A constructive version of Fitting's theorem on isomorphisms and equivalences of linear systems

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***Abstract*— Within the algebraic analysis approach to linear system theory, a multidimensional linear system can be studied by means of its associated finitely presented left module. Testing whether two linear systems/modules are isomorphic (the so-called equivalence problem) is an important issue in system/module theory. In this paper, we explicitly characterize the conditions for a homomorphism between two finitely presented left modules to define an isomorphism, and we give an explicit formula for the inverse of an isomorphism. Then, we constructively study Fitting's major theorem, which shows how to enlarge matrices presenting isomorphic modules by blocks of 0 and I to get equivalent matrices. The consequences of this result on the Auslander transposes and adjoints of the finitely presented left modules are given. The different results developed in this paper are implemented in the OREMORPHISMS package.**

I. INTRODUCTION

A linear multidimensional system (e.g., a linear system of ordinary differential (OD) equations, partial differential (PD) equations, OD time-delay equations, difference equations) can generally be written as $R\,\eta = 0$, where R is a $q \times p$ matrix with entries in a noncommutative polynomial ring D of functional operators (e.g., OD or PD operators, time-delay operators, shift operators, difference operators) and η is a vector of unknown functions. More precisely, if $\mathcal{F}$ is a left D-module, then we consider the *linear system* or *behaviour*:

$$\ker_{\mathcal{F}}(R.) = \{\eta \in \mathcal{F}^p \mid R\,\eta = 0\}.$$

The algebraic analysis approach to mathematical system theory (see, e.g., [3], [4], [14], [15], [17], [18], [22]) is based on the fact that the linear system $\ker_{\mathcal{F}}(R.)$ can be studied by means of the left D-module $M = D^{1\times p}/(D^{1\times q}\,R)$ *finitely presented by the matrix* R [21]. Indeed, Malgrange's remark [14] asserts that $\ker_{\mathcal{F}}(R.) \cong \hom_D(M, \mathcal{F})$, where $\hom_D(M, \mathcal{F})$ (resp., $\cong$) denotes the abelian group of left D-homomorphisms (i.e., left D-linear maps) from M to $\mathcal{F}$ (resp., an isomorphism). Hence, systemic properties of $\ker_{\mathcal{F}}(R.)$ can be studied by means of module properties of M and $\mathcal{F}$. Algorithms for checking certain module properties of M were recently developed in [4], [6], [17], [18], [22] based on constructive homological algebra for noncommutative polynomial rings D admitting Gröbner bases for admissible term orders. These algorithms were implemented in the packages and the computer algebra systems OREMODULES [5], OREMORPHISMS [7], Plural [13] and `homalg` [2].

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The purpose of this paper is to develop a constructive version of Fitting's result [10] (see [9], [11] for a modern formulation) which asserts that two matrices presenting isomorphic left D-modules can be enlarged by blocks of 0 and I (identity matrix) to get equivalent matrices. This important result in module theory explains the relations between the key concepts of isomorphism of modules and equivalence of matrices, and has many applications in linear system theory.

The paper is organized as follows. In Section II, we first recall the explicit characterization of a left D-homomorphism between two finitely presented left D-modules developed in [6]. Then, in Section III, we show how to explicitly characterize isomorphisms and we provide a formula for the inverse of an isomorphism. Section IV constructively studies Fitting's theorem. Finally, in Section V, we give a new constructive proof of the fact that the *Auslander transpose* [1] and the *adjoint* [4] of a finitely presented left D-module M depend only on M up to a *projective equivalence* [21]. This result, first due to Auslander [1] (see also [19]), plays an important role in the algebraic analysis approach.

II. HOMOMORPHISMS

In this section, we first recall the characterization of left D-homomorphisms of finitely presented left D-modules. In what follows, we suppose that D is a *noetherian ring* [21].

Lemma 1 ([6], [19], [21]): Let us consider the following *finite presentation* of the left D-modules M and M'

$$\begin{array}{ccccccc} D^{1\times q} & \xrightarrow{.R} & D^{1\times p} & \xrightarrow{\pi} & M & \longrightarrow 0, \\ D^{1\times q'} & \xrightarrow{.R'} & D^{1\times p'} & \xrightarrow{\pi'} & M' & \longrightarrow 0, \end{array}$$

namely, *exact sequences* [21] where $(.R)(\mu) = \mu\,R$ for all $\mu \in D^{1\times q}$, and π is the canonical projection onto the left D-module $M = D^{1\times p}/(D^{1\times q}\,R)$ (similarly for $.R'$ and π').

1) The existence of $f \in \hom_D(M, M')$ is equivalent to the existence of $P \in D^{p\times p'}$ and $Q \in D^{q\times q'}$ satisfying:

$$R\,P = Q\,R'. \quad (1)$$

Then, the following *commutative exact diagram* [21]

$$\begin{array}{ccccccc} D^{1\times q} & \xrightarrow{.R} & D^{1\times p} & \xrightarrow{\pi} & M & \longrightarrow 0 \\ \downarrow .Q & & \downarrow .P & & \downarrow f & \\ D^{1\times q'} & \xrightarrow{.R'} & D^{1\times p'} & \xrightarrow{\pi'} & M' & \longrightarrow 0 \end{array} \quad (2)$$

holds, where $f \in \hom_D(M, M')$ is defined by:

$$\forall\, \lambda \in D^{1\times p}, \quad f(\pi(\lambda)) = \pi'(\lambda\, P). \tag{3}$$

2) Let $R'_2 \in D^{q'_2 \times q'}$ be such that $\ker_D(.R') = D^{1\times q'_2}\, R'_2$ and let $P \in D^{p\times p'}$ and $Q \in D^{q\times q'}$ be two matrices satisfying $R\, P = Q\, R'$. Then, the matrices defined by

$$\begin{cases} \overline{P} = P + Z\, R', \\ \overline{Q} = Q + R\, Z + Z_2\, R'_2, \end{cases}$$

where $Z \in D^{p\times q'}$ and $Z_2 \in D^{q\times q'_2}$ are two arbitrary matrices, satisfy $R\, \overline{P} = \overline{Q}\, R'$ and:

$$\forall\, \lambda \in D^{1\times p}, \quad f(\pi(\lambda)) = \pi'(\lambda\, P) = \pi'(\lambda\, \overline{P}).$$

Let $f : M \longrightarrow M'$ be a left D-homomorphism of left D-modules. Then, we can define the following left D-modules:

$$\begin{cases} \ker f = \{m \in M \mid f(m) = 0\}, \\ \operatorname{im} f = \{m' \in M' \mid \exists\, m \in M : \; m' = f(m)\}, \\ \operatorname{coim} f = M/\ker f, \\ \operatorname{coker} f = M'/\operatorname{im} f. \end{cases}$$

Let us explicitly characterize the kernel, image, coimage and cokernel of $f \in \hom_D(M, M')$ when M and M' are two finitely presented left D-modules [6], [16].

Proposition 1 ([6]): Let $M = D^{1\times p}/(D^{1\times q}\, R)$ (resp., $M' = D^{1\times p'}/(D^{1\times q'}\, R')$) be a left D-module finitely presented by $R \in D^{q\times p}$ (resp., $R' \in D^{q'\times p'}$). Moreover, let $f \in \hom_D(M, M')$ be defined by $P \in D^{p\times p'}$ and $Q \in D^{q\times q'}$ satisfying $R\, P = Q\, R'$.

1) Let $S \in D^{r\times p}$ and $T \in D^{r\times q'}$ be such that

$$\ker_D\left(.(P^T \quad R'^T)^T\right) = D^{1\times r}\, (S \quad -T), \tag{4}$$

$L \in D^{q\times r}$ a matrix satisfying $R = L\, S$ and a matrix $S_2 \in D^{r_2\times r}$ such that $\ker_D(.S) = D^{1\times r_2}\, S_2$. Then:

$$\begin{aligned} \ker f &= (D^{1\times r}\, S)/(D^{1\times q}\, R) \\ &\cong D^{1\times r}/\left(D^{1\times (q+r_2)}\, \left(L^T \quad S_2^T\right)^T\right). \end{aligned}$$

2) With the above notations, we have:

$$\begin{aligned} \operatorname{coim} f &= D^{1\times p}/(D^{1\times r}\, S) \\ \cong \operatorname{im} f &= \left(D^{1\times (p+q')}\, \left(P^T \quad R'^T\right)^T\right)/(D^{1\times q'}\, R'). \end{aligned}$$

3) $\operatorname{coker} f = D^{1\times p'}/\left(D^{1\times (p+q')}\, \left(P^T \quad R'^T\right)^T\right)$, and thus the left D-module $\operatorname{coker} f$ admits the following beginning of a *finite free resolution* [21]:

$$D^{1\times r} \xrightarrow{.(S \quad -T)} D^{1\times (p+q')} \xrightarrow{.(P^T\ R'^T)^T} D^{1\times p'} \xrightarrow{\epsilon} \operatorname{coker} f \longrightarrow 0. \tag{5}$$

4) The following commutative exact diagram

$$\begin{tikzcd} & & 0 \arrow[d] & \\ D^{1\times r} \arrow[r, ".S"] \arrow[d, ".T"] & D^{1\times p} \arrow[r, "\kappa"] \arrow[d, ".P"] & \operatorname{coim} f \arrow[r] \arrow[d, "f^\sharp"] & 0 \\ D^{1\times q'} \arrow[r, ".R'"] & D^{1\times p'} \arrow[r, "\pi'"] & M' \arrow[r] \arrow[d] & 0 \\ & & \operatorname{coker} f \arrow[d] & \\ & & 0 & \end{tikzcd} \tag{6}$$

holds, where $f^\sharp : \operatorname{coim} f \longrightarrow M'$ is defined by:

$$\forall\, \lambda \in D^{1\times p}, \quad f^\sharp(\kappa(\lambda)) = \pi'(\lambda\, P).$$

III. Isomorphisms

In this section, we first recall a result of [6] which allows us to decide when $f \in \hom_D(M, M')$ is zero, injective, surjective or defines an isomorphism. See also [16].

Lemma 2 ([6]): With the notations of Proposition 1, the left D-homomorphism $f : M \longrightarrow M'$ is:

1) The zero homomorphism, i.e., $f = 0$, iff one of the following equivalent conditions holds:
   a) There exists $Z \in D^{p\times q'}$ such that $P = Z\, R'$. Then, there exists a matrix $Z' \in D^{q\times q'_2}$ such that $Q = R\, Z + Z'\, R'_2$, where $R'_2 \in D^{q'_2\times q'}$ is any matrix satisfying $\ker_D(.R') = D^{1\times q'_2}\, R'_2$.
   b) The matrix S admits a left inverse, i.e., there exists $X \in D^{p\times r}$ such that $X\, S = I_p$.
2) Injective, i.e., $\ker f = 0$, iff one of the following equivalent conditions holds:
   a) There exists $F \in D^{r\times q}$ such that $S = F\, R$. If $\rho : M \longrightarrow \operatorname{coim} f = M/\ker f$ is the canonical projection onto $\operatorname{coim} f$, then we have the following commutative exact diagram:

$$\begin{tikzcd} & 0 & 0 & \\ D^{1\times q} \arrow[r, ".R"] & D^{1\times p} \arrow[r, "\pi"] \arrow[u] & M \arrow[r] \arrow[u] & 0 \\ D^{1\times r} \arrow[r, ".S"] \arrow[u, ".F"] & D^{1\times p} \arrow[r, "\kappa"] \arrow[u, equal] & \operatorname{coim} f \arrow[r] \arrow[u, "\rho^{-1}"] & 0. \\ & 0 \arrow[u] & 0 \arrow[u] & \end{tikzcd}$$

   b) The matrix $(L^T \quad S_2^T)^T$ admits a left inverse.
3) Surjective, i.e., $\operatorname{im} f = M'$, iff $(P^T \quad R'^T)^T$ admits a left inverse. Then, the long exact sequence (5) *splits* [21], i.e., there exist $(X \quad Y) \in D^{p'\times (p+q')}$ and $(U^T \quad V^T)^T \in D^{(p+q')\times r}$, where $X \in D^{p'\times p}$, $Y \in D^{p'\times q'}$, $U \in D^{p\times r}$ and $V \in D^{q'\times r}$, such that the following identities hold:

$$\begin{cases} X\, P + Y\, R' = I_{p'}, \\ P\, X + U\, S = I_p, \\ P\, Y - U\, T = 0, \\ R'\, X + V\, S = 0, \\ R'\, Y - V\, T = I_{q'}. \end{cases} \tag{7}$$

We have the following commutative exact diagram:

$$\begin{array}{ccccccc}
 & & & & 0 & & \\
 & & & & \uparrow & & \\
D^{1\times r} & \xrightarrow{.S} & D^{1\times p} & \xrightarrow{\kappa} & \mathrm{coim}\, f & \longrightarrow & 0 \\
\uparrow .-V & & \uparrow .X & & \uparrow f^{\sharp -1} & & \\
D^{1\times q'} & \xrightarrow{.R'} & D^{1\times p'} & \xrightarrow{\pi'} & M' & \longrightarrow & 0. \\
 & & & & \uparrow & & \\
 & & & & 0 & &
\end{array}$$

4) An isomorphism, i.e., $M \cong M'$, iff the matrices $(L^T \quad S_2^T)^T$ and $(P^T \quad R'^T)^T$ admit a left inverse. The inverse f^{-1} of f is then defined by
$$\forall\, \lambda' \in D^{1\times p'}, \quad f^{-1}(\pi'(\lambda')) = \pi(\lambda'\, X),$$
where $X \in D^{p'\times p}$ is defined in 3. Moreover, we have the following commutative exact diagram:
$$\begin{array}{ccccccc}
D^{1\times q} & \xrightarrow{.R} & D^{1\times p} & \xrightarrow{\pi} & M & \longrightarrow & 0 \\
\uparrow .-V\,F & & \uparrow .X & & \uparrow f^{-1} & & \\
D^{1\times q'} & \xrightarrow{.R'} & D^{1\times p'} & \xrightarrow{\pi'} & M' & \longrightarrow & 0.
\end{array} \tag{8}$$

We can now characterize the inverse of an isomorphism.

Proposition 2: Let $R \in D^{q\times p}$, $R' \in D^{q'\times p'}$ and
$$\begin{array}{rccl}
f : M = D^{1\times p}/(D^{1\times q}\, R) & \longrightarrow & M' = D^{1\times p'}/(D^{1\times q'}\, R') \\
\pi(\lambda) & \longmapsto & \pi'(\lambda\, P)
\end{array}$$
be a left D-isomorphism, where $P \in D^{p\times p'}$ is a matrix such that $R\,P = Q\,R'$ for a certain matrix $Q \in D^{q\times q'}$.

1) f admits a right inverse $g \in \hom_D(M', M)$, namely $f \circ g = \mathrm{id}_{M'}$, i.e., we have $M \cong \ker f \oplus M'$, iff there exist three matrices $P' \in D^{p'\times p}$, $Q' \in D^{q'\times q}$ and $Z' \in D^{p'\times q'}$ satisfying the following relations:
$$\left\{\begin{array}{l} R'\,P' = Q'\,R, \\ P'\,P + Z'\,R' = I_{p'}. \end{array}\right. \tag{9}$$
Then, there exists a matrix $Z_2' \in D^{q'\times r'}$ satisfying $Q'\,Q + R'\,Z' + Z_2'\,R_2' = I_{q'}$, where $R_2' \in D^{r'\times q'}$ is a matrix such that $\ker_D(.R') = D^{1\times r'}\,R_2'$.

2) f admits a left inverse $g \in \hom_D(M', M)$, namely $g \circ f = \mathrm{id}_M$, i.e., we have $M' \cong M \oplus \mathrm{coker}\, f$, iff there exist three matrices $P' \in D^{p'\times p}$, $Q' \in D^{q'\times q}$ and $Z \in D^{p\times q}$ satisfying the following relations:
$$\left\{\begin{array}{l} R'\,P' = Q'\,R, \\ P\,P' + Z\,R = I_p. \end{array}\right. \tag{10}$$
Then, there exists a matrix $Z_2 \in D^{q\times r}$ satisfying $Q\,Q' + R\,Z + Z_2\,R_2 = I_q$, where $R_2 \in D^{r\times q}$ is a matrix such that $\ker_D(.R) = D^{1\times r}\,R_2$.

3) f is a left D-isomorphism, i.e., $f \in \mathrm{iso}_D(M, M')$, iff there exist $P' \in D^{p'\times p}$, $Q' \in D^{q'\times q}$, $Z \in D^{p\times q}$ and $Z' \in D^{p'\times q'}$ satisfying the following relations:
$$\left\{\begin{array}{l} R'\,P' = Q'\,R, \\ P\,P' + Z\,R = I_p, \\ P'\,P + Z'\,R' = I_{p'}. \end{array}\right. \tag{11}$$
Then, there exist $Z_2 \in D^{q\times r}$ and $Z_2' \in D^{q'\times r'}$ satisfying the following relations
$$\left\{\begin{array}{l} Q\,Q' + R\,Z + Z_2\,R_2 = I_q, \\ Q'\,Q + R'\,Z' + Z_2'\,R_2' = I_{q'}, \end{array}\right. \tag{12}$$
where $R_2 \in D^{r\times q}$ (resp., $R_2' \in D^{r'\times q'}$) is such that $\ker_D(.R) = D^{1\times r}\,R_2$ (resp., $\ker_D(.R') = D^{1\times r'}\,R_2'$).

Proof: 1. The existence of $g \in \hom_D(M', M)$ is equivalent to the existence of two matrices $P' \in D^{p'\times p}$ and $Q' \in D^{q'\times q}$ such that $R'\,P' = Q'\,R$ (see 1 of Lemma 1). Composing the following two commutative exact diagrams
$$\begin{array}{ccccccc}
D^{1\times q} & \xrightarrow{.R} & D^{1\times p} & \xrightarrow{\pi} & M & \longrightarrow & 0 \\
\downarrow .Q & & \downarrow .P & & \downarrow f & & \\
D^{1\times q'} & \xrightarrow{.R'} & D^{1\times p'} & \xrightarrow{\pi'} & M' & \longrightarrow & 0,
\end{array}$$
$$\begin{array}{ccccccc}
D^{1\times q} & \xrightarrow{.R} & D^{1\times p} & \xrightarrow{\pi} & M & \longrightarrow & 0 \\
\uparrow .Q' & & \uparrow .P' & & \uparrow g & & \\
D^{1\times q'} & \xrightarrow{.R'} & D^{1\times p'} & \xrightarrow{\pi'} & M' & \longrightarrow & 0,
\end{array}$$
and denoting by $\chi = \mathrm{id}_{M'} - f \circ g$, we obtain the following commutative exact diagram:
$$\begin{array}{ccccccc}
D^{1\times q'} & \xrightarrow{.R'} & D^{1\times p'} & \xrightarrow{\pi'} & M' & \longrightarrow & 0 \\
\uparrow .(I_{q'} - Q'\,Q) & & \uparrow .(I_{p'} - P'\,P) & & \uparrow \chi & & \\
D^{1\times q'} & \xrightarrow{.R'} & D^{1\times p'} & \xrightarrow{\pi'} & M' & \longrightarrow & 0.
\end{array}$$
By 1.a of Lemma 2, $\chi = 0$ iff there exists $Z' \in D^{p'\times q'}$ such that $I_{p'} - P'\,P = Z'\,R'$, i.e., $P'\,P + Z'\,R' = I_{p'}$, which proves the result since the following short exact sequence
$$0 \longrightarrow \ker f \longrightarrow M \;\underset{\xleftarrow{g}}{\xrightarrow{f}}\; M' \longrightarrow 0$$
then splits [21], namely, $M \cong \ker f \oplus M'$. According to 1.a of Lemma 2, there exists $Z_2' \in D^{q'\times r'}$ satisfying the relation $I_{q'} - Q'\,Q = R'\,Z' + Z_2'\,R_2'$, i.e., $Q'\,Q + R'\,Z' + Z_2'\,R_2' = I_{q'}$, where $R_2' \in D^{r'\times q'}$ is such that $\ker_D(.R') = D^{1\times r'}\,R_2'$.

2. Repeating the same arguments as in 1 with the left D-homomorphism $\delta = \mathrm{id}_M - g \circ f$, we obtain the following commutative exact diagram:
$$\begin{array}{ccccccc}
D^{1\times q} & \xrightarrow{.R} & D^{1\times p} & \xrightarrow{\pi} & M & \longrightarrow & 0 \\
\downarrow .(I_q - Q\,Q') & & \downarrow .(I_p - P\,P') & & \downarrow \delta & & \\
D^{1\times q} & \xrightarrow{.R} & D^{1\times p} & \xrightarrow{\pi} & M & \longrightarrow & 0.
\end{array}$$
By 1.a of Lemma 2, $\delta = 0$ iff there exists $Z \in D^{p\times q}$ such that $I_p - P\,P' = Z\,R$, i.e., $P\,P' + Z\,R = I_p$, which proves the result because the following short exact sequence
$$0 \longrightarrow M \;\underset{\xleftarrow{g}}{\xrightarrow{f}}\; M' \longrightarrow \mathrm{coker}\, f \longrightarrow 0$$
then splits, i.e., $M' \cong M \oplus \mathrm{coker}\, f$. Finally, using 1.a of Lemma 2, there exists $Z_2 \in D^{q\times r}$ satisfying the relation $I_q - Q\,Q' = R\,Z + Z_2\,R_2$, i.e., $Q\,Q' + R\,Z + Z_2\,R_2 = I_q$, where $R_2 \in D^{r\times q}$ is such that $\ker_D(.R) = D^{1\times r}\,R_2$.

3. This is a direct consequence of 1 and 2. ■

IV. Isomorphisms and equivalences

In this section, we constructively study Fitting's theorem [10] (see [9], [11] for a modern formulation) which explains the relations between isomorphisms of finitely presented left D-modules and equivalences of their presentation matrices.

Lemma 3: Let $R \in D^{q \times p}$, $R' \in D^{q' \times p'}$ and

$$\begin{array}{rcl} f : M = D^{1\times p}/(D^{1\times q}\, R) & \longrightarrow & M' = D^{1\times p'}/(D^{1\times q'}\, R') \\ \pi(\lambda) & \longmapsto & \pi'(\lambda\, P) \end{array}$$

be a left D-isomorphism, where $P \in D^{p \times p'}$ is a matrix such that $R\,P = Q\,R'$ for a certain matrix $Q \in D^{q \times q'}$. Then, there exist four matrices $P' \in D^{p' \times p}$, $Q' \in D^{q' \times q}$, $Z \in D^{p \times q}$ and $Z' \in D^{p' \times q'}$ such that $R'\,P' = Q'\,R$, $I_p = P\,P' + Z\,R$ and $I_{p'} = P'\,P + Z'\,R'$. Moreover, the following results hold:

1) The following two matrices

$$U = \begin{pmatrix} I_p & P \\ -P' & I_{p'} - P'\,P \end{pmatrix}, \; V = \begin{pmatrix} I_p - P\,P' & -P \\ P' & I_{p'} \end{pmatrix},$$

are *unimodular*, namely invertible, i.e.,

$$\begin{array}{l} U,\, V \in \mathrm{GL}_{p+p'}(D) = \{X \in D^{(p+p')\times(p+p')} \mid \\ \exists\, Y \in D^{(p+p')\times(p+p')} :\; X\,Y = Y\,X = I_{p+p'}\}, \end{array}$$

and are such that $V = U^{-1}$.

2) The following commutative exact diagram holds

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ D^{1\times(q+p')} & \xrightarrow{.\mathrm{diag}(R, I_{p'})} & D^{1\times(p+p')} & \xrightarrow{\pi \oplus 0_{p'}} & M & \longrightarrow & 0 \\ \downarrow .W & & \downarrow .U & & \downarrow f & & \\ D^{1\times(p+q')} & \xrightarrow{.\mathrm{diag}(I_p, R')} & D^{1\times(p+p')} & \xrightarrow{0_p \oplus \pi'} & M' & \longrightarrow & 0, \\ & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & \end{array}$$

with the following notations

$$W = \begin{pmatrix} R & Q \\ -P' & Z' \end{pmatrix} \in D^{(q+p')\times(p+q')},$$

$$\mathrm{diag}(R, I_{p'}) = \begin{pmatrix} R & 0 \\ 0 & I_{p'} \end{pmatrix}, \mathrm{diag}(I_p, R') = \begin{pmatrix} I_p & 0 \\ 0 & R' \end{pmatrix},$$

$$\begin{array}{ccccccc} D^{1\times(p+p')} & \xrightarrow{\pi \oplus 0_{p'}} & M & & D^{1\times(p'+p)} & \xrightarrow{0_p \oplus \pi'} & M' \\ (\lambda \quad \lambda') & \longmapsto & \pi(\lambda), & & (\lambda \quad \lambda') & \longmapsto & \pi'(\lambda'). \end{array} \tag{13}$$

3) If the matrices R and R' have *full row rank*, i.e., $\ker_D(.R) = 0$ and $\ker_D(.R') = 0$, then $q + p' = p + q'$ and the matrix W defined in 2 is unimodular, i.e.:

$$W \in \mathrm{GL}_{(q+p')}(D), \quad W^{-1} = \begin{pmatrix} Z & -P \\ Q' & R' \end{pmatrix}.$$

Finally, $\mathrm{diag}(R, I_{p'})$ and $\mathrm{diag}(I_p, R')$ are *equivalent*:

$$\mathrm{diag}(I_p, R') = W^{-1}\, \mathrm{diag}(R, I_{p'})\, U. \tag{14}$$

Proof: Since $f \in \mathrm{iso}_D(M, M')$ is defined by two matrices $P \in D^{p\times p'}$ and $Q \in D^{q \times q'}$ such that $R\,P = Q\,R'$, using 3 of Proposition 2, the existence of the inverse f^{-1} of f is equivalent to the existence of four matrices $P' \in D^{p'\times p}$, $Q' \in D^{q'\times q}$, $Z \in D^{p\times q}$ and $Z' \in D^{p'\times q'}$ such that $R'\,P' = Q'\,R$, $I_p = P\,P' + Z\,R$ and $I_{p'} = P'\,P + Z'\,R'$. Moreover, we have

$$\begin{aligned} \mathrm{diag}(R, I_{p'})\, U &= \begin{pmatrix} R & 0 \\ 0 & I_{p'} \end{pmatrix} \begin{pmatrix} I_p & P \\ -P' & I_{p'} - P'\,P \end{pmatrix} \\ &= \begin{pmatrix} R & R\,P \\ -P' & I_{p'} - P'\,P \end{pmatrix}, \\ W\, \mathrm{diag}(I_p, R') &= \begin{pmatrix} R & Q \\ -P' & Z' \end{pmatrix} \begin{pmatrix} I_p & 0 \\ 0 & R' \end{pmatrix} \\ &= \begin{pmatrix} R & Q\,R' \\ -P' & Z'\,R' \end{pmatrix}, \end{aligned}$$

which yields $\mathrm{diag}(R, I_{p'})\, U = W\, \mathrm{diag}(I_p, R')$ by the above relations. Moreover, we can check that $U\,V = I_{p+p'}$ and $V\,U = I_{p+p'}$, i.e., $U \in \mathrm{GL}_{p+p'}(D)$, and

$$\begin{aligned} & D^{1\times(p+p')}/(D^{1\times(q+p')}\, \mathrm{diag}(R, I_{p'})) \\ \cong\; & [D^{1\times p}/(D^{1\times q}\, R)] \oplus [D^{1\times p'}/(D^{1\times p'}\, I_{p'})] = D^{1\times p}/(D^{1\times q}\, R), \\ & D^{1\times(p+p')}/(D^{1\times(p+q')}\, \mathrm{diag}(I_p, R')) \\ \cong\; & [D^{1\times p}/(D^{1\times p}\, I_p)] \oplus [D^{1\times p'}/(D^{1\times q'}\, R')] = D^{1\times p'}/(D^{1\times q'}\, R'), \end{aligned}$$

which proves 1 and 2. Moreover, we can easily check that

$$\begin{aligned} Z'\,Q'\,R = Z'\,R'\,P' &= (I_{p'} - P'\,P)\,P' \\ &= P'\,(I_p - P\,P') = P'\,Z\,R, \end{aligned}$$

i.e., $(Z'\,Q' - P'\,Z)\,R = 0$, which yields $Z'\,Q' = P'\,Z$ when R has full row rank. Using 3 of Proposition 2, the identities $Q\,Q' + R\,Z = I_q$ and $Q'\,Q + R'\,Z' = I_{q'}$ hold, and thus

$$\begin{aligned} & \begin{pmatrix} R & Q \\ -P' & Z' \end{pmatrix} \begin{pmatrix} Z & -P \\ Q' & R' \end{pmatrix} \\ &= \begin{pmatrix} R\,Z + Q\,Q' & -R\,P + Q\,R' \\ -P'\,Z + Z'\,Q' & P'\,P + Z'\,R' \end{pmatrix} = I_{q+p'}. \end{aligned}$$

If R' has full row-rank, a similar computation shows that

$$\begin{aligned} & \begin{pmatrix} Z & -P \\ Q' & R' \end{pmatrix} \begin{pmatrix} R & Q \\ -P' & Z' \end{pmatrix} \\ &= \begin{pmatrix} Z\,R + P\,P' & Z\,Q - P\,Z' \\ Q'\,R - R'\,P' & Q'\,Q + R'\,Z' \end{pmatrix} = I_{p+q'}, \end{aligned}$$

which yields $q + p' = q' + p$, $W \in \mathrm{GL}_{q+p'}(D)$ and (14). ■

Example 1: Let us consider two linear PD systems used in the theory of linear elasticity, namely the Lie derivative of the euclidean metric of $\mathbb{R}^2$ and its *Spencer operator* [17]:

$$\left\{ \begin{array}{l} \partial_1\,\xi_1 = 0, \\ \frac{1}{2}\,(\partial_2\,\xi_1 + \partial_1\,\xi_2) = 0, \\ \partial_2\,\xi_2 = 0, \end{array} \right. \qquad \left\{ \begin{array}{l} \partial_1\,\zeta_1 = 0, \\ \partial_2\,\zeta_1 - \zeta_2 = 0, \\ \partial_1\,\zeta_2 = 0, \\ \partial_1\,\zeta_3 + \zeta_2 = 0, \\ \partial_2\,\zeta_3 = 0, \\ \partial_2\,\zeta_2 = 0. \end{array} \right.$$

Let $D = \mathbb{Q}[\partial_1, \partial_2]$ be the commutative polynomial ring of PD operators in ∂_1 and ∂_2 with rational constant coefficients,

$$R = \begin{pmatrix} \partial_1 & 0 \\ \frac{1}{2}\,\partial_2 & \frac{1}{2}\,\partial_1 \\ 0 & \partial_2 \end{pmatrix} \in D^{3\times 2},$$
$$R' = \begin{pmatrix} \partial_1 & \partial_2 & 0 & 0 & 0 & 0 \\ 0 & -1 & \partial_1 & 1 & 0 & \partial_2 \\ 0 & 0 & 0 & \partial_1 & \partial_2 & 0 \end{pmatrix}^T \in D^{6\times 3}, \tag{15}$$

$M = D^{1\times 2}/(D^{1\times 3}\, R)$ and $M' = D^{1\times 3}/(D^{1\times 6}\, R')$. In Example 3.2 of [6], we proved that the matrices

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \; Q = \frac{1}{2}\begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \end{pmatrix},$$

define $f \in \mathrm{iso}_D(M, M')$. Applying Lemma 3, we get:

$$U = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 \\ -\partial_2 & 0 & -\partial_2 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 \end{pmatrix} \in \mathrm{GL}_5(D). \tag{16}$$

Moreover, the matrix W defined in Lemma 3 has the form:

$$W = \begin{pmatrix} \partial_1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2}\,\partial_2 & \frac{1}{2}\,\partial_1 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & \partial_2 & 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\partial_2 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \in D^{6\times 8}.$$

Then, we have $\mathrm{diag}(R, I_3)\, U = W\, \mathrm{diag}(I_2, R')$, where the matrix W is not unimodular because it is not a square matrix. The computations can be done using the package OREMORPHISMS [7] built upon OREMODULES [5].

If $R \in D^{q\times p}$ (resp., $R' \in D^{q'\times p'}$) is a presentation matrix of the left D-module M (resp., M') and $M \cong M'$, then we now prove that R and R' can always be enlarged by blocks of 0 and I such that the resulting matrices $L \in D^{(q+p'+p+q')\times(p+p')}$ and $L' \in D^{(q+p'+p+q')\times(p+p')}$ respectively define a finite presentation of M and M', i.e.

$$M = D^{1\times(p+p')}/(D^{1\times(q+p'+p+q')}\, L),$$
$$M' = D^{1\times(p+p')}/(D^{1\times(q+p'+p+q')}\, L'),$$

and L and L' are equivalent, i.e., there exist $X \in \mathrm{GL}_{p+p'}(D)$ and $Y \in \mathrm{GL}_{q+p'+p+q'}(D)$ satisfying $L' = Y^{-1}\, L\, X$. This result is first due to Fitting [10]. For a modern formulation, see [9], [11]. The novelty of our proof is that it is completely constructive in the sense that the matrices X and Y are explicitly given in terms of the isomorphism $M \cong M'$.

Theorem 1: Let $R \in D^{q\times p}$, $R' \in D^{q'\times p'}$ and

$$\begin{array}{rcl} f: M = D^{1\times p}/(D^{1\times q}\, R) & \longrightarrow & M' = D^{1\times p'}/(D^{1\times q'}\, R') \\ \pi(\lambda) & \longmapsto & \pi'(\lambda\, P), \end{array}$$

be a left D-isomorphism, where $P \in D^{p\times p'}$ is a matrix such that $R\, P = Q\, R'$ for a certain matrix $Q \in D^{q\times q'}$. Moreover, let $R_2 \in D^{r\times q}$ (resp., $R_2' \in D^{r'\times q'}$) be a matrix such that $\ker_D(.R) = D^{1\times r}\, R_2$ (resp., $\ker_D(.R') = D^{1\times r'}\, R_2'$). Then, there exist 6 matrices $P' \in D^{p'\times p}$, $Q' \in D^{q'\times q}$, $Z \in D^{p\times q}$, $Z' \in D^{p'\times q'}$, $Z_2 \in D^{q\times r}$ and $Z_2' \in D^{q'\times r'}$ satisfying (11) and (12), and such that the following results hold:

1) With the notation $s = q + p' + p + q'$, we have:

$$X = \begin{pmatrix} I_p & P \\ -P' & I_{p'} - P'\, P \end{pmatrix} \in \mathrm{GL}_{p+p'}(D),$$
$$Y = \begin{pmatrix} I_q & 0 & R & Q \\ 0 & I_{p'} & -P' & Z' \\ -Z & P & 0 & P\, Z' - Z\, Q \\ -Q' & -R' & 0 & Z_2'\, R_2' \end{pmatrix} \in \mathrm{GL}_s(D),$$
$$X^{-1} = \begin{pmatrix} I_p - P\, P' & -P \\ P' & I_{p'} \end{pmatrix},$$
$$Y^{-1} = \begin{pmatrix} Z_2\, R_2 & 0 & -R & -Q \\ P'\, Z - Z'\, Q' & 0 & P' & -Z' \\ Z & -P & I_p & 0 \\ Q' & R' & 0 & I_{q'} \end{pmatrix}. \tag{17}$$

2) The following commutative exact diagram holds

$$\begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\ & D^{1\times(q+p'+p+q')} \arrow[r, ".L"] \arrow[d, ".Y"] & D^{1\times(p+p')} \arrow[r, "\pi\oplus 0_{p'}"] \arrow[d, ".X"] & M \arrow[r] \arrow[d, "f"] & 0 \\ & D^{1\times(q+p'+p+q')} \arrow[r, ".L'"] \arrow[d] & D^{1\times(p+p')} \arrow[r, "0_p\oplus\pi'"] \arrow[d] & M' \arrow[r] \arrow[d] & 0, \\ & 0 & 0 & 0 & \end{tikzcd} \tag{18}$$

where $\pi \oplus 0_{p'}$ and $0_p \oplus \pi'$ are defined by (13), and

$$L = \begin{pmatrix} R & 0 \\ 0 & I_{p'} \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \in D^{(q+p'+p+q')\times(p+p')},$$
$$L' = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ I_p & 0 \\ 0 & R' \end{pmatrix} \in D^{(q+p'+p+q')\times(p+p')},$$

i.e., we have $L\, X = Y\, L'$, and thus:

$$L' = Y^{-1}\, L\, X \quad \Leftrightarrow \quad L = Y\, L'\, X^{-1}.$$

Proof: Since $f \in \mathrm{iso}_D(M, M')$ is defined by the matrices $P \in D^{p\times p'}$ and $Q \in D^{q\times q'}$ satisfying $R\, P = Q\, R'$, then, according to 3 of Proposition 2, the existence of f^{-1} is equivalent to the existence of four matrices $P' \in D^{p'\times p}$, $Q' \in D^{q'\times q}$, $Z \in D^{p\times q}$ and $Z' \in D^{p'\times q'}$ satisfying:

$$\begin{cases} R\, P = Q\, R', \\ R'\, P' = Q'\, R, \end{cases} \qquad \begin{cases} P\, P' + Z\, R = I_p, \\ P'\, P + Z'\, R' = I_{p'}. \end{cases}$$

The fact that X belongs to $\mathrm{GL}_{p+p'}(D)$ can be proved as in Lemma 3. Moreover, 3 of Proposition 2 shows that two matrices $Z_2 \in D^{q \times r}$ and $Z_2' \in D^{q' \times r'}$ exist such that:

$$\left\{ \begin{array}{l} Q\, Q' + R\, Z + Z_2\, R_2 = I_q \\ Q'\, Q + R'\, Z' + Z_2'\, R_2' = I_{q'}. \end{array} \right.$$

Using the above relations, we can prove the identities defined in Fig. 1. We can then check the identity given by Fig. 2, i.e., $Y\, K = I_{q+p'+p+q'}$, where K is the matrix defined in the right-hand side of (17). Similarly, we can easily check that $K\, Y = I_{q+p'+p+q'}$, which proves that $K = Y^{-1}$.

Moreover, we can easily check that:

$$\begin{pmatrix} R & 0 \\ 0 & I_{p'} \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} I_p & P \\ -P' & I_{p'} - P'\, P \end{pmatrix} = \begin{pmatrix} R & R\, P \\ -P' & I_{p'} - P'\, P \\ 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Since $(P\, Z' - Z\, Q)\, R' = 0$, we then have

$$\begin{pmatrix} I_q & 0 & R & Q \\ 0 & I_{p'} & -P' & Z' \\ -Z & P & 0 & P\, Z' - Z\, Q \\ -Q' & -R' & 0 & Z_2'\, R_2' \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ I_p & 0 \\ 0 & R' \end{pmatrix} = \begin{pmatrix} R & R'\, Q \\ -P' & Z'\, R' \\ 0 & (P\, Z' - Z\, Q)\, R' \\ 0 & Z_2'\, R_2'\, R' \end{pmatrix} = \begin{pmatrix} R & R\, P \\ -P' & I_{p'} - P'\, P \\ 0 & 0 \\ 0 & 0 \end{pmatrix},$$

which yields $L\, X = Y\, L'$. Finally, using Lemma 3, we get:

$$\begin{array}{c} D^{1 \times (p+p')}/(D^{1 \times (q+p'+p+q')}\, L) \\ = D^{1 \times (p+p')}/(D^{1 \times (q+p')}\, \mathrm{diag}(R, I_{p'})) \cong M, \\ D^{1 \times (p+p')}/(D^{1 \times (q+p'+p+q')}\, L') \\ = D^{1 \times (p+p')}/(D^{1 \times (p+q')}\, \mathrm{diag}(I_p, R')) \cong M'. \end{array}$$

■

The matrices X, X^{-1}, Y and Y^{-1} can be computed when the pairs of matrices (P, Q) and (P', Q') respectively defining f and f^{-1} are known. The computation of the matrices X and Y and their inverses has been implemented in the OREMORPHISMS package [7].

Example 2: We consider again Example 1. With the notations of Theorem 1, the matrix $X = U \in \mathrm{GL}_5(D)$ is defined by (16) and the matrix $Y \in \mathrm{GL}_{14}(D)$ is defined by Fig. 3. Then, the matrices $L = (\mathrm{diag}(R, I_3)^T \quad 0^T)^T \in D^{14 \times 5}$ and $L' = (0^T \quad \mathrm{diag}(I_2, R')^T) \in D^{14 \times 5}$ are equivalent, namely:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \partial_1 & 0 & 0 \\ 0 & 0 & \partial_2 & -1 & 0 \\ 0 & 0 & 0 & \partial_1 & 0 \\ 0 & 0 & 0 & 1 & \partial_1 \\ 0 & 0 & 0 & 0 & \partial_2 \\ 0 & 0 & 0 & \partial_2 & 0 \end{pmatrix} = Y^{-1} \begin{pmatrix} \partial_1 & 0 & 0 & 0 & 0 \\ \frac{1}{2}\, \partial_2 & \frac{1}{2}\, \partial_1 & 0 & 0 & 0 \\ 0 & \partial_2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} X.$$

Using Theorem 1, we find again *Schanuel's lemma* [21]:

$$\begin{array}{c} \ker_D(.L) = \ker_D(.R) \oplus D^{1 \times (p+q')} \\ \cong \ker_D(.L') = \ker_D(.R') \oplus D^{1 \times (q+p')}. \end{array}$$

V. AUSLANDER TRANSPOSES AND ADJOINTS

In this section, we study the consequences of Theorem 1 on the so-called *Auslander transposes* of two isomorphic finitely generated left D-modules M and M' [1], [4]. Let $M = D^{1 \times p}/(D^{1 \times q}\, R)$ and $M' = D^{1 \times p'}/(D^{1 \times q'}\, R')$ be two left D-modules finitely presented respectively by $R \in D^{q \times p}$ and $R' \in D^{q' \times p'}$, and $f : M \longrightarrow M'$ a left D-isomorphism. Moreover, let $N = D^q/(R\, D^p)$ (resp., $N' = D^{q'}/(R'\, D^{p'})$) be the *Auslander transpose* right D-module of M (resp., M') [1], [4], and $\kappa : D^q \longrightarrow N$ (resp., $\kappa' : D^{q'} \longrightarrow N'$) the canonical projection onto N (resp., N'). With the notations of Theorem 1, we then get:

$$\begin{array}{lcl} \mathrm{coker}_D(L.) & = & D^{(q+p'+p+q')}/(L\, D^{(p+p')}) \\ & \cong & D^q/(R\, D^p) \oplus D^{(p'+p+q')}/(D^{p'}) \\ & \cong & N \oplus D^{(p+q')}, \\ \mathrm{coker}_D(L'.) & = & D^{(q+p'+p+q')}/(L'\, D^{(p+p')}) \\ & \cong & D^{(q+p'+p)}/(D^p) \oplus D^{q'}/(R'\, D^{p'}) \\ & \cong & D^{(q+p')} \oplus N'. \end{array}$$

Now, applying the *contravariant left exact functor* $\hom_D(\cdot\,, D)$ [21] to the commutative exact diagram (18), we get the one given by Fig. 4. Since $Y \in \mathrm{GL}_{(q+p'+p+q')}(D)$, (19) induces the following right D-isomorphism

$$\begin{array}{rcl} \gamma : D^{(q+p')} \oplus N' & \longrightarrow & N \oplus D^{(p+q')} \\ (\mathrm{id}_{q+p'} \oplus 0_p \oplus \kappa')(\lambda') & \longmapsto & (\kappa \oplus 0_{p'} \oplus \mathrm{id}_{p+q'})(Y\, \lambda'), \end{array} \tag{20}$$

which proves that $N \oplus D^{(p+q')} \cong N' \oplus D^{(q+p')}$. We have just constructively proved a result first due to Auslander [1].

Theorem 2 ([1], [19]): Let us consider two finite presentations of a left D-module M:

$$\begin{array}{ccccccc} D^{1 \times q} & \xrightarrow{.R} & D^{1 \times p} & \xrightarrow{\pi} & M & \longrightarrow 0, \\ D^{1 \times q'} & \xrightarrow{.R'} & D^{1 \times p'} & \xrightarrow{\pi'} & M & \longrightarrow 0, \end{array}$$

$$
\begin{gathered}
Z_2\,R_2+R\,Z+Q\,Q'=I_q, \quad -R\,P+Q\,R'=0, \quad P'\,P+Z'\,R'=I_{p'}, \quad Z\,R+P\,P'=I_p,\\
Z_2'\,R_2'\,R'=0, \quad Q'\,R-R'\,P'=0, \quad Q'\,Q+R'\,Z'+Z_2'\,R_2'=I_{q'},\\
(P\,Z'-Z\,Q)\,R'=P\,Z'\,R'-Z\,Q\,R'=P\,Z'\,R'-Z\,R\,P=P\,(I_{p'}-P'\,P)-(I_p-P\,P')\,P=0,
\end{gathered}
$$

$$
\begin{aligned}
-Z\,Z_2\,R_2+P\,(P'\,Z-Z'\,Q')+(P\,Z'-Z\,Q)\,Q' &= -Z\,Z_2\,R_2+(P\,P')\,Z-Z\,(Q\,Q')\\
&= -Z\,Z_2\,R_2+(I_p-Z\,R)\,Z-Z\,(I_q-R\,Z-Z_2\,R_2)=0,\\
-Q'\,Z_2\,R_2-R'\,(P'\,Z-Z'\,Q')+Z_2'\,R_2'\,Q' &= -Q'\,(Z_2\,R_2)-(R'\,P')\,Z+R'\,Z'\,Q'+(Z_2'\,R_2')\,Q'\\
&= -Q'\,(I_q-Q\,Q'-R\,Z)-Q'\,R\,Z+R'\,Z'\,Q'+(I_{q'}-Q'\,Q-R'\,Z')\,Q'=0.
\end{aligned}
$$

Fig. 1. Identities

$$
\begin{gathered}
\begin{pmatrix}
I_q & 0 & R & Q\\
0 & I_{p'} & -P' & Z'\\
-Z & P & 0 & P\,Z'-Z\,Q\\
-Q' & -R' & 0 & Z_2'\,R_2'
\end{pmatrix}
\begin{pmatrix}
Z_2\,R_2 & 0 & -R & -Q\\
P'\,Z-Z'\,Q' & 0 & P' & -Z'\\
Z & -P & I_p & 0\\
Q' & R' & 0 & I_{q'}
\end{pmatrix}\\
=\\
\begin{pmatrix}
Z_2\,R_2+R\,Z+Q\,Q' & -R\,P+Q\,R' & 0 & 0\\
0 & P'\,P+Z'\,R' & 0 & 0\\
-Z\,Z_2\,R_2+P\,(P'\,Z-Z'\,Q')+(P\,Z'-Z\,Q)\,Q' & (P\,Z'-Z\,Q)\,R' & Z\,R+P\,P' & 0\\
-Q'\,Z_2\,R_2-R'\,(P'\,Z-Z'\,Q')+Z_2'\,R_2'\,Q' & Z_2'\,R_2'\,R' & Q'\,R-R'\,P' & Q'\,Q+R'\,Z'+Z_2'\,R_2'
\end{pmatrix}\\
=I_{q+p'+p+q'},
\end{gathered}
$$

Fig. 2. Identity

$$
Y=\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & \partial_1 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & 0 & 0 & 0 & \frac{1}{2}\,\partial_2 & \frac{1}{2}\,\partial_1 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0\\
0 & 0 & 1 & 0 & 0 & 0 & 0 & \partial_2 & 0 & 0 & 0 & 0 & 1 & 0\\
0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 1 & 0 & -\partial_2 & 0 & 0 & -1 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
-1 & 0 & 0 & -\partial_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & -\partial_2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
-\partial_2 & 0 & 0 & 0 & -\partial_1 & 0 & 0 & 0 & -\partial_2 & \partial_1 & 1 & 0 & 0 & 0\\
0 & -2 & 0 & 0 & -1 & -\partial_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & -1 & 0 & 0 & -\partial_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & -2\,\partial_2 & \partial_1 & 0 & -\partial_2 & 0 & 0 & 0 & 0 & 0 & 0 & -\partial_2 & \partial_1 & 1
\end{pmatrix}\in \mathrm{GL}_{14}(D).
$$

Fig. 3. Unimodular matrix Y

$$
\begin{array}{ccccccccccc}
 & & & & 0 & & 0 & & 0 & & \\
 & & & & \uparrow & & \uparrow & & \uparrow & & \\
0 \longleftarrow & N\oplus D^{(p+q')} & \xleftarrow{\kappa\oplus 0_{p'}\oplus \mathrm{id}_{p+q'}} & & D^{(q+p'+p+q')} & \xleftarrow{L.} & D^{(p+p')} & \longleftarrow & \hom_D(M,D) & \longleftarrow 0 & \\
 & & & & \uparrow Y. & & \uparrow X. & & \uparrow f^\star & & \qquad (19)\\
0 \longleftarrow & D^{(q+p')}\oplus N' & \xleftarrow{\mathrm{id}_{q+p'}\oplus 0_p\oplus \kappa'} & & D^{(q+p'+p+q')} & \xleftarrow{L'.} & D^{(p+p')} & \longleftarrow & \hom_D(M',D) & \longleftarrow 0. & \\
 & & & & \uparrow & & \downarrow & & \uparrow & & \\
 & & & & 0 & & 0 & & 0 & &
\end{array}
$$

Fig. 4. Commutative exact diagram

If $N = D^q/(R\,D^p)$ and $N' = D^{q'}/(R'\,D^{p'})$ are the corresponding Auslander transposes, then the right D-isomorphism γ defined by (20) holds and yields

$$N \oplus D^{(p+q')} \cong N' \oplus D^{(q+p')},$$

i.e., N and N' are *projectively equivalent* right D-modules [21]. If D is a domain and $\mathrm{rank}_D(\cdot)$ is the *rank* [21], then:

$$\mathrm{rank}_D(N) + q' + p = \mathrm{rank}_D(N') + q + p'.$$

The fact that the Auslander transpose N depends only on M up to a projective equivalence plays a fundamental role in the characterization of the properties of M [4], [18].

If θ is an *involution* of D, namely an anti-isomorphism of D of order 2 [4], [21], and $\theta(R) = (\theta(R_{ij}))^T \in D^{p\times q}$, then the finitely presented right D-modules $N = D^q/(R\,D^p)$ and $N' = D^{q'}/(R'\,D^{p'})$ can be turned into the left D-modules $\widetilde{N} = D^{1\times q}/(D^{1\times p}\,\theta(R))$ and $\widetilde{N'} = D^{1\times q'}/(D^{1\times p'}\,\theta(R'))$. For more details, see [4]. In particular, if D is a ring of PD operators, then $\theta(R)$ is the formal adjoint $\widetilde{R}$ of R in the sense of the theory of distributions [17], [19]. The left D-module module $\widetilde{N}$, called *adjoint* of M, plays a fundamental role in the algebraic analysis approach to mathematical system theory [4], [17], [18], [22], mathematical physics [6], [17], variational and optimal control problems [17], [19].

Corollary 1: With the notations of Theorem 2, if θ is an involution of D, then the following isomorphism holds:

$$\widetilde{N} \oplus D^{1\times(p+q')} \cong \widetilde{N'} \oplus D^{1\times(q+p')}.$$

In particular, $\mathrm{rank}_D(\widetilde{N}) + q' + p = \mathrm{rank}_D(\widetilde{N'}) + q + p'$.

Example 3: Let us consider again Example 1. Using the trivial involution $\theta = \mathrm{id}_D$ of the commutative polynomial ring D, Corollary 1 then yields that the Auslander transposes and adjoints $N = D^3/(R\,D^2) \cong \widetilde{N} = D^{1\times 3}/(D^{1\times 2}\,R^T)$ of $M = D^{1\times 2}/(D^{1\times 3}\,R)$ and $N' = D^6/(R'\,D^3) \cong \widetilde{N'} = D^{1\times 6}/(D^{1\times 3}\,R'^T)$ of $M' = D^{1\times 3}/(D^{1\times 6}\,R')$ satisfy:

$$N \oplus D^8 \cong N' \oplus D^6, \quad \widetilde{N} \oplus D^{1\times 8} \cong \widetilde{N'} \oplus D^{1\times 6}.$$

The above D-isomorphisms are defined by (20), where the matrix $Y \in \mathrm{GL}_{14}(D)$ is defined in Example 2 (see Fig. 3). The finitely presented D-module $\widetilde{N}$ corresponds to the following linear PD system

$$R^T \begin{pmatrix} \sigma^{11} \\ 2\,\sigma^{12} \\ \sigma^{22} \end{pmatrix} = 0 \quad \Leftrightarrow \quad \begin{cases} \partial_1\,\sigma^{11} + \partial_2\,\sigma^{12} = 0, \\ \partial_1\,\sigma^{12} + \partial_2\,\sigma^{22} = 0, \end{cases} \tag{21}$$

where $(\sigma^{11}, \sigma^{12}, \sigma^{22})$ is the *symmetric stress tensor* [12], [17]. The D-module $\widetilde{N'}$ corresponds to the linear PD system

$$R'^T \begin{pmatrix} \sigma^{11} \\ \sigma^{12} \\ \mu^1 \\ \sigma^{21} \\ \sigma^{22} \\ \mu^2 \end{pmatrix} = 0 \;\Leftrightarrow\; \begin{cases} \partial_1\,\sigma^{11} + \partial_2\,\sigma^{12} = 0, \\ \partial_1\,\mu^1 + \partial_2\,\mu^2 + \sigma^{21} - \sigma^{12} = 0, \\ \partial_1\,\sigma^{21} + \partial_2\,\sigma^{22} = 0, \end{cases} \tag{22}$$

where $(\sigma^{11}, \sigma^{12}, \sigma^{21}, \sigma^{22})$ is a non-symmetric stress tensor and (μ^1, μ^2) a *couple-stress* [12], [17]. If the couple-stress vanishes, i.e., $\mu^1 = \mu^2 = 0$, then (22) becomes (21). (21) corresponds to the equilibrium of the stress tensor (i.e., without couple-stress and *density of forces* [12], [17]), and (22) corresponds to the equilibrium of the stress and couple-stress tensors (i.e., without density of forces and *volume density of couple*) [12], [17]. This last system was first discovered by E. and F. Cosserat in 1909 [8] and it is nowadays used in the study of liquid crystals, rocks and granular media. See [17] for a modern variational formulation of Cosserat's equations based on the *Spencer operator*, and *Lie pseudogroups*, and for extensions of Cosserat's ideas to electromagnetism, thermodynamics and general relativity.

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