```
> restart:
> with(LinearAlgebra):
> with(OreModules):
> march('open',
  "/Users/aquadrat/Documents/Travail/maple/Personnel/RankFactorizationProblem.mla"):
```

libname := "/Users/aquadrat/Documents/Travail/maple/Personnel", "/Library/Frameworks/Maple.framework/Versions/2023/lib", "/Users/aquadrat/Documents/Travail/maple/Personnel/RankFactorizationProblem.mla" **(1)**

```
> with(RankFactorizationProblem);
```

[*AntiDiagonal*, *CentroHermitian*, *Factorization*, *FiniteFreeResolution*, *FittingIdeal*, *IsCentroHermitian*, *IsInvertible*, *IsNilpotent*, *IsSolution*, *LeeMatrix*, *LeftLift*, *Lift*, *LocalSyzygyModule*, *RankFactorization*, *ReducedSyzygies*, *RightLift*, *Saturation*, *Simplification*, *Solutions*, *Syzygies*] **(2)**

```
> R := DefineOreAlgebra(seq(diff=[x[i],t[i]],i=1..4),polynom=[seq(t[i],i=1..4)]):
```

Fitting ideals

```
> M := Matrix([[a,b,c],[d,e,f],[g,h,j],[l,m,n]]);
```

$$M := \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & j \\ l & m & n \end{bmatrix} \quad \textbf{(3)}$$

```
> FittingIdeal(M,0,R);
```

$[a\,e\,j - a\,f\,h - b\,d\,j + b\,f\,g + c\,d\,h - c\,e\,g, a\,e\,n - a\,f\,m - b\,d\,n + b\,f\,l + c\,d\,m - c\,e\,l, a\,h\,n - a\,j\,m - b\,g\,n + b\,j\,l + c\,g\,m - c\,h\,l, d\,h\,n - d\,j\,m - e\,g\,n + e\,j\,l + f\,g\,m - f\,h\,l]$ **(4)**

```
> FittingIdeal(M,1,R);
```

$[a\,e - b\,d, a\,h - b\,g, a\,m - b\,l, a\,f - c\,d, a\,j - c\,g, a\,n - c\,l, b\,f - c\,e, b\,j - c\,h, b\,n - c\,m, d\,h - e\,g, d\,m - e\,l, d\,j - f\,g, d\,n - f\,l, e\,j - f\,h, e\,n - f\,m, g\,m - h\,l, g\,n - j\,l, h\,n - j\,m]$ **(5)**

```
> FittingIdeal(M,2,R);
```

$$[a, b, c, d, e, f, g, h, j, l, m, n] \quad \textbf{(6)}$$

```
> FittingIdeal(M,3,R);
```

$$[1] \quad \textbf{(7)}$$

```
> FittingIdeal(M,4,R);
```

$$[1] \quad (8)$$

```
> M := Matrix([[0,-x[3],0,-x[2]],[0,x[2],0,x[3]],[x[1]+x[4],0,x[1]+x
  [4],0]]);
```

$$M := \begin{bmatrix} 0 & -x_3 & 0 & -x_2 \\ 0 & x_2 & 0 & x_3 \\ x_1 + x_4 & 0 & x_1 + x_4 & 0 \end{bmatrix} \quad (9)$$

```
> FittingIdeal(M,0,R);
```

$$[0] \quad (10)$$

```
> FittingIdeal(M,1,R);
```

$$\left[0, (x_1 + x_4)\left(x_2^2 - x_3^2\right), -x_2^2 x_1 + x_3^2 x_1 - x_2^2 x_4 + x_3^2 x_4\right] \quad (11)$$

```
> FittingIdeal(M,1,R,"reduced");
```

$$\left[x_2^2 x_1 - x_3^2 x_1 + x_2^2 x_4 - x_3^2 x_4\right] \quad (12)$$

```
> FittingIdeal(M,2,R);
```

$$\left[0, x_2 (x_1 + x_4), x_3 (x_1 + x_4), -x_2 (x_1 + x_4), -x_3 (x_1 + x_4), x_2^2 - x_3^2\right] \quad (13)$$

```
> FittingIdeal(M,2,R,"reduced");
```

$$\left[x_3 x_1 + x_3 x_4, x_2^2 - x_3^2, x_2 x_1 + x_2 x_4\right] \quad (14)$$

```
> FittingIdeal(M,3,R);
```

$$\left[0, x_2, x_3, -x_2, -x_3, x_1 + x_4\right] \quad (15)$$

```
> FittingIdeal(M,3,R,"reduced");
```

$$\left[x_3, x_2, x_1 + x_4\right] \quad (16)$$

```
> FittingIdeal(M,4,R);
```

$$[1] \quad (17)$$

Saturation of an ideal by a polynomial

```
> Saturation(x[1]*x[2],[x[1]*x[2]*x[3]],R);
```

$$[x_3] \quad (18)$$

```
> Saturation(x[1],[x[2]^2,x[1]*x[3]-x[2]^2],R);
```

$$\left[x_3, x_2^2\right] \quad (19)$$

```
> Saturation(x[3],[x[1]^5*x[3]^3,x[1]*x[2]*x[3],x[2]*x[3]^4],R);
```

$$\left[x_2, x_1^5\right] \quad (20)$$

```
> Saturation(x[1],[x[1]^2],R);
```

$$[1] \quad (21)$$

```
> Saturation(x[1]*x[2],[x[1]^2, x[2]^2],R);
```

$$[1] \tag{22}$$

```
> Saturation(x[4],[x[1]*x[2]-x[3]*x[4],x[1]^2*x[3]-x[2]^2*x[4],x[2]^3
-x[1]*x[3]^2, x[1]^3-x[2]*x[3]^2],R);
```

$$\left[x_2 x_1 - x_3 x_4, x_1^2 x_3 - x_2^2 x_4, -x_1 x_3^2 + x_2^3, x_1^3 - x_3^2 x_2, x_3^4 - x_4^2 x_3^2, x_2 x_3^3 - x_2 x_3 x_4^2, x_2^2 x_3^2 - x_2^2 x_4^2\right] \tag{23}$$

```
> Saturation(0,[x[1]*x[2]],R);
```

$$[1] \tag{24}$$

```
> Saturation(x[1]*x[2],[0],R);
```

$$[0] \tag{25}$$

Test whether or not an element in a factor polynomial ring is nilpotent

```
> IsNilpotent(x[1],Matrix([[x[1]^2]]),R);
```

$$\mathit{true} \tag{26}$$

```
> IsNilpotent(x[1]+1,Matrix([[x[1]^2]]),R);
```

$$\mathit{false} \tag{27}$$

```
> IsNilpotent(x[1],Matrix([[x[1]^3],[x[1]^2*x[2]],[x[1]*x[2]^2],[x[2]
^3]]),R);
```

$$\mathit{true} \tag{28}$$

```
> IsNilpotent(x[1]*x[2],Matrix([[x[1]^3],[x[1]^2*x[2]],[x[1]*x[2]^2],
[x[2]^3]]),R);
```

$$\mathit{true} \tag{29}$$

```
> IsNilpotent(x[1]*x[2]+x[1],Matrix([[x[1]^3],[x[1]^2*x[2]],[x[1]*x
[2]^2],[x[2]^3]]),R);
```

$$\mathit{true} \tag{30}$$

```
> Factorize(Matrix([[(x[1]*x[2]+x[1])^3]]),Matrix([[x[1]^3],[x[1]^2*x
[2]],[x[1]*x[2]^2],[x[2]^3]]),R);
```

$$\begin{bmatrix} 1 & 3\,x_1 & 3\,x_1^2 & x_1^3 \end{bmatrix} \tag{31}$$

```
> IsNilpotent(x[2],Matrix([[x[1]^3],[x[1]^2*x[2]],[x[1]*x[2]^2],[x[2]
^3]]),R);
```

$$\mathit{true} \tag{32}$$

```
> IsNilpotent(0,Matrix([[x[1]^3],[x[1]^2*x[2]],[x[1]*x[2]^2],[x[2]^3]
]),R);
```

$$\mathit{true} \tag{33}$$

Test whether or not an element in a factor polynomial ring is invertible

```
> IsInvertible(x[1]+1,Matrix([[x[1]^2]]),R);
```

$$true \quad (34)$$

```
> IsInvertible(x[1],Matrix([[x[1]^2]]),R);
```

$$false \quad (35)$$

```
> IsInvertible(x[1]+1,Matrix([[x[1]^3],[x[1]^2*x[2]],[x[1]*x[2]^2],[x[2]^3]]),R);
```

$$true \quad (36)$$

```
> LeftInverse(Matrix([[x[1]+1],[x[1]^3],[x[1]^2*x[2]],[x[1]*x[2]^2],[x[2]^3]]),R);
```

$$\begin{bmatrix} x_1^2 - x_1 + 1 & -1 & 0 & 0 & 0 \end{bmatrix} \quad (37)$$

Computation of syzygies for a finitely presented module over a factor polynomial ring

```
> M := Matrix([[x[1]+1,0],[0,x[1]-1]]);
```

$$M := \begin{bmatrix} x_1 + 1 & 0 \\ 0 & x_1 - 1 \end{bmatrix} \quad (38)$$

```
> Syzygies(M,[],R);
```

$$INJ(2) \quad (39)$$

```
> Syzygies(M,Matrix([[0]]),R);
```

$$INJ(2) \quad (40)$$

```
> K := Syzygies(M,Matrix([[x[1]^2-1]]),R);
```

$$K := \begin{bmatrix} x_1 - 1 & 0 \\ 0 & x_1 + 1 \end{bmatrix} \quad (41)$$

```
> Mult(K,M,R);
```

$$\begin{bmatrix} x_1^2 - 1 & 0 \\ 0 & x_1^2 - 1 \end{bmatrix} \quad (42)$$

```
> Simplification(%,Matrix([[x[1]^2-1]]),R);
```

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad (43)$$

```
> Syzygies(K,Matrix([[x[1]^2-1]]),R);
```

$$\begin{bmatrix} x_1 + 1 & 0 \\ 0 & x_1 - 1 \end{bmatrix} \quad (44)$$

```
> L := Syzygies(K,Matrix([[x[1]^2-1],[x[2]*(x[1]+1)-1]]),R);
```

$$L := \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} \quad (45)$$

```
> ReducedSyzygies(K,Matrix([[x[1]^2-1],[x[2]*(x[1]+1)-1]]),R);
```

$$\begin{bmatrix} 2 & 0 \end{bmatrix} \quad (46)$$

```
> L := Syzygies(K,Matrix([[x[1]^2-1],[x[2]*(x[1]-1)-1]]),R);
```

$$L := \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix} \quad (47)$$

```
> K2 := Syzygies(M,Matrix([[x[1]^2-1],[x[2]*(x[1]-1)-1]]),R);
```

$$K2 := \begin{bmatrix} 1 & 0 \\ -2 & 0 \end{bmatrix} \quad (48)$$

```
> P2 := Mult(K2,M,R);
```

$$P2 := \begin{bmatrix} x_1 + 1 & 0 \\ -2\,x_1 - 2 & 0 \end{bmatrix} \quad (49)$$

```
> Simplification(%,Matrix([[x[1]^2-1],[x[2]*(x[1]-1)-1]]),R);
```

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad (50)$$

Example 6 of Inria Report 9438

```
> Q := Matrix([[x[1],x[2],0],[x[2],x[1],0]]);
```

$$Q := \begin{bmatrix} x_1 & x_2 & 0 \\ x_2 & x_1 & 0 \end{bmatrix} \quad (51)$$

```
> Q_t := Transpose(Q);
```

$$Q\_t := \begin{bmatrix} x_1 & x_2 \\ x_2 & x_1 \\ 0 & 0 \end{bmatrix} \quad (52)$$

```
> J0 := FittingIdeal(Q,0,R,"reduced");
```

$$J0 := [0] \quad (53)$$

```
> J1 := FittingIdeal(Q,1,R,"reduced");
```

$$J1 := [x_1^2 - x_2^2] \quad (54)$$

```
> J2 := FittingIdeal(Q,2,R,"reduced");
```
$$J2 := [x_2, x_1] \tag{55}$$

```
> J3 := FittingIdeal(Q,3,R,"reduced");
```
$$J3 := [1] \tag{56}$$

```
> K0 := Transpose(Syzygies(Q_t,Transpose(convert(J0,Matrix)),R));
```
$$K0 := \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{57}$$

```
> K1 := Transpose(Syzygies(Q_t,Transpose(convert(J1,Matrix)),R));
```
$$K1 := \begin{bmatrix} -x_2 & x_1 & 0 \\ x_1 & -x_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{58}$$

```
> Simplification(Mult(Q,K1,R),Transpose(convert(J1,Matrix)),R);
```
$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{59}$$

```
> K2 := Transpose(Syzygies(Q_t,Transpose(convert(J2,Matrix)),R));
```
$$K2 := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{60}$$

Example 7 of Inria Report 9438

```
> Q := Matrix([[x[1],x[2],2*x[1]+x[2]],[x[2],x[1],x[1]+2*x[2]]]);
```
$$Q := \begin{bmatrix} x_1 & x_2 & 2\,x_1 + x_2 \\ x_2 & x_1 & x_1 + 2\,x_2 \end{bmatrix} \tag{61}$$

```
> Q_t := Transpose(Q);
```
$$Q\_t := \begin{bmatrix} x_1 & x_2 \\ x_2 & x_1 \\ 2\,x_1 + x_2 & x_1 + 2\,x_2 \end{bmatrix} \tag{62}$$

```
> J0 := FittingIdeal(Q,0,R,"reduced");
```
$$J0 := [0] \tag{63}$$

```
> J1 := FittingIdeal(Q,1,R,"reduced");
```
$$J1 := [x_1^2 - x_2^2] \tag{64}$$

```
> J2 := FittingIdeal(Q,2,R,"reduced");
```

$$J2 := [x_2, x_1] \quad (65)$$

```
> J3 := FittingIdeal(Q,3,R,"reduced");
```

$$J3 := [1] \quad (66)$$

```
> K0 := Transpose(Syzygies(Q_t,Transpose(convert(J0,Matrix)),R));
```

$$K0 := \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \quad (67)$$

```
> K1 := Transpose(Syzygies(Q_t,Transpose(convert(J1,Matrix)),R));
```

$$K1 := \begin{bmatrix} 2 & 0 & 0 \\ 1 & -3\,x_2 & 2\,x_1 + x_2 \\ -1 & 2\,x_1 - x_2 & -x_2 \end{bmatrix} \quad (68)$$

```
> K2 := Transpose(Syzygies(Q_t,Transpose(convert(J2,Matrix)),R));
```

$$K2 := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (69)$$

Computation of left/right/generalized inverses for a finitely presented module over a factor polynomial ring

```
> LeftInverse(Q_t,R);
```

$$[\ ] \quad (70)$$

```
> S := LeftLift(Q_t,Matrix([[x[4]*J1[1]-1]]),R);
```

$$S := \begin{bmatrix} 0 & -\frac{x_4\,(x_1 + 2\,x_2)}{2} & \frac{x_1\,x_4}{2} \\ 0 & \frac{x_4\,(2\,x_1 + x_2)}{2} & -\frac{x_4\,x_2}{2} \end{bmatrix} \quad (71)$$

```
> simplify(S.Q_t);
```

$$\begin{bmatrix} (x_1^2 - x_2^2)\,x_4 & 0 \\ 0 & (x_1^2 - x_2^2)\,x_4 \end{bmatrix} \quad (72)$$

```
> Simplification(S.Q_t,Matrix([[x[4]*J1[1]-1]]),R);
```

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad (73)$$

```
> T := RightLift(Q,Matrix([[x[4]*J1[1]-1]]),R);
```

$$T := \begin{bmatrix} 0 & 0 \\ -\frac{x_4 (x_1 + 2 x_2)}{2} & \frac{x_4 (2 x_1 + x_2)}{2} \\ \frac{x_1 x_4}{2} & -\frac{x_4 x_2}{2} \end{bmatrix} \quad (74)$$

```
> Simplification(Q.T,Matrix([[x[4]*J1[1]-1]]),R);
```

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad (75)$$

```
> LeftLift(Q_t,Matrix([[x[4]*J2[1]-1]]),R);
```

$$[\,] \quad (76)$$

```
> LeftLift(Q_t,Matrix([[x[4]*J2[2]-1]]),R);
```

$$[\,] \quad (77)$$

```
> M := Matrix([[x[1]],[x[2]],[x[3]]]);
```

$$M := \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad (78)$$

```
> LeftLift(M,Matrix([[x[4]*x[1]-1]]),R);
```

$$\begin{bmatrix} x_4 & 0 & 0 \end{bmatrix} \quad (79)$$

```
> LeftLift(M,Matrix([[x[4]*x[2]-1]]),R);
```

$$\begin{bmatrix} 0 & x_4 & 0 \end{bmatrix} \quad (80)$$

```
> LeftLift(M,Matrix([[x[4]*x[3]-1]]),R);
```

$$\begin{bmatrix} 0 & 0 & x_4 \end{bmatrix} \quad (81)$$

```
> N := SyzygyModule(M,R);
```

$$N := \begin{bmatrix} -x_3 & 0 & x_1 \\ -x_2 & x_1 & 0 \\ 0 & -x_3 & x_2 \end{bmatrix} \quad (82)$$

```
> LeftLift(N,Matrix([[x[4]*x[1]-1]]),R);
```

$$[\,] \quad (83)$$

```
> RightLift(N,Matrix([[x[4]*x[1]-1]]),R);
```

$$[\,] \tag{84}$$

```
> P := Lift(N,Matrix([[x[4]*x[1]-1]]),R);
```

$$P := \begin{bmatrix} 0 & 0 & 0 \\ 0 & x_4 & 0 \\ x_4 & 0 & 0 \end{bmatrix} \tag{85}$$

```
> Simplification(N.P.N-N,Matrix([[x[4]*x[1]-1]]),R);
```

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{86}$$

```
> P2 := Lift(N,Matrix([[x[4]*x[2]-1]]),R);
```

$$P2 := \begin{bmatrix} 0 & -x_4 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & x_4 \end{bmatrix} \tag{87}$$

```
> Simplification(N.P2.N-N,Matrix([[x[4]*x[2]-1]]),R);
```

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{88}$$

```
> P3 := Lift(N,Matrix([[x[4]*x[3]-1]]),R);
```

$$P3 := \begin{bmatrix} -x_4 & 0 & 0 \\ 0 & 0 & -x_4 \\ 0 & 0 & 0 \end{bmatrix} \tag{89}$$

```
> Simplification(N.P3.N-N,Matrix([[x[4]*x[3]-1]]),R);
```

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{90}$$