Computing effectively stabilizing controllers for a class of nD systems

Yacine Bouzidi * **Thomas Cluzeau** ** **Guillaume Moroz** ***
Alban Quadrat *

* *INRIA Lille-Nord Europe, Lille, France*
(e-mail: {yacine.bouzidi,alban.quadrat}@inria.fr).
** *University of Limoges ; CNRS ; XLIM UMR 7252, Limoges, France*
(e-mail: thomas.cluzeau@unilim.fr).
*** *INRIA Nancy-Grand Est, Nancy, France*
(e-mail: guillaume.moroz@inria.fr)

Abstract: In this paper, we study the internal stabilizability and internal stabilization problems for multidimensional (nD) systems. Within the fractional representation approach, a multidimensional system can be studied by means of matrices with entries in the integral domain of structurally stable rational fractions, namely the ring of rational functions which have no poles in the closed unit polydisc $\overline{\mathbb{U}}^n = \{z = (z_1, \ldots, z_n) \in \mathbb{C}^n \mid |z_1| \leqslant 1, \ldots, |z_n| \leqslant 1\}$. It is known that the internal stabilizability of a multidimensional system can be investigated by studying a certain polynomial ideal $I = \langle p_1, \ldots, p_r \rangle$ that can be explicitly described in terms of the transfer matrix of the plant. More precisely the system is stabilizable if and only if $V(I) = \{z \in \mathbb{C}^n \mid p_1(z) = \cdots = p_r(z) = 0\} \cap \overline{\mathbb{U}}^n = \emptyset$. In the present article, we consider the specific class of linear nD systems (which includes the class of 2D systems) for which the ideal I is zero-dimensional, i.e., the p_i's have only a finite number of common complex zeros. We propose effective symbolic-numeric algorithms for testing if $V(I) \cap \overline{\mathbb{U}}^n = \emptyset$, as well as for computing, if it exists, a stable polynomial $p \in I$ which allows the effective computation of a stabilizing controller. We finally illustrate our algorithms on an example.

Keywords: nD systems, stability, stabilization, polynomial ideals, symbolic-numeric methods.

1. INTRODUCTION

Multidimensional or nD systems (Bose (1984)) are systems of functional equations whose unknown functions depend on n independent variables. The internal stabilizability and stabilization problems (see Quadrat (2003)) are fundamental issues in the study of multidimensional systems in control theory. Nowadays, the problems of stabilizability and stabilization are well-understood in the case of 1D systems whereas progress for nD systems with $n \geqslant 2$ are rather slow. One approach for handling stabilizability or stabilization issues in systems theory is the *fractional representation approach* (Vidyasagar (2011)) in which a plant is represented by its transfer matrix $P \in K^{q \times r}$ where $K = \mathbb{R}(z_1, \ldots, z_n)$. This transfer matrix admits a left factorization $P = D^{-1} N$ (also called fractional representation of P), where the matrices $D \in A^{q \times q}$ satisfying $\det(D) \neq 0$ and $N \in A^{q \times r}$ have entries in the integral domain $A = \mathbb{R}(z_1, \ldots, z_n)_S$ of structurally stable rational fractions, namely the ring of rational functions in $z_1, \ldots, z_n$ which have no poles in the closed unit polydisc of $\mathbb{C}^n$ defined by:

$$\overline{\mathbb{U}}^n = \{z = (z_1, \ldots, z_n) \in \mathbb{C}^n \mid |z_1| \leqslant 1, \ldots, |z_n| \leqslant 1\}.$$

Introducing the matrix $R = (D \quad -N) \in A^{q \times (q+r)}$, it is known (see Quadrat (2003)) that the multidimensional system given by the transfer matrix $P = D^{-1} N$ is then internally stabilizable if and only if the A-module $A^{1 \times (q+r)} / \overline{A^{1 \times q} R}$ is a projective A-module of rank r, where the closure $\overline{A^{1 \times q} R}$ of $A^{1 \times q} R$ in $A^{1 \times (q+r)}$ is defined by: $\overline{A^{1 \times q} R} = \{\lambda \in A^{1 \times (q+r)} \mid \exists\, a \in A \setminus \{0\} : a\, \lambda \in A^{1 \times q} R\}$. This projectivity condition is in turn equivalent to the fact that the reduced minors of the matrix R do not have common zeros in $\overline{\mathbb{U}}^n$ (see also Lin (1998)). In other terms, if we denote by $p_1, \ldots, p_r$ the *reduced minors* of R, i.e., the $q \times q$ minors of R divided by their gcd, by $I = \langle p_1, \ldots, p_r \rangle$ the polynomial ideal generated by the p_i's, and by $V(I) = \{z \in \mathbb{C}^n \mid p_1(z) = \cdots = p_r(z) = 0\}$ the associated algebraic variety, then the system is internally stabilizable if and only if $V(I) \cap \overline{\mathbb{U}}^n = \emptyset$.

The first contribution of this paper is an effective algorithm for testing the stabilizability condition $V(I) \cap \overline{\mathbb{U}}^n = \emptyset$ for the class of nD systems for which the ideal I is zero-dimensional, i.e., the p_i's have only a finite number of common complex zeros (i.e., $V(I)$ consists of a finite number of complex points). Note that this class includes the class of 2D systems. Our main idea is to take advantage of the univariate representation for zero-dimensional ideals (Canny (1988); Rouillier (1999)). This concept, yields a one-to-one mapping between the elements of $V(I)$ and the zeros of a univariate polynomial f. Numerical techniques can thus

* All but the third author were supported by the ANR-13-BS03-0005 (MS-DOS).

be applied to compute certified numerical approximations of the roots of f and then of the elements of $V(I)$.

In the case of a stabilizable plant, the next step consists in computing a stabilizing controller which can be achieved by computing a stable (i.e., devoid from zeros in $\overline{\mathbb{U}}^n$) polynomial $s \in I$ (see Lin (1988)). The *polydisc Nullstellensatz*, proved by Bridges, Mines, Richman and Schuster (see Bridges et al. (2004)), shows that the existence of a stable polynomial $s \in I$ is equivalent to $V(I) \cap \overline{\mathbb{U}}^n = \emptyset$. Several proofs of this result have been investigated in the literature, mainly for the case where I is a zero-dimensional ideal (see Xu et al. (1994) and reference therein). Nevertheless none of them is effective in the sense that it provides an algorithm for computing s using calculations that can be performed in an exact way by a computer. Indeed, starting from polynomials with rational coefficients ($I \subset \mathbb{Q}[z_1, \ldots, z_n]$), these algorithms are built on *spectral factorization*, i.e., factorization of polynomials in $\mathbb{Q}[z]$ into stable and instable factors. For irreducible polynomials in $\mathbb{Q}[z]$, this factorization requires the explicit computation of the complex roots of the polynomials, which can be done only approximately. This leads to approximate (stable) polynomials that do not belong to the polynomial ideal. As a consequence, these algorithms are able to solve the aforementioned problem only for few simple systems (see Sections 4 and 5 for details).

Our second contribution is an effective algorithm for computing a stable polynomial $s = \sum_{i=1}^{r} u_i\, p_i \in I$ for the class of systems for which $I \subset \mathbb{Q}[z_1, \ldots, z_n]$ is a zero-dimensional ideal. Our symbolic-numeric method roughly follows the lines of that proposed in Xu et al. (1994) but once again we take advantage of the univariate representation of zero-dimensional ideals to control the numeric precision required to achieve our goal.

The paper is organized as follows. In Section 2, we recall some classical results on the complex zeros of polynomials and polynomial systems. We also introduce the univariate representation of zero-dimensional ideals which will be our main tool in what follows. In Section 3, we provide an effective stabilizability test. In Section 4, we provide an effective *polydisc Nullstellensatz*. Finally, in Section 5, we illustrate our methods on one example.

2. PRELIMINARIES ON ALGEBRAIC SYSTEMS

The *bit-size* of an integer is the number of bits in its representation and for a rational number (resp., a polynomial with rational coefficients) the term *bit-size* refers to the maximum bit-size of its numerator and denominator (resp., of its coefficients). For a complex number $z \in \mathbb{C}$, we denote by $\Re(z) \in \mathbb{R}$ (resp., $\Im(z) \in \mathbb{R}$) its real (resp., imaginary) part. If $z_1, z_2 \in \mathbb{C}$, we write $z_1 < z_2$ if both $\Re(z_1) < \Re(z_2)$ and $\Im(z_1) < \Im(z_2)$. For $z_1, z_2 \in \mathbb{C}$ such that $z_1 < z_2$, we shall consider the *axes-parallel open box* or *box* for short $B = B(z_1, z_2) = \{z \in \mathbb{C} \mid z_1 < z < z_2\}$ and its *width* is defined by

$$w(B) = \max\{|\Re(z_2 - z_1)|, |\Im(z_2 - z_1)|\}.$$

We also introduce the non-negative real number $|B| = \max\{|\Re(z_1)|, |\Im(z_1)|, |\Re(z_2)|, |\Im(z_2)|\}$. In the following, we only consider boxes with *rational endpoints* that is to say, $\Re(z_1), \Im(z_1), \Re(z_2), \Im(z_2) \in \mathbb{Q}$. In that case, a box is given as a product of two real intervals. Finally, the box B is called *isolating* for a given polynomial $f \in \mathbb{Q}[z]$ if it contains exactly one complex zero of f.

The following result concerns the isolation of the zeros of a univariate polynomial (see Yap and Sagraloff (2011)).

Lemma 1. Let $f \in \mathbb{Q}[z]$ be a squarefree polynomial of degree d. Then, for all $\epsilon > 0$, one can compute disjoint axes-parallel open boxes $B_1, \ldots, B_d$, with rational endpoints such that each B_i contains exactly one complex root of f and satisfies $w(B_i) \leqslant \epsilon$.

In the algorithms given in Sections 3 and 4 below, we shall use a routine called `Isolate` which takes as input a univariate polynomial f, a box B, and a precision $\epsilon > 0$ and computes isolating boxes $B_1, \ldots, B_l$ with rational endpoints for the complex roots of f that lie inside the given box B and such that $\max_{i=1,\ldots,l} w(B_i) \leqslant \epsilon$. If B (resp., ϵ) is not specified in the input, all the complex roots in $\mathbb{C}$ are considered (resp., the boxes are computed up to a sufficient precision for isolation).

Let us now recall a standard property about width expansion through interval arithmetic in polynomial evaluation. Here we consider *exact interval arithmetic*, that is, the arithmetic operations on the interval endpoints are considered exact (see Cheng et al. (2010)). If $f \in \mathbb{Q}[x_1, x_2]$ is a bivariate polynomial of two real variables x_1 and x_2 and B a box, which consists in a product of two real intervals, we denote by $\Box f(B)$ the interval resulting from the evaluation of the polynomial f at the box B.

Lemma 2. (Cheng et al. (2010), Lemma 8). Let B be a box with rational endpoints satisfying $|B| \leqslant 2^{\sigma}$ and let $f \in \mathbb{Q}[x_1, x_2]$ be a bivariate polynomial of two real variables x_1 and x_2 of degree d and bit-size τ. Then, f can be evaluated at the box B by interval arithmetic into an interval $\Box f(B)$ of width at most $2^{\tau + d\,\sigma + 1}\, d^3\, w(B)$.

In particular, a direct consequence of Lemma 2 is that if $w(B) \leqslant \epsilon\, 2^{-\tau - d\,\sigma - 1 - 3\log_2(d)}$, then we have $w(\Box f(B)) \leqslant \epsilon$.

We now consider a set of polynomials $p_1, \ldots, p_r$ in $\mathbb{Q}[z_1, \ldots, z_n]$. We denote by $I = \langle p_1, \ldots, p_r \rangle$ the ideal generated by the p_i's and by

$$V(I) = \{z \in \mathbb{C}^n \mid p_1(z) = \cdots = p_r(z) = 0\} \subseteq \mathbb{C}^n,$$

the complex variety of their common zeros. In the sequel, we shall always assume that the ideal I under consideration is a *zero-dimensional* ideal, that is, that $V(I)$ consists of a finite number of complex points. Methods for computing certified numerical approximations for the elements of $V(I)$ usually proceed in two steps. First, a formal representation (as for instance, a *Gröbner basis*, a *triangular decomposition* or a *univariate representation*) of the set $V(I)$ is computed. Then, this formal representation is used to compute, more or less easily, numerical approximations of the elements of $V(I)$. The convenient representation of the variety of zero-dimensional ideals that we shall use in the sequel is the so-called *univariate representation* introduced in Rouillier (1999).

Definition 3. With the previous notation and assumptions, a *univariate representation* of $V(I)$ is the datum of a linear form $t = a_1\, z_1 + \cdots + a_n\, z_n$, with $a_1, \ldots, a_n \in \mathbb{Q}$ as well as $n+1$ univariate polynomials $f, g_{z_1}, \ldots, g_{z_n} \in \mathbb{Q}[t]$ such that the following two applications

$$\begin{array}{rcl} V(I) & \longrightarrow & V(f) = \{z \in \mathbb{C} \mid f(z) = 0\} \\ z = (z_1, \ldots, z_n) & \longmapsto & a_1\, z_1 + \cdots + a_n\, z_n, \end{array}$$

and

$$\begin{array}{rcl} V(f) & \longrightarrow & V(I) \\ z & \longmapsto & (g_{z_1}(z), \ldots, g_{z_n}(z)), \end{array}$$

provide a one-to-one correspondence between the elements of $V(I)$ and the zeros of f.

One of the main advantage of the univariate representation is that it permits to study the element of $V(I)$ through the roots of the univariate polynomial f. In particular, if we are interested in computing certified numerical approximations of the complex elements of $V(I)$, a first step consists in isolating the complex roots of the polynomial f. Then, numerical approximations of the coordinates $z_1, \ldots, z_n$ of $z \in V(I)$ are obtained by evaluating $g_{z_1}, \ldots, g_{z_n}$ at the resulting boxes. Lemmas 1 and 2 above show that we can thus obtain a given precision for the elements of $V(I)$.

In the algorithms given in Sections 3 and 4 below, we shall use a routine called `Univ_R` which takes as input a zero-dimensional polynomial ideal $I = \langle p_1, \ldots, p_r\rangle$ and computes a univariate representation of the variety $V(I)$, following the algorithm described in Rouillier (1999). This algorithm proceeds by pre-computing a Gröbner basis G of I (Cox et al. (1992)), and then performing linear algebra calculations in the quotient vector space $\mathbb{Q}[z_1, \ldots, z_n]/I$ which admits a basis composed of the monomials that are irreducible modulo the Gröbner basis G.

3. AN EFFECTIVE STABILIZABILITY TEST

Using the fractional representation approach to multidimensional systems, in Section 1, we have seen that a plant is stabilizable if and only if $V(I) \cap \overline{\mathbb{U}}^n = \emptyset$ for a certain polynomial ideal $I = \langle p_1, \ldots, p_r\rangle$ that can be explicitly described in terms of the transfer matrix of the plant.

Given an ideal $I = \langle p_1, \ldots, p_r\rangle \subset \mathbb{Q}[z_1, \ldots, z_n]$, the purpose of this section is to provide an effective algorithm to decide whether or not $V(I) \cap \overline{\mathbb{U}}^n = \emptyset$. To achieve this, we start by computing a univariate representation of $V(I)$. As explained in Section 2, such a representation allows to describe formally the elements $z = (z_1, \ldots, z_n)$ of $V(I)$ as

$$\{f(t) = 0, \quad z_1 = g_{z_1}(t), \quad \ldots, \quad z_n = g_{z_n}(t)\}, \tag{1}$$

where $f, g_{z_1}, \ldots, g_{z_n} \in \mathbb{Q}[t]$. In what follows, the degree of f is denoted by d, and those of $g_{z_1}, \ldots, g_{z_n}$ are then smaller than d (see Rouillier (1999)).

Using a univariate representation, one can compute a set of hypercubes in $\mathbb{R}^{2n}$ isolating the elements of $V(I)$. Each coordinate is represented by a box in $\mathbb{R}^2$ obtained from the intervals containing its real and imaginary parts. Moreover, from Lemmas 1 and 2, these hypercubes can be refined up to an arbitrary precision. We shall now consider the intersection between those hypercubes and $\overline{\mathbb{U}}^n$.

Below, for any $g_{z_i} \in \mathbb{Q}[t]$, we shall denote by $\mathcal{C}(g_{z_i})$ the bivariate polynomial $\Re(g_{z_i})^2 + \Im(g_{z_i})^2 - 1 \in \mathbb{Q}[x_1, x_2]$, where $\Re(g_{z_i})$ (resp., $\Im(g_{z_i})$) is the real (resp., complex) part of the polynomial resulting from $g_{z_i}(x_1 + i\, x_2)$.

From the definition of $\overline{\mathbb{U}}^n$, one can see that the situation is easier when $V(I)$ does not contain elements $z \in \mathbb{C}^n$ with $|z_i| = 1$ for some $i \in \{1, \ldots, r\}$. Indeed, we have:

Theorem 4. With the previous notations, let us consider $z = (z_1, \ldots, z_n) \in V(I)$ such that, for all $i \in \{1, \ldots, n\}$, $|z_i| \neq 1$. Let B be an isolating box for the corresponding root of f in the univariate representation of $V(I)$. Then, there exists $\epsilon > 0$ such that if $w(B) \leqslant \epsilon$, then, $\forall k \in \{1, \ldots, n\}$, the interval $\Box\mathcal{C}(g_{z_k})(B)$ does not contain zero.

Proof. Let d (resp., τ) denote an upper bound on the degree (resp., bit-size) of the polynomials g_{z_k}, $k = 1, \ldots, n$. For $k \in \{1, \ldots, n\}$, the real (resp., imaginary) part of $g_{z_k}(x_1 + i\, x_2)$ is a bivariate polynomial in x_1 and x_2 of degree (resp., bit-size) bounded by d (resp., $d + \tau$). Consequently, the bivariate polynomial $\mathcal{C}(g_{z_k}) \in \mathbb{Q}[x_1, x_2]$ has degree and bit-size bounded by $2\, d$ and $(d + \tau)\, d$ respectively. Now, for $z = (z_1, \ldots, z_n) \in V(I)$, let $m = \min_{k=1,\ldots,n} ||z_k| - 1| > 0$ and let B be an isolating box for the root of f corresponding to z in the univariate representation (1) of $V(I)$, and such that $|B| \leqslant 2^\sigma$. From Lemma 2, if we refine B so that $w(B) \leqslant \epsilon$, where $\epsilon = m\, 2^{-(d+\tau)\, d - 2\, d\, \sigma - 1 - 3\log_2(2\, d)}$, then, for all $k \in \{1, \ldots, n\}$, the interval $\Box\mathcal{C}(g_{z_k})(B)$ satisfies $w(\Box\mathcal{C}(g_{z_k})(B)) \leqslant m$ so that, by definition of m, it does not contain zero.

Therefore, if $V(I)$ does not contain elements $z \in \mathbb{C}^n$ with one coordinate z_i in the unit circle, one can easily test the stabilizability condition $V(I) \cap \overline{\mathbb{U}}^n = \emptyset$. Indeed, with the previous notations, we isolate the roots of f inside boxes $B_1, \ldots, B_d$. Then, for $i \in \{1, \ldots, d\}$, we refine B_i until, for all $k \in \{1, \ldots, n\}$, the interval $\Box\mathcal{C}(g_{z_k})(B_i)$ does not contain zero. If one of the intervals $\Box\mathcal{C}(g_{z_k})(B_i)$ is included in $\mathbb{R}_+$, we proceed to the next box B_i, otherwise we have found an element in $V(I) \cap \overline{\mathbb{U}}^n$ so that the system is certainly not stabilizable. After having investigated all the boxes B_i, we can then conclude about the stabilizability.

We shall now consider the case where $V(I)$ contains (at least) one element having some coordinates on the unit circle. In this case, we cannot proceed numerically as before since if $|z_k| = 1$, then, using the above notations, we cannot fulfill the condition that the interval $\Box\mathcal{C}(g_{z_k})(B)$ does not contain zero. To guarantee the termination of the algorithm, we shall then have to compute, for each variable z_k, the number of elements in $V(I)$ satisfying $|z_k| = 1$.

Lemma 5. Let $I \subset \mathbb{Q}[z_1, \ldots, z_n]$ be a zero-dimensional ideal and $V(I)$ the associated algebraic variety. Then, for all $k \in \{1, \ldots, n\}$, one can compute the non-negative integer $l_k = \sharp\{z = (z_1, \ldots, z_n) \in V(I) \mid |z_k| = 1\}$.

Proof. Let $\{f(t), z_1 - g_{z_1}(t), \ldots, z_n - g_{z_n}(t)\}$ be a univariate representation of $V(I)$ and $k \in \{1, \ldots, n\}$. Computing the resultant of the polynomials f and $z_k - g_{z_k}$ with respect to the variable t we get a univariate polynomial that can be written $r_k = \prod_{\alpha \in V(I)} (z_k - \alpha_k)^{\mu_{\alpha_k}}$, where the multiplicity μ_{α_k} corresponds to $\sharp\{z \in V(I) \mid z_k = \alpha_k\}$. Then, using the classical *Bistritz* test (see Bistritz (2002)), one can compute the number of complex roots counted with multiplicity of r_k that lie on the unit circle and obtain the non-negative integer l_k.

Using Lemma 5, we can test the stabilizability condition $V(I) \cap \overline{\mathbb{U}}^n = \emptyset$ as follows. We start with the variable z_1. We refine the isolating boxes $B_1, \ldots, B_d$ for the roots of f until exactly l_1 intervals $\square\mathcal{C}(g_{z_1})(B_i)$ contain zero. We throw away the boxes B_i's such that the interval $\square\mathcal{C}(g_{z_1})(B_i)$ is included in $\mathbb{R}_+$ and we proceed similarly with the next variable z_2. If at some point we have thrown away all the boxes B_i's, then the system is stabilizable. Otherwise it is not stabilizable.

We summarize our symbolic-numeric method for testing stabilizability in the following `IsStabilizable` algorithm.

Algorithm 1 `IsStabilizable`

Input: A set of r polynomials $p_1, \ldots, p_r \subset \mathbb{Q}[z_1, \ldots, z_n]$.
Output: `True` if $V(\langle p_1, \ldots, p_r \rangle) \cap \overline{\mathbb{U}}^n = \emptyset$, else `False`.
Begin
$\diamond$ $\{f, g_{z_1}, \ldots, g_{z_n}\} :=$ `Univ_R`$(\{p_1, \ldots, p_r\})$;
$\diamond$ $\{B_1, \ldots, B_d\} :=$ `Isolate`(f);
$\diamond$ $L_B := \{B_1, \ldots, B_d\}$ and $\epsilon := \min_{i=1,\ldots,d} w(B_i)$;
For k from 1 to n **do**
  $\diamond$ $l_k := \sharp\{z \in V(I) \mid |z_k| = 1\}$ (see Lemma 5);
  While $\sharp\{i \mid 0 \in \square\mathcal{C}(g_{z_k})(B_i)\} > l_k$ **do**
    $\diamond$ $\epsilon := \epsilon/2$;
    $\diamond$ For $i = 1, \ldots, d$, set $B_i :=$`Isolate`(f, B_i, ϵ);
  End While
  $\diamond$ $L_B := L_B \setminus \{B_i \mid \square\mathcal{C}(g_{z_k})(B_i) \subset \mathbb{R}_+\}$;
  $\diamond$ If $L_B = \{\}$, then **Return** `True` **End If**;
End For
Return `False`.
End

4. AN EFFECTIVE STABILIZATION ALGORITHM

When a multidimensional system is stabilizable, one is then interested in computing a stabilizing controller. Within the fractional representation approach, this problem reduces to the following task (see Section 1 or Lin (1988)): given a polynomial ideal $I = \langle p_1, \ldots, p_r \rangle$ satisfying $V(I) \cap \overline{\mathbb{U}}^n = \emptyset$, compute a polynomial $s \in \mathbb{Q}[z_1, \ldots, z_n]$ such that s belongs to the ideal I and s is a *stable polynomial*, i.e., s is devoid from zeros in $\overline{\mathbb{U}}^n$.

For an ideal $I \subset \mathbb{Q}[z_1, \ldots, z_n]$, the *polydisc Nullstellensatz* (see Bridges et al. (2004)) asserts that the existence of a stable polynomial $s \in I$ is equivalent to $V(I) \cap \overline{\mathbb{U}}^n = \emptyset$.

Theorem 6. (Polydisc Nullstellensatz). Let us consider a polynomial ideal $I = \langle p_1, \ldots, p_r \rangle \subset \mathbb{Q}[z_1, \ldots, z_n]$ such that $V(I) \cap \overline{\mathbb{U}}^n = \emptyset$. Then, there exist $r+1$ polynomials $s, u_1, \ldots, u_r$ in $\mathbb{Q}[z_1, \ldots, z_n]$ such that:

$$s = \sum_{i=1}^{r} u_i\, p_i \text{ and } V(s) \cap \overline{\mathbb{U}}^n = \emptyset.$$

Several proofs of Theorem 6 have been investigated in the literature. Nevertheless none of them is effective in the sense that it provides an algorithm for computing s and the *cofactors* u_i's using calculations that can be performed in an exact way by a computer. In Xu et al. (1994), the authors study 2D systems (i.e., $n = 2$) for which the ideal I under consideration is zero-dimensional. The idea of their method for computing a stable polynomial $s \in I$ is to compute univariate elimination polynomials $r_{z_1} \in \mathbb{Q}[z_1]$ and $r_{z_2} \in \mathbb{Q}[z_2]$ with respect to each variable z_1 and z_2 and to factorize them into a stable and an unstable factor, i.e., for $i = 1, 2$, $r_{z_i} = r_{z_i}^{(s)}\, r_{z_i}^{(u)}$, where the roots of $r_{z_i}^{(s)}$ (resp., $r_{z_i}^{(u)}$) are outside (resp., inside) the closed unit disc $\overline{\mathbb{U}}$, and then, to compute the stable polynomial as $s = r_{z_1}^{(s)}\, r_{z_2}^{(s)}$. However, this approach presents a major drawback with respect to the effectiveness aspect. Indeed, when the elimination polynomial r_{z_1} (resp., r_{z_2}) is an irreducible polynomial in $\mathbb{Q}[z_1]$ (resp. $\mathbb{Q}[z_2]$), its stable factor $r_{z_1}^{(s)}$ (resp., $r_{z_2}^{(s)}$) could not be computed exactly since it will have coefficients in $\mathbb{C}$, and thus, only an approximation of this polynomial can be obtained. As a consequence, the polynomial $s = r_{z_1}^{(s)}\, r_{z_2}^{(s)}$ will not belong to the ideal I.

In the sequel, we present a symbolic-numeric algorithm for computing s and the u_i's that follows roughly the approach of Xu et al. (1994) while we provide a way for tackling the effectiveness issue. Our main ingredient is the univariate representation of zero-dimensional ideals which allows us to compute and refine approximate factorizations over $\mathbb{Q}$ of the elimination polynomials r_{z_i}'s.

Let $I = \langle p_1, \ldots, p_r \rangle \subset \mathbb{Q}[z_1, \ldots, z_n]$ be a zero-dimensional ideal such that $V(I) \cap \overline{\mathbb{U}}^n = \emptyset$. For simplicity, we further assume that the ideal I is *radical*, i.e., $\sqrt{I} = \{p \in \mathbb{Q}[z_1, \ldots, z_n] \mid \exists m \in \mathbb{N}^*,\, p^m \in I\} = I$. The elements of $V(I)$ are given by a univariate representation

$$\{f(t) = 0, \quad z_1 = g_{z_1}(t), \quad \ldots, \quad z_n = g_{z_n}(t)\},$$

where $t = a_1 z_1 + \cdots + a_n z_n$, $a_k \in \mathbb{Q}$ for $k = 1, \ldots, n$, $f, g_{z_1}, \ldots, g_{z_n} \in \mathbb{Q}[t]$, and $\deg(f) = d$. Since I is a radical ideal, the polynomial f is a squarefree polynomial and $f(t) = \prod_{i=1}^{d} (t - \gamma_i)$ for distincts $\gamma_1 \ldots, \gamma_d \in \mathbb{C}$. Moreover, from Definition 3, if we introduce the polynomial ideal $I_r = \langle f(t), z_1 - g_{z_1}(t), \ldots, z_n - g_{z_n}(t) \rangle \subset \mathbb{Q}[t, z_1, \ldots, z_n]$, then we have $I_r = I \cap \langle t - \sum_{k=1}^{n} a_k\, z_k \rangle$. In particular, if $p(t, z_1, \ldots, z_n) \in I_r$, then $p(\sum_{k=1}^{n} a_k\, z_k, z_1, \ldots, z_n) \in I$.

Let us first explain how we can compute approximations $\tilde{r}_{z_k}^{(s)}$ of the stable polynomials $r_{z_k}^{(s)}$ appearing in the method of Xu et al. (1994) sketched above. Using Lemma 1, we can compute a set of boxes $B_1, \ldots, B_d$ with rational endpoints, isolating the distinct complex roots $\gamma_1, \ldots, \gamma_d$ of f. Then, according to the stabilizability condition $V(I) \cap \overline{\mathbb{U}}^n = \emptyset$, for all $i \in \{1, \ldots, d\}$, the box B_i can be refined so that there exists $k \in \{1, \ldots, n\}$ satisfying $\square\mathcal{C}(g_{z_k})(B_i) \subset \mathbb{R}_+$. We then set $\tilde{\gamma}_i \in \mathbb{Q}$ to the midpoint of the refined box B_i and we add the factor $z_k - g_{z_k}(\tilde{\gamma}_i)$ to the polynomial $\tilde{r}_k^{(s)}$. We finally obtain a set of stable univariate polynomials $\tilde{r}_k^{(s)} \in \mathbb{Q}[z_k]$, $k = 1, \ldots, n$ such that $\sum_{k=1}^{n} \deg(\tilde{r}_k^{(s)}) = d$.

Let us now introduce the polynomial $\tilde{s} = \prod_{k=1}^{n} \tilde{r}_k^{(s)}$. By construction $\tilde{s} \in \mathbb{Q}[z_1, \ldots, z_n]$ has rational coefficients and it vanishes on $V(\tilde{I}_r)$, where the polynomial ideal $\tilde{I}_r$ is defined by $\tilde{I}_r = \langle \tilde{f}(t), z_1 - g_{z_1}(t), \ldots, z_n - g_{z_n}(t) \rangle$ with $\tilde{f}(t) = \prod_{i=1}^{d} (t - \tilde{\gamma}_i) \in \mathbb{Q}[t]$. Hence, according to the classical *Nullstellensatz theorem* (Cox et al. (1992)), $\tilde{s}$ belongs to the ideal $\tilde{I}_r$ so that there exist polynomials $\tilde{h}_0, \tilde{h}_1, \ldots, \tilde{h}_n \in \mathbb{Q}[t, z_1, \ldots, z_n]$ such that $\tilde{s} = \tilde{h}_0\, \tilde{f} + \sum_{k=1}^{n} \tilde{h}_k\, (z_k - g_{z_k})$. Moreover $\tilde{h}_0$ can be explicitly computed as the quotient of the Euclidean division of $\tilde{s}(g_{z_1}(t), \ldots, g_{z_n}(t))$ by $\tilde{f}(t)$.

We shall now show that if we refine enough the boxes B_i's isolating the roots γ_i's of f, then the stable polynomial $s \in I$ that we are seeking for can be obtained from the polynomials $\tilde{s}$ and $\tilde{h}_0$ constructed as explained above. For $\epsilon > 0$ [1], we denote $\tilde{\gamma}_{i,\epsilon}$, $\tilde{s}_\epsilon$, $\tilde{f}_\epsilon(t) = \prod_{i=1}^{d} (t - \tilde{\gamma}_{i,\epsilon})$, and $\tilde{h}_{i,\epsilon}$ the objects constructed by the previous process where the roots of f are isolated up to precision ϵ (i.e., $w(B_i) \leqslant \epsilon$, for all $i \in \{1, \dots, d\}$). Using the previous notations, the main result of this section can be stated as follows:

Theorem 7. The polynomial $s = \tilde{s}_\epsilon - \tilde{h}_{0,\epsilon} (\tilde{f}_\epsilon - f)$ belongs to the ideal I_r. Moreover, there exists $\epsilon > 0$ such that the polynomial $s(\sum_{i=1}^{n} a_i z_i, z_1, \dots, z_n)$ is a stable polynomial.

The proof of Theorem 7, requires the following lemma.

Lemma 8. For $0 < \epsilon < 1$, the polynomial $\tilde{h}_{0,\epsilon} (\tilde{f}_\epsilon - f)$ has coefficients bounded by $\epsilon \rho$, where ρ is a positive real number that does not depend on ϵ.

Proof. Let $f = \sum_{i=0}^{d} a_i t^i$ and $\tilde{f}_\epsilon = \sum_{i=0}^{d} b_i t^i$, with $a_d = b_d = 1$, denote the expansion of the polynomials f and $\tilde{f}_\epsilon$ on the monomial basis. By the standard *Vieta's formulas*, for all $i \in \{1, \dots, d-1\}$, we have:

$$a_i - b_i = \sum_{1 \leqslant k_1 < \cdots < k_i \leqslant d} (\gamma_{k_1} \cdots \gamma_{k_i}) - (\tilde{\gamma}_{k_1,\epsilon} \cdots \tilde{\gamma}_{k_i,\epsilon}).$$

By assumption, $|\gamma_i - \tilde{\gamma}_{i,\epsilon}| \leqslant \epsilon$ for all $i \in \{1, \dots, d\}$, so that

$$|a_i - b_i| \leqslant \left| \sum_{1 \leqslant k_1 < \cdots < k_i \leqslant d} (\gamma_{k_1} \cdots \gamma_{k_i}) - ((\gamma_{k_1} - \epsilon) \cdots (\gamma_{k_i} - \epsilon)) \right|.$$

Now, we can write $(\gamma_{k_1} - \epsilon) \cdots (\gamma_{k_i} - \epsilon) = \sum_{l=0}^{i} \sigma_l \epsilon^l$, where the σ_l's denote the *symmetric functions* associated to $\gamma_{k_1}, \dots, \gamma_{k_i}$ and, in particular, $\sigma_0 = \gamma_{k_1} \dots \gamma_{k_i}$. Consequently, since $\epsilon < 1$, we get:

$$|a_i - b_i| \leqslant \epsilon \left| \sum_{1 \leqslant k_1 < \cdots < k_i \leqslant d} \sum_{l=1}^{i} \sigma_l \epsilon^{l-1} \right| \leqslant \epsilon \underbrace{\left| \sum_{1 \leqslant k_1 < \cdots < k_i \leqslant d} \sum_{l=1}^{i} \sigma_l \right|}_{\rho_1}.$$

On the other hand, the polynomial $\tilde{h}_{0,\epsilon}$ can be computed as the quotient of the Euclidean division of $\tilde{s}_\epsilon(g_{z_1}(t), \dots, g_{z_n}(t))$ by $\tilde{f}_\epsilon(t)$. Formally, these polynomials can be considered as polynomials in $\mathbb{Q}[\tilde{\gamma}_{1,\epsilon}, \dots, \tilde{\gamma}_{d,\epsilon}][t]$, where $\tilde{\gamma}_{1,\epsilon}, \dots, \tilde{\gamma}_{d,\epsilon}$ are considered as new indeterminates so that their quotient denoted by h_0 can be computed independently from ϵ. The coefficients of h_0 are thus bounded by a certain positive real number δ_1 that does not depend on ϵ (use, for instance, *Mignotte's bound* Mignotte (1989)). Now, for $i \in \{1, \dots, d\}$, since $\epsilon < 1$, we have $|\tilde{\gamma}_{i,\epsilon}| < |\gamma_i + \epsilon| < |\gamma_i + 1|$. Thus if we denote $\delta_2 = \max_{i=1,\dots,d} |\gamma_i + 1|$, then the coefficients of the evaluation $\tilde{h}_{0,\epsilon} \in \mathbb{Q}[t]$ of h_0 for particular values of $\tilde{\gamma}_{1,\epsilon}, \dots, \tilde{\gamma}_{d,\epsilon}$ in $\mathbb{Q}$ are bounded by $\rho_2 = \delta_1 \delta_2^d$. Finally, we have proved that the coefficients of $h_{0,\epsilon}(\tilde{f}_\epsilon - f)$ are bounded by $\epsilon \rho_1 \rho_2$, which ends the proof.

We are now in position to give a proof of Theorem 7.

Proof. With the previous notations, for all $\epsilon > 0$, we have $s = \tilde{s}_\epsilon - \tilde{h}_{0,\epsilon} (\tilde{f}_\epsilon - f) = \sum_{k=1}^{n} \tilde{h}_{k,\epsilon} (z_k - g_{z_k}) + \tilde{h}_{0,\epsilon} f$ so that s vanishes on $V(I_r)$, which implies $s \in I_r$. Let us now prove that we can choose ϵ so that s, viewed as $s(\sum_{i=1}^{n} a_i z_i, z_1, \dots, z_n) \in \mathbb{Q}[z_1, \dots, z_n]$, is stable, i.e., $\forall \lambda \in \overline{\mathbb{U}}^n$, $|s(\lambda)| > 0$. According to Lemma 8, for $\epsilon < 1$, we have $|\tilde{h}_{0,\epsilon}(\lambda)(\tilde{f}_\epsilon(\lambda) - f(\lambda))| \leqslant \epsilon \rho \delta$, where ρ (resp., δ) denotes a positive real number (resp., the degree of the polynomial $\tilde{h}_{0,\epsilon}(\tilde{f}_\epsilon - f)$) that does not depend on ϵ. On the other hand, the polynomial $\tilde{s}_\epsilon \in \mathbb{Q}[z_1, \dots, z_n]$ can be written as $\tilde{s}_\epsilon = \prod_{k=1}^{n} \tilde{r}_k^{(s)} = \prod_{i=1}^{d} (z_{k_i} - g_{z_{k_i}}(\tilde{\gamma}_{i,\epsilon}))$ for some $k_i \in \{1, \dots, n\}$ not necessarily distinct. Thus, for all $\lambda \in \overline{\mathbb{U}}^n$, we have $|\tilde{s}_\epsilon(\lambda)| \geqslant \prod_{i=1}^{d} \left| |\lambda_{k_i}| - |g_{z_{k_i}}(\tilde{\gamma}_{i,\epsilon})| \right|$. Now, if we denote by $m = \min_{z \in V(I)} (|z|_\infty - 1)$, the minimum distance between the elements of $V(I)$ and $\overline{\mathbb{U}}^n$, then we have $\left| |\lambda_{k_i}| - |g_{z_{k_i}}(\tilde{\gamma}_{i,\epsilon})| \right| \geqslant m - \epsilon$, which yields:

$$\forall\, \lambda \in \overline{\mathbb{U}}^n,\ |\tilde{s}_\epsilon(\lambda)| \geqslant (m - \epsilon)^d.$$

Finally, for sufficiently small ϵ, we have $(m - \epsilon)^d > \epsilon \rho \delta$ so that:

$$\begin{aligned} \forall \lambda \in \overline{\mathbb{U}}^n,\ |s(\lambda)| &\geqslant |\tilde{s}_\epsilon(\lambda)| - |\tilde{h}_{0,\epsilon}(\lambda)(\tilde{f}_\epsilon(\lambda) - f(\lambda))| \\ &\geqslant (m - \epsilon)^d - \epsilon \rho \delta \\ &> 0, \end{aligned}$$

which ends the proof.

The following `StablePolynomial` algorithm summarizes our method for computing a stable polynomial in a zero-dimensional ideal I satisfying $V(I) \cap \overline{\mathbb{U}}^n = \emptyset$. The routine `IsStable` is used to test if a polynomial $p \in \mathbb{Q}[z_1, \dots, z_n]$ is stable, i.e., if $V(p) \cap \overline{\mathbb{U}}^n = \emptyset$ (see Bouzidi et al. (2015)).

Algorithm 2 `StablePolynomial`

Input: $I := \langle p_1, \dots, p_r \rangle$ be such that $V(I) \cap \overline{\mathbb{U}}^n = \emptyset$.
Output: $s \in I$ such that $V(s) \cap \overline{\mathbb{U}}^n = \emptyset$.
Begin
◇ $\{f, g_{z_1}, \dots, g_{z_n}\}$:= `Univ_R`($\{p_1, \dots, p_r\}$);
◇ $\{B_1, \dots, B_d\}$:= `Isolate`(f);
◇ $\epsilon := \min_{i=1,\dots,d} w(B_i)$;
Do
  ◇ $[r_1, \dots, r_n] := [1, \dots, 1]$ and $\tilde{f} := 1$;
  ◇ outside := `False`;
  For each B in $\{B_1, \dots, B_d\}$ **do**
    While (outside=`False`) **do**
      For i from 1 to n **do**
        If $\square\mathcal{C}(g_{z_k})(B) \subset \mathbb{R}_+$ then
          ◇ γ := `midpoint`(B);
          ◇ $r_i := r_i\,(z_i - g_{z_i}(\gamma))$;
          ◇ outside := `True` and **Break For**;
        End If
      End For
      ◇ $\epsilon := \epsilon/2$; B :=`Isolate`(f, B, ϵ);
    End While
    ◇ $\tilde{f} := \tilde{f}\,(t - \gamma)$; outside := `False`;
  End ForEach
  ◇ $\tilde{s} := \prod_{i=1}^{n} r_i$;
  ◇ $\tilde{s}_t := \tilde{s}$ evaluated at $z_i = g_{z_i}(t)$;
  ◇ h_0 := `quotient`($\tilde{s}_t, \tilde{f}$) in $\mathbb{Q}[t]$;
  ◇ $s := \tilde{s} - h_0\,(\tilde{f} - f)$ evaluated at $t = \sum_{k=1}^{n} a_k z_k$;
While (`IsStable`(s)=`False`)
◇ Return s.
End

[1] small enough so that the previous process can be applied.

5. AN EXAMPLE

Let us illustrate the algorithm of Section 4 on the 2D SISO system given by the following transfer function:

$$P = \frac{N}{D} = \frac{z_1^2 - 2\,z_1 - 2}{z_1 + z_2 - 2}.$$

We would like to compute (if it exists) a controller $C = \frac{X}{Y}$ such that the closed-loop system is stable, which means that $1 + C\,P$ is stable. In other words, we are seeking for two polynomials $X, Y \in \mathbb{Q}[z_1, z_2]$ such that the polynomial $s = X\,N + Y\,D$ is stable. To be consistent with the notation of Section 4, we denote the polynomials N, D by p_1, p_2 and $I = \langle p_1, p_2 \rangle$ is the corresponding ideal.

The associated variety $V(I)$ contains two elements, namely, $(1-\sqrt{3}, 1+\sqrt{3})$ and $(1+\sqrt{3}, 1-\sqrt{3})$, so that the stabilizability condition $V(I) \cap \overline{\mathbb{U}}^2 = \emptyset$ is clearly fulfilled. Yet, p_1 and p_2 are both unstable polynomials. For $i \in \{1, 2\}$, the univariate elimination polynomial (i.e., resultant of p_1 and p_2) $r_{z_i} \in \mathbb{Q}[z_i]$ is given by $r_{z_i} = z_i^2 - 2\,z_i - 2$. The polynomial $z_i^2 - 2\,z_i - 2$ being irreducible in $\mathbb{Q}[z_i]$, this makes the approach of Xu et al. (1994) impracticable. Let us apply the algorithm of Section 4 for computing a stable polynomial $s \in I$. We start by computing a univariate representation of $V(I)$. We get:

$$f(t) := t^2 - 2\,t - 2 = 0, \quad z_1 = t, \quad z_2 = 2 - t.$$

The roots of $f(t)$ are given by $\gamma_1 \approx -0.73$ and $\gamma_2 \approx 2.73$ and choosing the precision $\epsilon = \frac{1}{2}$, we get the approximate roots (in $\mathbb{Q}$) $\tilde{\gamma}_1 = -\frac{1}{2}$ and $\tilde{\gamma}_2 = 3$. Consequently, the algorithm of Section 4 yields

$$\tilde{f}(t) = \left(t + \frac{1}{2}\right)(t-3), \quad \tilde{s}(z_1, z_2) = (z_1 - 3)\left(z_2 - \frac{5}{2}\right),$$

which then leads to:

$$h_0(t) = -1, \quad (\tilde{f} - f)(t) = -\frac{1}{2}\,t + \frac{1}{2}.$$

Finally, after substituting $t = z_1$ in $\tilde{f} - f$, we get:

$$s(z_1, z_2) = z_1\,z_2 - 3\,z_1 - 3\,z_2 + 8.$$

We can then check that this polynomial is stable (see Bouzidi et al. (2015)). As a result, we obtain the corresponding cofactors, that is,

$$s = -p_1 + (z_1 - 3)\,p_2 \in I,$$

and thus a stabilizing controller is given by $C = \frac{-1}{(z_1-3)}$.

Figure 1 shows the stable polynomial obtained by an exact factorization of r_{z_1} and r_{z_2} in dots, the approximate factorization $\tilde{s}$ used in our algorithm in dash, and finally the stable polynomial s obtained after adding the correcting term represented by the solid curve.

We have implemented two routines `IsStabilizable` and `StablePolynomial`, which correspond to the algorithms given in Sections 3 and 4, in the computer algebra system Maple. Our code is available at Bouzidi et al. (2016).

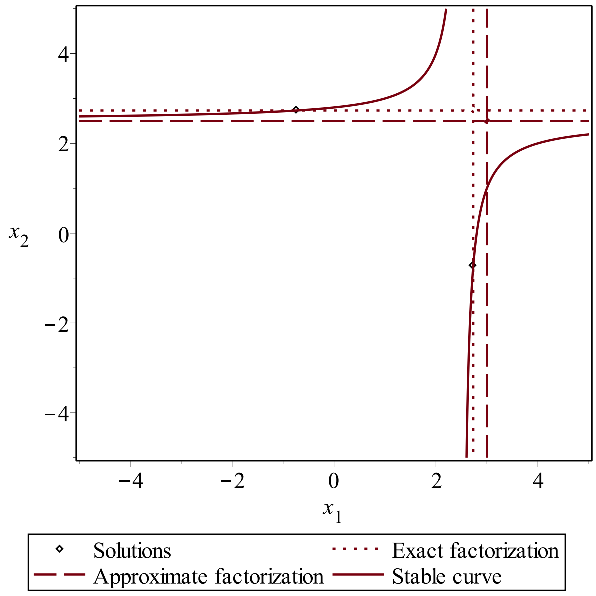

Fig. 1. Stabilizing polynomial for the variety $V(I)$ corresponding to $I = \langle z_1^2 - 2\,z_1 - 2,\ z_2^2 - 2\,z_2 - 2 \rangle$

REFERENCES

Bistritz, Y. (2002). Zero location of polynomials with respect to the unit-circle unhampered by nonessential singularities. *IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications*, 49(3), 305–314.

Bose, N.K. (1984). *Multidimensional System Theory.* D. Reidel Publishing Comp.

Bouzidi, Y., Cluzeau, T., Moroz, G., and Quadrat, A. (2016). `https://sites.google.com/site/algebrya/software/stabilization`. [Online; accessed 11-November-2016].

Bouzidi, Y., Quadrat, A., and Rouillier, F. (2015). Computer algebra methods for testing the structural stability of multidimensional systems. In *Proceedings of the 9th international Workshop on Multidimensional (nD) Systems (nDS'15), Vila Real (Portugal).*

Bridges, D., Mines, R., Richman, F., and Schuster, P. (2004). The polydisk nullstellensatz. *Proceedings of the American Mathematical Society*, 2133–2140.

Canny, J. (1988). Some algebraic and geometric computations in pspace. In *Proceedings of the twentieth annual ACM symposium on Theory of computing*, 460–467. ACM.

Cheng, J., Lazard, S., Peñaranda, L., Pouget, M., Rouillier, F., and Tsigaridas, E. (2010). On the topology of real algebraic plane curves. *Mathematics in Computer Science*, 4(1), 113–137.

Cox, D., Little, J., and O'shea, D. (1992). *Ideals, Varieties, and Algorithms*, volume 3. Springer.

Lin, Z. (1988). Feedback stabilization of multivariable two-dimensional linear systems. *International Journal of Control*, 48(3), 1301–1317.

Lin, Z. (1998). Feedback stabilizability of MIMO nD linear systems. *Multidimensional Systems and Signal Processing*, 9(2), 149–172.

Mignotte, M. (1989). *Mathématiques pour le calcul formel.* Presses Universitaires de France, Addison-Wesley Publishing Company.

Quadrat, A. (2003). The fractional representation approach to synthesis problems: An algebraic analysis viewpoint part ii: Internal stabilization. *SIAM Journal on Control and Optimization*, 42(1), 300–320.

Rouillier, F. (1999). Solving zero-dimensional systems through the rational univariate representation. *Applicable Algebra in Engineering, Communication and Computing*, 9(5), 433–461.

Vidyasagar, M. (2011). *Control system synthesis: a factorization approach.* Morgan & Claypool Publishers.

Xu, L., Saito, O., and Abe, K. (1994). Output feedback stabilizability and stabilization algorithms for 2D systems. *Multidimensional Systems and Signal Processing*, 5(1), 41–60.

Yap, C.K. and Sagraloff, M. (2011). A simple but exact and efficient algorithm for complex root isolation. In *Proceedings of the 36th International Symposium on Symbolic and Algebraic Computation*, ISSAC '11, 353–360. ACM, New York, NY, USA.