \right. \Rightarrow \left\{ \begin{array}{l} z = x_1 - x_2, \\ \left(\frac{d}{dt} + 1\right) z = 0. \end{array} \right.$$

• **Theorem**: If $\mathcal{F}$ is an injective cogenerator left D-module, $R \in D^{q \times p}$, $M = D^{1 \times p}/(D^{1 \times q}\, R)$ and $\mathcal{B} = \ker_{\mathcal{F}}(R.)$, then we have:

autonomous elts of $\mathcal{B} \stackrel{1-1}{\longleftrightarrow}$ **torsion elts** of M.

• **Corollary**: $\mathcal{B}$ is **parametrizable iff** $\mathcal{B}$ **has no autonomous elements**.

• What about systems with autonomous elements?

Monge problem (1784)

• **Question**: Monge asked when it was possible to parametrize all solutions of a 1-D underdetermined system by means of arbitrary functions and constants.

• **Monge problem** was popularized by Hadamard in:

J. Hadamard, "Sur l'équilibre des plaques élastiques circulaires libres ou appuyées et celui de la sphère isotrope", Annales scientifiques de l'ENS, 18 (1901), 313-342.

$\Rightarrow$ Hadamard considered the linear case (1901).

$\Rightarrow$ Hilbert, Cartan, Zervos studied the non-linear case.

• **Extension of Monge problem for partial differential equations** by Goursat (1930).

The parametrizations can depend on arbitrary functions of all the independent variables $x_1, \ldots, x_n$, on functions of $n-1$ independent variables, ..., on functions of 1 independent variable x_i and constants.

"J'espère [que mes résultats] pourront contribuer à appeler l'attention de quelques jeunes mathématiciens sur un sujet difficile et bien peu étudié." E. Goursat.

Results

Solution for 1-D linear systems

• Let us consider $D = \mathbb{R}\left[\frac{d}{dt}\right]$ or $D = \mathbb{R}(t)\left[\frac{d}{dt}\right]$ and a **full row rank matrix** $R \in D^{q \times p}$ ($1 \leq q \leq p$).

1. We compute a **Smith/Jacobson form** of R,

$$R = U\,(\mathrm{diag}(d_1, \ldots, d_q),\ 0)\,V,$$

where $U \in D^{q \times q}$ and $V \in D^{p \times p}$ are **unimodular**.

2. We define the matrices:

$$\begin{cases} R'' = U\,\mathrm{diag}(d_1, \ldots, d_q), \\ R' = (I_q \quad 0)\,V, \end{cases} \quad \Rightarrow R = R''\,R'.$$

3. $V = (V_1^T \quad V_2^T)^T, \quad V_1 \in D^{q \times p} \quad \Rightarrow V_1 = R'$.

4. If we denote by $V^{-1} = (S \quad Q) \in D^{p \times p}$, then:

$$\begin{pmatrix} R' \\ V_2 \end{pmatrix} (S \quad Q) = I_p, \quad (S \quad Q) \begin{pmatrix} R' \\ V_2 \end{pmatrix} = I_p.$$

Solution for 1-D linear systems

5. $R\,\eta = 0 \Leftrightarrow R''\,(R'\,\eta) = 0 \Leftrightarrow \left\{ \begin{array}{l} R'\,\eta = \tau, \\ R''\,\tau = 0. \end{array} \right.$

6. $R''\,\tau = 0 \Leftrightarrow d_1\,\tau_1 = 0, \ldots, d_q\,\tau_q = 0$ $(\star)$.

Let $\overline{\tau} \in \mathcal{F}^q$ be a **fundamental solution** of $(\star)$.

7. Integration of the **inhomogenous system** $R'\,\eta = \overline{\tau}$:

a. **General solution** of $R'\,\eta = 0$.

$$R'\,\eta = 0 \Leftrightarrow \eta = Q\,\xi, \quad \forall\,\xi \in \mathcal{F}^{p-q}.$$

b. **Particular solution**: $\overline{\eta} = S\,\overline{\tau}$ $((R'\,S)\,\overline{\tau} = \overline{\tau})$.

$$\boxed{R\,\eta = 0 \;\Leftrightarrow\; \eta = S\,\overline{\tau} + Q\,\xi, \;\forall\,\xi \in \mathcal{F}^{p-q}.}$$

Hadamard's example

• We consider the system ($\lambda,\ \mu \in \mathbb{R}$)

$$\begin{cases} \rho\,\frac{\partial\theta}{\partial\rho} + \frac{1}{2}\,(\theta + K) + \frac{\lambda+\mu}{2}\,\left(\rho\,\frac{\partial\sigma}{\partial\rho} - \sigma\right) = 0, \\ 2\,\rho\,\frac{\partial\theta}{\partial\rho} + \rho\,\frac{\partial K}{\partial\rho} + 3\,K - (3\,\lambda + 2\,\mu)\,\sigma = 0, \\ \lambda\,\sigma + 2\,\mu\,\left(G + \rho\,\frac{\partial G}{\partial\rho}\right) - \rho\,\frac{\partial\theta}{\partial\rho} - K = 0, \end{cases} \qquad (\star)$$

where σ, G, θ, K are functions of $\rho = \sqrt{x^2 + y^2 + z^2}$.

• We consider the **eulerian operator** $d = \rho\,\frac{\partial}{\partial\rho}$, the commutative ring $D = \mathbb{R}(\lambda,\mu)[d]$ and:

$$R = \begin{pmatrix} d + \frac{1}{2} & \frac{(\lambda+\mu)}{2}\,(d-1) & \frac{1}{2} & 0 \\ 2\,d & -3\,\lambda - 2\,\mu & d + 3 & 0 \\ -d & \lambda & -1 & 2\,\mu\,(d+1) \end{pmatrix}.$$

• Computing a **Smith form** of R, we then obtain:

$$R = U\,\mathrm{Diag}\,V, \quad \mathrm{Diag} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & d+1 & 0 \end{pmatrix}.$$

• We need to solve the system in $\mathcal{F} = C^{\infty}(\mathbb{R}_+)$:

$$\tau_1 = 0, \quad \tau_2 = 0, \quad (d+1)\,\tau_3 = \rho\,\frac{\partial\tau_3}{\partial\rho} + \tau_3 = 0.$$

$$\boxed{\Rightarrow \overline{\tau_1} = 0, \quad \overline{\tau_2} = 0, \quad \overline{\tau_3} = C/\rho,\ C \in \mathbb{R}.}$$

Hadamard's example

- We obtain that $V^{-1} = (S \quad Q)$ is defined by:

$$S = \left(\begin{array}{ccc} 0 & 0 & \frac{2\,(\lambda+\mu)\,(3\,\mu\,d+\lambda\,d+5\,\mu+2\,\lambda)}{\mu\,(2\,\mu+\lambda)\,(7\,\mu+3\,\lambda)} \\ 0 & 1 & -\frac{4\,(d+1)\,(3\,\mu+\lambda)}{\mu\,(2\,\mu+\lambda)\,(7\,\mu+3\,\lambda)} \\ 1 & -(\lambda+\mu)\,(d-1) & -\frac{2\,(\lambda+\mu)\,(13\,\mu\,d+5\,\lambda\,d+4\,\lambda+11\,\mu)}{\mu\,(2\,\mu+\lambda)\,(7\,\mu+3\,\lambda)} \\ 0 & 0 & \frac{3\,\mu^2\,d+4\,\mu\,\lambda\,d+\lambda^2\,d+\lambda^2-4\,\mu^2+\lambda\,\mu}{\mu^2\,(7\,\mu+3\,\lambda)\,(2\,\mu+\lambda)} \end{array}\right)$$

$$Q = \left(\begin{array}{c} \mu\,(-(\lambda+\mu)\,d^2-2\,(\lambda+\mu)\,d+\mu) \\ \mu\,(2\,d+3)\,(d+1) \\ \mu\,(2\,(\lambda+\mu)\,d^2+2\,(2\,\lambda+\mu)\,d+3\,\lambda+2\,\mu) \\ -(d+1)\,((\lambda+\mu)\,d-2\,\mu)/2 \end{array}\right).$$

- **All $\mathcal{F}$-solutions of $R\,\eta = 0$ are of the form**

$$\left(\begin{array}{c} \theta(\rho) \\ \sigma(\rho) \\ K(\rho) \\ G(\rho) \end{array}\right) = \left(\begin{array}{c} \frac{2\,(\lambda+\mu)\,C}{\mu\,(3\,\lambda+7\,\mu)\,\rho} \\ 0 \\ \frac{2\,(\lambda+\mu)\,C}{\mu\,(3\,\lambda+7\,\mu)\,\rho} \\ -\frac{C}{\mu\,(\lambda+2\,\mu)\,\rho} \end{array}\right) + Q\left(\rho\frac{\partial}{\partial\rho}\right)\,\xi(\rho),$$

where ξ is **any function** of $\mathcal{F}$ and C is any constant.

- See Hadamard for **applications to elasticity**.

Module theory

Ore algebras

- **Definition**: A non-commutative polynomial ring $D = A[\partial; \sigma, \delta]$ in ∂ is called **skew** if

$$\boxed{\partial\, a = \sigma(a)\, \partial + \delta(a), \quad a \in A,}$$

where $\sigma : A \rightarrow A$ satisfies $\forall\, a, b \in A$:

$$\begin{cases} \sigma(1) = 1, \\ \sigma(a + b) = \sigma(a) + \sigma(b), \\ \sigma(a\, b) = \sigma(a)\, \sigma(b), \end{cases}$$

and $\delta : A \rightarrow A$ is such that $\forall\, a, b \in A$:

$$\begin{cases} \delta(a + b) = \delta(a) + \delta(b), \\ \delta(a\, b) = \sigma(a)\, \delta(b) + \delta(a)\, b. \end{cases}$$

- $P = \sum_{i=0}^{r} a_i\, \partial^i \in D, \quad a_i \in A$.

- **Definition**: (Chyzak-Salvy): The skew ring

$$D = k[x_1, \ldots, x_n][\partial_1; \sigma_1, \delta_1] \ldots [\partial_m; \sigma_m, \delta_m]$$

is called an **Ore algebra** if :

$$\boxed{\begin{cases} \sigma_i\, \delta_j = \delta_j\, \sigma_i, \quad 1 \leq i, j \leq m, \\ \sigma_i(\partial_j) = \partial_j, \quad \delta_i(\partial_j) = 0, \quad j < i. \end{cases}}$$

$\Rightarrow D$ is a **non-commutative ring**.

Examples of Ore algebras

- **Ordinary differential operators**:

$$D = A\left[\tfrac{d}{dt}; 1, \tfrac{d}{dt}\right], \quad A = k[t], k(t), \ldots$$

$$P = \textstyle\sum_{i=0}^{m} a_i(t)\, \tfrac{d^i}{dt^i} \in D, \quad \tfrac{d}{dt}\, a(t) = \dot{a}(t).$$

- **Time-delay (time-advance) operators**:

$$D = A[\delta_h; \sigma_h, 0], \quad A = k[t], k(t), \ldots$$

$$P = \textstyle\sum_{i=0}^{m} a_i(t)\, \delta_h^i \in D, \quad \sigma_h\, a(t) = a(t-h).$$

- **Shift operators**:

$$D = A[\delta; \sigma, 0], \quad A = k[n], k(n), \ldots$$

$$P = \textstyle\sum_{i=0}^{m} a_i(n)\, \delta^i \in D, \quad \sigma\, a(n) = a(n+1).$$

- **Differential time-delay operators**:

$$D = A\left[\tfrac{d}{dt}; 1, \tfrac{d}{dt}\right][\delta_h; \sigma_h, 0], \quad A = k[t], k(t), \ldots$$

$$P = \textstyle\sum_{0 \le i+j \le m}^{n} a_{ij}(t)\, \tfrac{d^i}{dt^i}\, \delta_h^j \in D.$$

- **Partial differential operators**:

$$D = A[d_1; 1, \partial_1] \ldots [d_n; 1, \partial_n], \; A = k[x_1, \ldots, x_n],$$

$$P = \textstyle\sum_{0 \le |\mu| \le m} a_\mu(x)\, d^\mu, \; d^\mu = d_1^{\mu_1} \ldots d_n^{\mu_n}, \; \partial_i = \tfrac{\partial}{\partial x_i}.$$

A few properties

• **Proposition**: If A is a **left noetherian ring** and σ is an automorphism, then the skew polynomial ring $A[\partial; \sigma, \delta]$ is a **left noetherian ring**.

• **Proposition**: If D is a **left noetherian ring**, then D has the **left Ore property**, namely,

$$\forall\, (P_1,\, P_2) \in D^2,\; \exists\, (0,\, 0) \neq (Q_1,\, Q_2) \in D^2 :$$

$$\boxed{Q_1\, P_1 = Q_2\, P_2.}$$

• **Theorem**: (Kredel): Let D be an Ore algebra s.t.

$$\sigma_i(x_j) = a_{ij}\, x_j + b_{ij}, \quad \delta_i(x_j) = c_{ij},$$

$$0 \neq a_{ij},\, b_{ij} \in k,\; c_{ij} \in k[x_1, \ldots, x_n],\; \deg(c_{ij}) \leq 1,$$

then, **Gröbner bases** w.r.t. any term order can be computed algorithmically.

$\Rightarrow$ **Maple Ore_algebra** (Chyzak, ALGO, INRIA).

`http://algo.inria.fr/libraries/`

Systems-Modules

- Let the system defined by

$$\ker_{\mathcal{F}}(R.) = \{\eta \in \mathcal{F}^p \mid R\,\eta = 0\}, \quad R \in D^{q\times p},$$

where D is an **Ore algebra** and $\mathcal{F}$ a **left** D**-module**.

As in **algebraic geometry**, we associate with $\ker_{\mathcal{F}}(R.)$ the **left** D-module:

$$\boxed{M = D^{1\times p}/(D^{1\times q}\, R)}$$

- **Example**: The wind tunnel model (Manitius 84):

$$\begin{cases} \dot{x}_1(t) = -a\,x_1(t) + k\,a\,x_2(t-h), \\ \dot{x}_2(t) = x_3(t), \\ \dot{x}_3(t) = -\omega^2\,x_2(t) - 2\,\zeta\,\omega\,x_3(t) + \omega^2\,u(t). \end{cases} \qquad (\star)$$

- The system $(\star)$ is equivalent to:

$$\underbrace{\begin{pmatrix} \frac{d}{dt} + a & -k\,a\,\delta_h & 0 & 0 \\ 0 & \frac{d}{dt} & -1 & 0 \\ 0 & \omega^2 & \frac{d}{dt} + 2\,\zeta\,\omega & -\omega^2 \end{pmatrix}}_{R \in D^{3\times 4}} \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ u(t) \end{pmatrix} = 0$$

$$D = \mathbb{R}(a, k, \omega, \zeta)\left[\frac{d}{dt}, \delta_h\right], \quad \boxed{M = D^{1\times 4}/(D^{1\times 3}\, R)}.$$

Classification of modules

• **Definition**:

a. M is **free** if $\exists\, r \in \mathbb{Z}_+ :\ M \cong D^r$.

b. M is **projective** if $\exists\, r \in \mathbb{Z}_+$ and a D-module P:

$$M \oplus P \cong D^r.$$

c. M is **reflexive** if ϵ is an isomorphism:

$$\begin{array}{rcl} \epsilon: \ M & \longrightarrow & M^{\star\star}, \\ m & \longmapsto & \epsilon(m), \qquad \epsilon(m)(f) = f(m). \end{array}$$

d. M is **torsion-free** if:

$$t(M) = \{m \in M \,|\, \exists\, 0 \neq P \in D : P\, m = 0\} = 0.$$

$m \in t(M)$ is called a **torsion element** of M.

e. M is **torsion** if $M = t(M)$.

• **Theorem**:

1. **free** $\Rightarrow$ **projective** $\Rightarrow$ .. $\Rightarrow$ **reflexive** $\Rightarrow$ **torsion-free**.

2. If D **is a principal domain** (e.g., $K\left[\frac{d}{dt}\right]$), then:

torsion-free = free.

3. If $D = k[x_1, \ldots, x_n]$, where k is a field:

projective = free (Th. Quillen-Suslin).

Homological algebra

- $R \in D^{q \times p}, \quad M = D^{1 \times p}/(D^{1 \times q}\, R), \quad \widetilde{N} = D^{1 \times q}/(D^{1 \times p}\, \theta(R)).$

Module M	**Homological algebra**	$d(\widetilde{N})^{\star}$
with torsion	$t(M) \cong \mathrm{ext}_D^1(\widetilde{N}, D) \neq 0$	$n-1$
torsion-free	$\mathrm{ext}_D^1(\widetilde{N}, D) = 0$	$n-2$
reflexive	$\mathrm{ext}_D^i(\widetilde{N}, D) = 0,$	$n-3$
...	...	...
projective	$\mathrm{ext}_D^i(\widetilde{N}, D) = 0,$ $1 \leq i \leq n$	-1
free	$\emptyset$ (Quillen-Suslin th.**)	

$\star$ R has full row rank, i.e., left D-linearly independent rows.
$d(\widetilde{N})$ is the Krull dimension of the characteristic variety of $\widetilde{N}$.

$\star\star$ D is a commutative polynomial ring over the field $\mathbb{C}$.

Involution

• **Definition**: An **involution** of an Ore algebra D is a k-linear map $\theta : D \to D$ satisfying:

1. $\theta(a_1\, a_2) = \theta(a_2)\, \theta(a_1), \quad a_1, a_2 \in D,$

2. $\theta^2 = id_D$.

• If $R \in D^{q\times p}$, then we have:

$$\boxed{\theta(R) \triangleq (\theta(R_{ij}))^T \in D^{p\times q}}.$$

• **Example**: 1. If $D = k[x_1, \ldots, x_n]$, then $\theta = id_D$.

2. If $D = k[t]\left[\frac{d}{dt},\, \delta_h,\, \delta_{-h}\right]$, then an involution of D is defined by:

$$t \mapsto t, \quad \frac{d}{dt} \mapsto -\frac{d}{dt}, \quad \delta_h \mapsto \delta_{-h}, \quad \delta_{-h} \mapsto \delta_h.$$

Let $R = (t\,\frac{d}{dt} \quad - t^2\,\delta_h) \in D^{1\times 2}$, then we have:

$$\theta(R) = \left(\begin{array}{c} -\frac{d}{dt}\, t \\ -\delta_{-h}\, t^2 \end{array}\right) = \left(\begin{array}{c} -t\,\frac{d}{dt} + 1 \\ -(t+h)^2\,\delta_{-h} \end{array}\right).$$

• **Right** D-module $N \overset{\theta}{\Longleftrightarrow}$ **Left** D-module $\widetilde{N}$:

$$\forall\, P \in D,\ \forall\, n \in \widetilde{N}:\ P \circ n = n\, \theta(P).$$

Free resolutions

• **Definition**: A sequence of D-morphisms

$$M' \xrightarrow{f} M \xrightarrow{g} M''$$

is **a complex** at M if $\operatorname{im} f \subseteq \ker g$.

The **defect of exactness at** M is defined by:

$$H(M) = \ker g / \operatorname{im} f.$$

A complex is **exact at** M if $\operatorname{im} f = \ker g$.

• **Definition**: A **free resolution** of a left D-module M is an exact sequence of the form

$$\ldots \xrightarrow{.R_3} D^{1\times l_2} \xrightarrow{.R_2} D^{1\times l_1} \xrightarrow{.R_1} D^{1\times l_0} \xrightarrow{\pi} M \longrightarrow 0,$$

where $R_i \in D^{l_i \times l_{i-1}}$ and:

$$\begin{array}{ccc} D^{1\times l_i} & \xrightarrow{.R_i} & D^{1\times l_{i-1}} \\ (P_1 \ \ldots \ P_{l_i}) & \longmapsto & (P_1 \ \ldots \ P_{l_i})\, R_i. \end{array}$$

$\ker_D(.R_i)$ is called the i^{th} **syzygy module** of M.

• **Algorithm**: Find **a basis of the compatibility conditions** of $R_i\, y = u$ by eliminating of y:

$$\forall\, P \in \ker_D(.R_i), \quad P\,(R_i\, y) = P\, u \Rightarrow P\, u = 0.$$

Example

- Let $R = (d_1\, d_2 \quad d_1^2)^T$ and the $D = \mathbb{R}[d_1, d_2]$-module $M = D/(D^{1\times 2}\, R)$ defined by:

$$\begin{cases} d_1\, d_2\, y = 0, \\ d_1^2\, y = 0. \end{cases}$$

- We have the following **exact sequence**:

$$0 \longrightarrow \ker_D(.R) \longrightarrow D^{1\times 2} \xrightarrow{.R} D \xrightarrow{\pi} M \longrightarrow 0.$$

- We have the following **equality** (D is a **GCDD**):

$$\begin{aligned} &\ker_D(.R) \\ &= \{(P_1 \quad P_2) \in D^{1\times 2} \mid P_1\, d_1\, d_2 = -P_2\, d_1^2\} \\ &= \{(P_1 \quad P_2) \in D^{1\times 2} \mid P_1\, d_2 = -P_2\, d_1\} \\ &= \{(Q\, d_1 \quad -Q\, d_2) \mid Q \in D\} \\ &= D\, R_2, \quad \text{where } R_2 = (d_1 \quad -d_2), \end{aligned}$$

$\Rightarrow$ **we have the following free resolution of** M:

$$0 \longrightarrow D \xrightarrow{.R_2} D^{1\times 2} \xrightarrow{.R} D \xrightarrow{\pi} M \longrightarrow 0.$$

Extension functor

- Let $\mathcal{F}$ be a **left D-module**.

- **Definition**: Let M be a left D-module and

$$\ldots \xrightarrow{.R_3} D^{1\times l_2} \xrightarrow{.R_2} D^{1\times l_1} \xrightarrow{.R_1} D^{1\times l_0} \xrightarrow{\pi} M \longrightarrow 0$$

a free resolution of M.

- We define the **reduced free resolution** of M:

$$\ldots \xrightarrow{.R_3} D^{1\times l_2} \xrightarrow{.R_2} D^{1\times l_1} \xrightarrow{.R_1} D^{1\times l_0} \longrightarrow 0 \quad (\star).$$

- Applying $\hom_D(\cdot, \mathcal{F})$ to $(\star)$, we obtain:

$$\ldots \xleftarrow{R_3.} \mathcal{F}^{l_2} \xleftarrow{R_2.} \mathcal{F}^{l_1} \xleftarrow{R_1.} \mathcal{F}^{l_0} \longleftarrow 0 \quad (\star\star)$$

$$\text{where} \quad \begin{array}{ccc} \mathcal{F}^{l_i} & \xleftarrow{R_i.} & \mathcal{F}^{l_{i-1}} \\ R_i\,\eta & \longleftarrow\!\shortmid & \eta. \end{array}$$

- We denote **the defects of exactness** of $(\star\star)$ by:

$$\boxed{\begin{cases} \mathrm{ext}_D^0(M,\mathcal{F}) = \hom_D(M,\mathcal{F}) = \ker_{\mathcal{F}}(R_1.), \\ \mathrm{ext}_D^i(M,\mathcal{F}) = \ker_{\mathcal{F}}(R_{i+1}.)\,/\,\mathrm{im}_{\mathcal{F}}(R_i.), \;\; i \geq 1. \end{cases}}$$

- **Theorem**: The abelian group $\mathrm{ext}_D^i(M,\mathcal{F})$ **only depends on** M and $\mathcal{F}$ and **not on the resolution** $(\star)$.

Example

- Let $R = (d_1\, d_2 \quad d_1^2)^T$ and the $D = \mathbb{R}[d_1, d_2]$-module $M = D/(D^{1\times 2}\, R)$ defined by:

$$d_1\, d_2\, y = 0, \;\; d_1^2\, y = 0.$$

- We have the following **free resolution** for M:

$$0 \longrightarrow D \xrightarrow{.R_2} D^{1\times 2} \xrightarrow{.R} D \xrightarrow{\pi} M \longrightarrow 0 \quad (\star),$$

where $R = (d_1\, d_2 \quad d_1^2)^T$ and $R_2 = (d_1 \quad -d_2)$.

- The **transposed of the reduced resolution** is:

$$0 \longleftarrow D \xleftarrow{R_2.} D^2 \xleftarrow{R.} D \longleftarrow 0.$$

- We have the following **defects of exactness**:

$$\begin{cases} \mathrm{ext}^1_D(M, D) = \ker_D(R_2.) \,/\, (R\, D), \\ \mathrm{ext}^2_D(M, D) = D/(R_2\, D^2). \end{cases}$$

- $$\begin{aligned} \ker_D(R_2.) &= \{(P_1 \quad P_2)^T \in D^2 \mid d_1\, P_1 = d_2\, P_2\} \\ &= \{(d_2\, P \quad d_1\, P)^T \mid P \in D\} \\ &= (d_2 \quad d_1)^T\, D. \end{aligned}$$

$$\mathrm{ext}^1_D(M, D) = ((d_2 \quad d_1)^T\, D)/((d_1\, d_2 \quad d_1^2)^T\, D) \neq 0$$

$$z = \kappa((d_2 \quad d_1)), \quad d_1\, z = \kappa((d_1\, d_2 \quad d_1^2)) = 0.$$

- $1 \notin I = (d_1, d_2) \Rightarrow \mathrm{ext}^2_D(M, D) = D/I \neq 0.$

Classification

• **Theorem**: Let $M = D^{1\times p}/(D^{1\times q}\, R)$ and $\widetilde{N} = D^{1\times q}/(D^{1\times p}\, \widetilde{R})$ be the left D-modules defined by the finite presentations:

$$\begin{array}{ccccccc} & & D^{1\times q} & \xrightarrow{.R} & D^{1\times p} & \longrightarrow M \longrightarrow 0, \\ 0 \longleftarrow \widetilde{N} \longleftarrow & & D^{1\times q} & \xleftarrow{.\widetilde{R}} & D^{1\times p}, & \end{array}$$

where $\widetilde{R} = (\theta(R_{ij}))^T$ is the **adjoint** of R, then:

1. **torsion submodule** $t(M) \cong \mathrm{ext}^1_D(\widetilde{N}, D)$.

2. M is **torsion-free** $\Leftrightarrow \mathrm{ext}^1_D(\widetilde{N}, D) = 0$,

3. M is **reflexive** $\Leftrightarrow \mathrm{ext}^i_D(\widetilde{N}, D) = 0, \; i = 1, 2$,

4. M is **projective** $\Leftrightarrow \mathrm{ext}^i_D(\widetilde{N}, D) = 0, \; i = 1, \ldots, n$.

• **Theorem**: We have the exact sequence

$$D^{1\times q} \xrightarrow{.R} D^{1\times p} \xrightarrow{.R_{-1}} D^{1\times l_{-1}} \xrightarrow{.R_{-2}} \ldots \xrightarrow{.R_{-r+1}} D^{1\times l_{-r}}$$

$$\Longleftrightarrow \quad \mathrm{ext}^i_D(\widetilde{N}, D) = 0, \quad i = 1, \ldots, r.$$

Algorithm

- We follow steps 1, 2, 3 and 4:

$$\begin{array}{ccc} 4.\quad \theta(Q)\,\xi = \eta & \Longrightarrow & R\,\eta = 0 \quad 1. \\ \Uparrow & & \Downarrow \\ \text{involution } \theta & & \text{involution } \theta \\ \Uparrow & & \Downarrow \\ 3.\quad 0 = Q\,\mu & \overset{\text{G.B.}}{\Longleftrightarrow} & \theta(R)\,\lambda = \mu \quad 2. \end{array}$$

$$\begin{aligned} Q \circ \theta(R) = 0 \Rightarrow \theta(Q \circ \theta(R)) &= \theta^2(R) \circ \theta(Q) \\ &= R \circ \theta(Q) = 0. \end{aligned}$$

- **The last step**:

$$\theta(Q)\,\xi = \eta \overset{\text{G.B.}}{\Longleftrightarrow} R'\,\eta = 0$$

$$\boxed{\text{ext}^1_D(\widetilde{N}, D) = (D^{1\times q'}\,R')/(D^{1\times q}\,R)} \qquad \text{(G.B.)}$$

where $\widetilde{N} = D^{1\times q}/(D^{1\times p}\,\theta(R)) \Longleftrightarrow \theta(R)\,\lambda = 0.$

The wind tunnel model

• **Is the wind tunnel model parametrizable?**

$$\begin{cases} \dot{x}_1(t) + a\,x_1(t) - k\,a\,x_2(t-h) = 0, \\ \dot{x}_2(t) - x_3(t) = 0, \\ \dot{x}_3(t) + \omega^2\,x_2(t) + 2\,\zeta\,\omega\,x_3(t) - \omega^2\,u(t) = 0. \end{cases} \quad (1)$$

$$\begin{cases} \left(\frac{d}{dt} + a\right)\,\lambda_1 = \mu_1, \\ -k\,a\,\delta_h\,\lambda_1 + \frac{d}{dt}\,\lambda_2 + \omega^2\,\lambda_3 = \mu_2, \\ -\lambda_2 + \left(\frac{d}{dt} + 2\,\zeta\,\omega\right)\,\lambda_3 = \mu_3, \\ -\omega^2\,\lambda_3 = \mu_4. \end{cases} \quad (2)$$

$$\omega^2\,k\,a\,\delta_h\,\mu_1 + \left(\omega^2\,\frac{d}{dt} - \omega^2\,a\right)\,\mu_2 + \left(\omega^2\,\frac{d^2}{dt^2} + \omega^2\,a\,\frac{d}{dt}\right)\,\mu_3$$
$$+ \left(\frac{d^3}{dt^3} + 2\,\zeta\,\omega\,\frac{d^2}{dt^2} + a\,\frac{d^2}{dt^2} + \omega^2\,\frac{d}{dt} + 2\,a\,\zeta\,\omega\,\frac{d}{dt} + a\,\omega^2\right)\,\mu_4 = 0. \quad (3)$$

$$\begin{cases} x_1(t) = & -\omega^2\,k\,a\,\xi(t-h), \\ x_2(t) = & -\omega^2\,\dot{\xi}(t) + a\,\omega^2\,\xi(t), \\ x_3(t) = & \omega^2\,\ddot{\xi}(t) - \omega^2\,a\,\dot{\xi}(t), \\ u(t) = & -\xi(t)^{(3)} + (2\,\zeta\,\omega + a)\,\ddot{\xi}(t) \\ & -(\omega^2 + 2\,a\,\omega\,\zeta)\,\dot{\xi}(t) + a\,\omega\,\xi(t). \end{cases} \quad (4)$$

The compatibility conditions of (4) are exactly generated by (1) $\Rightarrow$ **the system is parametrized by (4).**

Flexible rod

$$\begin{cases} \frac{\partial^2 z}{\partial t^2}(x,t) = \frac{\partial^2 z}{\partial x^2}(x,t), & y_1(t) = z(0,t), \\ \frac{\partial z}{\partial x}(0,t) = -u(t), & y_2(t) = z(1,t), \\ \frac{\partial z}{\partial x}(1,t) = 0, & \end{cases}$$

$$\Rightarrow \begin{cases} \dot{y}_1(t) - \dot{y}_2(t-1) - u(t) = 0, \\ 2\,\dot{y}_1(t-1) - \dot{y}_2(t) - \dot{y}_2(t-2) = 0. \end{cases} \qquad (1)$$

$$\begin{cases} \frac{d}{dt}\lambda_1 + 2\,\delta\,\frac{d}{dt}\lambda_2 = \mu_1, \\ -\delta\,\frac{d}{dt}\lambda_1 - \frac{d}{dt}(\delta^2+1)\,\lambda_2 = \mu_2, \\ -\lambda_1 = \mu_3. \end{cases} \qquad (2)$$

$$(\delta^2+1)\,\mu_1 + 2\,\delta\,\mu_2 - \frac{d}{dt}(1-\delta^2)\,\mu_3 = 0. \qquad (3)$$

$$\begin{cases} y_1(t) = \xi(t) + \xi(t-2), \\ y_2(t) = 2\,\xi(t-1), \\ u(t) = \dot{\xi}(t) - \dot{\xi}(t-2). \end{cases} \qquad (4)$$

$$\begin{cases} \dot{y}_1(t) - \dot{y}_2(t-1) - u(t) = 0, \\ 2\,y_1(t-1) - y_2(t) - y_2(t-2) = 0, \\ \dot{y}_1(t-1) - \dot{y}_2(t-1) = 0, \end{cases} \qquad (5)$$

$$\Rightarrow \theta(t) = 2\,y_1(t-1) - y_2(t) - y_2(t-1), \quad \dot{\theta}(t) = 0.$$

$\Rightarrow (-c/2, -c, 0)^T$ **is not parametrized by** (4).

Injective modules

• **Definition**: A left D-module $\mathcal{F}$ is **injective** if, for every finitely generated left ideal $I = (P_1, \ldots, P_m)$ of D, there exists $y \in \mathcal{F}$ which satisfies

$$\left\{ \begin{array}{rcl} P_1\, y & = & u_1, \\ \vdots & & \vdots \\ P_m\, y & = & u_m, \end{array} \right.$$

where $u_1, \ldots, u_m \in \mathcal{F}$ satisfy the relations of I, i.e.:

$$\sum_{i=1}^{m} Q_i\, P_i = 0 \Rightarrow \sum_{i=1}^{m} Q_i\, u_i = 0.$$

• **Proposition**: $\mathcal{F}$ is an **injective left D-module**, iff, for every left D-module M, we have:

$$\boxed{\mathrm{ext}_D^i(M, \mathcal{F}) = 0, \; i \geq 1.}$$

• **Definition**: A left D-module $\mathcal{F}$ is called **cogenerator** if $\hom_D(M, \mathcal{F}) = 0 \Rightarrow M = 0$.

• **Definition**: If $\mathcal{F}$ is an **injective cogenerator** left D-module, then we have the **following equivalences**:

1. $\mathcal{F}^q \xleftarrow{R.} \mathcal{F}^p \xleftarrow{Q.} \mathcal{F}^m$ is **exact sequence**.

2. $D^{1\times q} \xrightarrow{.R} D^{1\times p} \xrightarrow{.Q} D^{1\times m}$ is **exact sequence**.

Parametrizability problem

- Let $\mathcal{F}$ be an **injective cogenerator** left D-module.

- **Example**: $D = \mathbb{R}\left[\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right]$, $\mathcal{F} = C^\infty(\mathbb{R}^n)$.

- **Example**: $D = \mathbb{R}\left[\delta_1, \ldots, \delta_n\right]$, $\mathcal{F} = k^{\mathbb{N}^n}$.

- **Example**: $D = \mathbb{R}(t)\left[\frac{d}{dt}\right]$, $\mathcal{F}$ is the set of smooth functions except in a finite number of points.

- **Theorem**: We have the following equivalences:

1. $\exists\, Q \in D^{p \times m}: \ \ker_{\mathcal{F}}(R.) = Q\, \mathcal{F}^m$.

2. $\exists\, Q \in D^{p \times m}: \ \ker_D(.Q) = D^{1 \times p}\, R$,

   where $\ker_D(.Q) = \{\lambda \in D^{1 \times p} \mid \lambda\, Q = 0\}$.

3. The left D-module $M = D^{1 \times p}/(D^{1 \times q}\, R)$ is **torsion-free**, namely:

$$t(M) = \{m \in M \mid \exists\, 0 \neq P \in D: \ P\, m = 0\} = 0.$$

4. $t(M) \cong \mathrm{ext}^1_D(\widetilde{N}, D) = 0$, where:

$$\widetilde{N} = D^{1 \times q}/(D^{1 \times q}\, (\theta(R_{ij}))^T).$$

Factorization

- We have the **exact sequence**:

$$0 \longrightarrow t(M) \xrightarrow{\iota} M \xrightarrow{\rho} M/t(M) \longrightarrow 0.$$

- **Theorem**: There exist two matrices $R'' \in D^{q \times q'}$ and $R' \in D^{q' \times p}$ such that:

$$\begin{cases} R = R''\, R', \\ t(M) = (D^{1 \times q'}\, R')/(D^{1 \times q}\, R), \\ M/t(M) = D^{1 \times p}/(D^{1 \times q'}\, R'). \end{cases}$$

- R' and R'' can be computed using OREMODULES.

- We have the **commutative exact diagram** ($\star$):

$$\begin{array}{ccccccccc} & & & & & & 0 & & \\ & & & & & & \downarrow & & \\ & & & & & & t(M) & & \\ & & & & & & \downarrow \iota & & \\ & & D^{1 \times q} & \xrightarrow{.R} & D^{1 \times p} & \xrightarrow{\pi} & M & \longrightarrow & 0, \\ & & \downarrow .R'' & & \| & & \downarrow \rho & & \\ D^{1 \times r'} & \xrightarrow{.T} & D^{1 \times q'} & \xrightarrow{.R'} & D^{1 \times p} & \xrightarrow{\pi'} & M/t(M) & \longrightarrow & 0. \\ & & & & & & \downarrow & & \\ & & & & & & 0 & & \end{array}$$

Autonomous behaviour

• Applying $\hom_D(\cdot, \mathcal{F})$ to ($\star$), we obtain the **commutative exact diagram**:

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \uparrow & & \\
 & & & & & & \hom_D(t(M), \mathcal{F}) & & \\
 & & & & & & i^\star \uparrow & & \\
 & & \mathcal{F}^q & \xleftarrow{R.} & \mathcal{F}^p & \longleftarrow & \ker_{\mathcal{F}}(R.) & \longleftarrow & 0 \\
 & & R''. \uparrow & & \| & & \rho^\star \uparrow & & \\
\mathcal{F}^{r'} & \xleftarrow{T.} & \mathcal{F}^{q'} & \xleftarrow{R'.} & \mathcal{F}^p & \longleftarrow & \ker_{\mathcal{F}}(R'.) & \longleftarrow & 0. \\
 & & & & & & \uparrow & & \\
 & & & & & & 0 & &
\end{array}$$

• $\ker_{\mathcal{F}}(R'.)$ is the **parametrizable sub-behaviour**

$$\boxed{\Rightarrow \exists\, Q \in D^{p \times m}: \quad \ker_{\mathcal{F}}(R'.) = Q\, \mathcal{F}^m.}$$

• **Theorem**: Let $T \in D^{r' \times q'}$ be a matrix such that $\ker_D(.R') = D^{1 \times r'}\, T$. Then, we have:

$$\begin{aligned}
\hom_D(t(M), \mathcal{F}) &= \{\tau \in \mathcal{F}^{q'} \mid R''\, \tau = 0, \; T\, \tau = 0\} \\
&= \ker_{\mathcal{F}}(R''.) \cap \ker_{\mathcal{F}}(T.).
\end{aligned}$$

• $\hom_D(t(M), \mathcal{F})$ is called **autonomous behaviour**.

• T **can be computed** using OREMODULES.

General result

1. **Compute** $R' \in D^{q' \times p}$ and $R'' \in D^{q \times q'}$ such that:

$$\left\{ \begin{array}{l} R = R'' \, R', \\ t(M) = (D^{1 \times q'} \, R') / (D^{1 \times q} \, R), \\ M / t(M) = D^{1 \times p} / (D^{1 \times q'} \, R'). \end{array} \right.$$

2. **Compute** $Q \in D^{p \times m}$ such that:

$$\ker_D(.Q) = D^{1 \times q'} \, R' \Rightarrow \ker_{\mathcal{F}}(R'.) = Q \, \mathcal{F}^m.$$

3. **Compute** $T \in D^{r' \times q'}$ such that:

$$\ker_D(.R') = D^{1 \times r'} \, T \Rightarrow \ker_{\mathcal{F}}(T.) = R' \, \mathcal{F}^p.$$

4. **Find a fundamental solution** $\overline{\tau} \in \mathcal{F}^{q'}$ of:

$$\left\{ \begin{array}{l} R'' \, \tau = 0, \\ T \, \tau = 0. \end{array} \right.$$

5. **Find the general solution** of the system:

$$R' \, \eta = \overline{\tau}, \quad \eta \in \mathcal{F}^p.$$

a. **Find a particular solution** $\overline{\eta} \in \mathcal{F}^p$: $\quad R' \, \overline{\eta} = \overline{\tau}$.

b. **Find the general solution** of $R' \, \eta = 0$. We have:

$$\ker_{\mathcal{F}}(R'.) = Q \, \mathcal{F}^m.$$

6. **The general solution of** $R \, \eta = 0$ **is of the form:**

$$\boxed{\eta = \overline{\eta} + Q \, \xi, \quad \forall \, \xi \in \mathcal{F}^m.}$$

Particular case

• When is it possible to algebraically obtain a particular solution of $R'\,\eta = \overline{\tau}$ (**variation of constants**)?

• **Theorem**: Let us consider $R \in D^{q\times p}$ and the left D-module $M = D^{1\times p}/(D^{1\times q}\,R)$.

Then, we have $\boxed{M \cong t(M) \oplus M/t(M)}$

iff there exist $S \in D^{p\times q'}$ and $V \in D^{q'\times q}$ such that:

$$\boxed{R' - R'\,S\,R' = V\,R.} \qquad (\star)$$

• $(\star)$ **always holds** for $D = K\left[\frac{d}{dt}\right]$, $K = \mathbb{R}$, $\mathbb{R}(t), \mathbb{R}[t]$, or when $M/t(M)$ is a projective left D-module.

• **Algorithms are implemented** in OREMODULES.

• **Corollary**: Let $\overline{\tau} \in \mathcal{F}^{q'}$ be a solution of the system:

$$R''\,\tau = 0, \quad T\,\tau = 0.$$

Then, $\overline{\eta} \triangleq S\,\overline{\tau}$ is **a particular solution** of $R'\,\eta = \overline{\tau}$.

$\Rightarrow$ The **general solution** of $R\,\eta = 0$ is of the form:

$$\boxed{\eta = S\,\overline{\tau} + Q\,\xi, \quad \forall\,\xi \in \mathcal{F}^m.}$$

Algorithm I

- Let D be a **commutative polynomial ring**.

- **Lemma**: If $U \in D^{a\times b}$, $V \in D^{b\times c}$ and $W \in D^{c\times d}$, then we have

$$\boxed{U\,V\,W = (V_1 \ldots V_b)\,(U^T \otimes W)}$$

with the notations:

$$\left\{\begin{array}{l} X \otimes Y = \begin{pmatrix} X_{11}\,Y & \ldots & X_{1a}\,Y \\ \vdots & X_{ij}\,Y & \vdots \\ X_{b1}\,Y & \ldots & X_{ba}\,Y \end{pmatrix}, \\ V = (V_1^T \ldots V_b^T)^T, \quad V_i \in D^{1\times c}. \end{array}\right.$$

- We have $R' - R'\,S\,R' = V\,R \Leftrightarrow f\,T = g$,

where $$\left\{\begin{array}{l} f = (S_1 \ldots S_p \quad V_1 \ldots V_{q'}) \in D^{1\times(p+q)\,q'}, \\ T = \begin{pmatrix} R'^T \otimes R' \\ I_{q'} \otimes R \end{pmatrix} \in D^{(p+q)\,q'\times p\,q'}, \\ g = (R'_1 \ldots R'_{q'}) \in D^{1\times p\,q'}. \end{array}\right.$$

- **Algorithm**:

1. Compute a **Gröbner basis** G of the rows of T.
2. Compute the **normal form** r of g modulo G.

If $r = 0$, then $\exists\, f \in D^{1\times(p+q)\,q'}$ such that $f\,T = g$, else there is no solution.

Algorithm II

- We consider $A_n(\mathbb{Q}) = \mathbb{Q}[x_1, \ldots, x_n][d_1, \ldots, d_n]$.

- We can **search for a matrix** $S \in D^{p \times q'}$ **of a given order** d which satisfies:

$$\exists\, V \in D^{q' \times q} : \quad R' - R'\, S\, R' = V\, R.$$

- **Algorithm**:

1. Compute a **Gröbner basis** G of

$$\left\{ \sum_{i=1}^{p} R_{ij}\, \lambda_j - \mu_i \mid i = 1, \ldots, q \right\}$$

in the free left D-module

$$\bigoplus_{i=1}^{p} D\, \lambda_i \oplus \bigoplus_{i=1}^{q} D\, \mu_i \oplus \bigoplus_{1 \leq i \leq p,\, 1 \leq j \leq q,\, 0 \leq |k| \leq d} D\, a_k^{(i,j)},$$

w.r.t. an **order which eliminates the λ_i's**.

2. Let $S_{ij} = \sum_{0 \leq |k| \leq d,\, 0 \leq |l| \leq d} a_k^{(i,j)}\, x^k\, \partial^l$.

3. **Compute** $X = R' - R'\, S\, R'$.

4. For $i = 1, \ldots, q'$, compute the **normal form** F_i of $\sum_{j=1}^{p} X_{ij}\, \lambda_j$ modulo G and **collect** in H the non-zero coefficients of $x^k\, \partial^l\, \lambda_j$ in F_i.

5. **Solve** H for $a_k^{(i,j)}$.

OREMODULES

- OREMODULES is a tool-box developed in *Maple*.

- OREMODULES uses *Mgfun* developed by F. Chyzak

`http://algo.inria.fr/chyzak/mgfun.html.`

- OREMODULES is developed by Chyzak-Q.-Robertz.

- OREMODULES can handle linear systems of ODEs, PDEs, differential time-delay systems, multidimensional discrete systems. . .

- OREMODULES computes:

1. autonomous elements, non-controllable elements,

2. parametrizations of under-determined systems,

3. left-/right-/generalized inverses,

4. flat outputs, π-polynomials,

5. first integrals of motion,

6. Euler-Lagrange equations. . .

- A **second release is available** on the web page:

`http://wwwb.math.rwth-aachen.de/OreModules.`

List of the functions

Main functions

`Parametrization`
`MinimalParametrization(s)`
`AutonomousElements`
`LeftInverse(Rat)`
`LocalLeftInverse`
`GeneralizedInverse(Rat)`
`PiPolynomial`
`Complement`
`FirstIntegral`
`LQEquations`

Module theory

`TorsionElements`
`Exti(Rat)`
`Extn(Rat)`
`Quotient(Rat)`
`SyzygyModule(Rat)`
`Resolution(Rat)`
`FreeResolution(Rat)`
`OreRank`

Some low-level functions

`DefineOreAlgebra`
`Involution`
`Factorize`
`Mult`
`ApplyMatrix`

- **Library of examples**: More than 30 examples.

Examples

Bipendulum

- Let $D = \mathbb{R}(g,l)\left[\frac{d}{dt}\right]$, $\mathcal{F} = C^{\infty}(\mathbb{R})$ and:

$$R = \begin{pmatrix} \frac{d^2}{dt^2} + \frac{g}{l} & 0 & -\frac{g}{l} \\ 0 & \frac{d^2}{dt^2} + \frac{g}{l} & -\frac{g}{l} \end{pmatrix} \in D^{2\times 3}.$$

- Using OreModules, we obtain:

$$R'' = \begin{pmatrix} \frac{d^2}{dt^2} + \frac{g}{l} & \frac{1}{l} \\ 0 & \frac{1}{l} \end{pmatrix}, \quad R' = \begin{pmatrix} 1 & -1 & 0 \\ 0 & l\,\frac{d^2}{dt^2} + g & -g \end{pmatrix},$$

$$Q = \begin{pmatrix} g & g & l\,\frac{d^2}{dt^2} + g \end{pmatrix}^T, \quad S = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & -\frac{1}{g} \end{pmatrix}.$$

- The matrix S satisfies $R'\,S = I_2 \Rightarrow R' - R'\,S\,R' = 0$.

- The **autonomous elements** $\tau = R'\,\eta$ satisfy:

$$\begin{cases} l\,\ddot{\tau}_1 + g\,\tau_1 = 0, \\ \tau_2 = 0, \end{cases} \Leftrightarrow \begin{cases} \tau_1 = c_1\,\sin\left(\sqrt{\frac{g}{l}}\,t\right) + c_2\,\cos\left(\sqrt{\frac{g}{l}}\,t\right), \\ \tau_2 = 0. \end{cases}$$

- Then, **all elements of** $\ker_{\mathcal{F}}(R.)$ **are of the form**:

$$\begin{cases} \eta_1(t) = c_1\,\sin\left(\sqrt{\frac{g}{l}}\,t\right) + c_2\,\cos\left(\sqrt{\frac{g}{l}}\,t\right) + g\,\xi(t), \\ \eta_2(t) = g\,\xi(t), \\ \eta_3(t) = l\,\ddot{\xi}(t) + g\,\xi(t). \end{cases}$$

Singular OD system

• **Let us parametrize all $\mathcal{F} = C^{\infty}(\mathbb{R})$-solutions of:**

$$\ddot{y}(t) - t\,\dot{u}(t) - u(t) = 0. \quad (1)$$

• We consider $D = A_1(\mathbb{R}) = \mathbb{R}[t]\left[\frac{d}{dt}\right]$ and:

$$R = \begin{pmatrix} \frac{d^2}{dt^2} & -t\,\frac{d}{dt} - 1 \end{pmatrix} \in D^{1\times 2}.$$

• We easily obtain that $R'' = \frac{d}{dt}$ and $R' = \begin{pmatrix} \frac{d}{dt} & -t \end{pmatrix}$.

$\Rightarrow$ **We need to find all $\mathcal{F}$-solutions of:**

$$\dot{y}(t) - t\,u(t) = C, \quad C \in \mathbb{R}. \quad (2)$$

• A **right-inverse** S of R' is defined by $S = \begin{pmatrix} t & \frac{d}{dt} \end{pmatrix}^T$

$$\Rightarrow R' - R'\,S\,R' = 0 \Rightarrow (y_\star \quad u_\star)^T = (C\,t \quad 0)^T$$

is a **particular solution** of (2).

• Using a parametrization of all $\mathcal{F}$-solutions of the homogeneous part of (2), **we finally obtain**:

$$\begin{cases} y(t) = \dot{y}(0)\,t + t^2\,\xi_1(t) + t\,\dot{\xi}_2(t) - \xi_2(t), \\ u(t) = t\,\dot{\xi}_1(t) + 2\,\xi_1(t) + \ddot{\xi}_2(t), \end{cases}$$

for all $\xi_1,\ \xi_2 \in \mathcal{F}$.

$\boxed{P(\partial)\,\mathrm{div}}$

- $D = \mathbb{R}[\partial_1, \partial_2, \partial_3]$, $\mathrm{div} = (\partial_1 \quad \partial_2 \quad \partial_3)$.

- **Let us parametrize all $\mathcal{F} = C^\infty(\mathbb{R}^3)$-solutions of:**

$P(\partial)\,(\partial_1\, B_1 + \partial_2 B_2 + \partial_3 B_3) = 0, \quad 0 \neq P(\partial) \in D.$

- $\left\{ \begin{array}{l} \tau = \partial_1 B_1 + \partial_2\, B_2 + \partial_3\, B_3, \\ P(\partial)\,\tau = 0, \end{array} \right. \Rightarrow \left\{ \begin{array}{l} R' = \mathrm{div}, \\ R'' = P(\partial). \end{array} \right.$

Let $\overline{\tau} \in \mathcal{F}^3$ be a **fundamental solution** of $P(\partial)\,\tau = 0$.

- We need to solve the **inhomogeneous system**:

$$\mathrm{div}\vec{B} = \overline{\tau}. \quad (\star)$$

a. Find a **particular solution** $\vec{B}_\star$ of $(\star)$:

$$R' - R'\,S\,R' = V\,R \Leftrightarrow \textstyle\sum_{i=1}^{3} \partial_i S_i - V\,P(\partial) = 1,$$

$$\Leftrightarrow (\partial_1, \partial_2, \partial_3, P(\partial)) = D \Leftrightarrow P(0) \neq 0.$$

$\Rightarrow \vec{B}_\star = (S_1\ S_2\ S_3)^T\,\overline{\tau}$ is a **particular solution**:

$$(\partial_1 \quad \partial_2 \quad \partial_3)\,\vec{B}_\star = (1 + V\,P(\partial))\,\overline{\tau} = \overline{\tau}.$$

b. Find a **general solution** of $\mathrm{div}\,\vec{A} = 0$:

$$\mathrm{div}\,\vec{B} = 0 \Leftrightarrow \vec{B} = \mathrm{curl}\,\vec{\Psi},\ \forall\,\vec{\Psi} \in \mathcal{F}^3.$$

$$\boxed{\vec{B} = (S_1 \quad S_2 \quad S_3)^T\,\overline{\tau} + \mathrm{curl}\,\vec{\Psi}, \quad \forall\,\vec{\Psi} \in \mathcal{F}^3.}$$

$\boxed{\partial_1 \, \mathrm{div}}$

• Let us consider $\mathrm{div} = (\partial_1 \quad \partial_2 \quad \partial_3)$.

• **Let us parametrize all $C^\infty(\mathbb{R}^3)$-solutions of**:

$$R\,\vec{B} = \partial_1 \, \mathrm{div}\,\vec{B} = 0. \quad (1)$$

• We obtain $R' = \mathrm{div}$ and $R'' = \partial_1$.

• Hence, we need to find a **particular solution** of:

$$\mathrm{div}\,\vec{B} = \Phi(x_2, x_3), \quad \Phi \in C^\infty(\mathbb{R}^2). \quad (2)$$

• As $R(0) = 0$, we obtain that there is **no splitting** $M \cong t(M) \oplus M/t(M)$ over the ring $\mathbb{R}[\partial_1, \partial_2, \partial_3]$.

• If we consider $A_3 = \mathbb{R}[x_1, x_2, x_3][\partial_1, \partial_2, \partial_3]$, then we obtain that $S = (x_1 \quad 0 \quad 0)^T$ satisfies:

$$R' - R'\,S\,R' = x_1\,R.$$

$\Rightarrow$ a **splitting** $M \cong t(M) \oplus M/t(M)$ exists over A_3.

• A **particular solution** of (2) is given by:

$$\vec{B}_\star = (x_1\,\Phi(x_2, x_3) \quad 0 \quad 0)^T.$$

•**All $C^\infty(\mathbb{R}^3)$-solutions of (1) are given by:**

$$\boxed{\vec{B} = \vec{B}_\star + \mathrm{curl}\,\vec{\Psi}, \quad \forall\,\vec{\Psi} \in C^\infty(\mathbb{R}^3)^3.}$$

grad ∘ div

- Let us consider the system $\text{grad}\,(\text{div}\,\vec{B}) = \vec{0}$, i.e.:

$$\left\{ \begin{array}{l} \partial_1\,(\partial_1\,B_1 + \partial_2\,B_2 + \partial_3\,B_3) = 0, \\ \partial_2\,(\partial_1\,B_1 + \partial_2\,B_2 + \partial_3\,B_3) = 0, \\ \partial_3\,(\partial_1\,B_1 + \partial_2\,B_2 + \partial_3\,B_3) = 0. \end{array} \right. \quad (1)$$

- We obtain the matrices:

$$R = \text{grad}\,(\text{div}), \quad R' = \text{div}, \quad R'' = \text{grad}, \quad T = 0.$$

- **There is one autonomous element** in system (1):

$$\tau = \text{div}\,\vec{B}, \quad \partial_1\,\tau = 0, \quad \partial_2\,\tau = 0, \quad \partial_3\,\tau = 0.$$

- We need to **parametrize all $C^\infty(\mathbb{R}^3)$-solutions** of:

$$\text{div}\,\vec{B} = C, \quad C \in \mathbb{R}.$$

- If we consider $A_3 = \mathbb{R}[x_1, x_2, x_3][\partial_1, \partial_2, \partial_3]$, then

$$S = (x_1 \quad 0 \quad 0)^T : \quad R'\,S + (-x_1 \quad 0 \quad 0)\,R'' = 1.$$

$\Rightarrow$ a **splitting** $M \cong t(M) \oplus M/t(M)$ exists over A_3.

$\Rightarrow$ a **particular solution** of $(\star)$ is given by:

$$\vec{B}_\star = (C\,x_1 \quad 0 \quad 0)^T. \quad (2)$$

- **All $C^\infty(\mathbb{R}^3)$-solutions of (1) are given by:**

$$\boxed{\vec{B} = \vec{B}_\star + \text{curl}\,\vec{\Psi}, \quad \forall\,\vec{\Psi} \in C^\infty(\mathbb{R}^3)^3.}$$

Linear elasticity $n=2$

• In linear elasticity, we need to solve the equation

$$\Delta\,\Delta\,A = c\,\Delta\,V, \quad \Delta = d_1^2 + d_2^2, \quad c \in \mathbb{R}^\star, \quad (\star)$$

where A is the **Airy function** and V **potential**.

• We obtain the following matrices:

$$R = (\Delta\,\Delta \quad -c\,\Delta), \quad R'' = \Delta, \quad R' = (\Delta \quad -c),$$

$$T = 0, \quad P = (1 \quad \Delta/c)^T,$$

$$\Rightarrow \left\{ \begin{array}{l} \tau = \Delta\,A - c\,V, \\ \Delta\,\tau = 0, \end{array} \right. \quad \Rightarrow \overline{\tau} = \ln\left(1/\sqrt{x_1^2 + x_2^2}\right),$$

fundamental solution of $\Delta\,\tau = 0$ in $\mathcal{F} = \mathcal{D}'(\mathbb{R}^2)$.

• The matrix $S = (0 \quad -1/c)^T$ satisfies:

$$R'\,S = 1 \Rightarrow R' - R'\,S\,R' = 0 \Rightarrow V = 0,$$

$$\Rightarrow \left(\begin{array}{c} A_\star \\ V_\star \end{array} \right) = S\,\overline{\tau} = \left(\begin{array}{c} 0 \\ -\frac{\overline{\tau}}{c} \end{array} \right)$$

is a **particular solution** of $\Delta\,A - c\,V = \overline{\tau}$.

• **All $\mathcal{F}$-solutions** of $(\star)$ are given by:

$$\boxed{\left\{ \begin{array}{l} A = A, \\ V = \frac{1}{c}(\Delta\,A - \overline{\tau}), \end{array} \right. \quad \forall\, A \in \mathcal{F}.}$$

Saint Venant's equations

• **Linearized model** around the Riemann invariants (Dubois-Petit-Rouchon, ECC99):

$$\begin{cases} y_1(t-2\,\Delta)+y_2(t)-2\,\dot{y}_3(t-\Delta)=0, \\ y_1(t)+y_2(t-2\,\Delta)-2\,\dot{y}_3(t-\Delta)=0. \end{cases}$$

• We consider $D=\mathbb{R}\left[\frac{d}{dt},\delta\right]$ and the matrix:

$$R=\begin{pmatrix} \delta^2 & 1 & -2\,\delta\,\frac{d}{dt} \\ 1 & \delta^2 & -2\,\delta\,\frac{d}{dt} \end{pmatrix}\in D^{2\times 3}.$$

• $R'=\begin{pmatrix} 1 & -1 & 0 \\ 0 & -\delta^2-1 & 2\,\delta\,\frac{d}{dt} \end{pmatrix}, \quad R''=\begin{pmatrix} \delta^2 & -1 \\ 1 & -1 \end{pmatrix}.$

• As $\ker_D(.R')=0$, the **autonomous elements**

$$\begin{cases} \tau_1=y_1-y_2, \\ \tau_2=-y_2(t-2\,\Delta)-y_2+2\,\dot{y}_3(t-\Delta), \end{cases}$$

satisfy the system $\begin{cases} \tau_1(t-2\,\Delta)-\tau_2(t)=0, \\ \tau_1(t)-\tau_2(t)=0, \end{cases}$

$\Rightarrow \tau_1=\tau_2$ is a $2\,\Delta$**-periodic function**.

• Let $\mathcal{F}=C^\infty(\mathbb{R})$. We then have:

$$\ker_{\mathcal{F}}(R'.)=Q\,\mathcal{F}, \quad Q=\left(2\,\delta\,\frac{d}{dt},\ 2\,\delta\,\frac{d}{dt},\ 1+\delta^2\right)^T.$$

Saint Venant's equations

• The matrices

$$S = \begin{pmatrix} \frac{1}{2} & 0 \\ -\frac{1}{2} & 0 \\ 0 & 0 \end{pmatrix}, \quad V = \begin{pmatrix} 0 & 0 \\ -\frac{1}{2} & -\frac{1}{2} \end{pmatrix},$$

satisfy the relation:

$$R' - R'\, S\, R' = V\, R.$$

$$\Rightarrow \begin{pmatrix} \overline{y_1} \\ \overline{y_2} \\ \overline{y_3} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 \\ -\frac{1}{2} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \overline{\tau_1} \\ \overline{\tau_2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\,\overline{\tau_1} \\ -\frac{1}{2}\,\overline{\tau_1} \\ 0 \end{pmatrix}$$

is a **particular solution** of $R'\, y = \overline{\tau}$.

• A **parametrization** of $\ker_{\mathcal{F}}(R.)$ is then defined by

$$\begin{cases} y_1(t) = \frac{1}{2}\,\overline{\tau_1}(t) + 2\,\dot{\xi}(t - \Delta), \\ y_2(t) = -\frac{1}{2}\,\overline{\tau_1}(t) + 2\,\dot{\xi}(t - \Delta), \\ y_3(t) = \xi(t) + \xi(t - 2\,\Delta), \end{cases}$$

where ξ is **any element of** $\mathcal{F}$ and $\overline{\tau_1}$ is **any** $2\,\Delta$-**periodic function** of $\mathcal{F}$.

Flexible rod

- We consider the system defined by (Mounier 95):

$$\begin{cases} \dot{y}_1(t) - \dot{y}_2(t-1) - u(t) = 0, \\ 2\,\dot{y}_1(t-1) - \dot{y}_2(t) - \dot{y}_2(t-2) = 0. \end{cases}$$

- We consider $D = \mathbb{R}\left[\frac{d}{dt}, \delta\right]$ and:

$$R = \begin{pmatrix} \frac{d}{dt} & -\delta\,\frac{d}{dt} & -1 \\ 2\,\delta\,\frac{d}{dt} & -\frac{d}{dt} - \delta^2\,\frac{d}{dt} & 0 \end{pmatrix}.$$

- **Using the algorithms**, we obtain the matrices:

$$R' = \begin{pmatrix} -2\,\delta & 1+\delta^2 & 0 \\ -\frac{d}{dt} & \delta\,\frac{d}{dt} & 1 \\ \delta\,\frac{d}{dt} & -\frac{d}{dt} & \delta \end{pmatrix}, \quad R'' = \begin{pmatrix} 0 & -1 & 0 \\ 0 & -\delta & 1 \end{pmatrix},$$

$$T = \begin{pmatrix} \frac{d}{dt} & -\delta & 1 \end{pmatrix}, \quad Q = \begin{pmatrix} 1+\delta^2 \\ 2\,\delta \\ -\delta^2\,\frac{d}{dt} + \frac{d}{dt} \end{pmatrix}.$$

- The **autonomous elements** are defined by:

$$\begin{cases} -\tau_2 = 0, \\ -\delta\,\tau_2 + \tau_3 = 0, \\ \frac{d}{dt}\,\tau_1 - \delta\,\tau_2 + \tau_3 = 0, \end{cases} \Leftrightarrow \begin{cases} \tau_1 = C \in \mathbb{R}, \\ \tau_1 = -2\,\delta y_1 + (1+\delta^2)\,y_2, \end{cases}$$

$\Rightarrow \overline{\tau} = (\tau_1,\, \tau_2,\, \tau_3)^T = (C,\, 0,\, 0)^T$ where $C \in \mathbb{R}$.

Flexible rod

- The matrices

$$S = \begin{pmatrix} \frac{1}{2}\,\delta & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad V = \begin{pmatrix} 0 & 0 \\ -1 & \frac{1}{2}\,\delta \\ -\delta & \frac{1}{2}\,\delta^2 \end{pmatrix},$$

satisfy the relation:

$$R' - R'\,S\,R' = V\,R.$$

$$\Rightarrow \begin{pmatrix} \overline{y_1} \\ \overline{y_2} \\ \overline{u} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\,\delta & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} C \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\,C \\ C \\ 0 \end{pmatrix}$$

is a **particular solution** of $R'\,y = \overline{\tau}$.

- Let $\mathcal{F} = C^{\infty}(\mathbb{R})$. A **parametrization** of $\ker_{\mathcal{F}}(R.)$ is then defined by

$$\begin{cases} y_1(t) = \frac{1}{2}\,C + \xi(t) + \xi(t-2), \\ y_2(t) = C + 2\,\xi(t), \\ u(t) = \dot{\xi}(t-2) + \dot{\xi}(t), \end{cases}$$

where ξ is **any element of** $\mathcal{F}$ and $C \in \mathbb{R}$.

Applications

Controllability

- **Two pendula mounted on a car**:

$$\left\{\begin{array}{l} m_1\,L_1\,\ddot{w}_1(t)+m_2\,L_2\,\ddot{w}_2(t)-w_3(t)+(M+m_1+m_2)\,\ddot{w}_4(t)=0,\\ m_1\,L_1^2\,\ddot{w}_1(t)-m_1\,L_1\,g\,w_1(t)+m_1\,L_1\,\ddot{w}_4(t)=0,\\ m_2\,L_2^2\,\ddot{w}_2(t)-m_2\,L_2\,g\,w_2(t)+m_2\,L_2\,\ddot{w}_4(t)=0.\end{array}\right. \qquad (\star)$$

- $(\star)$ is **parametrizable** iff $L_1 \neq L_2$.

- A **parametrization** of $(\star)$ is given by:

$$\left\{\begin{array}{ll} w_1(t)= & -L_2\,\xi^{(4)}(t)+g\,\ddot{\xi}(t),\\ w_2(t)= & -L_1\,\xi^{(4)}(t)+g\,\ddot{\xi}(t),\\ w_3(t)= & L_1\,L_2\,M\,\xi^{(6)}(t)\\ & -(L_1\,m_2+L_2\,m_1+g\,(L_1+L_2)\,M)\,\xi^{(4)}(t)\\ & +g^2\,(m_1+m_2+M)\,\xi^{(2)(t)}\\ w_4(t)= & L_1\,L_2\,\xi^{(4)}(t)-g\,(L_1+L_2)\,\ddot{\xi}(t)+g^2\,\xi(t).\end{array}\right.$$

$$\xi(t)=\frac{1}{g^2\,(L_1-L_2)}\,(L_1^2\,w_1(t)-L_2^2\,w_2(t)+(L_1-L_2)\,w_4(t)).$$

- **Patching problem** $\Leftrightarrow$ **controllability**: $\quad T>0$.

$w^p=(w_1^p,w_2^p,w_3^p,w_4^p)$ a **past trajectory** of $(\star)$ on $]-\infty,0[$.

$w^f=(w_1^f,w_2^f,w_3^f,w_4^f)$ a **future trajectory** of $(\star)$ on $]T,+\infty[$.

$$\Rightarrow \exists\, w=(w_1,w_2,w_3,w_4)\in C^\infty(\mathbb{R})^4 \text{ trajectory of } (\star):$$

$$\left\{\begin{array}{l} w_{]-\infty,0[}=w^p,\\ w_{]T,+\infty[}=w^f.\end{array}\right.$$

It is **enough to find** $\xi\in C^\infty(\mathbb{R})$ such that:

$$\xi_{]-\infty,0[}=\xi^p \quad \& \quad \xi_{]T,+\infty[}=\xi^f.$$

Motion planning

• **Flexible rod with a torque** (Mounier 95):

$$\begin{cases} \sigma^2 \frac{\partial^2 q(\tau,x)}{\partial \tau^2} - \frac{\partial^2 q(\tau,x)}{\partial x^2} = 0, \\ \frac{\partial q}{\partial x}(\tau, 0) = -u(\tau), \\ \frac{\partial q}{\partial x}(\tau, L) = -J \, \frac{\partial^2 q}{\partial \tau^2}(\tau, L), \\ y(\tau) = q(\tau, L). \end{cases} \qquad (\star)$$

• Using **d'Alembert formula**

$$\begin{cases} q(\tau, x) = \phi(\tau + \sigma\, x) + \psi(\tau - \sigma\, x), \\ t = (\sigma/J)\, \tau, \\ v = (2\, J/\sigma^2)\, u, \end{cases}$$

we obtain ($D = \mathbb{R}\left[\frac{d}{dt}, \delta, \tau\right]$, $\mathcal{F} = C^\infty(\mathbb{R})$):

$$(\star) \Rightarrow \ddot{y}(t+1) + \ddot{y}(t-1) + \dot{y}(t+1) - \dot{y}(t-1) = v(t)$$

$$\Leftrightarrow \begin{cases} y(t) = \xi(t-1), \\ v(t) = \ddot{\xi}(t) + \ddot{\xi}(t-2) + \dot{\xi}(t) - \dot{\xi}(t-2). \end{cases}$$

• If y_r is a **desired trajectory**

$$\xi_r(t-1) = y_r(t) \Rightarrow \xi_r(t) = y_r(t+1),$$

thus, we obtain the **open-loop control law**:

$$v_r(t) = \ddot{\xi}_r(t) + \ddot{\xi}_r(t-2) + \dot{\xi}_r(t) - \dot{\xi}_r(t-2)$$

$$= \ddot{y}_r(t+1) + \ddot{y}_r(t-1) + \dot{y}_r(t+1) - \dot{y}_r(t-1).$$

Optimal control

- **Problem**: Let us minimize the **cost function**

$$\frac{1}{2}\int_0^T (x(t)^2 + u(t)^2)\,dt$$

where $\dot{x}(t) + x(t) - u(t) = 0$, $x(0) = x_0$.

- $\dot{x}(t) + x(t) - u(t) = 0$ is **parametrized** by:

$$\left\{\begin{array}{l} \xi(t) = x(t), \\ \dot{\xi}(t) + \xi(t) = u(t). \end{array}\right. \qquad (\star)$$

- By substitution of $(\star)$ in the cost, we are led to:

minimize $\frac{1}{2}\int_0^T (\xi(t)^2 + (\dot{\xi}(t) + \xi(t))^2)\,dt$.

We obtain the system

$$\left\{\begin{array}{l} \xi(t) = x(t), \\ \dot{\xi}(t) + \xi(t) = u(t), \\ \ddot{\xi}(t) - 2\,\xi(t) = 0, \\ \dot{\xi}(T) + \xi(T) = 0, \\ \xi(0) = x_0, \end{array}\right.$$

which, by integrations, gives the **optimal controller**:

$$u(t) = \frac{-e^{\sqrt{2}(t-T)} + e^{-\sqrt{2}(t-T)}}{(1-\sqrt{2})\,e^{\sqrt{2}(t-T)} - (1+\sqrt{2})\,e^{-\sqrt{2}(t-T)}}\,x(t).$$

Optimal control

- **Let us extremize the cost**

$$I = \int_0^T \frac{1}{2} \left(x_1^2(t) + x_2^2(t) + u^2(t) \right) dt, \quad (1)$$

under the **differential constraint**:

$$\begin{cases} \dot{x}_1 = x_2 + u, & x_1(0) = x_1^0, \\ \dot{x}_2 = x_1 + u, & x_2(0) = x_2^0. \end{cases} \quad (2)$$

- The system (2) can be **parametrized** by:

$$\begin{cases} x_1(t) = (x_1^0 - x_2^0)\, e^{-t} + \xi(t), \\ x_2(t) = \xi(t), \\ u(t) = (x_1^0 - x_2^0)\, e^{-t} + \dot{\xi}(t) - \xi(t). \end{cases} \quad (3)$$

- **Substituting (3) into (2), we obtain a variational problem without differential constraint.**

$\Rightarrow$ **Euler-Lagrange equations** give:

$$\begin{cases} \ddot{\xi}(t) - 3\,\xi(t) = (x_1^0 - x_2^0)\, e^{-t}, \\ \dot{\xi}(T) - \xi(T) = (x_1^0 - x_2^0)\, e^{-T}, \\ \xi(0) = x_2^0, \end{cases} \quad (3)$$

- **Integrating** (3), we obtain:

$$\xi(t) = -\frac{1}{2} \frac{(-2\, e^{-\sqrt{3}t} + \sqrt{3}\, e^{-\sqrt{3}t} - e^{\sqrt{3}(t-2T)} + e^{-t-2\sqrt{3}T} + 2\, e^{-t} - e^{-t}\sqrt{3})}{e^{-2\sqrt{3}T} + 2 - \sqrt{3}} (x_1^0 - x_2^0)$$
$$- \frac{1}{2} \frac{(-4\, e^{-\sqrt{3}t} - 2\, e^{\sqrt{3}(t-2T)} + 2\sqrt{3}\, e^{-\sqrt{3}t})}{e^{-2\sqrt{3}T} + 2 - \sqrt{3}}\, x_2^0.$$

Optimal control

- **Substituting** ξ into (3), we obtain

$$\left\{ \begin{array}{rcl} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} & = & P(t) \begin{pmatrix} x_1^0 - x_2^0 \\ x_2^0 \end{pmatrix}, \\ u(t) & = & Q(t) \begin{pmatrix} x_1^0 - x_2^0 \\ x_2^0 \end{pmatrix}, \end{array} \right. \qquad (4)$$

where P is $1/2$ of the matrix

$$\begin{pmatrix} \frac{2\,e^{-t} - e^{-t}\sqrt{3} + 2\,e^{-\sqrt{3}\,t} - \sqrt{3}\,e^{-\sqrt{3}\,t} + e^{\sqrt{3}\,(t-2\,T)} + e^{-t-2\sqrt{3}\,T}}{e^{-2\sqrt{3}\,T} + 2 - \sqrt{3}} & \frac{2\,e^{\sqrt{3}\,(t-2\,T)} + 4\,e^{-\sqrt{3}\,t} - 2\sqrt{3}\,e^{-\sqrt{3}\,t}}{e^{-2\sqrt{3}\,T} + 2 - \sqrt{3}} \\ \frac{2\,e^{-\sqrt{3}\,t} - \sqrt{3}\,e^{-\sqrt{3}\,t} + e^{\sqrt{3}\,(t-2\,T)} - e^{-t-2\sqrt{3}\,T} - 2\,e^{-t} + e^{-t}\sqrt{3}}{e^{-2\sqrt{3}\,T} + 2 - \sqrt{3}} & \frac{4\,e^{-\sqrt{3}\,t} + 2\,e^{\sqrt{3}\,(t-2\,T)} - 2\sqrt{3}\,e^{-\sqrt{3}\,t}}{e^{-2\sqrt{3}\,T} + 2 - \sqrt{3}} \end{pmatrix},$$

and Q is $1/2$ of the matrix:

$$\begin{pmatrix} \frac{-\sqrt{3}\,e^{-\sqrt{3}\,t} - e^{\sqrt{3}\,(t-2\,T)} + \sqrt{3}\,e^{\sqrt{3}\,(t-2\,T)} + e^{-\sqrt{3}\,t}}{e^{-2\sqrt{3}\,T} + 2 - \sqrt{3}} & \frac{-2\sqrt{3}\,e^{-\sqrt{3}\,t} - 2\,e^{\sqrt{3}\,(t-2\,T)} + 2\,e^{-\sqrt{3}\,t} + 2\sqrt{3}\,e^{\sqrt{3}\,(t-2\,T)}}{e^{-2\sqrt{3}\,T} + 2 - \sqrt{3}} \end{pmatrix}.$$

- **Eliminating** $x_1^0 - x_2^0$ and x_2^0 from (4), we obtain the **optimal controller**

$$u(t) = K(t) \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix},$$

where $K = Q\,P^{-1}$ is defined by:

$$K = -\frac{1}{2} \begin{pmatrix} \frac{-\sqrt{3}\,e^{-\sqrt{3}\,t} - e^{\sqrt{3}\,(t-2\,T)} + \sqrt{3}\,e^{\sqrt{3}\,(t-2\,T)} + e^{-\sqrt{3}\,t}}{-2\,e^{-\sqrt{3}\,t} - e^{\sqrt{3}\,(t-2\,T)} + \sqrt{3}\,e^{-\sqrt{3}\,t}} & \frac{-\sqrt{3}\,e^{-\sqrt{3}\,t} - e^{\sqrt{3}\,(t-2\,T)} + \sqrt{3}\,e^{\sqrt{3}\,(t-2\,T)} + e^{-\sqrt{3}\,t}}{-2\,e^{-\sqrt{3}\,t} - e^{\sqrt{3}\,(t-2\,T)} + \sqrt{3}\,e^{-\sqrt{3}\,t}} \end{pmatrix}.$$

Variational problems

• We extremize **the electromagnetism Lagragian**

$$\int \left(\frac{1}{2\,\mu_0} \parallel \vec{B} \parallel^2 - \frac{\epsilon_0}{2} \parallel \vec{E} \parallel^2 \right) dx_1\, dx_2\, dx_3\, dt \quad (1)$$

where $\vec{B}$ and $\vec{E}$ satisfy:

$$\begin{cases} \vec{\nabla}.\vec{B} = 0, \\ \vec{\nabla} \wedge \vec{E} + \dfrac{\partial \vec{B}}{\partial t} = 0. \end{cases} \qquad (2)$$

• System (2) is **parametrizable**:

$$\begin{cases} \frac{\partial \vec{B}}{\partial t} + \vec{\nabla} \wedge \vec{E} = \vec{0}, \\ \vec{\nabla}\,.\,\vec{B} = 0, \end{cases} \Leftrightarrow \begin{cases} \vec{E} = -\vec{\nabla}\,V - \frac{\partial \vec{A}}{\partial t}, \\ \vec{B} = \vec{\nabla} \wedge \vec{A}. \end{cases} \qquad (3)$$

• Substituting (3) in (1), we obtain a **variational problem in $\vec{A}$ and V without differential constraint.**

The variation of this problem gives ($c^2 = 1/(\epsilon_0\,\mu_0)$)

$$\begin{cases} \dfrac{1}{c^2}\dfrac{\partial^2 \vec{A}}{\partial t^2} - \Delta\,\vec{A} = 0, \\ \dfrac{1}{c^2}\dfrac{\partial^2 V}{\partial t^2} - \Delta\,V = 0, \\ \vec{\nabla} \wedge \vec{A} = \vec{B}, \\ -\vec{\nabla} V - \dfrac{\partial \vec{A}}{\partial t} = \vec{E}, \end{cases} \quad \text{(\textbf{electromagnetic waves})}.$$

up to the **gauge condition** $\vec{\nabla}.\vec{A} + \frac{1}{c^2}\frac{\partial V}{\partial t} = 0.$

Variational problems

• Let us consider the linear system of PDEs:

$$(\partial_1 + 1)\,(\partial_1\, y_1(x) + \partial_2\, y_2(x)) = 0. \quad (1)$$

• **Let us extremize**

$$I = \frac{1}{2} \int\int_{\Omega} (y_1^2(x) + y_2^2(x))\, dx_1\, dx_2, \quad (2)$$

under the **differential constraint** (1).

• **All $\mathcal{F} = C^{\infty}(\mathbb{R}^2)$-solutions** of (1) are given by

$$\begin{cases} y_1(x) = \Phi(x_2)\, e^{-x_1} - \partial_2\, \xi(x), \\ y_2(x) = \partial_1\, \xi(x), \end{cases} \quad (3)$$

for all $\xi \in \mathcal{F}$ and $\Phi \in C^{\infty}(\mathbb{R})$.

• **Substituting (3) into (2), we obtain a variational problem without differential constraint.**

$\Rightarrow$ **Euler-Lagrange equations** give:

$$\begin{cases} \Delta\, \xi(x) = \dot{\Phi}(x_2)\, e^{-x_1}, \\ y_1(x) = \Phi(x_2)\, e^{-x_1} - \partial_2\, \xi(x), \\ y_2(x) = \partial_1\, \xi(x). \end{cases}$$

Conclusion

• Using the **parametrization of controllable multidimensional systems**, **classical problems have recently been revisited**:

⋆ **Controllability:** a patching solution problem.

⋆ **Dynamic placement** by solving Bézout equations.

⋆ **Optimal control & variational problems**.

• Using the **general parametrization of uncontrollable multidimensional systems** developed here, we can extend the previous results.

⇒ For more details, see:

A. Q., D. Robertz, "**Parametrizing all solutions of uncontrollable multidimensional linear systems**", proceedings of the 16^{th} IFAC World Congress, Prague (04-08/07/05),

A. Q., D. Robertz, "**On the Monge problem for multidimensional linear systems**", submitted for publication.

⇒ For optimal control problems, see the **implementation in** OREMODULES for 1-D linear systems.