

```
[> restart:
```

```
[> with(LinearAlgebra):
```

```
[> with(OreModules):
```

```
> march('open',  
"/Users/aquadrat/Documents/Travail/maple/Personnel/RankFactorizationProblem.mla"):
```

```
libname := "/Users/aquadrat/Documents/Travail/maple/Personnel", (1)
```

```
"/Library/Frameworks/Maple.framework/Versions/2023/lib",
```

```
"/Users/aquadrat/Documents/Travail/maple/Personnel/RankFactorizationProblem.mla"
```

```
> with(RankFactorizationProblem);
```

```
[AntiDiagonal, CentroHermitian, Factorization, FiniteFreeResolution, FittingIdeal, (2)
```

```
IsCentroHermitian, IsInvertible, IsNilpotent, IsSolution, LeeMatrix, LeftLift, Lift,
```

```
LocalSyzygyModule, RankFactorization, ReducedSyzygies, RightLift, Saturation, Simplification,
```

```
Solutions, Syzygies]
```

```
[> R := DefineOreAlgebra(seq(diff=[x[i],t[i]],i=1..4),polynom=[seq(t  
[i],i=1..4)]):
```

```
[Example 1 of Inria Report 9438
```

```
> M := Matrix([[1,0$2,1],[0$4],[0$4],[1,0$2,1]]);
```

$$M := \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

(3)

```
> D1 := Matrix([[1,0$3],[0$4],[0$4],[0$3,-1]]);
```

$$D1 := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

(4)

```
> D2 := Matrix([[0$4],[0,1,0$2],[0$2,-1,0],[0$4]]);
```

(5)

$$D2 := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (5)$$

> D3 := Matrix([[0\$3,1],[0\$4],[0\$4],[-1,0\$3]]);

$$D3 := \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \quad (6)$$

> D4 := Matrix([[0\$4],[0\$2,1,0],[0,-1,0\$2],[0\$4]]);

$$D4 := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (7)$$

> RF0 := RankFactorization(M,[D1,D2,D3,D4],0);

$$RF0 := \left[[0], \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, [1 \ 0 \ 0 \ 1], [x_1 - x_4], \text{table}([1 = [_y]]), \text{table}([1 = []]), \right. \quad (8)$$

[Ore_algebra, ["diff", "diff", "diff", "diff", "diff"], [t_1, t_2, t_3, t_4, _t], [x_1, x_2, x_3, x_4, _y], [t_1, t_2, t_3,

t_4, _t], [], 0, [], [], [t_1, t_2, t_3, t_4, _t], [], [], [diff = [x_1, t_1], diff = [x_2, t_2], diff = [x_3, t_3], diff

= [x_4, t_4], diff = [_y, _t]], [-a -> _a x_1 - \left(\frac{\partial}{\partial t_1} _a \right), -a -> _a x_2 - \left(\frac{\partial}{\partial t_2} _a \right), -a -> _a x_3

$$- \left(\frac{\partial}{\partial t_3} - a \right),_{-a \rightarrow -a} x_4 - \left(\frac{\partial}{\partial t_4} - a \right),_{-a \rightarrow -a} y - \left(\frac{\partial}{\partial t} - a \right) \right] \Bigg],$$

$$\begin{bmatrix} x_1 & 0 & x_4 & 0 \\ 0 & x_2 & 0 & x_3 \\ 0 & -x_3 & 0 & -x_2 \\ -x_4 & 0 & -x_1 & 0 \end{bmatrix}$$

> RF0bis := RankFactorization(M, [D1, D2, D3, D4], 0, "reduced");

$$RF0bis := \left[[0], \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, [1 \ 0 \ 0 \ 1], [x_1 - x_4], \text{table}([1 = [_y]]), \text{table}([1 = [\]]), \right. \quad (9)$$

$$\left[Ore_algebra, ["diff", "diff", "diff", "diff", "diff"], [t_1, t_2, t_3, t_4, _t], [x_1, x_2, x_3, x_4, _y], [t_1, t_2, t_3,$$

$$t_4, _t], [\], 0, [\], [\], [t_1, t_2, t_3, t_4, _t], [\], [\], [diff = [x_1, t_1], diff = [x_2, t_2], diff = [x_3, t_3], diff$$

$$= [x_4, t_4], diff = [_y, _t]], \left[-a \rightarrow -a x_1 - \left(\frac{\partial}{\partial t_1} - a \right),_{-a \rightarrow -a} x_2 - \left(\frac{\partial}{\partial t_2} - a \right),_{-a \rightarrow -a} x_3$$

$$- \left(\frac{\partial}{\partial t_3} - a \right),_{-a \rightarrow -a} x_4 - \left(\frac{\partial}{\partial t_4} - a \right),_{-a \rightarrow -a} y - \left(\frac{\partial}{\partial t} - a \right) \right] \Bigg],$$

$$\begin{bmatrix} x_1 & 0 & x_4 & 0 \\ 0 & x_2 & 0 & x_3 \\ 0 & -x_3 & 0 & -x_2 \\ -x_4 & 0 & -x_1 & 0 \end{bmatrix}$$

> **Sol0 := Solutions(M, [D1, D2, D3, D4], 0);**

$$\text{Sol0} := \left[[0], [x_1 - x_4], \begin{bmatrix} x_1 & 0 & x_4 & 0 \\ 0 & x_2 & 0 & x_3 \\ 0 & -x_3 & 0 & -x_2 \\ -x_4 & 0 & -x_1 & 0 \end{bmatrix}, \text{table} \left(\left[1 = \begin{bmatrix} _y & 0 & 0 & _y \\ 0 & 0 & 0 & 0 \\ -_y & 0 & 0 & -_y \\ 0 & 0 & 0 & 0 \end{bmatrix} \right] \right), \right. \quad (10)$$

$$\begin{aligned} & \text{table} \left(\left[1 = \left[\text{Ore_algebra}, ["diff", "diff", "diff", "diff", "diff"], [t_1, t_2, t_3, t_4, _t], [x_1, x_2, x_3, x_4, _y], [t_1, t_2, t_3, t_4, _t], [], 0, [], [], [t_1, t_2, t_3, t_4, _t], [], [], [diff = [x_1, t_1], diff = [x_2, t_2], diff = [x_3, t_3], diff = [x_4, t_4], diff = [_y, _t]], \left[_a \rightarrow _a x_1 - \left(\frac{\partial}{\partial t_1} _a \right), _a \rightarrow _a x_2 - \left(\frac{\partial}{\partial t_2} _a \right), _a \rightarrow _a x_3 - \left(\frac{\partial}{\partial t_3} _a \right), _a \rightarrow _a x_4 - \left(\frac{\partial}{\partial t_4} _a \right), _a \rightarrow _a _y - \left(\frac{\partial}{\partial _t} _a \right) \right] \right] \right) \end{aligned}$$

> **Solutions(M, [D1, D2, D3, D4], 0, "reduced");**

$$\left[[0], [x_1 - x_4], \begin{bmatrix} x_1 & 0 & x_4 & 0 \\ 0 & x_2 & 0 & x_3 \\ 0 & -x_3 & 0 & -x_2 \\ -x_4 & 0 & -x_1 & 0 \end{bmatrix}, \text{table} \left(\left[1 = \begin{bmatrix} _y & 0 & 0 & _y \\ 0 & 0 & 0 & 0 \\ -_y & 0 & 0 & -_y \\ 0 & 0 & 0 & 0 \end{bmatrix} \right] \right), \text{table} \left(\left[1 \right. \right. \quad (11)$$

$$\begin{aligned} & \left. \left[\text{Ore_algebra}, ["diff", "diff", "diff", "diff", "diff"], [t_1, t_2, t_3, t_4, _t], [x_1, x_2, x_3, x_4, _y], [t_1, t_2, t_3, t_4, _t], [], 0, [], [], [t_1, t_2, t_3, t_4, _t], [], [], [diff = [x_1, t_1], diff = [x_2, t_2], diff = [x_3, t_3], diff = [x_4, t_4], diff = [_y, _t]], \left[_a \rightarrow _a x_1 - \left(\frac{\partial}{\partial t_1} _a \right), _a \rightarrow _a x_2 - \left(\frac{\partial}{\partial t_2} _a \right), _a \rightarrow _a x_3 - \left(\frac{\partial}{\partial t_3} _a \right), _a \rightarrow _a x_4 - \left(\frac{\partial}{\partial t_4} _a \right), _a \rightarrow _a _y - \left(\frac{\partial}{\partial _t} _a \right) \right] \right] \right) \end{aligned}$$

> **IsSolution(Sol0);**

$$\text{table}([1 = [true]]) \quad (12)$$

> **RF1 := RankFactorization(M, [D1, D2, D3, D4], 1);**

$$RFI := \left[\begin{array}{c} [x_2^2 x_1 - x_3^2 x_1 + x_2^2 x_4 - x_3^2 x_4], \\ \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & -x_1 x_3 - x_4 x_3 & x_2 x_1 + x_4 x_2 & 0 \\ -1 & 0 & 0 & x_2^2 - x_3^2 \\ 0 & x_2 x_1 + x_4 x_2 & -x_1 x_3 - x_4 x_3 & 0 \end{array} \right] \end{array} \right], \quad (13)$$

$$\left[\begin{array}{cccc} 1 & 0 & 0 & 1 \end{array} \right], [x_1 - x_4, x_2^2 x_4 - x_3^2 x_4], \text{table} \left(\left(\left[\begin{array}{c} _y \\ 0 \\ 0 \\ 0 \end{array} \right], 2 = \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ _y \end{array} \right] \right) \right), \text{table} \left(\left[\begin{array}{cccc} x_2^2 - x_3^2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & x_1 + x_4 \end{array} \right], 2 = \left[\begin{array}{cccc} 1 & x_2^2 - x_3^2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 2_y x_4 & 2 & 0 & 0 \end{array} \right] \right)$$

$$, \left[\text{Ore_algebra}, ["diff", "diff", "diff", "diff", "diff"], [t_1, t_2, t_3, t_4, _t], [x_1, x_2, x_3, x_4, _y], [t_1, t_2,$$

$$t_3, t_4, _t], [], 0, [], [], [t_1, t_2, t_3, t_4, _t], [], [], [diff = [x_1, t_1], diff = [x_2, t_2], diff = [x_3, t_3], diff$$

$$= [x_4, t_4], diff = [_y, _t], \left[_a \rightarrow _a x_1 - \left(\frac{\partial}{\partial t_1} _a \right), _a \rightarrow _a x_2 - \left(\frac{\partial}{\partial t_2} _a \right), _a \rightarrow _a x_3$$

$$- \left(\frac{\partial}{\partial t_3} _a \right), _a \rightarrow _a x_4 - \left(\frac{\partial}{\partial t_4} _a \right), _a \rightarrow _a _y - \left(\frac{\partial}{\partial _t} _a \right) \right]$$

$$\begin{bmatrix} x_1 & 0 & x_4 & 0 \\ 0 & x_2 & 0 & x_3 \\ 0 & -x_3 & 0 & -x_2 \\ -x_4 & 0 & -x_1 & 0 \end{bmatrix}$$

> **Soll := Solutions(M, [D1, D2, D3, D4], 1);**

$$Soll := \left[\begin{array}{l} [x_2^2 x_1 - x_1 x_3^2 + x_2^2 x_4 - x_3^2 x_4], [x_1 - x_4, x_2^2 x_4 - x_3^2 x_4], \\ \begin{bmatrix} x_1 & 0 & x_4 & 0 \\ 0 & x_2 & 0 & x_3 \\ 0 & -x_3 & 0 & -x_2 \\ -x_4 & 0 & -x_1 & 0 \end{bmatrix} \end{array} \right], \quad (14)$$

table([1 = [[_y + (x₂² - x₃²) y_{1,1}, (x₂² - x₃²) y_{1,2}, (x₂² - x₃²) y_{1,3}, _y + (x₂² - x₃²) y_{1,4}],

[(-x₁ x₃ - x₃ x₄) y_{2,1} + (x₁ x₂ + x₂ x₄) y_{3,1}, (-x₁ x₃ - x₃ x₄) y_{2,2} + (x₁ x₂ + x₂ x₄) y_{3,2}, (-x₁ x₃ - x₃ x₄) y_{2,3} + (x₁ x₂ + x₂ x₄) y_{3,3}, (-x₁ x₃ - x₃ x₄) y_{2,4} + (x₁ x₂ + x₂ x₄) y_{3,4}],

[-_y - (x₂² - x₃²) y_{1,1} + (x₂² - x₃²) (2 y_{1,1} + (x₁ + x₄) y_{4,1} + (2 _y x₄ + 1) y_{5,1} + (-2 _y x₄ - 1) y_{6,1}), -(x₂² - x₃²) y_{1,2} + (x₂² - x₃²) (2 y_{1,2} + (x₁ + x₄) y_{4,2} + (2 _y x₄ + 1) y_{5,2} + (-2 _y x₄ - 1) y_{6,2}), -(x₂² - x₃²) y_{1,3} + (x₂² - x₃²) (2 y_{1,3} + (x₁ + x₄) y_{4,3} + (2 _y x₄ + 1) y_{5,3} + (-2 _y x₄ - 1) y_{6,3}), -_y - (x₂² - x₃²) y_{1,4} + (x₂² - x₃²) (2 y_{1,4} + (x₁ + x₄) y_{4,4} + (2 _y x₄ + 1) y_{5,4} + (-2 _y x₄ - 1) y_{6,4})],

[(x₁ x₂ + x₂ x₄) y_{2,1} + (-x₁ x₃ - x₃ x₄) y_{3,1}, (x₁ x₂ + x₂ x₄) y_{2,2} + (-x₁ x₃ - x₃ x₄) y_{3,2},

(x₁ x₂ + x₂ x₄) y_{2,3} + (-x₁ x₃ - x₃ x₄) y_{3,3}, (x₁ x₂ + x₂ x₄) y_{2,4} + (-x₁ x₃ - x₃ x₄) y_{3,4}]], 2

= [[y_{1,1} + (x₂² - x₃²) y_{2,1}, y_{1,2} + (x₂² - x₃²) y_{2,2}, y_{1,3} + (x₂² - x₃²) y_{2,3}, y_{1,4} + (x₂² - x₃²) y_{2,4}],

[(-x₁ x₃ - x₃ x₄) y_{3,1} + (x₁ x₂ + x₂ x₄) y_{4,1}, (-x₁ x₃ - x₃ x₄) y_{3,2} + (x₁ x₂ + x₂ x₄) y_{4,2}, (

$$\begin{aligned}
& -x_1 x_3 - x_3 x_4) y_{3,3} + (x_1 x_2 + x_2 x_4) y_{4,3}, (-x_1 x_3 - x_3 x_4) y_{3,4} + (x_1 x_2 + x_2 x_4) y_{4,4}], \\
& [-y_{1,1} - (x_2^2 - x_3^2) y_{2,1} + (x_2^2 - x_3^2) (2 _y x_4 y_{1,1} + _y + 2 y_{2,1}), -y_{1,2} - (x_2^2 - x_3^2) y_{2,2} \\
& + (x_2^2 - x_3^2) (2 _y x_4 y_{1,2} + 2 y_{2,2}), -y_{1,3} - (x_2^2 - x_3^2) y_{2,3} + (x_2^2 - x_3^2) (2 _y x_4 y_{1,3} \\
& + 2 y_{2,3}), -y_{1,4} - (x_2^2 - x_3^2) y_{2,4} + (x_2^2 - x_3^2) (2 _y x_4 y_{1,4} + _y + 2 y_{2,4})], \\
& [(x_1 x_2 + x_2 x_4) y_{3,1} + (-x_1 x_3 - x_3 x_4) y_{4,1}, (x_1 x_2 + x_2 x_4) y_{3,2} + (-x_1 x_3 - x_3 x_4) y_{4,2}, \\
& (x_1 x_2 + x_2 x_4) y_{3,3} + (-x_1 x_3 - x_3 x_4) y_{4,3}, (x_1 x_2 + x_2 x_4) y_{3,4} + (-x_1 x_3 - x_3 x_4) y_{4,4}]]] \\
& , \text{table} \left(\left[1 = \left[\text{Ore_algebra}, ["diff", "diff", "diff", "diff", "diff"], [t_1, t_2, t_3, t_4, _t], [x_1, x_2, x_3, x_4, _y], [t_1, t_2, t_3, t_4, _t], [y_{1,1}, y_{1,2}, y_{1,3}, y_{1,4}, y_{2,1}, y_{2,2}, y_{2,3}, y_{2,4}, y_{3,1}, y_{3,2}, y_{3,3}, y_{3,4}, y_{4,1}, y_{4,2}, y_{4,3}, y_{4,4}], 0, [], [], [t_1, t_2, t_3, t_4, _t], [], [], [diff = [x_1, t_1], diff = [x_2, t_2], diff = [x_3, t_3], diff = [x_4, t_4], diff = [_y, _t]], \left[_a \rightarrow _a x_1 - \left(\frac{\partial}{\partial t_1} _a \right), _a \rightarrow _a x_2 - \left(\frac{\partial}{\partial t_2} _a \right), _a \rightarrow _a x_3 - \left(\frac{\partial}{\partial t_3} _a \right), _a \rightarrow _a x_4 - \left(\frac{\partial}{\partial t_4} _a \right), _a \rightarrow _a _y - \left(\frac{\partial}{\partial _t} _a \right) \right], 2 = \left[\text{Ore_algebra}, ["diff", "diff", "diff", "diff", "diff"], [t_1, t_2, t_3, t_4, _t], [x_1, x_2, x_3, x_4, _y], [t_1, t_2, t_3, t_4, _t], [y_{1,1}, y_{1,2}, y_{1,3}, y_{1,4}, y_{2,1}, y_{2,2}, y_{2,3}, y_{2,4}, y_{3,1}, y_{3,2}, y_{3,3}, y_{3,4}, y_{4,1}, y_{4,2}, y_{4,3}, y_{4,4}], 0, [], [], [t_1, t_2, t_3, t_4, _t], [], [], [diff = [x_1, t_1], diff = [x_2, t_2], diff = [x_3, t_3], diff = [x_4, t_4], diff = [_y, _t]], \left[_a \rightarrow _a x_1 - \left(\frac{\partial}{\partial t_1} _a \right), _a \rightarrow _a x_2 - \left(\frac{\partial}{\partial t_2} _a \right), _a \rightarrow _a x_3 - \left(\frac{\partial}{\partial t_3} _a \right), _a \rightarrow _a x_4 - \left(\frac{\partial}{\partial t_4} _a \right), _a \rightarrow _a _y - \left(\frac{\partial}{\partial _t} _a \right) \right]]]]] \right)
\end{aligned}$$

> **IsSolution(Sol1);**

table([1 = [true], 2 = [true]])

(15)

> **RF1bis := RankFactorization(M, [D1, D2, D3, D4], 1, "reduced");**

$$RF1bis := \begin{bmatrix} [x_2^2 x_1 - x_1 x_3^2 + x_2^2 x_4 - x_3^2 x_4], & \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -x_1 x_3 - x_3 x_4 & x_1 x_2 + x_2 x_4 & 0 \\ -1 & 0 & 0 & x_2^2 - x_3^2 \\ 0 & x_1 x_2 + x_2 x_4 & -x_1 x_3 - x_3 x_4 & 0 \end{bmatrix} \end{bmatrix},$$

(16)

$$\begin{aligned}
 & [1 \ 0 \ 0 \ 1], [x_1 - x_4, x_2^2 x_4 - x_3^2 x_4], \text{table} \left(\left(\left[\begin{array}{c} -y \\ 0 \\ 0 \\ 0 \end{array} \right], 2 = \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ -y \end{array} \right] \right) \right), \text{table} \left(\left[\begin{array}{c} 1 \end{array} \right] \right) \\
 & = \left(\left[\begin{array}{cccc} x_2^2 - x_3^2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 0 \end{array} \right], 2 = \left[\begin{array}{cccc} x_2^2 - x_3^2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 0 \end{array} \right] \right), \left[\text{Ore_algebra}, ["diff", "diff", "diff", "diff"], \right.
 \end{aligned}$$

"diff"], [t₁, t₂, t₃, t₄, -t], [x₁, x₂, x₃, x₄, -y], [t₁, t₂, t₃, t₄, -t], [], 0, [], [], [t₁, t₂, t₃, t₄, -t], [],

[], [diff = [x₁, t₁], diff = [x₂, t₂], diff = [x₃, t₃], diff = [x₄, t₄], diff = [-y, -t]], [-a → -a x₁

- (∂/∂t₁ -a), -a → -a x₂ - (∂/∂t₂ -a), -a → -a x₃ - (∂/∂t₃ -a), -a → -a x₄ - (∂/∂t₄ -a), -a

→ -a -y - (∂/∂-t -a)]]],

$$\left[\begin{array}{cccc} x_1 & 0 & x_4 & 0 \\ 0 & x_2 & 0 & x_3 \\ 0 & -x_3 & 0 & -x_2 \\ -x_4 & 0 & -x_1 & 0 \end{array} \right]$$

> Sol1bis := Solutions(M, [D1, D2, D3, D4], 1, "reduced");

$$\text{Sollbis} := \left[\begin{array}{l} [x_2^2 x_1 - x_1 x_3^2 + x_2^2 x_4 - x_3^2 x_4], [x_1 - x_4, x_2^2 x_4 - x_3^2 x_4], \\ \left[\begin{array}{cccc} x_1 & 0 & x_4 & 0 \\ 0 & x_2 & 0 & x_3 \\ 0 & -x_3 & 0 & -x_2 \\ -x_4 & 0 & -x_1 & 0 \end{array} \right] \end{array} \right], \quad (17)$$

$$\text{table}([1 = [[-y + (x_2^2 - x_3^2) y_{1,1}, (x_2^2 - x_3^2) y_{1,2}, (x_2^2 - x_3^2) y_{1,3}, -y + (x_2^2 - x_3^2) y_{1,4}],$$

$$[(-x_1 x_3 - x_3 x_4) y_{2,1} + (x_1 x_2 + x_2 x_4) y_{3,1}, (-x_1 x_3 - x_3 x_4) y_{2,2} + (x_1 x_2 + x_2 x_4) y_{3,2}, (-x_1 x_3 - x_3 x_4) y_{2,3} + (x_1 x_2 + x_2 x_4) y_{3,3}, (-x_1 x_3 - x_3 x_4) y_{2,4} + (x_1 x_2 + x_2 x_4) y_{3,4}],$$

$$[-y + (x_2^2 - x_3^2) y_{1,1}, (x_2^2 - x_3^2) y_{1,2}, (x_2^2 - x_3^2) y_{1,3}, -y + (x_2^2 - x_3^2) y_{1,4}],$$

$$[(x_1 x_2 + x_2 x_4) y_{2,1} + (-x_1 x_3 - x_3 x_4) y_{3,1}, (x_1 x_2 + x_2 x_4) y_{2,2} + (-x_1 x_3 - x_3 x_4) y_{3,2},$$

$$(x_1 x_2 + x_2 x_4) y_{2,3} + (-x_1 x_3 - x_3 x_4) y_{3,3}, (x_1 x_2 + x_2 x_4) y_{2,4} + (-x_1 x_3 - x_3 x_4) y_{3,4}]]], 2$$

$$= [[(x_2^2 - x_3^2) y_{1,1}, (x_2^2 - x_3^2) y_{1,2}, (x_2^2 - x_3^2) y_{1,3}, (x_2^2 - x_3^2) y_{1,4}],$$

$$[(-x_1 x_3 - x_3 x_4) y_{2,1} + (x_1 x_2 + x_2 x_4) y_{3,1}, (-x_1 x_3 - x_3 x_4) y_{2,2} + (x_1 x_2 + x_2 x_4) y_{3,2}, (-x_1 x_3 - x_3 x_4) y_{2,3} + (x_1 x_2 + x_2 x_4) y_{3,3}, (-x_1 x_3 - x_3 x_4) y_{2,4} + (x_1 x_2 + x_2 x_4) y_{3,4}],$$

$$[-(x_2^2 - x_3^2) y_{1,1} + (x_2^2 - x_3^2) (-y + 2 y_{1,1}), (x_2^2 - x_3^2) y_{1,2}, (x_2^2 - x_3^2) y_{1,3}, -(x_2^2 - x_3^2) y_{1,4} + (x_2^2 - x_3^2) (-y + 2 y_{1,4})],$$

$$[(x_1 x_2 + x_2 x_4) y_{2,1} + (-x_1 x_3 - x_3 x_4) y_{3,1}, (x_1 x_2 + x_2 x_4) y_{2,2} + (-x_1 x_3 - x_3 x_4) y_{3,2},$$

$$(x_1 x_2 + x_2 x_4) y_{2,3} + (-x_1 x_3 - x_3 x_4) y_{3,3}, (x_1 x_2 + x_2 x_4) y_{2,4} + (-x_1 x_3 - x_3 x_4) y_{3,4}]]]]]$$

```
, table([ 1 = [ Ore_algebra, ["diff", "diff", "diff", "diff", "diff"], [t1, t2, t3, t4, _t], [x1, x2, x3, x4,
_y], [t1, t2, t3, t4, _t], [y1,1, y1,2, y1,3, y1,4, y2,1, y2,2, y2,3, y2,4, y3,1, y3,2, y3,3, y3,4], 0, [ ], [ ], [t1,
t2, t3, t4, _t], [ ], [ ], [diff= [x1, t1], diff= [x2, t2], diff= [x3, t3], diff= [x4, t4], diff= [_y, _t]],
[ -a->-a x1 - (∂/∂t1 -a), -a->-a x2 - (∂/∂t2 -a), -a->-a x3 - (∂/∂t3 -a), -a->-a x4
- (∂/∂t4 -a), -a->-a _y - (∂/∂_t -a) ]], 2 = [ Ore_algebra, ["diff", "diff", "diff", "diff",
"diff"], [t1, t2, t3, t4, _t], [x1, x2, x3, x4, _y], [t1, t2, t3, t4, _t], [y1,1, y1,2, y1,3, y1,4, y2,1, y2,2, y2,3,
y2,4, y3,1, y3,2, y3,3, y3,4], 0, [ ], [ ], [t1, t2, t3, t4, _t], [ ], [ ], [diff= [x1, t1], diff= [x2, t2], diff
= [x3, t3], diff= [x4, t4], diff= [_y, _t]], [ -a->-a x1 - (∂/∂t1 -a), -a->-a x2 - (∂/∂t2 -a), -a
->-a x3 - (∂/∂t3 -a), -a->-a x4 - (∂/∂t4 -a), -a->-a _y - (∂/∂_t -a) ]]]])
```

```
> IsSolution(Sol1bis);
```

```
table([ 1 = [true], 2 = [true]])
```

(18)

```
> RF2 := RankFactorization(M, [D1, D2, D3, D4], 2);
```

```
RF2 := [ [x1 x3 + x3 x4, x2^2 - x3^2, x1 x2 + x2 x4], [ 1 0 0 0 0 0 0
0 -x3 x2 x1 + x4 0 0 0
-1 0 0 0 x3 x2 0
0 x2 -x3 0 0 0 x1 + x4 ] ]
```

(19)

$$[1 \ 0 \ 0 \ 1], [x_1 - x_4, x_3 x_4, x_2 x_4], \text{table} \left(\left(\left[\begin{array}{c} _y \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{array} \right], \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ _y \\ 0 \\ 0 \end{array} \right], \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ _y \\ 0 \end{array} \right] \right) \right), \text{table} \left(\left[\begin{array}{c} 1 \end{array} \right] \right)$$

$$= \left[\begin{array}{cccccccccccc} x_3 & x_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 2 & 0 & 0 & 0 & 0 & -x_3 & x_2 & x_1 + x_4 & 2_y x_4 + 1 & -2_y x_4 - 1 & \dots \\ 0 & 2 & 0 & 0 & 0 & x_2 & -x_3 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \end{array} \right], 2$$

7 × 14 Matrix

$$= \left[\begin{array}{cccccccc} x_3 & x_2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 2 & 0 & 2_y x_4 & 0 & 0 & 0 & -x_3 & x_2 \\ 0 & 2 & 0 & 0 & 0 & 0 & x_2 & -x_3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right], 3$$

$$= \left(\left(\left[\begin{array}{cccccccc} x_3 & x_2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & -x_3 & x_2 & _y x_3 x_4 \\ 0 & 2 & 2_y x_4 & 0 & 0 & 0 & x_2 & -x_3 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right] \right) \right), \left[\text{Ore_algebra}, ["diff", "diff"], \right]$$

"diff", "diff", "diff"], [t1, t2, t3, t4, _t], [x1, x2, x3, x4, _y], [t1, t2, t3, t4, _t], [], 0, [], [], [t1, t2,

t3, t4, _t], [], [], [diff = [x1, t1], diff = [x2, t2], diff = [x3, t3], diff = [x4, t4], diff = [_y, _t]], [-a

->_a x1 - (∂/∂t1 -a), -a->-a x2 - (∂/∂t2 -a), -a->-a x3 - (∂/∂t3 -a), -a->-a x4

- (∂/∂t4 -a), -a->-a _y - (∂/∂_t -a)]]],
$$\begin{bmatrix} x_1 & 0 & x_4 & 0 \\ 0 & x_2 & 0 & x_3 \\ 0 & -x_3 & 0 & -x_2 \\ -x_4 & 0 & -x_1 & 0 \end{bmatrix}$$

> Sol2 := Solutions(M, [D1, D2, D3, D4], 2);

Sol2 :=
$$\left[\begin{array}{l} [x_1 x_3 + x_3 x_4, x_2^2 - x_3^2, x_1 x_2 + x_2 x_4], [x_1 - x_4, x_3 x_4, x_2 x_4], \\ \begin{bmatrix} x_1 & 0 & x_4 & 0 \\ 0 & x_2 & 0 & x_3 \\ 0 & -x_3 & 0 & -x_2 \\ -x_4 & 0 & -x_1 & 0 \end{bmatrix} \end{array} \right], \quad (20)$$

table([1 = [[x2 y2,1 + x3 y1,1 + _y, x2 y2,2 + x3 y1,2, x2 y2,3 + x3 y1,3, x2 y2,4 + x3 y1,4
+ _y],
[-x3 y3,1 + x2 y4,1 + (x1 + x4) y5,1, -x3 y3,2 + x2 y4,2 + (x1 + x4) y5,2, -x3 y3,3 + x2 y4,3
+ (x1 + x4) y5,3, -x3 y3,4 + x2 y4,4 + (x1 + x4) y5,4],
[-x2 y2,1 - x3 y1,1 - _y + x3 (2 y1,1 - x3 y6,1 + x2 y7,1 + (x1 + x4) y8,1 + (2 _y x4
+ 1) y9,1 + (-2 _y x4 - 1) y10,1) + x2 (2 y2,1 + x2 y6,1 - x3 y7,1 + (x1 + x4) y11,1
+ (2 _y x4 + 1) y12,1 + (-2 _y x4 - 1) y13,1), -x2 y2,2 - x3 y1,2 + x3 (2 y1,2 - x3 y6,2
+ x2 y7,2 + (x1 + x4) y8,2 + (2 _y x4 + 1) y9,2 + (-2 _y x4 - 1) y10,2) + x2 (2 y2,2
+ x2 y6,2 - x3 y7,2 + (x1 + x4) y11,2 + (2 _y x4 + 1) y12,2 + (-2 _y x4 - 1) y13,2),

$$\begin{aligned}
& -x_2 y_{2,3} - x_3 y_{1,3} + x_3 (2 y_{1,3} - x_3 y_{6,3} + x_2 y_{7,3} + (x_1 + x_4) y_{8,3} + (2 _y x_4 + 1) y_{9,3} + \\
& -2 _y x_4 - 1) y_{10,3}) + x_2 (2 y_{2,3} + x_2 y_{6,3} - x_3 y_{7,3} + (x_1 + x_4) y_{11,3} + (2 _y x_4 + 1) y_{12,3} \\
& + (-2 _y x_4 - 1) y_{13,3}), -x_2 y_{2,4} - x_3 y_{1,4} - _y + x_3 (2 y_{1,4} - x_3 y_{6,4} + x_2 y_{7,4} + (x_1 \\
& + x_4) y_{8,4} + (2 _y x_4 + 1) y_{9,4} + (-2 _y x_4 - 1) y_{10,4}) + x_2 (2 y_{2,4} + x_2 y_{6,4} - x_3 y_{7,4} \\
& + (x_1 + x_4) y_{11,4} + (2 _y x_4 + 1) y_{12,4} + (-2 _y x_4 - 1) y_{13,4})],
\end{aligned}$$

$$[x_2 y_{3,1} - x_3 y_{4,1} + (x_1 + x_4) y_{14,1}, x_2 y_{3,2} - x_3 y_{4,2} + (x_1 + x_4) y_{14,2}, x_2 y_{3,3} - x_3 y_{4,3} + (x_1$$

$$\begin{aligned}
& + x_4) y_{14,3}, x_2 y_{3,4} - x_3 y_{4,4} + (x_1 + x_4) y_{14,4}]], 2 = [[x_2 y_{2,1} + x_3 y_{1,1} + y_{3,1}, \\
& x_2 y_{2,2} + x_3 y_{1,2} + y_{3,2}, x_2 y_{2,3} + x_3 y_{1,3} + y_{3,3}, x_2 y_{2,4} + x_3 y_{1,4} + y_{3,4}], \\
& [-x_3 y_{4,1} + x_2 y_{5,1} + (x_1 + x_4) y_{6,1}, -x_3 y_{4,2} + x_2 y_{5,2} + (x_1 + x_4) y_{6,2}, -x_3 y_{4,3} + x_2 y_{5,3} \\
& + (x_1 + x_4) y_{6,3}, -x_3 y_{4,4} + x_2 y_{5,4} + (x_1 + x_4) y_{6,4}], \\
& [-x_2 y_{2,1} - x_3 y_{1,1} - y_{3,1} + x_3 (2 _y x_4 y_{3,1} + x_2 y_{8,1} - x_3 y_{7,1} + _y + 2 y_{1,1}) + x_2 (x_2 y_{7,1} \\
& - x_3 y_{8,1} + 2 y_{2,1}), -x_2 y_{2,2} - x_3 y_{1,2} - y_{3,2} + x_3 (2 _y x_4 y_{3,2} + x_2 y_{8,2} - x_3 y_{7,2} + 2 y_{1,2}) \\
& + x_2 (x_2 y_{7,2} - x_3 y_{8,2} + 2 y_{2,2}), -x_2 y_{2,3} - x_3 y_{1,3} - y_{3,3} + x_3 (2 _y x_4 y_{3,3} + x_2 y_{8,3} \\
& - x_3 y_{7,3} + 2 y_{1,3}) + x_2 (x_2 y_{7,3} - x_3 y_{8,3} + 2 y_{2,3}), -x_2 y_{2,4} - x_3 y_{1,4} - y_{3,4} \\
& + x_3 (2 _y x_4 y_{3,4} + x_2 y_{8,4} - x_3 y_{7,4} + _y + 2 y_{1,4}) + x_2 (x_2 y_{7,4} - x_3 y_{8,4} + 2 y_{2,4})],
\end{aligned}$$

$$[x_2 y_{4,1} - x_3 y_{5,1} + (x_1 + x_4) y_{9,1}, x_2 y_{4,2} - x_3 y_{5,2} + (x_1 + x_4) y_{9,2}, x_2 y_{4,3} - x_3 y_{5,3} + (x_1$$

$$\begin{aligned}
& + x_4) y_{9,3}, x_2 y_{4,4} - x_3 y_{5,4} + (x_1 + x_4) y_{9,4}]], 3 = [[x_2 y_{2,1} + x_3 y_{1,1} + y_{3,1}, x_2 y_{2,2} \\
& + x_3 y_{1,2} + y_{3,2}, x_2 y_{2,3} + x_3 y_{1,3} + y_{3,3}, x_2 y_{2,4} + x_3 y_{1,4} + y_{3,4}], \\
& [-x_3 y_{4,1} + x_2 y_{5,1} + (x_1 + x_4) y_{6,1}, -x_3 y_{4,2} + x_2 y_{5,2} + (x_1 + x_4) y_{6,2}, -x_3 y_{4,3} + x_2 y_{5,3} \\
& + (x_1 + x_4) y_{6,3}, -x_3 y_{4,4} + x_2 y_{5,4} + (x_1 + x_4) y_{6,4}], \\
& [-x_2 y_{2,1} - x_3 y_{1,1} - y_{3,1} + x_3 (_y x_3 x_4 y_{9,1} + x_2 y_{8,1} - x_3 y_{7,1} + 2 y_{1,1}) + x_2 (2 _y x_4 y_{3,1} \\
& + x_2 y_{7,1} - x_3 y_{8,1} + _y + 2 y_{2,1} - y_{9,1}), -x_2 y_{2,2} - x_3 y_{1,2} - y_{3,2} + x_3 (_y x_3 x_4 y_{9,2} \\
& + x_2 y_{8,2} - x_3 y_{7,2} + 2 y_{1,2}) + x_2 (2 _y x_4 y_{3,2} + x_2 y_{7,2} - x_3 y_{8,2} + 2 y_{2,2} - y_{9,2}), -x_2 y_{2,3} \\
& - x_3 y_{1,3} - y_{3,3} + x_3 (_y x_3 x_4 y_{9,3} + x_2 y_{8,3} - x_3 y_{7,3} + 2 y_{1,3}) + x_2 (2 _y x_4 y_{3,3} + x_2 y_{7,3} \\
& - x_3 y_{8,3} + 2 y_{2,3} - y_{9,3}), -x_2 y_{2,4} - x_3 y_{1,4} - y_{3,4} + x_3 (_y x_3 x_4 y_{9,4} + x_2 y_{8,4} - x_3 y_{7,4}
\end{aligned}$$

$$\begin{aligned}
& + 2 y_{1,4}) + x_2 (2 _y x_4 y_{3,4} + x_2 y_{7,4} - x_3 y_{8,4} + _y + 2 y_{2,4} - y_{9,4})], \\
& [x_2 y_{4,1} - x_3 y_{5,1} + (x_1 + x_4) y_{10,1}, x_2 y_{4,2} - x_3 y_{5,2} + (x_1 + x_4) y_{10,2}, x_2 y_{4,3} - x_3 y_{5,3} + (x_1 \\
& + x_4) y_{10,3}, x_2 y_{4,4} - x_3 y_{5,4} + (x_1 + x_4) y_{10,4}]]), \text{table} \left(\left[1 = \left[\text{Ore_algebra}, ["diff", "diff", \right. \right. \right. \\
& \text{"diff", "diff", "diff"}], [t_1, t_2, t_3, t_4, _t], [x_1, x_2, x_3, x_4, _y], [t_1, t_2, t_3, t_4, _t], [y_{1,1}, y_{1,2}, y_{1,3}, y_{1,4}, \\
& y_{2,1}, y_{2,2}, y_{2,3}, y_{2,4}, y_{3,1}, y_{3,2}, y_{3,3}, y_{3,4}, y_{4,1}, y_{4,2}, y_{4,3}, y_{4,4}, y_{5,1}, y_{5,2}, y_{5,3}, y_{5,4}, y_{6,1}, y_{6,2}, y_{6,3}, \\
& y_{6,4}, y_{7,1}, y_{7,2}, y_{7,3}, y_{7,4}, y_{8,1}, y_{8,2}, y_{8,3}, y_{8,4}, y_{9,1}, y_{9,2}, y_{9,3}, y_{9,4}, y_{10,1}, y_{10,2}, y_{10,3}, y_{10,4}, y_{11,1}, \\
& y_{11,2}, y_{11,3}, y_{11,4}, y_{12,1}, y_{12,2}, y_{12,3}, y_{12,4}, y_{13,1}, y_{13,2}, y_{13,3}, y_{13,4}, y_{14,1}, y_{14,2}, y_{14,3}, y_{14,4}], 0, [\] \\
& [\], [t_1, t_2, t_3, t_4, _t], [\], [\], [diff = [x_1, t_1], diff = [x_2, t_2], diff = [x_3, t_3], diff = [x_4, t_4], diff \\
& = [_y, _t]], \left[_a \rightarrow _a x_1 - \left(\frac{\partial}{\partial t_1} _a \right), _a \rightarrow _a x_2 - \left(\frac{\partial}{\partial t_2} _a \right), _a \rightarrow _a x_3 - \left(\frac{\partial}{\partial t_3} _a \right), _a \rightarrow _a x_4 - \left(\frac{\partial}{\partial t_4} _a \right), _a \rightarrow _a _y - \left(\frac{\partial}{\partial _t} _a \right) \right]], 2 = \left[\text{Ore_algebra}, ["diff", "diff", "diff", \\
& \text{"diff", "diff"}], [t_1, t_2, t_3, t_4, _t], [x_1, x_2, x_3, x_4, _y], [t_1, t_2, t_3, t_4, _t], [y_{1,1}, y_{1,2}, y_{1,3}, y_{1,4}, y_{2,1}, \\
& y_{2,2}, y_{2,3}, y_{2,4}, y_{3,1}, y_{3,2}, y_{3,3}, y_{3,4}, y_{4,1}, y_{4,2}, y_{4,3}, y_{4,4}, y_{5,1}, y_{5,2}, y_{5,3}, y_{5,4}, y_{6,1}, y_{6,2}, y_{6,3}, y_{6,4}, \\
& y_{7,1}, y_{7,2}, y_{7,3}, y_{7,4}, y_{8,1}, y_{8,2}, y_{8,3}, y_{8,4}, y_{9,1}, y_{9,2}, y_{9,3}, y_{9,4}], 0, [\], [\], [t_1, t_2, t_3, t_4, _t], [\], [\], \\
& [diff = [x_1, t_1], diff = [x_2, t_2], diff = [x_3, t_3], diff = [x_4, t_4], diff = [_y, _t]], \left[_a \rightarrow _a x_1 \right. \\
& - \left(\frac{\partial}{\partial t_1} _a \right), _a \rightarrow _a x_2 - \left(\frac{\partial}{\partial t_2} _a \right), _a \rightarrow _a x_3 - \left(\frac{\partial}{\partial t_3} _a \right), _a \rightarrow _a x_4 - \left(\frac{\partial}{\partial t_4} _a \right), _a \\
& \rightarrow _a _y - \left(\frac{\partial}{\partial _t} _a \right) \left. \right]], 3 = \left[\text{Ore_algebra}, ["diff", "diff", "diff", "diff", "diff"], [t_1, t_2, t_3, t_4, \\
& _t], [x_1, x_2, x_3, x_4, _y], [t_1, t_2, t_3, t_4, _t], [y_{1,1}, y_{1,2}, y_{1,3}, y_{1,4}, y_{2,1}, y_{2,2}, y_{2,3}, y_{2,4}, y_{3,1}, y_{3,2}, y_{3,3}, \\
& y_{3,4}, y_{4,1}, y_{4,2}, y_{4,3}, y_{4,4}, y_{5,1}, y_{5,2}, y_{5,3}, y_{5,4}, y_{6,1}, y_{6,2}, y_{6,3}, y_{6,4}, y_{7,1}, y_{7,2}, y_{7,3}, y_{7,4}, y_{8,1}, y_{8,2}, \\
& y_{8,3}, y_{8,4}, y_{9,1}, y_{9,2}, y_{9,3}, y_{9,4}, y_{10,1}, y_{10,2}, y_{10,3}, y_{10,4}], 0, [\], [\], [t_1, t_2, t_3, t_4, _t], [\], [\], [diff
\end{aligned}$$

$$= [x_1, t_1], \text{diff} = [x_2, t_2], \text{diff} = [x_3, t_3], \text{diff} = [x_4, t_4], \text{diff} = [_y, _t], \left[\begin{array}{l} \text{--}a \rightarrow \text{--}a x_1 - \left(\frac{\partial}{\partial t_1} \text{--}a \right), \\ \text{--}a \rightarrow \text{--}a x_2 - \left(\frac{\partial}{\partial t_2} \text{--}a \right), \text{--}a \rightarrow \text{--}a x_3 - \left(\frac{\partial}{\partial t_3} \text{--}a \right), \text{--}a \rightarrow \text{--}a x_4 - \left(\frac{\partial}{\partial t_4} \text{--}a \right), \text{--}a \rightarrow \text{--}a \text{--}y \\ - \left(\frac{\partial}{\partial \text{--}t} \text{--}a \right) \end{array} \right] \right]$$

> **IsSolution(Sol2);**

table([1 = [true], 2 = [true], 3 = [true]])

(21)

> **RF2bis := RankFactorization(M, [D1, D2, D3, D4], 2, "reduced");**

$$RF2bis := \left[\begin{array}{l} [x_1 x_3 + x_3 x_4, x_2^2 - x_3^2, x_1 x_2 + x_2 x_4], \\ \left[\begin{array}{cccccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -x_3 & x_2 & x_1 + x_4 & 0 & 0 \\ -1 & 0 & 0 & 0 & x_3 & x_2 \\ 0 & x_2 & -x_3 & 0 & 0 & 0 \\ & & & & & x_1 + x_4 \end{array} \right] \end{array} \right], \quad (22)$$

$$\left[\begin{array}{cccc} 1 & 0 & 0 & 1 \end{array} \right], [x_1 - x_4, x_3 x_4, x_2 x_4], \text{table} \left(\left[\begin{array}{c} \text{--}y \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{array} \right], 1 = \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{array} \right], 2 = \left[\begin{array}{c} 0 \\ \text{--}y \\ 0 \\ 0 \\ 0 \end{array} \right], 3 = \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ \text{--}y \\ 0 \end{array} \right] \right), \text{table} \left[\begin{array}{c} 1 \\ \end{array} \right]$$

$$= \left[\begin{array}{cccccc} x_3 & x_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right], 2 = \left[\begin{array}{cccccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 2 \text{--}y x_4 & 0 & 0 & 0 & x_2 & 0 \\ 0 & 0 & 0 & 0 & -x_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right], 3$$

$$= \left(\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -y x_3 x_4 & 0 \\ 2 - y x_4 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \right), \left[\text{Ore_algebra}, ["diff", "diff", "diff", "diff", "diff"], \right.$$

$$\left. [t_1, t_2, t_3, t_4, -t], [x_1, x_2, x_3, x_4, -y], [t_1, t_2, t_3, t_4, -t], [], 0, [], [], [t_1, t_2, t_3, t_4, -t], [], [], [diff$$

$$= [x_1, t_1], diff = [x_2, t_2], diff = [x_3, t_3], diff = [x_4, t_4], diff = [-y, -t], \left[-a \rightarrow -a x_1 - \left(\frac{\partial}{\partial t_1} - a \right), \right.$$

$$\left. -a \rightarrow -a x_2 - \left(\frac{\partial}{\partial t_2} - a \right), -a \rightarrow -a x_3 - \left(\frac{\partial}{\partial t_3} - a \right), -a \rightarrow -a x_4 - \left(\frac{\partial}{\partial t_4} - a \right), -a \rightarrow -a - y \right.$$

$$\left. - \left(\frac{\partial}{\partial -t} - a \right) \right], \left(\begin{bmatrix} x_1 & 0 & x_4 & 0 \\ 0 & x_2 & 0 & x_3 \\ 0 & -x_3 & 0 & -x_2 \\ -x_4 & 0 & -x_1 & 0 \end{bmatrix} \right)$$

> Sol2bis := Solutions(M, [D1, D2, D3, D4], 2, "reduced");

$$\text{Sol2bis} := \left[[x_1 x_3 + x_3 x_4, x_2^2 - x_3^2, x_1 x_2 + x_2 x_4], [x_1 - x_4, x_3 x_4, x_2 x_4], \right.$$

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$$\begin{bmatrix} x_1 & 0 & x_4 & 0 \\ 0 & x_2 & 0 & x_3 \\ 0 & -x_3 & 0 & -x_2 \\ -x_4 & 0 & -x_1 & 0 \end{bmatrix}, \text{table}([1 = [[x_2 y_{2,1} + x_3 y_{1,1} + _y, x_2 y_{2,2} + x_3 y_{1,2},$$

$$x_2 y_{2,3} + x_3 y_{1,3}, x_2 y_{2,4} + x_3 y_{1,4} + _y],$$

$$[-x_3 y_{3,1} + x_2 y_{4,1} + (x_1 + x_4) y_{5,1}, -x_3 y_{3,2} + x_2 y_{4,2} + (x_1 + x_4) y_{5,2}, -x_3 y_{3,3} + x_2 y_{4,3} \\ + (x_1 + x_4) y_{5,3}, -x_3 y_{3,4} + x_2 y_{4,4} + (x_1 + x_4) y_{5,4}],$$

$$[x_2 y_{2,1} + x_3 y_{1,1} - _y, x_2 y_{2,2} + x_3 y_{1,2}, x_2 y_{2,3} + x_3 y_{1,3}, x_2 y_{2,4} + x_3 y_{1,4} - _y],$$

$$[x_2 y_{3,1} - x_3 y_{4,1} + (x_1 + x_4) y_{6,1}, x_2 y_{3,2} - x_3 y_{4,2} + (x_1 + x_4) y_{6,2}, x_2 y_{3,3} - x_3 y_{4,3} + (x_1$$

$$+ x_4) y_{6,3}, x_2 y_{3,4} - x_3 y_{4,4} + (x_1 + x_4) y_{6,4}]], 2 = [[y_{1,1}, y_{1,2}, y_{1,3}, y_{1,4}],$$

$$[-x_3 y_{2,1} + x_2 y_{3,1} + (x_1 + x_4) y_{4,1}, -x_3 y_{2,2} + x_2 y_{3,2} + (x_1 + x_4) y_{4,2}, -x_3 y_{2,3} + x_2 y_{3,3} \\ + (x_1 + x_4) y_{4,3}, -x_3 y_{2,4} + x_2 y_{3,4} + (x_1 + x_4) y_{4,4}],$$

$$[-y_{1,1} + x_3 (2 _y x_4 y_{1,1} + x_2 y_{5,1} + _y) - x_2 x_3 y_{5,1}, -y_{1,2} + x_3 (2 _y x_4 y_{1,2} + x_2 y_{5,2}) \\ - x_2 x_3 y_{5,2}, -y_{1,3} + x_3 (2 _y x_4 y_{1,3} + x_2 y_{5,3}) - x_2 x_3 y_{5,3}, -y_{1,4} + x_3 (2 _y x_4 y_{1,4} \\ + x_2 y_{5,4} + _y) - x_2 x_3 y_{5,4}],$$

$$[x_2 y_{2,1} - x_3 y_{3,1} + (x_1 + x_4) y_{6,1}, x_2 y_{2,2} - x_3 y_{3,2} + (x_1 + x_4) y_{6,2}, x_2 y_{2,3} - x_3 y_{3,3} + (x_1$$

$$+ x_4) y_{6,3}, x_2 y_{2,4} - x_3 y_{3,4} + (x_1 + x_4) y_{6,4}]], 3 = [[y_{1,1}, y_{1,2}, y_{1,3}, y_{1,4}],$$

$$[-x_3 y_{2,1} + x_2 y_{3,1} + (x_1 + x_4) y_{4,1}, -x_3 y_{2,2} + x_2 y_{3,2} + (x_1 + x_4) y_{4,2}, -x_3 y_{2,3} + x_2 y_{3,3} + (x_1 + x_4) y_{4,3}, -x_3 y_{2,4} + x_2 y_{3,4} + (x_1 + x_4) y_{4,4}],$$

$$[-y_{1,1} + _y x_3^2 x_4 y_{5,1} + x_2 (2 _y x_4 y_{1,1} + _y - y_{5,1}), -y_{1,2} + _y x_3^2 x_4 y_{5,2} + x_2 (2 _y x_4 y_{1,2} - y_{5,2}), -y_{1,3} + _y x_3^2 x_4 y_{5,3} + x_2 (2 _y x_4 y_{1,3} - y_{5,3}), -y_{1,4} + _y x_3^2 x_4 y_{5,4} + x_2 (2 _y x_4 y_{1,4} + _y - y_{5,4})],$$

$$[x_2 y_{2,1} - x_3 y_{3,1} + (x_1 + x_4) y_{6,1}, x_2 y_{2,2} - x_3 y_{3,2} + (x_1 + x_4) y_{6,2}, x_2 y_{2,3} - x_3 y_{3,3} + (x_1$$

$$+ x_4) y_{6,3}, x_2 y_{2,4} - x_3 y_{3,4} + (x_1 + x_4) y_{6,4}]]], \text{table}\left(\left[1 = \left[Ore_algebra, ["diff", "diff",$$

$$"diff", "diff", "diff"], [t_1, t_2, t_3, t_4, _t], [x_1, x_2, x_3, x_4, _y], [t_1, t_2, t_3, t_4, _t], [y_{1,1}, y_{1,2}, y_{1,3}, y_{1,4},$$

$$y_{2,1}, y_{2,2}, y_{2,3}, y_{2,4}, y_{3,1}, y_{3,2}, y_{3,3}, y_{3,4}, y_{4,1}, y_{4,2}, y_{4,3}, y_{4,4}, y_{5,1}, y_{5,2}, y_{5,3}, y_{5,4}, y_{6,1}, y_{6,2}, y_{6,3},$$

$$y_{6,4}], 0, [], [], [t_1, t_2, t_3, t_4, _t], [], [], [diff = [x_1, t_1], diff = [x_2, t_2], diff = [x_3, t_3], diff = [x_4,$$

$$t_4], diff = [_y, _t]], \left[-a \rightarrow _a x_1 - \left(\frac{\partial}{\partial t_1} _a\right), -a \rightarrow _a x_2 - \left(\frac{\partial}{\partial t_2} _a\right), -a \rightarrow _a x_3$$

$$- \left(\frac{\partial}{\partial t_3} _a\right), -a \rightarrow _a x_4 - \left(\frac{\partial}{\partial t_4} _a\right), -a \rightarrow _a _y - \left(\frac{\partial}{\partial _t} _a\right)\right], 2 = \left[Ore_algebra,$$

$$["diff", "diff", "diff", "diff", "diff"], [t_1, t_2, t_3, t_4, _t], [x_1, x_2, x_3, x_4, _y], [t_1, t_2, t_3, t_4, _t], [y_{1,1},$$

$$y_{1,2}, y_{1,3}, y_{1,4}, y_{2,1}, y_{2,2}, y_{2,3}, y_{2,4}, y_{3,1}, y_{3,2}, y_{3,3}, y_{3,4}, y_{4,1}, y_{4,2}, y_{4,3}, y_{4,4}, y_{5,1}, y_{5,2}, y_{5,3}, y_{5,4},$$

$$y_{6,1}, y_{6,2}, y_{6,3}, y_{6,4}], 0, [], [], [t_1, t_2, t_3, t_4, _t], [], [], [diff = [x_1, t_1], diff = [x_2, t_2], diff = [x_3,$$

$$t_3], diff = [x_4, t_4], diff = [_y, _t]], \left[-a \rightarrow _a x_1 - \left(\frac{\partial}{\partial t_1} _a\right), -a \rightarrow _a x_2 - \left(\frac{\partial}{\partial t_2} _a\right), -a \rightarrow _a x_3$$

$$- \left(\frac{\partial}{\partial t_3} _a\right), -a \rightarrow _a x_4 - \left(\frac{\partial}{\partial t_4} _a\right), -a \rightarrow _a _y - \left(\frac{\partial}{\partial _t} _a\right)\right], 3 = \left[Ore_algebra,$$

$$\left[-a \rightarrow_a x_3 - \left(\frac{\partial}{\partial t_3} - a \right), -a \rightarrow_a x_4 - \left(\frac{\partial}{\partial t_4} - a \right), -a \rightarrow_a y - \left(\frac{\partial}{\partial t} - a \right) \right],$$

$$\begin{bmatrix} x_1 & 0 & x_4 & 0 \\ 0 & x_2 & 0 & x_3 \\ 0 & -x_3 & 0 & -x_2 \\ -x_4 & 0 & -x_1 & 0 \end{bmatrix}$$

> RankFactorization(M, [D1,D2,D3,D4], 3, "reduced");

$$\left[[x_3, x_2, x_1 + x_4], \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, [1 \ 0 \ 0 \ 1], [x_4], \text{table} \left(\left[\begin{array}{c} 1 = \begin{bmatrix} 0 \\ 0 \\ -y \\ 0 \end{bmatrix} \end{array} \right] \right), \text{table} \left(\begin{bmatrix} 1 \end{bmatrix} \right) \right] \quad (26)$$

$$= \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right), \left[\text{Ore_algebra}, ["diff", "diff", "diff", "diff", "diff"], [t_1, t_2, t_3, t_4, -t], [x_1, x_2,$$

$$x_3, x_4, -y], [t_1, t_2, t_3, t_4, -t], [], 0, [], [], [t_1, t_2, t_3, t_4, -t], [], [], [diff = [x_1, t_1], diff = [x_2, t_2],$$

$$diff = [x_3, t_3], diff = [x_4, t_4], diff = [-y, -t]], \left[-a \rightarrow_a x_1 - \left(\frac{\partial}{\partial t_1} - a \right), -a \rightarrow_a x_2 - \left(\frac{\partial}{\partial t_2} - a \right),$$

$$\left[-a \rightarrow_a x_3 - \left(\frac{\partial}{\partial t_3} - a \right), -a \rightarrow_a x_4 - \left(\frac{\partial}{\partial t_4} - a \right), -a \rightarrow_a y - \left(\frac{\partial}{\partial t} - a \right) \right],$$

$$\begin{aligned}
&= \left(\begin{array}{cccc} y_{1,1} & y_{1,2} & y_{1,3} & y_{1,4} \\ y_{2,1} & y_{2,2} & y_{2,3} & y_{2,4} \\ -y + y_{1,1} & y_{1,2} & y_{1,3} & -y + y_{1,4} \\ y_{3,1} & y_{3,2} & y_{3,3} & y_{3,4} \end{array} \right), \text{table} \left(\left[1 = \left[\text{Ore_algebra}, ["diff", "diff", "diff", \right. \right. \right. \right. \\
&\text{"diff", "diff"}], [t_1, t_2, t_3, t_4, -t], [x_1, x_2, x_3, x_4, -y], [t_1, t_2, t_3, t_4, -t], [y_{1,1}, y_{1,2}, y_{1,3}, y_{1,4}, y_{2,1}, \\
&y_{2,2}, y_{2,3}, y_{2,4}, y_{3,1}, y_{3,2}, y_{3,3}, y_{3,4}], 0, [], [], [t_1, t_2, t_3, t_4, -t], [], [], [diff = [x_1, t_1], diff = [x_2, \\
&t_2], diff = [x_3, t_3], diff = [x_4, t_4], diff = [-y, -t]], \left[-a \rightarrow -a x_1 - \left(\frac{\partial}{\partial t_1} - a \right), -a \rightarrow -a x_2 \right. \\
&\left. - \left(\frac{\partial}{\partial t_2} - a \right), -a \rightarrow -a x_3 - \left(\frac{\partial}{\partial t_3} - a \right), -a \rightarrow -a x_4 - \left(\frac{\partial}{\partial t_4} - a \right), -a \rightarrow -a -y - \left(\frac{\partial}{\partial -t} - a \right) \right] \right] \right) \\
&> \text{IsSolution(Sol3bis);} \\
&\qquad \qquad \qquad \text{table}([1 = [true]]) \qquad \qquad \qquad (30)
\end{aligned}$$

[Example 2 of Inria Report 9438

$$\begin{aligned}
&> M := \text{Matrix}([[1,0],[0,1]]); \\
&\qquad \qquad \qquad M := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \qquad \qquad (31)
\end{aligned}$$

$$\begin{aligned}
&> D1 := \text{Matrix}([-1,-2],[1,2]); \\
&\qquad \qquad \qquad D1 := \begin{bmatrix} -1 & -2 \\ 1 & 2 \end{bmatrix} \qquad \qquad \qquad (32)
\end{aligned}$$

$$\begin{aligned}
&> D2 := \text{Matrix}([-3,-4],[3,4]); \\
&\qquad \qquad \qquad D2 := \begin{bmatrix} -3 & -4 \\ 3 & 4 \end{bmatrix} \qquad \qquad \qquad (33)
\end{aligned}$$

$$\begin{aligned}
&> Sol0 := \text{Solutions}(M,[D1,D2],0); \\
&\qquad \qquad \qquad Sol0 := "No solutions" \qquad \qquad \qquad (34)
\end{aligned}$$

$$\begin{aligned}
&> Sol1 := \text{Solutions}(M,[D1,D2],1); \\
&\qquad \qquad \qquad Sol1 := "No solutions" \qquad \qquad \qquad (35)
\end{aligned}$$

[Example 3 of Inria Report 9438

> M := Matrix([[15,14,13],[24,20,16]]);

$$M := \begin{bmatrix} 15 & 14 & 13 \\ 24 & 20 & 16 \end{bmatrix} \quad (36)$$

> D1 := Matrix([[1,-1],[1,1]]);

$$D1 := \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \quad (37)$$

> D2 := Matrix([[1,2],[-1,2]]);

$$D2 := \begin{bmatrix} 1 & 2 \\ -1 & 2 \end{bmatrix} \quad (38)$$

> D3 := Matrix([[1,3],[4,3]]);

$$D3 := \begin{bmatrix} 1 & 3 \\ 4 & 3 \end{bmatrix} \quad (39)$$

> RF0 := RankFactorization(M,[D1,D2,D3],0);

$$RF0 := \left[\begin{matrix} [x_2^2, x_1 x_2, x_1^2], \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 15 & 14 & 13 \\ 24 & 20 & 16 \end{bmatrix}, [x_2^2, x_1 x_2, x_1^2], \text{table} \end{matrix} \right] \quad (40)$$

$$= \left[\begin{matrix} -\frac{y(35x_1 + 194x_2)}{388} & -\frac{y(85x_1 + 194x_2)}{388} \\ \frac{y(21x_1 + 31x_2)}{388} & -\frac{y(51x_1 - 91x_2)}{388} \\ \frac{y(7x_1 + 22x_2)}{194} & -\frac{y(17x_1 - 2x_2)}{194} \end{matrix} \right], 2$$

$$= \left[\begin{matrix} -\frac{55yx_1}{194} & -\frac{5yx_1}{194} \\ \frac{3y(11x_1 + 7x_2)}{194} & \frac{3y(x_1 - 17x_2)}{194} \\ \frac{y(11x_1 - 7x_2)}{97} & \frac{y(x_1 + 17x_2)}{97} \end{matrix} \right], 3$$

$$= \left(\begin{array}{cc} \frac{66 _y x_1}{97} & \frac{6 _y x_1}{97} \\ \frac{_y (38 x_1 + 33 x_2)}{97} & -\frac{_y (23 x_1 - 3 x_2)}{97} \\ -\frac{_y (7 x_1 + 22 x_2)}{97} & \frac{_y (17 x_1 - 2 x_2)}{97} \end{array} \right), \text{table} \left(\begin{array}{c} 1 \\ \end{array} \right)$$

$$= \left[\begin{array}{c} 5 x_1^2 + 12 x_1 x_2 \\ -3 x_1^2 + 5 x_1 x_2 + 6 x_2^2 \\ -2 x_1^2 - 4 x_2^2 \end{array} \right], 2 = \left[\begin{array}{cc} 5 x_1^2 + 12 x_1 x_2 & 5 x_1 + 12 x_2 \\ -3 x_1^2 + 5 x_1 x_2 + 6 x_2^2 & 6 _y x_2^3 - 3 x_1 + 5 x_2 \\ -2 x_1^2 - 4 x_2^2 & -4 _y x_2^3 - 2 x_1 \end{array} \right], 3$$

$$= \left(\begin{array}{ccc} 5 x_1^2 + 12 x_1 x_2 & 12 _y x_1 x_2 + 5 & 5 x_1 + 12 x_2 \\ -3 x_1^2 + 5 x_1 x_2 + 6 x_2^2 & 5 _y x_1 x_2 + 6 _y x_2^2 - 3 & 6 _y x_1 x_2^2 - 3 x_1 + 5 x_2 \\ -2 x_1^2 - 4 x_2^2 & -4 _y x_2^2 - 2 & -4 _y x_1 x_2^2 - 2 x_1 \end{array} \right),$$

[Ore_algebra, ["diff", "diff", "diff"], [t1, t2, _t], [x1, x2, _y], [t1, t2, _t], [], 0, [], [], [t1, t2,

_t], [], [], [diff= [x1, t1], diff= [x2, t2], diff= [_y, _t]], [-a->_a x1 - (∂/∂t1 -a), -a->_a x2

- (∂/∂t2 -a), -a->_a _y - (∂/∂_t -a)]]], [x1 - x2 x1 + 2 x2 x1 + 3 x2]

[x1 + x2 -x1 + 2 x2 4 x1 + 3 x2]

> Sol0 := Solutions(M, [D1, D2, D3], 0);

Sol0 := [[x2^2, x1 x2, x1^2], [x2^2, x1 x2, x1^2], [x1 - x2 x1 + 2 x2 x1 + 3 x2] , table ([1 = [[(41)

- 15 _y (35 x1 + 194 x2) / 388 + 6 _y (85 x1 + 194 x2) / 97 + (5 x1^2 + 12 x1 x2) y1,1,

- 7 _y (35 x1 + 194 x2) / 194 + 5 _y (85 x1 + 194 x2) / 97 + (5 x1^2 + 12 x1 x2) y1,2,

- 13 _y (35 x1 + 194 x2) / 388 + 4 _y (85 x1 + 194 x2) / 97 + (5 x1^2 + 12 x1 x2) y1,3] ,

[15 _y (21 x1 + 31 x2) / 388 - 6 _y (51 x1 - 91 x2) / 97 + (-3 x1^2 + 5 x1 x2 + 6 x2^2) y1,1,

7 _y (21 x1 + 31 x2) / 194 - 5 _y (51 x1 - 91 x2) / 97 + (-3 x1^2 + 5 x1 x2 + 6 x2^2) y1,2,

$$\left[\frac{13 _y (21 x_1 + 31 x_2)}{388} - \frac{4 _y (51 x_1 - 91 x_2)}{97} + (-3 x_1^2 + 5 x_1 x_2 + 6 x_2^2) y_{1,3} \right],$$

$$\left[\frac{15 _y (7 x_1 + 22 x_2)}{194} - \frac{12 _y (17 x_1 - 2 x_2)}{97} + (-2 x_1^2 - 4 x_2^2) y_{1,1}, \right.$$

$$\left. \frac{7 _y (7 x_1 + 22 x_2)}{97} - \frac{10 _y (17 x_1 - 2 x_2)}{97} + (-2 x_1^2 - 4 x_2^2) y_{1,2}, \right.$$

$$\left. \frac{13 _y (7 x_1 + 22 x_2)}{194} - \frac{8 _y (17 x_1 - 2 x_2)}{97} + (-2 x_1^2 - 4 x_2^2) y_{1,3} \right], 2 = \left[\left[\right. \right.$$

$$- \frac{945 _y x_1}{194} + (5 x_1^2 + 12 x_1 x_2) y_{1,1} + (5 x_1 + 12 x_2) y_{2,1}, - \frac{435 _y x_1}{97} + (5 x_1^2$$

$$+ 12 x_1 x_2) y_{1,2} + (5 x_1 + 12 x_2) y_{2,2}, - \frac{795 _y x_1}{194} + (5 x_1^2 + 12 x_1 x_2) y_{1,3} + (5 x_1$$

$$+ 12 x_2) y_{2,3} \left. \right],$$

$$\left[\frac{45 _y (11 x_1 + 7 x_2)}{194} + \frac{36 _y (x_1 - 17 x_2)}{97} + (-3 x_1^2 + 5 x_1 x_2 + 6 x_2^2) y_{1,1} + (6 _y x_2^3$$

$$- 3 x_1 + 5 x_2) y_{2,1}, \frac{21 _y (11 x_1 + 7 x_2)}{97} + \frac{30 _y (x_1 - 17 x_2)}{97} + (-3 x_1^2 + 5 x_1 x_2 + 6$$

$$x_2^2) y_{1,2} + (6 _y x_2^3 - 3 x_1 + 5 x_2) y_{2,2}, \frac{39 _y (11 x_1 + 7 x_2)}{194} + \frac{24 _y (x_1 - 17 x_2)}{97} + ($$

$$-3 x_1^2 + 5 x_1 x_2 + 6 x_2^2) y_{1,3} + (6 _y x_2^3 - 3 x_1 + 5 x_2) y_{2,3} \right],$$

$$\begin{aligned}
& \left[\frac{15 _y (11 x_1 - 7 x_2)}{97} + \frac{24 _y (x_1 + 17 x_2)}{97} + (-2 x_1^2 - 4 x_2^2) y_{1,1} + (-4 _y x_2^3 \right. \\
& - 2 x_1) y_{2,1}, \frac{14 _y (11 x_1 - 7 x_2)}{97} + \frac{20 _y (x_1 + 17 x_2)}{97} + (-2 x_1^2 - 4 x_2^2) y_{1,2} + (\\
& -4 _y x_2^3 - 2 x_1) y_{2,2}, \frac{13 _y (11 x_1 - 7 x_2)}{97} + \frac{16 _y (x_1 + 17 x_2)}{97} + (-2 x_1^2 - 4 x_2^2) y_{1,3} \\
& \left. + (-4 _y x_2^3 - 2 x_1) y_{2,3} \right], 3 = \left[\left[\frac{1134 _y x_1}{97} + (5 x_1^2 + 12 x_1 x_2) y_{1,1} \right. \right. \\
& + (12 _y x_1 x_2 + 5) y_{2,1} + (5 x_1 + 12 x_2) y_{3,1}, \frac{1044 _y x_1}{97} + (5 x_1^2 + 12 x_1 x_2) y_{1,2} \\
& + (12 _y x_1 x_2 + 5) y_{2,2} + (5 x_1 + 12 x_2) y_{3,2}, \frac{954 _y x_1}{97} + (5 x_1^2 + 12 x_1 x_2) y_{1,3} \\
& \left. + (12 _y x_1 x_2 + 5) y_{2,3} + (5 x_1 + 12 x_2) y_{3,3} \right], \\
& \left[\frac{15 _y (38 x_1 + 33 x_2)}{97} - \frac{24 _y (23 x_1 - 3 x_2)}{97} + (-3 x_1^2 + 5 x_1 x_2 + 6 x_2^2) y_{1,1} \right. \\
& + (5 _y x_1 x_2 + 6 _y x_2^2 - 3) y_{2,1} + (6 _y x_1 x_2^2 - 3 x_1 + 5 x_2) y_{3,1}, \frac{14 _y (38 x_1 + 33 x_2)}{97} \\
& - \frac{20 _y (23 x_1 - 3 x_2)}{97} + (-3 x_1^2 + 5 x_1 x_2 + 6 x_2^2) y_{1,2} + (5 _y x_1 x_2 + 6 _y x_2^2 - 3) y_{2,2} \\
& \left. + (6 _y x_1 x_2^2 - 3 x_1 + 5 x_2) y_{3,2}, \frac{13 _y (38 x_1 + 33 x_2)}{97} - \frac{16 _y (23 x_1 - 3 x_2)}{97} + (-3 \right.
\end{aligned}$$

$$\begin{aligned}
& x_1^2 + 5x_1x_2 + 6x_2^2)y_{1,3} + (5y_{1,1}x_1x_2 + 6y_{1,2}x_2^2 - 3)y_{2,3} + (6y_{1,1}x_1x_2^2 - 3x_1 + 5x_2)y_{3,3} \Big], \\
& \left[-\frac{15y_{1,1}(7x_1 + 22x_2)}{97} + \frac{24y_{1,2}(17x_1 - 2x_2)}{97} + (-2x_1^2 - 4x_2^2)y_{1,1} + (-4y_{1,1}x_2^2 \right. \\
& - 2)y_{2,1} + (-4y_{1,1}x_1x_2^2 - 2x_1)y_{3,1}, -\frac{14y_{1,1}(7x_1 + 22x_2)}{97} + \frac{20y_{1,2}(17x_1 - 2x_2)}{97} \\
& + (-2x_1^2 - 4x_2^2)y_{1,2} + (-4y_{1,2}x_2^2 - 2)y_{2,2} + (-4y_{1,1}x_1x_2^2 - 2x_1)y_{3,2}, \\
& \left. -\frac{13y_{1,1}(7x_1 + 22x_2)}{97} + \frac{16y_{1,2}(17x_1 - 2x_2)}{97} + (-2x_1^2 - 4x_2^2)y_{1,3} + (-4y_{1,2}x_2^2 \right. \\
& \left. - 2)y_{2,3} + (-4y_{1,1}x_1x_2^2 - 2x_1)y_{3,3} \right] \Big], \text{table} \left(\left[1 = \left[\text{Ore_algebra}, ["diff", "diff"], \right. \right. \right. \\
& \text{"diff"}, [t_1, t_2, -t], [x_1, x_2, -y], [t_1, t_2, -t], [y_{1,1}, y_{1,2}, y_{1,3}], 0, [], [], [t_1, t_2, -t], [], [], [diff \\
& = [x_1, t_1], diff = [x_2, t_2], diff = [-y, -t]], \left[-a \rightarrow -a x_1 - \left(\frac{\partial}{\partial t_1} - a \right), -a \rightarrow -a x_2 - \left(\frac{\partial}{\partial t_2} - a \right), -a \right. \\
& \left. \rightarrow -a -y - \left(\frac{\partial}{\partial -t} - a \right) \right] \Big], 2 = \left[\text{Ore_algebra}, ["diff", "diff", "diff"], [t_1, t_2, -t], [x_1, x_2, -y], [t_1, \\
& t_2, -t], [y_{1,1}, y_{1,2}, y_{1,3}, y_{2,1}, y_{2,2}, y_{2,3}], 0, [], [], [t_1, t_2, -t], [], [], [diff = [x_1, t_1], diff = [x_2, \\
& t_2], diff = [-y, -t]], \left[-a \rightarrow -a x_1 - \left(\frac{\partial}{\partial t_1} - a \right), -a \rightarrow -a x_2 - \left(\frac{\partial}{\partial t_2} - a \right), -a \rightarrow -a -y \right. \\
& \left. - \left(\frac{\partial}{\partial -t} - a \right) \right] \Big], 3 = \left[\text{Ore_algebra}, ["diff", "diff", "diff"], [t_1, t_2, -t], [x_1, x_2, -y], [t_1, t_2, -t], \\
& [y_{1,1}, y_{1,2}, y_{1,3}, y_{2,1}, y_{2,2}, y_{2,3}, y_{3,1}, y_{3,2}, y_{3,3}], 0, [], [], [t_1, t_2, -t], [], [], [diff = [x_1, t_1], diff \\
& = [x_2, t_2], diff = [-y, -t]], \left[-a \rightarrow -a x_1 - \left(\frac{\partial}{\partial t_1} - a \right), -a \rightarrow -a x_2 - \left(\frac{\partial}{\partial t_2} - a \right), -a \rightarrow -a -y \right. \\
& \left. - \left(\frac{\partial}{\partial -t} - a \right) \right] \Big] \Big] \Big]
\end{aligned}$$

> **IsSolution(Sol0);**

$$\text{table}([1 = [true], 2 = [true], 3 = [true]]) \quad (42)$$

> **RF0bis := RankFactorization(M, [D1, D2, D3], 0, "reduced");**

(43)

$$RF0bis := \left([x_2^2, x_1 x_2, x_1^2], \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 15 & 14 & 13 \\ 24 & 20 & 16 \end{bmatrix}, [x_2^2, x_1 x_2, x_1^2], \text{table} \begin{bmatrix} 1 \end{bmatrix} \right) \quad (43)$$

$$= \begin{bmatrix} -\frac{y(35x_1 + 194x_2)}{388} & -\frac{y(85x_1 + 194x_2)}{388} \\ \frac{y(21x_1 + 31x_2)}{388} & -\frac{y(51x_1 - 91x_2)}{388} \\ \frac{y(7x_1 + 22x_2)}{194} & -\frac{y(17x_1 - 2x_2)}{194} \end{bmatrix}, 2$$

$$= \begin{bmatrix} -\frac{55yx_1}{194} & -\frac{5yx_1}{194} \\ \frac{3y(11x_1 + 7x_2)}{194} & \frac{3y(x_1 - 17x_2)}{194} \\ \frac{y(11x_1 - 7x_2)}{97} & \frac{y(x_1 + 17x_2)}{97} \end{bmatrix}, 3$$

$$= \begin{bmatrix} \frac{66yx_1}{97} & \frac{6yx_1}{97} \\ \frac{y(38x_1 + 33x_2)}{97} & -\frac{y(23x_1 - 3x_2)}{97} \\ -\frac{y(7x_1 + 22x_2)}{97} & \frac{y(17x_1 - 2x_2)}{97} \end{bmatrix}], \text{table} \begin{bmatrix} 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5x_1^2 + 12x_1x_2 \\ -3x_1^2 + 5x_1x_2 + 6x_2^2 \\ -2x_1^2 - 4x_2^2 \end{bmatrix}, 2 = \begin{bmatrix} 5x_1 + 12x_2 \\ 6yx_2^3 - 3x_1 + 5x_2 \\ -4yx_2^3 - 2x_1 \end{bmatrix}, 3 = \begin{bmatrix} 5x_1 + 12x_2 \\ 6yx_1x_2^2 - 3x_1 + 5x_2 \\ -4yx_1x_2^2 - 2x_1 \end{bmatrix}$$

$$), [Ore_algebra, ["diff", "diff", "diff"], [t_1, t_2, -t], [x_1, x_2, -y], [t_1, t_2, -t], [], 0, [], [], [t_1,$$

$$-\frac{945}{194} \frac{y x_1}{194} + (5x_1 + 12x_2) y_{1,1}, -\frac{435}{97} \frac{y x_1}{97} + (5x_1 + 12x_2) y_{1,2}, -\frac{795}{194} \frac{y x_1}{194} + (5x_1 + 12x_2) y_{1,3},$$

$$\left[\frac{45}{194} \frac{y (11x_1 + 7x_2)}{194} + \frac{36}{97} \frac{y (x_1 - 17x_2)}{97} + (6y x_2^3 - 3x_1 + 5x_2) y_{1,1}, \right. \\ \frac{21}{97} \frac{y (11x_1 + 7x_2)}{97} + \frac{30}{97} \frac{y (x_1 - 17x_2)}{97} + (6y x_2^3 - 3x_1 + 5x_2) y_{1,2}, \\ \left. \frac{39}{194} \frac{y (11x_1 + 7x_2)}{194} + \frac{24}{97} \frac{y (x_1 - 17x_2)}{97} + (6y x_2^3 - 3x_1 + 5x_2) y_{1,3} \right],$$

$$\left[\frac{15}{97} \frac{y (11x_1 - 7x_2)}{97} + \frac{24}{97} \frac{y (x_1 + 17x_2)}{97} + (-4y x_2^3 - 2x_1) y_{1,1}, \right.$$

$$\frac{14}{97} \frac{y (11x_1 - 7x_2)}{97} + \frac{20}{97} \frac{y (x_1 + 17x_2)}{97} + (-4y x_2^3 - 2x_1) y_{1,2},$$

$$\left. \frac{13}{97} \frac{y (11x_1 - 7x_2)}{97} + \frac{16}{97} \frac{y (x_1 + 17x_2)}{97} + (-4y x_2^3 - 2x_1) y_{1,3} \right], 3 = \left[\right.$$

$$\left[\frac{1134}{97} \frac{y x_1}{97} + (5x_1 + 12x_2) y_{1,1}, \frac{1044}{97} \frac{y x_1}{97} + (5x_1 + 12x_2) y_{1,2}, \frac{954}{97} \frac{y x_1}{97} + (5x_1 + 12x_2) y_{1,3} \right],$$

$$\left[\frac{15}{97} \frac{y (38x_1 + 33x_2)}{97} - \frac{24}{97} \frac{y (23x_1 - 3x_2)}{97} + (6y x_1 x_2^2 - 3x_1 + 5x_2) y_{1,1}, \right.$$

$$\frac{14}{97} \frac{y (38x_1 + 33x_2)}{97} - \frac{20}{97} \frac{y (23x_1 - 3x_2)}{97} + (6y x_1 x_2^2 - 3x_1 + 5x_2) y_{1,2},$$

$$\left. \frac{13}{97} \frac{y (38x_1 + 33x_2)}{97} - \frac{16}{97} \frac{y (23x_1 - 3x_2)}{97} + (6y x_1 x_2^2 - 3x_1 + 5x_2) y_{1,3} \right],$$

$$\begin{aligned}
& \left[-\frac{15_y(7x_1 + 22x_2)}{97} + \frac{24_y(17x_1 - 2x_2)}{97} + (-4_yx_1x_2^2 - 2x_1)y_{1,1}, \right. \\
& -\frac{14_y(7x_1 + 22x_2)}{97} + \frac{20_y(17x_1 - 2x_2)}{97} + (-4_yx_1x_2^2 - 2x_1)y_{1,2}, \\
& \left. -\frac{13_y(7x_1 + 22x_2)}{97} + \frac{16_y(17x_1 - 2x_2)}{97} + (-4_yx_1x_2^2 - 2x_1)y_{1,3} \right] \Bigg], \text{table} \left(\left[\begin{array}{l} 1 \\ 0 \\ 0 \end{array} \right] \right. \\
& = \left[\text{Ore_algebra}, ["diff", "diff", "diff"], [t_1, t_2, -t], [x_1, x_2, -y], [t_1, t_2, -t], [y_{1,1}, y_{1,2}, y_{1,3}], 0, \right. \\
& \left. [], [], [t_1, t_2, -t], [], [], [diff = [x_1, t_1], diff = [x_2, t_2], diff = [-y, -t]], \left[-a \rightarrow -a x_1 \right. \right. \\
& \left. \left. - \left(\frac{\partial}{\partial t_1} -a \right), -a \rightarrow -a x_2 - \left(\frac{\partial}{\partial t_2} -a \right), -a \rightarrow -a y - \left(\frac{\partial}{\partial -t} -a \right) \right] \right], 2 = \left[\text{Ore_algebra}, \right. \\
& ["diff", "diff", "diff"], [t_1, t_2, -t], [x_1, x_2, -y], [t_1, t_2, -t], [y_{1,1}, y_{1,2}, y_{1,3}], 0, [], [], [t_1, t_2, -t], \\
& \left. [], [], [diff = [x_1, t_1], diff = [x_2, t_2], diff = [-y, -t]], \left[-a \rightarrow -a x_1 - \left(\frac{\partial}{\partial t_1} -a \right), -a \rightarrow -a x_2 \right. \right. \\
& \left. \left. - \left(\frac{\partial}{\partial t_2} -a \right), -a \rightarrow -a y - \left(\frac{\partial}{\partial -t} -a \right) \right] \right], 3 = \left[\text{Ore_algebra}, ["diff", "diff", "diff"], [t_1, t_2, \right. \\
& \left. -t], [x_1, x_2, -y], [t_1, t_2, -t], [y_{1,1}, y_{1,2}, y_{1,3}], 0, [], [], [t_1, t_2, -t], [], [], [diff = [x_1, t_1], diff \right. \\
& = [x_2, t_2], diff = [-y, -t]], \left[-a \rightarrow -a x_1 - \left(\frac{\partial}{\partial t_1} -a \right), -a \rightarrow -a x_2 - \left(\frac{\partial}{\partial t_2} -a \right), -a \rightarrow -a y \right. \\
& \left. \left. - \left(\frac{\partial}{\partial -t} -a \right) \right] \right] \Bigg]
\end{aligned}$$

```

> IsSolution(Sol0bis);
      table([1 = [true], 2 = [true], 3 = [true]])
(45)

```

```

> Sol1 := Solutions(M, [D1, D2, D3], 1);
      Sol1 := "No solutions"
(46)

```

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```

> D1 := Matrix([[0$3, 2], [3, 0$2, 1], [0$4], [0$3, 2]]);
(47)

```

$$D1 := \begin{bmatrix} 0 & 0 & 0 & 2 \\ 3 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \quad (47)$$

> D2 := Matrix([[5,3,0\$2],[0\$4],[0,5,2,0],[0,3,2,0]]);

$$D2 := \begin{bmatrix} 5 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 5 & 2 & 0 \\ 0 & 3 & 2 & 0 \end{bmatrix} \quad (48)$$

> M := Matrix([[30,0\$2],[0\$3],[12,0\$2],[12,0\$2]]);

$$M := \begin{bmatrix} 30 & 0 & 0 \\ 0 & 0 & 0 \\ 12 & 0 & 0 \\ 12 & 0 & 0 \end{bmatrix} \quad (49)$$

> Sol1 := Solutions(M,[D1,D2],1);

Sol1 := "No solutions" (50)

> RF0 := RankFactorization(M,[D1,D2],0);

$$RF0 := \begin{bmatrix} 2x_4x_1 - 3x_2x_4 - 2x_3x_4, 3x_1x_2 + x_2x_4, 2x_1^2 - 2x_1x_3 + x_2x_4, 9x_2^2x_4 + 6x_2x_3x_4 \\ + 2x_2x_4^2 \end{bmatrix} \quad (51)$$

$$\begin{bmatrix} x_2 & 2x_1 - 2x_3 & 0 & 0 \\ x_4 & 3x_4 & 3x_1 + x_4 & 9x_2x_4 + 6x_3x_4 + 2x_4^2 \end{bmatrix}, [6 \ 0 \ 0], [5x_2x_4$$

$$+ 2 x_3 x_4, 3 x_1 x_3 + x_3 x_4, 6 x_3^2 x_4 + 5 x_3 x_4^2], \text{table} \left(\left[\begin{array}{c} 0 \\ \frac{2-y}{3} \\ 0 \\ 0 \end{array} \right], 2 = \left[\begin{array}{c} 0 \\ 0 \\ -y \\ 0 \end{array} \right], 3 \right.$$

$$= \left[\begin{array}{c} 0 \\ -3-yx_3 \\ 0 \\ \frac{5-y}{2} \end{array} \right], \text{table} \left(\left[\begin{array}{c} 1 \\ \left[\left[\left[3, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \dots \right] \right] \right] \right] \right.$$

$$\left[-1, 18x_2 + 12x_3 + 4x_4, 9x_2 + 6x_3 + 2x_4, 24-yx_3x_4 + 20-yx_4^2 + 18, 0, 0, 0, 0, 0, 0, \dots \right],$$

$$\left[0, 0, -6x_4, 0, -2x_4, x_2, 9x_2 - 2x_4, 6x_3x_4 + 2x_4^2, 2, 4-yx_4^2, \dots \right],$$

$$\left[0, -6, 0, -30-yx_4, 1, 0, 1, -\frac{9x_2}{2} - 3x_3 - x_4, -3-yx_2 - 2-yx_3, -2-yx_4, \dots \right]],$$

$$\left[\begin{array}{c} 4 \times 16 \text{ Matrix} \\ \left[\left[\left[3, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \dots \right] \right] \right] \right], 2 = \left[\left[\left[3, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \dots \right] \right] \right],$$

$$\left[-1, 12x_3 + 4x_4, 6x_3 + 2x_4, -6, -2, -6, 4, 6, 0, 0, \dots \right],$$

$$\left[0, 0, -6x_4, -2-yx_4^2, -2-yx_4^2, -6-yx_3x_4 - 10-yx_4^2, 4-yx_4^2, 2-yx_4^2, -2x_4, -2x_4, \dots \right],$$

$$\left[0, -6, 0, 9-yx_3 + -yx_4, 3-yx_3 + -yx_4, 12-yx_3 + 5-yx_4, -6-yx_3 - 2-yx_4, -9-yx_3 - -yx_4, 1, 1, \dots \right]],$$

$$\left[\begin{array}{c} 4 \times 12 \text{ Matrix} \\ \left[\left[\left[3, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \dots \right] \right] \right] \right], 3 = \left[\left[\left[3, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \dots \right] \right] \right],$$

$$RFO := \left[\begin{array}{l} 2 x_4 x_1 - 3 x_2 x_4 - 2 x_3 x_4, 3 x_1 x_2 + x_2 x_4, 2 x_1^2 - 2 x_1 x_3 + x_2 x_4, 9 x_2^2 x_4 + 6 x_2 x_3 x_4 \end{array} \right] \quad (52)$$

$$+ 2 x_2 x_4^2, \left[\begin{array}{ccc} x_2 & 2 x_1 - 2 x_3 & 0 \\ x_4 & 3 x_4 & 3 x_1 + x_4 \end{array} \right], [6 \ 0 \ 0], [5 x_2 x_4 + 2 x_3 x_4, 3 x_1 x_3 + x_3 x_4],$$

$$\text{table} \left(\left(\left[\begin{array}{c} 0 \\ \frac{2_y}{3} \\ 0 \end{array} \right], 2 = \left[\begin{array}{c} 0 \\ 0 \\ _y \end{array} \right] \right) \right), \text{table} \left(\left(\left[\begin{array}{ccc} 3 & & 0 \\ -1 & 24_y x_3 x_4 + 20_y x_4^2 + 18 & \\ 0 & & -60_y x_4^2 \end{array} \right], 2 \right. \right.$$

$$\left. = \left[\begin{array}{ccc} 3 & & 0 \\ -1 & -6 x_3 - 2 x_4 & \\ 0 & & 6 x_4 \end{array} \right] \right), [Ore_algebra, ["diff", "diff", "diff", "diff", "diff"], [t_1, t_2, t_3, t_4,$$

$$_t], [x_1, x_2, x_3, x_4, _y], [t_1, t_2, t_3, t_4, _t], [], 0, [], [], [t_1, t_2, t_3, t_4, _t], [], [], [diff = [x_1, t_1],$$

$$diff = [x_2, t_2], diff = [x_3, t_3], diff = [x_4, t_4], diff = [_y, _t]], \left[_a \rightarrow _a x_1 - \left(\frac{\partial}{\partial t_1} _a \right), _a \rightarrow _a x_2$$

$$- \left(\frac{\partial}{\partial t_2} _a \right), _a \rightarrow _a x_3 - \left(\frac{\partial}{\partial t_3} _a \right), _a \rightarrow _a x_4 - \left(\frac{\partial}{\partial t_4} _a \right), _a \rightarrow _a _y - \left(\frac{\partial}{\partial _t} _a \right) \right],$$

$$\begin{bmatrix} 2x_4 & 5x_1 + 3x_2 \\ 3x_1 + x_4 & 0 \\ 0 & 5x_2 + 2x_3 \\ 2x_4 & 3x_2 + 2x_3 \end{bmatrix}$$

> **Solo := Solutions(M, [D1, D2], 0, "reduced");**

$$\text{Solo} := \left[\begin{array}{l} [2x_4x_1 - 3x_2x_4 - 2x_3x_4, 3x_1x_2 + x_2x_4, 2x_1^2 - 2x_1x_3 + x_2x_4, 9x_2^2x_4 + 6x_2x_3x_4 \\ + 2x_2x_4^2], [5x_2x_4 + 2x_3x_4, 3x_1x_3 + x_3x_4], \end{array} \right. \quad (53)$$

$$\left. \begin{array}{l} + 2x_2x_4^2], [5x_2x_4 + 2x_3x_4, 3x_1x_3 + x_3x_4], \end{array} \right[\begin{array}{cc} 2x_4 & 5x_1 + 3x_2 \\ 3x_1 + x_4 & 0 \\ 0 & 5x_2 + 2x_3 \\ 2x_4 & 3x_2 + 2x_3 \end{array} \right], \text{table}([1 = [\\ [3x_2y_{1,1} + (2x_1 - 2x_3)(4y - y_{1,1} + (24yx_3x_4 + 20yx_4^2 + 18)y_{2,1}), 3x_2y_{1,2} \\ + (2x_1 - 2x_3)(-y_{1,2} + (24yx_3x_4 + 20yx_4^2 + 18)y_{2,2}), 3x_2y_{1,3} + (2x_1 - 2x_3)(\\ -y_{1,3} + (24yx_3x_4 + 20yx_4^2 + 18)y_{2,3})], \\ [3x_4y_{1,1} + 3x_4(4y - y_{1,1} + (24yx_3x_4 + 20yx_4^2 + 18)y_{2,1}) - 60(3x_1 + x_4)y \\ x_4^2y_{2,1}, 3x_4y_{1,2} + 3x_4(-y_{1,2} + (24yx_3x_4 + 20yx_4^2 + 18)y_{2,2}) - 60(3x_1 + x_4)y \\ x_4^2y_{2,2}, 3x_4y_{1,3} + 3x_4(-y_{1,3} + (24yx_3x_4 + 20yx_4^2 + 18)y_{2,3}) - 60(3x_1 + x_4)y \\ x_4^2y_{2,3}]], 2 = [[3x_2y_{1,1} + (2x_1 - 2x_3)(-y_{1,1} + (-6x_3 - 2x_4)y_{2,1}), 3x_2y_{1,2} \\ + (2x_1 - 2x_3)(-y_{1,2} + (-6x_3 - 2x_4)y_{2,2}), 3x_2y_{1,3} + (2x_1 - 2x_3)(-y_{1,3} + (\\ -6x_3 - 2x_4)y_{2,3})], \\ [3x_4y_{1,1} + 3x_4(-y_{1,1} + (-6x_3 - 2x_4)y_{2,1}) + (3x_1 + x_4)(6x_4y_{2,1} + 6y), 3x_4y_{1,2} \\ + 3x_4(-y_{1,2} + (-6x_3 - 2x_4)y_{2,2}) + 6(3x_1 + x_4)x_4y_{2,2}, 3x_4y_{1,3} + 3x_4(-y_{1,3} + ($$

$$\begin{aligned}
& -6x_3 - 2x_4)y_{2,3} + 6(3x_1 + x_4)x_4y_{2,3}]]], \text{table}\left(\left[1 = \left[Ore_algebra, ["diff", "diff", \right. \right. \right. \\
& \text{"diff", "diff", "diff"}], [t_1, t_2, t_3, t_4, _t], [x_1, x_2, x_3, x_4, _y], [t_1, t_2, t_3, t_4, _t], [y_{1,1}, y_{1,2}, y_{1,3}, y_{2,1}, \\
& y_{2,2}, y_{2,3}], 0, [], [], [t_1, t_2, t_3, t_4, _t], [], [], [diff = [x_1, t_1], diff = [x_2, t_2], diff = [x_3, t_3], diff \\
& = [x_4, t_4], diff = [_y, _t]], \left[-a \rightarrow _a x_1 - \left(\frac{\partial}{\partial t_1} _a\right), -a \rightarrow _a x_2 - \left(\frac{\partial}{\partial t_2} _a\right), -a \rightarrow _a x_3 \right. \\
& \left. - \left(\frac{\partial}{\partial t_3} _a\right), -a \rightarrow _a x_4 - \left(\frac{\partial}{\partial t_4} _a\right), -a \rightarrow _a _y - \left(\frac{\partial}{\partial _t} _a\right)\right], 2 = \left[Ore_algebra, \right. \\
& ["diff", "diff", "diff", "diff", "diff"], [t_1, t_2, t_3, t_4, _t], [x_1, x_2, x_3, x_4, _y], [t_1, t_2, t_3, t_4, _t], [y_{1,1}, \\
& y_{1,2}, y_{1,3}, y_{2,1}, y_{2,2}, y_{2,3}], 0, [], [], [t_1, t_2, t_3, t_4, _t], [], [], [diff = [x_1, t_1], diff = [x_2, t_2], diff \\
& = [x_3, t_3], diff = [x_4, t_4], diff = [_y, _t]], \left[-a \rightarrow _a x_1 - \left(\frac{\partial}{\partial t_1} _a\right), -a \rightarrow _a x_2 - \left(\frac{\partial}{\partial t_2} _a\right), -a \right. \\
& \left. \rightarrow _a x_3 - \left(\frac{\partial}{\partial t_3} _a\right), -a \rightarrow _a x_4 - \left(\frac{\partial}{\partial t_4} _a\right), -a \rightarrow _a _y - \left(\frac{\partial}{\partial _t} _a\right)\right]]]]])
\end{aligned}$$

[Example 7 of Inria Report 9438

```
> M := Matrix([[0$3],[0$3]]);
```

$$M := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (54)$$

```
> D1 := DiagonalMatrix([1$2]);
```

$$D1 := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad (55)$$

```
> D2 := Matrix([[0,1],[1,0]]);
```

$$D2 := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad (56)$$

```
> D3 := Matrix([[2,1],[1,2]]);
```

$$D3 := \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \quad (57)$$

```
> RF0 := RankFactorization(M,[D1,D2,D3],0);
```

$$RF0 := \left[[0], \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, [0 \ 0 \ 0], [0], \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \left[Ore_algebra, ["diff", "diff"], [t_1, \right. \right. \quad (58)$$

$$t_2], [x_1, x_2], [t_1, t_2], [], 0, [], [], [t_1, t_2], [], [], [diff = [x_1, t_1], diff = [x_2, t_2]], \left[-a \rightarrow -a x_1 \right. \\ \left. - \left(\frac{\partial}{\partial t_1} - a \right), -a \rightarrow -a x_2 - \left(\frac{\partial}{\partial t_2} - a \right) \right], \begin{bmatrix} x_1 & x_2 & 2x_1 + x_2 \\ x_2 & x_1 & x_1 + 2x_2 \end{bmatrix} \right]$$

> **RF1 := RankFactorization(M, [D1, D2, D3], 1);**

$$RF1 := \left[[x_1^2 - x_2^2], \begin{bmatrix} 2 & 0 & 0 \\ 1 & -3x_2 & 2x_1 + x_2 \\ -1 & 2x_1 - x_2 & -x_2 \end{bmatrix}, [0 \ 0 \ 0], [0], \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad (59)$$

$$\left[Ore_algebra, ["diff", "diff"], [t_1, t_2], [x_1, x_2], [t_1, t_2], [], 0, [], [], [t_1, t_2], [], [], [diff$$

$$= [x_1, t_1], diff = [x_2, t_2], \left[-a \rightarrow -a x_1 - \left(\frac{\partial}{\partial t_1} - a \right), -a \rightarrow -a x_2 - \left(\frac{\partial}{\partial t_2} - a \right) \right],$$

$$\begin{bmatrix} x_1 & x_2 & 2x_1 + x_2 \\ x_2 & x_1 & x_1 + 2x_2 \end{bmatrix} \right]$$

> **RF2 := RankFactorization(M, [D1, D2, D3], 2);**

$$RF2 := \left[[x_2, x_1], \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, [0 \ 0 \ 0], [0], \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \left[Ore_algebra, ["diff", \quad (60)$$

$$"diff"], [t_1, t_2], [x_1, x_2], [t_1, t_2], [], 0, [], [], [t_1, t_2], [], [], [diff = [x_1, t_1], diff = [x_2, t_2]],$$

$$\left[-a \rightarrow -a x_1 - \left(\frac{\partial}{\partial t_1} - a \right), -a \rightarrow -a x_2 - \left(\frac{\partial}{\partial t_2} - a \right) \right], \begin{bmatrix} x_1 & x_2 & 2x_1 + x_2 \\ x_2 & x_1 & x_1 + 2x_2 \end{bmatrix} \right]$$

> **Sol0 := Solutions(M, [D1, D2, D3], 0);**

$$Sol0 := \left[[0], [0], \begin{bmatrix} x_1 & x_2 & 2x_1 + x_2 \\ x_2 & x_1 & x_1 + 2x_2 \end{bmatrix}, \text{table} \left(\left(1 = \begin{bmatrix} 2y_{1,1} & 2y_{1,2} & 2y_{1,3} \\ y_{1,1} & y_{1,2} & y_{1,3} \\ -y_{1,1} & -y_{1,2} & -y_{1,3} \end{bmatrix} \right) \right), \text{table} \left(\left(1 \quad (61) \right. \right.$$

$$= \left[Ore_algebra, ["diff", "diff"], [t_1, t_2], [x_1, x_2], [t_1, t_2], [y_{1,1}, y_{1,2}, y_{1,3}], 0, [], [], [t_1, t_2], [], [], [diff = [x_1, t_1], diff = [x_2, t_2]], \left[-a \rightarrow -a x_1 - \left(\frac{\partial}{\partial t_1} - a \right), -a \rightarrow -a x_2 - \left(\frac{\partial}{\partial t_2} - a \right) \right] \right]$$

> **IsSolution(Sol0);**

$$\text{table}([1 = [true]]) \quad (62)$$

> **Sol1 := Solutions(M, [D1, D2, D3], 1);**

$$Sol1 := \left[[x_1^2 - x_2^2], [0], \begin{bmatrix} x_1 & x_2 & 2x_1 + x_2 \\ x_2 & x_1 & x_1 + 2x_2 \end{bmatrix}, \text{table}([1 = [[2y_{1,1}, 2y_{1,2}, 2y_{1,3}], \right. \quad (63)$$

$$[y_{1,1} - 3x_2y_{2,1} + (2x_1 + x_2)y_{3,1}, y_{1,2} - 3x_2y_{2,2} + (2x_1 + x_2)y_{3,2}, y_{1,3} - 3x_2y_{2,3} + (2x_1 + x_2)y_{3,3}],$$

$$[-y_{1,1} + (2x_1 - x_2)y_{2,1} - x_2y_{3,1}, -y_{1,2} + (2x_1 - x_2)y_{2,2} - x_2y_{3,2}, -y_{1,3} + (2x_1$$

$$- x_2)y_{2,3} - x_2y_{3,3}]]], \text{table}([1 = [Ore_algebra, ["diff", "diff"], [t_1, t_2], [x_1, x_2], [t_1, t_2],$$

$$[y_{1,1}, y_{1,2}, y_{1,3}, y_{2,1}, y_{2,2}, y_{2,3}, y_{3,1}, y_{3,2}, y_{3,3}], 0, [], [], [t_1, t_2], [], [], [diff = [x_1, t_1], diff$$

$$= [x_2, t_2]], \left[-a \rightarrow -a x_1 - \left(\frac{\partial}{\partial t_1} - a \right), -a \rightarrow -a x_2 - \left(\frac{\partial}{\partial t_2} - a \right) \right] \right]$$

> **IsSolution(Sol1);**

$$\text{table}([1 = [true]]) \quad (64)$$

> **Sol2 := Solutions(M, [D1, D2, D3], 2);**

$$Sol2 := \left[[x_2, x_1], [0], \begin{bmatrix} x_1 & x_2 & 2x_1 + x_2 \\ x_2 & x_1 & x_1 + 2x_2 \end{bmatrix}, \text{table} \left(\left[1 = \begin{bmatrix} y_{1,1} & y_{1,2} & y_{1,3} \\ y_{2,1} & y_{2,2} & y_{2,3} \\ y_{3,1} & y_{3,2} & y_{3,3} \end{bmatrix} \right] \right), \text{table} \left([1 \right. \quad (65)$$

$$= [Ore_algebra, ["diff", "diff"], [t_1, t_2], [x_1, x_2], [t_1, t_2], [y_{1,1}, y_{1,2}, y_{1,3}, y_{2,1}, y_{2,2}, y_{2,3}, y_{3,1},$$

$$y_{3,2}, y_{3,3}], 0, [], [], [t_1, t_2], [], [], [diff = [x_1, t_1], diff = [x_2, t_2]], \left[-a \rightarrow -a x_1 - \left(\frac{\partial}{\partial t_1} - a \right),$$

$$-a \rightarrow -a x_2 - \left(\frac{\partial}{\partial t_2} - a \right) \right] \right]$$

> **IsSolution(Sol2);**

$$\text{table}([1 = [true]]) \quad (66)$$

> **M := Matrix([[30,0\$2],[0\$3],[12,0\$2],[12,0\$2]]);**

$$M := \begin{bmatrix} 30 & 0 & 0 \\ 0 & 0 & 0 \\ 12 & 0 & 0 \\ 12 & 0 & 0 \end{bmatrix} \quad (67)$$

> D1 := Matrix([[0\$3,2],[3,0\$2,1],[0\$4],[0\$3,2]]);

$$D1 := \begin{bmatrix} 0 & 0 & 0 & 2 \\ 3 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \quad (68)$$

> D2 := Matrix([[5,3,0\$2],[0\$4],[0,5,2,0],[0,3,2,0]]);

$$D2 := \begin{bmatrix} 5 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 5 & 2 & 0 \\ 0 & 3 & 2 & 0 \end{bmatrix} \quad (69)$$

> RF0 := RankFactorization(M,[D1,D2],0);

$$RF0 := \begin{bmatrix} 2x_4x_1 - 3x_2x_4 - 2x_3x_4, 3x_1x_2 + x_2x_4, 2x_1^2 - 2x_1x_3 + x_2x_4, 9x_2^2x_4 + 6x_2x_3x_4 \end{bmatrix} \quad (70)$$

$$+ 2x_2x_4^2, \begin{bmatrix} x_2 & 2x_1 - 2x_3 & 0 & 0 \\ x_4 & 3x_4 & 3x_1 + x_4 & 9x_2x_4 + 6x_3x_4 + 2x_4^2 \end{bmatrix}, \begin{bmatrix} 6 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 5x_2x_4 \end{bmatrix}$$

$$+ 2 x_3 x_4, 3 x_1 x_3 + x_3 x_4, 6 x_3^2 x_4 + 5 x_3 x_4^2], \text{table} \left(\left[\begin{array}{c} 0 \\ \frac{2_y}{3} \\ 0 \\ 0 \end{array} \right], 2 = \left[\begin{array}{c} 0 \\ 0 \\ _y \\ 0 \end{array} \right], 3 \right.$$

$$= \left[\begin{array}{c} 0 \\ -3_y x_3 \\ 0 \\ \frac{5_y}{2} \end{array} \right], \text{table} \left(\left[\begin{array}{c} 1 \\ \left[\left[\left[3, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \dots \right] \right] \right] \right] \right.$$

$$\left[-1, 18 x_2 + 12 x_3 + 4 x_4, 9 x_2 + 6 x_3 + 2 x_4, 24_y x_3 x_4 + 20_y x_4^2 + 18, 0, 0, 0, 0, 0, 0, \dots \right],$$

$$\left[0, 0, -6 x_4, 0, -2 x_4, x_2, 9 x_2 - 2 x_4, 6 x_3 x_4 + 2 x_4^2, 2, 4_y x_4^2, \dots \right],$$

$$\left[0, -6, 0, -30_y x_4, 1, 0, 1, -\frac{9 x_2}{2} - 3 x_3 - x_4, -3_y x_2 - 2_y x_3, -2_y x_4, \dots \right]],$$

$$\left[\begin{array}{c} 4 \times 16 \text{ Matrix} \\ \left[\left[\left[3, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \dots \right] \right] \right] \right],$$

$$\left[-1, 12 x_3 + 4 x_4, 6 x_3 + 2 x_4, -6, -2, -6, 4, 6, 0, 0, \dots \right],$$

$$\left[0, 0, -6 x_4, -2_y x_4^2, -2_y x_4^2, -6_y x_3 x_4 - 10_y x_4^2, 4_y x_4^2, 2_y x_4^2, -2 x_4, -2 x_4, \dots \right],$$

$$\left[0, -6, 0, 9_y x_3 + _y x_4, 3_y x_3 + _y x_4, 12_y x_3 + 5_y x_4, -6_y x_3 - 2_y x_4, -9_y x_3 - _y x_4, 1, 1, \dots \right]],$$

$$\left[\begin{array}{c} 4 \times 12 \text{ Matrix} \\ \left[\left[\left[3, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \dots \right] \right] \right] \right],$$

$$RF0bis := \left[\left[2 x_4 x_1 - 3 x_2 x_4 - 2 x_3 x_4, 3 x_1 x_2 + x_2 x_4, 2 x_1^2 - 2 x_1 x_3 + x_2 x_4, 9 x_2^2 x_4 + 6 x_2 x_3 x_4 \right] \right] \quad (71)$$

$$+ 2 x_2 x_4^2, \left[\begin{array}{ccc} x_2 & 2 x_1 - 2 x_3 & 0 \\ x_4 & 3 x_4 & 3 x_1 + x_4 \end{array} \right], \left[\begin{array}{ccc} 6 & 0 & 0 \end{array} \right], \left[5 x_2 x_4 + 2 x_3 x_4, 3 x_1 x_3 + x_3 x_4 \right],$$

$$\text{table} \left(\left(\left[\begin{array}{c} 0 \\ \frac{2_y}{3} \\ 0 \end{array} \right], 2 = \left[\begin{array}{c} 0 \\ 0 \\ _y \end{array} \right] \right) \right), \text{table} \left(\left(\left[\begin{array}{ccc} 3 & & 0 \\ -1 & 24 _y x_3 x_4 + 20 _y x_4^2 + 18 & \\ 0 & & -60 _y x_4^2 \end{array} \right], 2 \right. \right.$$

$$\left. \left. \left[\begin{array}{cc} 3 & 0 \\ -1 & -6 x_3 - 2 x_4 \\ 0 & 6 x_4 \end{array} \right] \right) \right), \left[\text{Ore_algebra}, ["diff", "diff", "diff", "diff", "diff"], [t_1, t_2, t_3, t_4,$$

$$_t], [x_1, x_2, x_3, x_4, _y], [t_1, t_2, t_3, t_4, _t], [], 0, [], [], [t_1, t_2, t_3, t_4, _t], [], [], [diff = [x_1, t_1],$$

$$diff = [x_2, t_2], diff = [x_3, t_3], diff = [x_4, t_4], diff = [_y, _t], \left[_a \rightarrow _a x_1 - \left(\frac{\partial}{\partial t_1} _a \right), _a \rightarrow _a x_2$$

$$- \left(\frac{\partial}{\partial t_2} _a \right), _a \rightarrow _a x_3 - \left(\frac{\partial}{\partial t_3} _a \right), _a \rightarrow _a x_4 - \left(\frac{\partial}{\partial t_4} _a \right), _a \rightarrow _a _y - \left(\frac{\partial}{\partial _t} _a \right) \right],$$

$$\left[\begin{array}{cc} 2x_4 & 5x_1 + 3x_2 \\ 3x_1 + x_4 & 0 \\ 0 & 5x_2 + 2x_3 \\ 2x_4 & 3x_2 + 2x_3 \end{array} \right]$$