Using symbolic computation to solve algebraic Riccati equations arising in invariant filtering

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***Abstract*— This paper proposes a new step in the development of invariant observers. In the past, this theory led to impressive simplifications of the error equations encountered in estimation problems, especially those related to navigation. This was used to reduce computation load or derive new theoretical properties. Here, we leverage this advantage to obtain closed-form solutions of the underlying algebraic Riccati equations through advanced symbolic computation methods.**

I. Introduction

Luenberger observers are of major importance in engineering applications, both for their simplicity and their strong theoretical properties described, for instance, in [19]. Originally introduced for linear systems, they have been adapted to non-linear problems through first-order expansions about a specific operating point. This strategy reaches its limits when the first-order error equation is not (or not enough) autonomous, i.e., depends on a specific trajectory. The development of invariant observers since the early 2000's − starting with [1], then going on with, for instance [11] and [9] − has given a way to obtain autonomous errors for invariant systems on Lie groups, making the gain tuning of Luenberger observers much easier. The approach has been also shown to work for combinations of left- and right-invariant systems such as the one considered in the present paper ([23]).

In [6], a new framework has been introduced replacing the notion of invariance with the satisfaction of a short equation turning out to encompass left-invariant and right-invariant systems, combinations of both, linear systems, and additional equations arising in inertial navigation. This reformulation made the theory much simpler, although not fully relying on the intuitive notion of invariance, and allowed unveiling a new and unexpected log-linear property of the error equations for some non-linear systems on Lie groups. These advances were used to revisit the idea of tuning the gains of invariant observers through the methodology of extended Kalman filtering, previously proposed in [8] and [10]. Striking industrial applications have been developed by Safran Group which registered several patents such as [**?**], and, in [4], invariant filtering was also shown to solve a classical false observability problem of EKF-SLAM. This field is still very active, driven by the benefits already obtained in term of robustness and performance.

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But, in early work on invariant observers ([22], [24]), the driving force was rather the reduction of the computation load compared to EKF, without reducing performance. An idea was to replace Kalman gains by their asymptotic value, which is made possible by the autonomous error equations produced by invariant observers. To go further, in the present paper, we propose to leverage the simplicity of the involved algebraic Riccati equations to solve them analytically. In this preliminary work, we consider a classical attitude estimation problem and show the symbolic computation methods do allow deriving a simple analytical solution.

Undoubtedly, from both theoretical and practical viewpoints, differential and algebraic Riccati equations are one of the most fundamental concepts in linear control theory (e.g., linear quadratic optimal control, H^∞ control theory, Kalman filtering). An algebraic Riccati equation is a quadratic equation in an unknown symmetric matrix, i.e., it defines a quadratic polynomial system. The explicit computation of the definite positive solution of algebraic Riccati equations for low order linear systems in terms of their coefficients, considered as unknown parameters, has recently been initiated in [15], [2], [26], [25], [27] based on real algebraic geometry and symbolic computation techniques such as *Gröbner bases* (see, e.g., [7] and the references therein), *Cylindrical Algebraic Decomposition* ([13]), *discriminant varieties* ([18]), ... These results have been used to the explicit computation of H_∞ controllers for general classes of lower dimensional linear systems ([26], [25], [27]) and successfully applied to mechanical systems such as, for instance, the standard benchmark in robust control defined by the two mass-spring system with damping ([27]). In this paper, we show how the results developed in [26], [25], [27] can also be applied to find the closed-form solution of an algebraic Riccati equation appearing in the estimation of the attitude of a very simple model of a *Unmanned Aerial Vehicle* (UAV). More sophisticated examples are under study and will be developed in forthcoming publications.

Note that the proposed system is not new, neither is the invariant observer we describe in Section III, which can also be found in [23]. Several other papers have addressed the problem of attitude filtering exploiting the Lie group structure of the space of rotations SO(3) and the invariances of the problem: see, for instance, [11], [12], [21] or [28], [16]. Most methods used a deterministic framework (i.e., where noises are turned off), as we do in this paper. Attempts to explicitly take into account some uncertainties have also led to interesting results ([20], [5]).

The plan of the paper is the following. Section II presents the considered problem, Section III describes an invariant observer for this system and derives the algebraic Riccati equation allowing efficient gain tuning. Section IV is the main contribution of this paper: it provides an analytical formula solving the algebraic Riccati equation obtained in the previous section. Finally, Section V discusses the range of the result and proposes further investigation on the issue raised by the present paper.

II. A CLASSICAL ATTITUDE ESTIMATION MODEL

In this paper, we consider the classical problem of estimating the attitude of a *Unmanned Aerial Vehicle* (UAV). The system is modeled by the state variables defined below:

- $R_t \in SO(3)$ is an orthogonal matrix representing the attitude of the vehicle. It is defined as the transition matrix from the body (or vehicle) frame to the geographic frame, i.e., for any vector whose coordinates are u_b in the body frame and u_g in the geographic frame we have:
$$R_t u_b = u_g.$$
- $v_t \in \mathbb{R}^3$ is the vehicle velocity vector in the geographic frame.

These variables can be propagated over time using the measurements of an *Inertial Measurement Unit* (IMU):

- $\omega_t \in \mathbb{R}^3$ is the instantaneous rotation vector, in the vehicle frame, returned by the gyroscopes. Note that the measurement bias is not modeled here. A reader having some experience with invariant observers knows the importance of this hypothesis in the theory and could legitimately wonder if the results obtained in Section IV hold if bias is modeled. This important issue will be discussed in Section V.
- $f_t \in \mathbb{R}^3$ is the specific force, in the vehicle frame, returned by the accelerometers. It is defined as the sum of all nongravitational forces applied to the vehicle. Here also, bias is not modeled. This choice is more common in the literature than ignoring gyro bias, mostly because observability of accelerometer bias is very dependent on the trajectory.

Now, we can write the equations followed by R_t and v_t:

$$\frac{d}{dt} R_t = R_t \left(\omega_t\right)_\times \tag{1}$$
$$\frac{d}{dt} v_t = g + R_t f_t \tag{2}$$

where $g \in \mathbb{R}^3$ is the gravity field and the notation $(u)_\times$ for $u \in \mathbb{R}^3$ denotes the skew-symmetric matrix defined by the relation:

$$\forall\, v \in \mathbb{R}^3, \quad (u)_\times v = u \times v.$$

A. Measurements

To keep an accurate attitude estimation over time, the drone uses two measurements:

- Earth magnetic field $b \in \mathbb{R}^3$, measured in the reference frame of the drone by a magnetometer:
$$Y_t^b = R_t^T b.$$
- Drone velocity v_t, also measured in its own reference frame:
$$Y_t^v = R_t^T v_t.$$
It can come from an anenometer, from depth-camera-based visual odometery, or even from a virtual sensor always returning zero. The latter case is actually a robust way to encode the *low velocity* prior, which makes more sense in most applications than the usual quasi-static (i.e., zero-acceleration) hypothesis.

Next section directly gives the equations of the invariant observer proposed in [23] for this problem, without going through the whole theory.

III. INVARIANT OBSERVER

Classical Luenberger observers theory applied to system (1) and (2) leads to an error equation depending on the current state estimate. This makes gain tuning difficult, even regardless of the linear approximation issues. This problem is circumvented by the theory of invariant observers, which leads to autonomous error variable when it can be applied. As shown in [23], it is the case here. Denoting by $\hat{R}_t$ and $\hat{v}_t$ the estimates of R_t and v_t respectively, an invariant observer for system (1) and (2) is defined by:

$$\frac{d}{dt}\hat{R}_t = \hat{R}_t \omega_t + \left(k_b^R \left(\hat{R}_t Y_t^b - b\right) + k_v^R \left(\hat{R}_t Y_t^v - v_t\right)\right)_\times \hat{R}_t, \tag{3}$$
$$\frac{d}{dt}\hat{v}_t = g + \hat{R}_t f_t + k_b^v \left(\hat{R}_t Y_t^b - b\right) + k_v^v \left(\hat{R}_t Y_t^v - v_t\right), \tag{4}$$

where $k_b^R, k_v^R, k_b^v, k_v^v \in \mathbb{R}^3$ are gain matrices whose value is the point of Section IV. Now, we introduce the invariant estimation error $\eta_t = \left(\eta_t^R, \eta_t^v\right)$, where the rotation part $\eta_t^R \in \mathbb{R}^3$ and the velocity part η_t^v are defined as:

$$\eta_t^R = \hat{R}_t R_t^T, \tag{5}$$
$$\eta_t^v = \hat{R}_t R_t^T v_t - \hat{v}_t. \tag{6}$$

Differentiating (5) and (6) with respect to time t, then injecting (1), (2), (3) and (4), we obtain

$$\frac{d}{dt}\eta_t^R = \left(k_b^R \left(\hat{R}_t Y_t^b - b\right) + k_v^R \left(\hat{R}_t Y_t^v - \hat{v}_t\right)\right)_\times \eta_t^R,$$
$$\frac{d}{dt}\eta_t^v = \left(\eta_t^R - I_3\right) g + k_b^v \left(\hat{R}_t Y_t^b - b\right) + k_v^v \left(\hat{R}_t Y_t^v - \hat{v}_t\right) + \left(k_b^R \left(\hat{R}_t Y_t^b - b\right) + k_v^R \left(\hat{R}_t Y_t^v - v_t\right)\right)_\times \eta_t^v,$$

and finally:

$$\frac{d}{dt}\eta_t^R = \left(k_b^R \left(\eta_t^b - I_3\right) b + k_v^R \eta_t^v\right)_\times \eta_t^R, \tag{7}$$
$$\frac{d}{dt}\eta_t^v = \left(\eta_t^R - I_3\right) g + k_b^v \left(\hat{R}_t Y_t^b - b\right) + k_v^v \eta_t^v + \left(k_b^R \left(\eta_t^b - I_3\right) b + k_v^R \eta_t^v\right)_\times \eta_t^v. \tag{8}$$

These error equations can be expanded up to the first order using a linearized attitude error variable $\xi_t \in \mathbb{R}^3$ defined by the relation $\eta_t^R = \exp\left(\left(\xi_t\right)_\times\right)$, where $\exp(\cdot)$ denotes the matrix exponential. Actually, only the first-order relation between ξ_t and η_t will be used:

$$\eta_t^R = I_3 + (\xi_t)_\times + \circ\left(\|\xi_t\|^2\right). \tag{9}$$

Introducing (9) in the left-hand term of (7), then keeping only the first-order terms in ξ_t and η_t^v in equations (7) and (8), we obtain the linearized equations (10) and(11) below:

$$\frac{d}{dt}\xi_t = -k_b^R\,(b)_\times\,\xi_t + k_v^R\eta_t^v, \tag{10}$$

$$\frac{d}{dt}\eta_t^v = -(g)_\times\,\xi_t - k_b^v\,(b)_\times\,\xi_t + k_v^v\eta_t^v. \tag{11}$$

The last step is to give this system of equations a canonical form by introducing the stacked linear error defined by:

$$e_t = \begin{pmatrix}\xi_t\\ \eta_t^v\end{pmatrix} \in \mathbb{R}^6.$$

Linearized equations (7) and (8) become:

$$\frac{d}{dt}e_t = (F - KH)\,e_t, \tag{12}$$

where the matrices F, K and H are defined by:

$$F = \begin{pmatrix} 0_3 & 0_3\\ -(g)_\times & 0_3\end{pmatrix},\ H = \begin{pmatrix}(b)_\times & 0\\ 0_3 & -I_3\end{pmatrix},\ K = \begin{pmatrix}k_b^R & k_b^v\\ k_v^R & k_v^v\end{pmatrix}$$

The system (12) can then be stabilized by choosing the gain matrix K as

$$K = PH^TR^{-1},$$

where $P \in \mathbb{R}^{6\times 6}$ is the solution of the following algebraic Riccati equation

$$FP + PF^T + Q - PH^TR^{-1}HP = 0, \tag{13}$$

where $Q \in \mathbb{R}^6$ and $R^{-1} \in \mathbb{R}^6$ are tuning parameters encoding the trust we have, respectively, in the dynamics and in the observations. The classical design procedure consists in solving once and for all the equation using existing methods (see, e.g., [17], [29] and the references therein), then keeping the obtained gain constant in the final embedded system. If evidence rises that some sensors are undergoing performance degradation, the only way to adapt is to have pre-computed several gain values and set a switching-mode condition. Indeed, usual electronics embedded on drones does not allow real-time resolution of algebraic Riccati equations. What next section will show is that the dramatic simplification of error equations allowed by invariant filtering makes analytic resolution possible, using symbolic computation methods.

IV. Resolution of the algebraic Riccati equation

In this section, we shall consider the following case

$$\begin{cases} b = (b_1,\ 0,\ 0)^T, \quad b_1 \neq 0,\\ H = \operatorname{diag}(r_1,\dots,r_6), \quad r_i > 0,\ i = 1,\dots,6,\\ Q = \operatorname{diag}(q_1,\dots,q_6), \quad q_i > 0,\ i = 1,\dots,6,\end{cases}$$

where the notation $\operatorname{diag}(r_1,\dots,r_6)$ stands for the matrix with the r_i's on the diagonal and 0 elsewhere.

Let us seek for a definite positive P of (13) of the form:

$$P = \begin{pmatrix} p_{11} & 0 & 0 & 0 & p_{15} & 0\\ 0 & p_{22} & 0 & p_{24} & 0 & 0\\ 0 & 0 & p_{33} & 0 & 0 & 0\\ 0 & p_{24} & 0 & p_{44} & 0 & 0\\ p_{15} & 0 & 0 & 0 & p_{55} & 0\\ 0 & 0 & 0 & 0 & 0 & p_{66}\end{pmatrix}. \tag{14}$$

We do not loose any generality in assuming this particular form for P since, as we shall see later, a definite positive solution of (13) of the form of (14) can be found and there exists a unique definite positive solution of the Riccati equation for an observable system (12).

Remark 1: In the case where we restrict the number of the unknowns parameters r_i's and q_j's by, e.g., assuming that some of them are equal or are fixed to certain values, using *Gröbner basis techniques* (see, e.g., [7] and the references therein), we can prove that (14) is a direct consequence of (13). For more details, see [3]. The above argument shows that it is also the case when all the parameters are considered unknown. But we were not able to check it due to the long computation required to obtain the corresponding Gröbner basis on a standard computer.

Substituting (14) into the algebraic Riccati equation (13), we obtain the following polynomial system:

$$\begin{cases} p_{15}^2 - q_1\,r_5 = 0,\\ p_{66}^2 - q_6\,r_6 = 0,\\ b_1^2\,p_{33}^2 - q_3\,r_2 = 0,\\ p_{15}\,p_{55} - g\,r_5\,p_{11} = 0,\\ b_1^2\,r_4\,p_{22}^2 + r_3\,p_{24}^2 - q_2\,r_3\,r_4 = 0,\\ p_{55}^2 - 2\,g\,r_5\,p_{15} - q_5\,r_5 = 0,\\ b_1^2\,r_4\,p_{22}\,p_{24} + r_3\,p_{24}\,p_{44} + g\,r_3\,r_4\,p_{22} = 0,\\ b_1^2\,r_4\,p_{24}^2 + r_3\,p_{44}^2 + 2\,g\,r_3\,r_4\,p_{24} - q_4\,r_3\,r_4 = 0.\end{cases} \tag{15}$$

Inspecting (15), we note that (15) can be split into the following four uncoupled polynomial systems:

$$\begin{cases} p_{15}^2 - q_1\,r_5 = 0,\\ p_{15}\,p_{55} - g\,r_5\,p_{11} = 0,\\ p_{55}^2 - 2\,g\,r_5\,p_{15} - q_5\,r_5 = 0,\end{cases} \tag{16}$$

$$b_1^2\,p_{33}^2 - q_3\,r_2 = 0, \tag{17}$$

$$p_{66}^2 - q_6\,r_6 = 0, \tag{18}$$

$$\begin{cases} b_1^2\,r_4\,p_{22}^2 + r_3\,p_{24}^2 - q_2\,r_3\,r_4 = 0,\\ b_1^2\,r_4\,p_{22}\,p_{24} + r_3\,p_{24}\,p_{44} + g\,r_3\,r_4\,p_{22} = 0,\\ r_3\,p_{44}^2 + b_1^2\,r_4\,p_{24}^2 + 2\,g\,r_3\,r_4\,p_{24} - q_4\,r_3\,r_4 = 0.\end{cases} \tag{19}$$

We note that (17) and (18) are independent equations which completely determine p_{33} and p_{66}. Also, the first equation of (16) can be used to obtain p_{15}, i.e., we have:

$$p_{15} = \pm\sqrt{q_1\,r_5}, \quad p_{33} = \pm\sqrt{\frac{q_3\,r_2}{b_1^2}}, \quad p_{66} = \pm\sqrt{q_6\,r_6}.$$

Note that p_{33} only exists when $b_1 \neq 0$, which is always the case since we assume the system (13) to observable.

Eliminating the variables p_{55} and p_{15} from (16) (e.g., by computing a Gröbner basis of (16) for the elimination term order $p_{55} > p_{15} > p_{11}$ ([7])), (16) is then equivalent to the following polynomial system:

$$\begin{cases} 2\,q_1^2\,p_{55} - g^2\,p_{11}^3 + q_1\,q_5\,p_{11} = 0, \\ 2\,g\,q_1\,p_{15} - g^2\,p_{11}^2 + q_1\,q_5 = 0,, \\ (g^2\,p_{11}^2 - q_1\,q_5)^2 - 4\,g^2\,q_1^3\,r_5 = 0. \end{cases}$$

The above *triangular system* can easily be solved:

$$\begin{cases} p_{15} = \frac{1}{2\,g\,q_1}\,(g^2\,p_{11}^2 - q_1\,q_5), \\ p_{55} = \frac{1}{2\,q_1^2}\,p_{11}\,(g^2\,p_{11}^2 - q_1\,q_5), \\ (g^2\,p_{11}^2 - q_1\,q_5)^2 - 4\,g^2\,q_1^3\,r_5 = 0. \end{cases} \tag{20}$$

Let us note $\pi_1 = (g^2\,p_{11}^2 - q_1\,q_5)^2 - 4\,g^2\,q_1^3\,r_5$. The solutions of (16) can be obtained by solving the univariate polynomial π_1 and then substituting its solutions into the first two identities of (20) to get p_{15} and p_{55}. Noticing that $g^2\,p_{11}^2 - q_1\,q_5 = \pm 2\,g\,q_1\,\sqrt{q_1\,r_5}$, we have:

$$\begin{cases} p_{15} = \pm\sqrt{q_1\,r_5}, \\ p_{55} = \pm\frac{g}{q_1}\,p_{11}\,\sqrt{q_1\,r_5}, \\ (g^2\,p_{11}^2 - q_1\,q_5)^2 - 4\,g^2\,q_1^3\,r_5 = 0. \end{cases} \tag{21}$$

The polynomial π_1 is even in p_{11}, i.e., if we set $p_{11}^2 = x$, then x satisfies the following quadratic equation:

$$(g^2\,x - q_1\,q_5)^2 - 4\,g^2\,q_1^3\,r_5 = 0.$$

If we note $u = 2\,g\,q_1\,\sqrt{q_1\,r_5}$ and $v = q_1\,q_5$, then the solutions of π_1 are defined by:

$$\frac{1}{g}\,\sqrt{u+v}, \quad -\frac{1}{g}\,\sqrt{u+v}, \quad \frac{1}{g}\,\sqrt{-u+v}, \quad -\frac{1}{g}\,\sqrt{-u+v}. \tag{22}$$

The first two solutions of π_1 are real, the first (resp., second) one is positive (resp., negative). Moreover, if we have $v \geq u$, i.e., $q_5 \geq 2\,g\,\sqrt{q_1\,r_5}$, then the third and fourth solutions of π_1 are also real (complex otherwise), and thus π_1 admits 4 real solutions. In this case, the third solution is positive, the fourth is negative and $\sqrt{u+v} \geq \sqrt{-u+v}$.

Similarly as for (16), eliminating p_{44} and p_{24} from (19) (e.g., by computing a Gröbner basis of (19) for the elimination term order $p_{44} > p_{24} > p_{22}$), with the notations

$$\begin{cases} \alpha = 2\,g\,q_2\,r_3^2, \\ \beta = -b_1^4\,q_2\,r_4 + b_1^2\,q_4\,r_3 + g^2\,r_3^2, \\ \gamma = q_2\,r_3\,(b_1^2\,q_2\,r_4 - q_4\,r_3), \\ \varepsilon = q_2\,r_3\,(q_2^2\,r_4^2\,b_1^6 - 2\,q_2\,q_4\,r_3\,r_4\,b_1^4 + q_4^2\,r_3^2\,b_1^2 \\ \qquad + g^2\,r_3^2\,(q_4\,r_3 - 3\,b_1^2\,q_2\,r_4)), \\ \nu = \gamma^2 - \alpha^2\,q_2\,r_4, \end{cases}$$

(19) is equivalent to the following polynomial system:

$$\begin{cases} \alpha\,p_{24} + \beta\,p_{22}^2 + \gamma = 0, \\ \frac{1}{2}\,\alpha^2\,p_{44} - \beta^2\,p_{22}^3 + \varepsilon\,p_{22} = 0, \\ \beta^2\,p_{22}^4 - 2\,\varepsilon\,p_{22}^2 + \nu = 0. \end{cases} \tag{23}$$

Thus, all the solutions of (19) are then defined by:

$$\begin{cases} p_{24} = -\frac{1}{\alpha}\,(\beta\,p_{22}^2 + \gamma), \\ p_{44} = \frac{2}{\alpha^2}\,p_{22}\,(\beta^2\,p_{22}^2 - \varepsilon), \\ \beta^2\,p_{22}^4 - 2\,\varepsilon\,p_{22}^2 + \nu = 0. \end{cases} \tag{24}$$

Let us note $\pi_2 = \beta^2\,p_{22}^4 - 2\,\varepsilon\,p_{22}^2 + \nu$. The solutions of (19) can be obtained by first solving the univariate polynomial π_2 and then substituting its solutions into the first two identities of (24). Note that π_2 is even in p_{22}, i.e., if we set $p_{22}^2 = y$, then y satisfies the quadratic equation:

$$\beta^2\,y^2 - 2\,\varepsilon\,y + \nu = 0. \tag{25}$$

The discriminant of the polynomial (25) is then:

$$\Delta = 4\,(\varepsilon^2 - \beta^2\,\nu) = 16\,g^4\,q_2^3\,r_3^7\,r_4\,(b_1^2\,q_4 + g^2\,r_3) \geq 0.$$

If $\delta = \Delta/4$, then the solutions of (25) are reals and:

$$y_1 = \frac{\varepsilon + \sqrt{\delta}}{\beta^2}, \quad y_2 = \frac{\varepsilon - \sqrt{\delta}}{\beta^2}.$$

We have $y_1 > y_2$. The fourth solutions of π_2 are given by:

$$\sqrt{y_1}, \quad -\sqrt{y_1}, \quad \sqrt{y_2}, \quad -\sqrt{y_2}. \tag{26}$$

If $y_2 < 0$, i.e., $\varepsilon \leq \sqrt{\delta}$, then π_2 has only two reals solutions, one positive and one negative.

Remark 2: In particular, this last condition holds if $\varepsilon \leq 0$. Alternatively, if $-\nu > 0$, then we get $\delta = \varepsilon^2 - \beta^2\,\nu \geq \varepsilon^2$, and thus, we have $\sqrt{\delta} \geq |\varepsilon|$. Since the case of $\varepsilon \leq 0$ has just been considered, let us suppose that $\varepsilon \geq 0$. Then, we get $\varepsilon - \sqrt{\delta} \leq 0$, which shows that $\sqrt{y_2}$ and $-\sqrt{y_2}$ are complex. Hence, if $-\nu > 0$, i.e.,

$$\left(q_2^2\,r_4^2\right)\,b_1^4 - (2\,q_2\,q_4\,r_3\,r_4)\,b_1^2 - \left(4\,q_2\,r_3^2\,r_4\right)\,g^2 + q_4^2\,r_3^2 \geq 0,$$

or if $\varepsilon \leq 0$, i.e.,

$$\left(q_2^2\,r_4^2\right)\,b_1^6 - 2\,\left(q_2\,q_4\,r_3\,r_4\right)\,b_1^4 + \left(q_4^2\,r_3^2\right)\,b_1^2 \\ + r_3^2\,(q_4\,r_3 - 3\,b_1^2\,q_2\,r_4)\,g^2 \leq 0,$$

then π_2 has only one positive real solution.

All the (complex) solutions P of the form of (14) of the algebraic Riccati equation (13) are thus polynomially parametrized by the solutions of the univariate polynomials π_1 and π_2. Considering the degrees of these polynomials and of (17) and (18), we obtain that the number of (complex) solutions of (15) is $2 \times 2 \times 4 \times 4 = 64 = 2^6$, where 6 is the order of the linear system (12).

Since the system is observable ($b_1 \neq 0$), one knows that there exists a definite positive real solution P of the Riccati equation (13). Let us now try to determine which one of these 64 solutions defines the positive definite matrix P.

To do that, we recall the standard result due to Sylvester.

Proposition 1 (See, e.g., [14]): A symmetric matrix $P \in \mathbb{R}^{n \times n}$ is definite positive if and only if all of *principal minors* of P, namely, the determinants of the upper left $i \times i$ corners of P for $i = 1, \ldots, n$, are positive.

The principal minors of (14) are respectively defined by:

$$\begin{cases} m_1 = p_{11}, \\ m_2 = p_{11}\,p_{22}, \\ m_3 = p_{11}\,p_{22}\,p_{33}, \\ m_4 = p_{11}\,p_{33}\,(p_{22}\,p_{44} - p_{24}^2), \\ m_5 = p_{33}\,(p_{22}\,p_{44} - p_{24}^2)\,(p_{11}\,p_{55} - p_{15}^2), \\ m_6 = p_{33}\,p_{66}\,(p_{22}\,p_{44} - p_{24}^2)\,(p_{11}\,p_{55} - p_{15}^2). \end{cases}$$

According to Sylvester's criterion, all the principal minors m_i's of $P > 0$ have to be positive. Hence, we must have:

$$\begin{cases} p_{11} > 0, \\ p_{22} > 0, \\ p_{33} > 0, \\ p_{22}\,p_{44} - p_{24}^2 > 0, \\ p_{11}\,p_{55} - p_{15}^2 > 0, \\ p_{66} > 0. \end{cases}$$

Therefore, we first have to consider the following solutions:

$$\begin{cases} p_{33} = \sqrt{\dfrac{q_3\,r_2}{b_1^2}}, \\ p_{66} = \sqrt{q_6\,r_6}. \end{cases} \tag{27}$$

Now, let us determine which solution (22) yields $p_{11} > 0$ and $p_{11}\,p_{55} - p_{15}^2 > 0$. Using (20), we get:

$$p_{11}\,p_{55} - p_{15}^2 = \frac{(g^2\,p_{11}^2 - q_1\,q_5)\,(g^2\,p_{11}^2 + q_1\,q_5)}{4\,g^2\,q_1^2}.$$

Thus, the condition $p_{11}\,p_{55} - p_{15}^2 > 0$ can be replaced by

$$g^2\,p_{11}^2 - q_1\,q_5 > 0,$$

which, using $(g^2\,p_{11}^2 - q_1\,q_5)^2 - 4\,g^2\,q_1^3\,r_5 = 0$, yields:

$$p_{11} = \frac{1}{g}\sqrt{(2\,\sqrt{q_1\,r_5}\,g + q_5)\,q_1}. \tag{28}$$

Hence, using (20) or (21), we obtain:

$$\begin{cases} p_{15} = \sqrt{q_1\,r_5}, \\ p_{55} = \frac{1}{q_1}\,\sqrt{q_1\,r_5}\,\sqrt{\left(2\,\sqrt{q_1\,r_5}\,g + q_5\right)\,q_1}. \end{cases} \tag{29}$$

Finally, let us determine which solution (26) of π_2 satisfies the last two conditions $p_{22} > 0$ and $p_{22}\,p_{44} - p_{24}^2 > 0$. Using the equation $\pi_2 = 0$, we first obtain:

$$\begin{aligned} p_{22}\,p_{44} - p_{24}^2 &= \frac{1}{\alpha^2}\,(\beta^2\,p_{22}^4 - 2\,(\varepsilon + \beta\,\gamma)\,p_{22}^2 - \gamma^2) \\ &= -\frac{1}{\alpha^2}\,(2\,\gamma\,\beta\,p_{22}^2 + \gamma^2 + \nu) \\ &= -\frac{2\,\gamma}{\alpha^2}\,\left(\beta\,p_{22}^2 + \frac{\gamma^2 + \nu}{2\,\gamma}\right). \end{aligned} \tag{30}$$

Thus, we have to determine the sign of the above expression when evaluated at the solutions (26) of $\pi_2 = 0$. Since the solutions (26) depend on the parameters, the sign of (30) can change with them. In particular, if one of the conditions given in Remark 2 is satisfied, then we know that there is a unique positive real solution $\sqrt{y_1}$ of π_2. Then, using (24), we obtain the corresponding values for p_{24} and p_{44} (see the forthcoming (32)), which completely defines the matrix $P > 0$ satisfying (13).

In the general case, we can try to repeat what we have done with $\pi_2 = 0$ by considering:

$$\left(\beta\,p_{22}^2 - \frac{\varepsilon}{\beta}\right)^2 = \frac{\varepsilon^2 - \beta^2\,\nu}{\beta^2}.$$

But, unfortunately, we then have

$$\left(\frac{\gamma^2 + \nu}{2\,\gamma}\right) - \left(-\frac{\varepsilon}{\beta}\right) = -\frac{2\,g^4\,q_2^3\,r_3^6\,r_4}{\gamma\,\beta},$$

i.e., the right-hand sided of the above equality is not reduced to 0. If so, it would have been quite easy to conclude. We have to consider another method.

To do that, we use the concept of a *discriminant variety* ([18]), which computes algebraic varieties in the parameter space

$$\mathcal{P} = \{b_1,\ q_1, \ldots, q_6,\ r_1, \ldots, r_6\}$$

outside which the number of real solutions of a polynomial system, depending of these parameters, is constant. Using the Maple command `CellDecomposition` of the package `RootFinding[Parametric]`, we can compute the cells of maximal dimension of the *Cylindrical Algebraic Decomposition* ([13]) of the following *semi-algebraic set* (namely, a set of equations and inequalities) defined by:

$$S = \{\beta^2\,p_{22}^4 - 2\,\varepsilon\,p_{22}^2 + \nu = 0,\ p_{22} > 0,\ p_{22}\,p_{44} - p_{24}^2 > 0, \\ g > 0, \quad q_i > 0, \quad r_i > 0 \quad i = 1, \ldots, 6\}.$$

Using (30), $p_{22}\,p_{44} - p_{24}^2 > 0$ can be replaced in S by:

$$2\,\gamma\,\beta\,p_{22}^2 + \gamma^2 + \nu < 0.$$

As shown in [3], we obtain 38 different cells defined in the parameter space $\mathcal{P}$ – by means of polynomial equations and inequalities – in which that there is only one real solution. This result is a consequence of the unicity of the definite positive solution P of the algebraic Riccati equation. For each cell, a witness point, i.e., a rational value of the parameters, is explicitly given. We can then evaluate the 4 solutions (26) of $\pi_2 = 0$ at these witness points and check if we obtain positive or negative real, or complex values. Since the nature of the solutions in each cell is the same, we obtain that

$$p_{22} = \sqrt{y_1} = \sqrt{\frac{\varepsilon + \sqrt{\delta}}{\beta^2}} \tag{31}$$

is always the unique positive real solution. Then, p_{24} and p_{44} can be obtained by the first two identities of (19), i.e.:

$$\begin{cases} p_{24} = -\frac{1}{\alpha}\,(\beta\,y_1 + \gamma), \\ p_{44} = \frac{2}{\alpha^2}\,\sqrt{y_1}\,(\beta^2\,y_1 - \varepsilon). \end{cases} \tag{32}$$

We can then substitute (27), (28), (29), (31) and (32) into (14) to obtain the definite positive solution P of (13). We point out that $P > 0$ is defined by a unique simple analytical expression over the whole parameter space $\mathcal{P}$.

Finally, since there exists a unique definite positive solution P of (13), the assumption that P is of the form of (14), i.e., with zero entries else than p_{11}, p_{15}, p_{22}, p_{22}, p_{24}, p_{33}, p_{44}, p_{55} and p_{66}, was not restrictive since, by unicity, we have just proved that they must be equal to 0.

V. CONCLUSION AND FUTURE WORK

In this paper, for a classical attitude estimation problem, we showed that the algebraic Riccati equation used to tune the gains of an invariant observer can be solved analytically as a function of the design parameters $(q_i)_{i=1,\dots,6}$ and $(r_i)_{i=1,\dots,6}$, and of the physical quantities g and b. A straightforward application is to build an observer whose gains adapt to a change of the sensors accuracy, measured, for instance, through an innovation test. Indeed, instead of dealing with the gain matrix K directly, the result of this paper allows the practitioner to design tuning politics for R and Q matrices, which have a clear physical meaning. The matrix K being then computed as the solution of the algebraic Riccati equation, stability of the obtained observer is ensured. Actually, stability is ensured for a given set of constant values for $(q_i)_{i=1,\dots,6}$ and $(r_i)_{i=1,\dots,6}$. If the values of these parameters change over the trajectory (which is the situation where the result of the present paper is useful), a more careful theoretical analysis has to be performed. Yet, this analysis is much easier if the gains are given by an analytical formula than if they are defined implicitly as the result of an algebraic Riccati equation.

The example we considered in this paper is extremely simple and the adding gyro biases to the model would certainly make the result more valuable. But it is well-known that error equation autonomy is lost in this case. Thus, we shall probably have to work on specific operating points.

From a theoretical point of view, the point raised by Remark 1 is very interesting: most variables of the equation can be zeroed by symbolic computation arguments, but a probabilistic reasoning on P, seen as a covariance matrix, gives the same result. Interpreting the properties of the solutions of algebraic Ricatti equations from different points of view (automatic control, probabilities and symbolic computation) seems a promising route for future work.

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