An algebraic analysis approach to mathematical system theory

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Plan

1. **Modelling problems**

⋆ 4 different models of a stirred tank

2. **Analysis problems**

⋆ Ore algebras

⋆ Gröbner basis

⋆ Algebraic analysis:

   a. Module theory

   b. Systems-Modules Dictionary

   c. Homological algebra

⋆ Maple package OREMODULES

3. **Synthesis problems**

4. **Conclusion**

Modelling problems

Model of a stirred tank

- H. Kwakernaak, R. Sivan, *Linear Optimal Control Systems*, Wiley, 1972.

- **Non-linear model** of a stirred tank (page 7):

$$\begin{cases} \dfrac{dV(t)}{dt} = -k\sqrt{\dfrac{V(t)}{S}} + F_1(t) + F_2(t), \\ \dfrac{d\left(c(t)\,V(t)\right)}{dt} = -c(t)\,k\sqrt{\dfrac{V(t)}{S}} + c_1\,F_1(t) + c_2\,F_2(t), \end{cases}$$

F_1, F_2: time-varying flow rates of two incoming flows feeding the tank,

c_1, c_2: constant concentrations,

c: concentration in the tank,

V: volume in the tank,

k: experimental constant,

S: constant cross-sectional area of the tank.

Model of a stirred tank

- H. Kwakernaak, R. Sivan, *Linear Optimal Control Systems*, Wiley, 1972.

- **Linearized model** in a neighbourhood of the steady-state situation (pages 8-9).

$$\begin{cases} \dot{x}_1(t) = -\dfrac{1}{2\,\theta}\, x_1(t) + u_1(t) + u_2(t), \\ \dot{x}_2(t) = -\dfrac{1}{\theta}\, x_2(t) + \left(\dfrac{c_1 - c_0}{V_0}\right) u_1(t) + \left(\dfrac{c_2 - c_0}{V_0}\right) u_2(t), \end{cases}$$

V_0: constant volume, c_0: constant concentration,

F_{10}, F_{20}: steady-state conditions,

$$\begin{cases} V(t) = V_0 + x_1(t), \\ c(t) = c_0 + x_2(t), \\ F_1(t) = F_{10} + u_1(t), \\ F_2(t) = F_{20} + u_2(t), \end{cases}$$

$F_0 = k\,\sqrt{(V_0/S)}$,

$\theta = V_0/F_0$: holdup time of the tank.

Model of a stirred tank

- H. Kwakernaak, R. Sivan, *Linear Optimal Control Systems*, Wiley, 1972.

- **Discrete-time model**: if we consider that the stirred tank is commanded by a process control computer, then the valve settings only change at discrete instants, remaining constant in between (page 449).

We then obtain:

$$\begin{cases} x_1(i+1) = e^{-\frac{\Delta}{2\theta}}\, x_1(i) + 2\,\theta\,(1 - e^{-\frac{\Delta}{2\theta}})\, u_1(i) \\ \qquad\qquad\qquad\qquad + 2\,\theta\,(1 - e^{-\frac{\Delta}{2\theta}})\, u_2(i), \\ x_2(i+1) = e^{-\frac{\Delta}{\theta}}\, x_2(i) + \dfrac{\theta\,(c_1 - c_0)}{V_0}\,(1 - e^{-\frac{\Delta}{\theta}})\, u_1(i) \\ \qquad\qquad\qquad\qquad + \dfrac{\theta\,(c_2 - c_0)}{V_0}\,(1 - e^{-\frac{\Delta}{\theta}})\, u_2(i), \end{cases}$$

Δ: constant length of time intervals.

Model of a stirred tank

- H. Kwakernaak, R. Sivan, *Linear Optimal Control Systems*, Wiley, 1972.

- **Differential time-delay model**: if we consider that there is a transport delay of amplitude occuring in the pipe, we then obtain (pages 449-451):

$$
\begin{cases}
\dot{x}_1(t) = -\frac{1}{2\,\theta}\,x_1(t) + u_1(t) + u_2(t), \\
\dot{x}_2(t) = -\frac{1}{\theta}\,x_2(t) + \left(\frac{c_1 - c_0}{V_0}\right) u_1(t-\tau) + \\
\qquad\qquad\qquad\qquad \left(\frac{c_2 - c_0}{V_0}\right) u_2(t-\tau),
\end{cases}
$$

$\tau > 0$: amplitude of the delay.

- **Other types of models**:

$\star$ discrete-time model of the previous differential time-delay system (pages 449-452).

$\star$ partial differential equations,

$\star$ hybrid systems...

Analysis problems

Ore algebras

• **Definition**: A non-commutative polynomial ring $D = A[\partial; \sigma, \delta]$ in ∂ is called **skew** if

$$\boxed{\partial\, a = \sigma(a)\, \partial + \delta(a), \quad a \in A,}$$

where $\sigma : A \rightarrow A$ satisfies $\forall\, a, b \in A$:

$$\begin{cases} \sigma(1) = 1, \\ \sigma(a + b) = \sigma(a) + \sigma(b), \\ \sigma(a\, b) = \sigma(a)\, \sigma(b), \end{cases}$$

and $\delta : A \rightarrow A$ is such that $\forall\, a, b \in A$:

$$\begin{cases} \delta(a + b) = \delta(a) + \delta(b), \\ \delta(a\, b) = \sigma(a)\, \delta(b) + \delta(a)\, b. \end{cases}$$

• $P = \sum_{i=0}^{r} a_i\, \partial^i \in D, \quad a_i \in A$.

• **Definition**: (Chyzak-Salvy): The skew ring

$$D = k[x_1, \ldots, x_n][\partial_1; \sigma_1, \delta_1] \ldots [\partial_m; \sigma_m, \delta_m]$$

is called an **Ore algebra** if :

$$\boxed{\begin{cases} \sigma_i\, \delta_j = \delta_j\, \sigma_i, \quad 1 \leq i, j \leq m, \\ \sigma_i(\partial_j) = \partial_j, \quad \delta_i(\partial_j) = 0, \quad j < i. \end{cases}}$$

$\Rightarrow D$ is a **non-commutative ring**.

Examples of Ore algebras

- **Ordinary differential operators**:

$$D = A\left[\frac{d}{dt}; 1, \frac{d}{dt}\right], \quad A = k[t], k(t), \ldots$$

$$P = \sum_{i=0}^{m} a_i(t)\, \frac{d^i}{dt^i} \in D, \quad \frac{d}{dt}\, a(t) = \dot{a}(t).$$

- **Time-delay (time-advance) operators**:

$$D = A[\delta_h; \sigma_h, 0], \quad A = k[t], k(t), \ldots$$

$$P = \sum_{i=0}^{m} a_i(t)\, \delta_h^i \in D, \quad \sigma_h\, a(t) = a(t-h).$$

- **Shift operators**:

$$D = A[\delta; \sigma, 0], \quad A = k[n], k(n), \ldots$$

$$P = \sum_{i=0}^{m} a_i(n)\, \delta^i \in D, \quad \sigma\, a(n) = a(n+1).$$

- **Differential time-delay operators**:

$$D = A\left[\frac{d}{dt}; 1, \frac{d}{dt}\right][\delta_h; \sigma_h, 0], \quad A = k[t], k(t), \ldots$$

$$P = \sum_{0 \leq i+j \leq m}^{n} a_{ij}(t)\, \frac{d^i}{dt^i}\, \delta_h^j \in D.$$

- **Partial differential operators**:

$$D = A[d_1; 1, \partial_1] \ldots [d_n; 1, \partial_n], \; A = k[x_1, \ldots, x_n], \ldots$$

$$P = \sum_{0 \leq |\mu| \leq m} a_\mu(x)\, d^\mu, \; d^\mu = d_1^{\mu_1} \ldots d_n^{\mu_n}, \; \partial_i = \frac{\partial}{\partial x_i}.$$

Example

The models of the stirred tank can be written as:

$$R\,z = 0, \quad z = (x_1, x_2, u_1, u_2)^T.$$

- **Differential model**:

$$R = \begin{pmatrix} \frac{d}{dt} + \frac{1}{2\,\theta} & 0 & -1 & -1 \\ 0 & \frac{d}{dt} + \frac{1}{\theta} & -\frac{(c_1 - c_0)}{V_0} & -\frac{(c_2 - c_0)}{V_0} \end{pmatrix} \in \mathbb{R}\left[\frac{d}{dt}\right]^{2\times 4}.$$

- **Discrete-time model**:

$$R = \begin{pmatrix} \delta - a & 0 & -b & -b \\ 0 & \delta - a & -c & -d \end{pmatrix} \in \mathbb{R}[\delta]^{2\times 4},$$

with the notations $a = e^{-\frac{\Delta}{2\,\theta}}, \quad b = 2\,\theta\,(1 - a),$

$$c = \frac{(c_1 - c_0)}{2\,V_0}\,b, \quad d = \frac{(c_2 - c_0)}{2\,V_0}\,b.$$

- **Differential time-delay model**:

$$R = \begin{pmatrix} \frac{d}{dt} + \frac{1}{2\,\theta} & 0 & -1 & -1 \\ 0 & \frac{d}{dt} + \frac{1}{\theta} & -\frac{(c_1 - c_0)}{V_0}\,\delta_\tau & -\frac{(c_2 - c_0)}{V_0}\,\delta_\tau \end{pmatrix} \in \mathbb{R}\left[\frac{d}{dt}, \delta_\tau\right]^{2\times 4}.$$

Gröbner bases

• **Theorem**: (Kredel): Let D be an Ore algebra s.t.

$$\sigma_i(x_j) = a_{ij}\, x_j + b_{ij}, \quad \delta_i(x_j) = c_{ij},$$

$0 \neq a_{ij},\, b_{ij} \in k,\; c_{ij} \in k[x_1, \ldots, x_n]$, $\deg(c_{ij}) \leq 1$, then, **Gröbner bases** w.r.t. any term order can be computed algorithmically.

• **Implementations**:

⇒ **Maple Ore_algebra** (Chyzak, ALGO, INRIA).

`http://algo.inria.fr/libraries/`

⇒ **Plural** (University of Kaiserslautern).

`http://www.singular.uni-kl.de/plural/`

⇒ **Mathematica NCAlgebra** (Helton and co.).

`http://www.math.ucsd.edu/ ncalg/`

Definition

• **Notation**: $D = k[x_1, \ldots, x_n]$, $k = \mathbb{Q}$, $\mathbb{R}$.

• **Definition**: A **monomial ordering** $\succ$ is a total ordering on $\{x^\alpha = x_1^{\alpha_1} \ldots x_n^{\alpha_n} \mid \alpha \in \mathbb{Z}_+^n\}$ satisfying:

$$\begin{cases} x^\alpha \neq 1 \Rightarrow x^\alpha \succ 1, \\ x^\alpha \succ x^\beta \Rightarrow x^\gamma \, x^\alpha \succ x^\gamma \, x^\beta, \quad \forall \, \gamma \in \mathbb{Z}_+^n. \end{cases}$$

• **Example**: degree lexicographical ordering $\succ_{\text{lexdeg}}$:

$x^\alpha \succ_{\text{lexdeg}} x^\beta \;\Leftrightarrow\; |\alpha| > |\beta| \quad \text{or} \quad |\alpha| = |\beta|,$

$\exists \, 1 \leq i \leq n: \; \alpha_1 = \beta_1, \ldots, \alpha_{i-1} = \beta_{i-1}, \; \alpha_i > \beta_i,$

where $|\alpha| = \sum_{i=1}^n \alpha_i$ is the length of α.

• **Definition**: Let $P = a_\alpha \, x^\alpha + a_\beta \, x^\beta + \ldots + a_\gamma \, x^\gamma \in D$, where $x^\alpha \succ x^\beta \succ \ldots$ and $a_\alpha, \, a_\beta, \ldots, a_\gamma \in k$.

Then, $\text{lm}(P) = x^\alpha$ is the **leading monomial** of P.

• **Definition**: Let I be an ideal, i.e., $I = \sum_{i=1}^r D \, P_i$, for certain $P_i \in D$. Then,

$G = \{G_1, \ldots, G_s\} \subset D \backslash 0$ is a **Gröbner basis** of I

$\Leftrightarrow \forall \, 0 \neq Q \in I, \quad \exists \, 1 \leq i \leq s: \quad \text{lm}(G_i) \mid \text{lm}(Q).$

Properties

• **Example**: Let $D = k[x, y]$, $\succ_{\text{lexdef}}$ and the ideal $I = (x^2, x\,y - 1) = \{P_1\,x^2 + P_2(x\,y - 1),\ P_i \in D\}$.

$G = \{x^2,\ x\,y - 1\}$ is **not a Gröbner basis** of I as:

$$\begin{cases} S(x^2, x\,y - 1) \triangleq y\,(x^2) - x\,(x\,y - 1) = x \in I, \\ x^2 \nmid x, \quad x\,y \nmid x. \end{cases}$$

• **Theorem**: Every ideal $I \subset D\backslash 0$ admits a Gröbner basis G for a fixed monomial order. Moreover,

if $G = \{G_1, \ldots, G_r\}$, then $I = \sum_{i=1}^{r} D\,G_i$.

• **Theorem**: A family $G = \{G_1, \ldots, G_r\}$ of generators of I is a Gröbner basis iff, for all $i \neq j$, the remainder of $S(G_i, G_j)$ by G is 0.

• **Example**: $\{x^2,\ x\,y - x\}$ **is a Gröbner basis** of the ideal $J = (x^2,\ x\,y - x)$ as:

$$\begin{cases} S(x^2, x\,y - x) \triangleq y\,(x^2) - x\,(x\,y - x) = x^2, \\ x^2 \mid S(x^2, x\,y - x). \end{cases}$$

• **Example**: We consider $\{x^2,\ x\,y - 1,\ x\}$ and

$$S(x, x\,y - 1) \triangleq y\,(x) - (x\,y - 1) = 1 \in I \Rightarrow I = D,$$

and $\{1\}$ is the Gröbner basis of $I = D$.

Extension to modules

• **Extension to D-submodules of D^r** by extending the monomial ordering into a **module ordering** on:

$$x^\alpha\, e_i = (0 \ldots x^\alpha \ldots 0)^T, \quad i = 1, \ldots, r,\ \alpha \in \mathbb{Z}_+^n.$$

$\Rightarrow$ **effective computations of syzygy modules**.

• **Example**: Consider $D = \mathbb{R}\left[\frac{d}{dt},\, \delta_h\right]$ and the **over-determined** differential time-delay linear system:

$$\left\{\begin{array}{l} \left(\frac{d}{dt} + a\right)\, \lambda_1 = \mu_1, \\ -k\, a\, \delta_h\, \lambda_1 + \frac{d}{dt}\, \lambda_2 + \omega^2\, \lambda_3 = \mu_2, \\ -\lambda_2 + \left(\frac{d}{dt} + 2\, \zeta\, \omega\right)\, \lambda_3 = \mu_3, \\ -\omega^2\, \lambda_3 = \mu_4. \end{array}\right. \qquad (\star).$$

$(\star)$ admits a **compatibility condition**, namely:

$$\omega^2\, k\, a\, \delta_h\, \mu_1 + \left(\omega^2\, \tfrac{d}{dt} - \omega^2\, a\right)\, \mu_2 + \left(\omega^2\, \tfrac{d^2}{dt^2} + \omega^2\, a\, \tfrac{d}{dt}\right)\, \mu_3$$
$$+ \left(\tfrac{d^3}{dt^3} + 2\, \zeta\, \omega\, \tfrac{d^2}{dt^2} + a\, \tfrac{d^2}{dt^2} + \omega^2\, \tfrac{d}{dt} + 2\, a\, \zeta\, \omega\, \tfrac{d}{dt} + a\, \omega^2\right)\, \mu_4 = 0.$$

Compatibility conditions can be obtained by means of a computation of a Gröbner basis for an elimination order.

Methodology

1. A **linear system** Σ is defined by a **matrix with entries** R **in a ring** D, i.e., we have $R\,\eta = 0$.

1. Using the matrix R, **we define a** D**-module** M.

2. We develop **a dictionary between the properties of the system** Σ **and the module** M.

3. We use **module theory** in order to classify the properties of the module M.

4. We use **homological algebra** in order to check the properties of the module M.

5. Using effective algebra, we develop **effective algorithms** which check the properties of the module M, and thus, of the system Σ.

6. **Implementation** in OreModules (Maple):

`http://wwwb.math.rwth-aachen.de/OreModules/`

Systems-Modules

- Let a system be defined by the **equations**

$$R\,\eta = 0, \quad R \in D^{q \times p},$$

where η is the **system variables** (inputs, outputs, states, latent variables...) and D is an **Ore algebra.**

Like in **algebraic geometry**, we associate with the system the **left** D-module:

$$\boxed{M = D^{1 \times p}/(D^{1 \times q}\,R)}$$

- **Example**: The wind tunnel model (Manitius 84):

$$\begin{cases} \dot{x}_1(t) = -a\,x_1(t) + k\,a\,x_2(t-h), \\ \dot{x}_2(t) = x_3(t), \\ \dot{x}_3(t) = -\omega^2\,x_2(t) - 2\,\zeta\,\omega\,x_3(t) + \omega^2\,u(t). \end{cases} \qquad (\star)$$

- The system $(\star)$ is equivalent to:

$$\underbrace{\begin{pmatrix} \frac{d}{dt} + a & -k\,a\,\delta_h & 0 & 0 \\ 0 & \frac{d}{dt} & -1 & 0 \\ 0 & \omega^2 & \frac{d}{dt} + 2\,\zeta\,\omega & -\omega^2 \end{pmatrix}}_{R \in D^{3 \times 4}} \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ u(t) \end{pmatrix} = 0$$

$$D = \mathbb{R}(a, k, \omega, \zeta)\left[\frac{d}{dt}, \delta_h\right], \quad \boxed{M = D^{1 \times 4}/(D^{1 \times 3}\,R)}.$$

Classification of modules

• **Definition**:

$a)$ M is **free** if $\exists\, r \in \mathbb{Z}_+ :\ M \cong D^r$.

$b)$ M is **projective** if $\exists\, r \in \mathbb{Z}_+$ and a D-module P:

$$M \oplus P \cong D^r.$$

$c)$ M is **reflexive** if ϵ is an isomorphism:

$$\begin{aligned} \epsilon :\ \ M &\longrightarrow\ \ M^{\star\star}, \\ m &\longmapsto\ \ \epsilon(m), \qquad \epsilon(m)(f) = f(m). \end{aligned}$$

$d)$ M is **torsion-free** if:

$$t(M) = \{m \in M \mid \exists\, 0 \neq P \in D : P\, m = 0\} = 0.$$

$m \in t(M)$ is called a **torsion element** of M.

• **Theorem**:

1. **free** $\Rightarrow$ **projective** $\Rightarrow$ .. $\Rightarrow$ **reflexive** $\Rightarrow$ **torsion-free**.

2. If D **is a principal domain** (e.g., $K\left[\frac{d}{dt}\right]$, $K[\delta]$), then we have:

torsion-free = free.

3. If $D = k[x_1, \ldots, x_n]$, where k is a field:

projective = free (Th. Quillen-Suslin).

Systems-Modules-Homological algebra

- $R \in D^{q \times p}, \quad M = D^{1\times p}/(D^{1\times q}\, R), \quad \widetilde{N} = D^{1\times q}/(D^{1\times p}\, \theta(R))$.

Systems	**Module** M	**Homological algebra**	$d(\widetilde{N})$
$\exists$ non-controllable elements	with torsion	$t(M) \cong \text{ext}^1_D(\widetilde{N}, D) \neq 0$	$n-1$
Controllability Parametrizability $\pi\left(\partial_{\sigma(1)}, \ldots, \partial_{\sigma(n-1)}\right)$-free	torsion-free	$\text{ext}^1_D(\widetilde{N}, D) = 0$	$n-2$
The parametrization is parametrizable $\pi\left(\partial_{\sigma(1)}, \ldots, \partial_{\sigma(n-2)}\right)$-free	reflexive	$\text{ext}^i_D(\widetilde{N}, D) = 0,$ $i = 1, 2$	$n-3$
...	...	...	...
Bézout Identities Chain of n parametrizations Internal stabilizability	projective	$\text{ext}^i_D(\widetilde{N}, D) = 0,$ $1 \leq i \leq n$	-1
Flatness Youla parametrization of stabilizing controllers	free	$\emptyset$ (Quillen-Suslin th.)	

Involution

• **Definition**: An **involution** of an Ore algebra D is a k-linear map $\theta : D \to D$ satisfying:

1. $\theta(a_1\, a_2) = \theta(a_2)\, \theta(a_1), \quad a_1, a_2 \in D,$

2. $\theta^2 = id_D$.

• If $R \in D^{q \times p}$, then we have:

$$\boxed{\theta(R) \triangleq (\theta(R_{ij}))^T \in D^{p \times q}}.$$

• **Example**: 1. If $D = k[x_1, \ldots, x_n]$, then $\theta = id_D$.

2. If $D = k[t]\left[\frac{d}{dt},\, \delta_h,\, \delta_{-h}\right]$, then an involution of D is defined by:

$$t \mapsto t, \quad \frac{d}{dt} \mapsto -\frac{d}{dt}, \quad \delta_h \mapsto \delta_{-h}, \quad \delta_{-h} \mapsto \delta_h.$$

Let $R = (t\, \frac{d}{dt} \quad -t^2\, \delta_h) \in D^{1 \times 2}$, then we have:

$$\theta(R) = \begin{pmatrix} -\frac{d}{dt}\, t \\ -\delta_{-h}\, t^2 \end{pmatrix} = \begin{pmatrix} -t\, \frac{d}{dt} + 1 \\ -(t+h)^2\, \delta_{-h} \end{pmatrix}.$$

• **Right** D-module $N \overset{\theta}{\Longleftrightarrow}$ **Left** D-module $\widetilde{N}$:

$$\forall\, P \in D,\ \forall\, n \in \widetilde{N} :\ P \circ n = n\, \theta(P).$$

Extension functor

- **Parametrizability problem**:

$$R\,\eta = 0 \overset{?}{\Longleftrightarrow} \exists\ P \in D^{p\times m}:\ \ \eta = P\,\xi,\ \forall\ \xi.$$

- **Hints**: We follow steps 1, 2, 3 and 4:

$$\begin{array}{ccc} 4.\quad \theta(Q)\,\xi = \eta & \Longrightarrow & R\,\eta = 0 \quad 1. \\ \Uparrow & & \Downarrow \\ \text{involution } \theta & & \text{involution } \theta \\ \Uparrow & & \Downarrow \\ 3.\quad 0 = Q\,\mu & \overset{\text{G.B.}}{\Longleftrightarrow} & \theta(R)\,\lambda = \mu \quad 2. \end{array}$$

$$\begin{aligned} Q \circ \theta(R) = 0 \Rightarrow \theta(Q \circ \theta(R)) &= \theta^2(R) \circ \theta(Q) \\ &= R \circ \theta(Q) = 0. \end{aligned}$$

- **The last step**:

$$\theta(Q)\,\xi = \eta \overset{\text{G.B.}}{\Longleftrightarrow} R'\,\eta = 0$$

$$\boxed{\text{ext}^1_D(\widetilde{N}, D) = (D^{1\times q'}\,R')/(D^{1\times q}\,R)} \qquad \text{(G.B.)}$$

where $\widetilde{N} = D^{1\times q}/(D^{1\times p}\,\theta(R)) \Longleftrightarrow \theta(R)\,\lambda = 0.$

The wind tunnel model

- **Is the wind tunnel model parametrizable?**

$$\begin{cases} \dot{x}_1(t) + a\,x_1(t) - k\,a\,x_2(t-h) = 0, \\ \dot{x}_2(t) - x_3(t) = 0, \\ \dot{x}_3(t) + \omega^2\,x_2(t) + 2\,\zeta\,\omega\,x_3(t) - \omega^2\,u(t) = 0. \end{cases} \quad (1)$$

$$\begin{cases} \left(\frac{d}{dt} + a\right)\,\lambda_1 = \mu_1, \\ -k\,a\,\delta_h\,\lambda_1 + \frac{d}{dt}\,\lambda_2 + \omega^2\,\lambda_3 = \mu_2, \\ -\lambda_2 + \left(\frac{d}{dt} + 2\,\zeta\,\omega\right)\,\lambda_3 = \mu_3, \\ -\omega^2\,\lambda_3 = \mu_4. \end{cases} \quad (2)$$

$$\omega^2\,k\,a\,\delta_h\,\mu_1 + \left(\omega^2\,\tfrac{d}{dt} - \omega^2\,a\right)\,\mu_2 + \left(\omega^2\,\tfrac{d^2}{dt^2} + \omega^2\,a\,\tfrac{d}{dt}\right)\,\mu_3$$
$$+ \left(\tfrac{d^3}{dt^3} + 2\,\zeta\,\omega\,\tfrac{d^2}{dt^2} + a\,\tfrac{d^2}{dt^2} + \omega^2\,\tfrac{d}{dt} + 2\,a\,\zeta\,\omega\,\tfrac{d}{dt} + a\,\omega^2\right)\,\mu_4 = 0. \quad (3)$$

$$\begin{cases} x_1(t) = & -\omega^2\,k\,a\,\xi(t-h), \\ x_2(t) = & -\omega^2\,\dot{\xi}(t) + a\,\omega^2\,\xi(t), \\ x_3(t) = & \omega^2\,\ddot{\xi}(t) - \omega^2\,a\,\dot{\xi}(t), \\ u(t) = & -\xi(t)^{(3)} + (2\,\zeta\,\omega + a)\,\ddot{\xi}(t) \\ & -(\omega^2 + 2\,a\,\omega\,\zeta)\,\dot{\xi}(t) + a\,\omega\,\xi(t). \end{cases} \quad (4)$$

The compatibility conditions of (4) are exactly generated by (1) $\Rightarrow$ **the system is parametrized by (4).**

Flexible rod

$$\begin{cases} \frac{\partial^2 z}{\partial t^2}(x,t) = \frac{\partial^2 z}{\partial x^2}(x,t), & y_1(t) = z(0,t), \\ \frac{\partial z}{\partial x}(0,t) = -u(t), & y_2(t) = z(1,t), \\ \frac{\partial z}{\partial x}(1,t) = 0, & \end{cases}$$

$$\Rightarrow \begin{cases} \dot{y}_1(t) - \dot{y}_2(t-1) - u(t) = 0, \\ 2\,\dot{y}_1(t-1) - \dot{y}_2(t) - \dot{y}_2(t-2) = 0. \end{cases} \qquad (1)$$

$$\begin{cases} \frac{d}{dt}\lambda_1 + 2\,\delta\,\frac{d}{dt}\lambda_2 = \mu_1, \\ -\delta\,\frac{d}{dt}\lambda_1 - \frac{d}{dt}(\delta^2+1)\,\lambda_2 = \mu_2, \\ -\lambda_1 = \mu_3. \end{cases} \qquad (2)$$

$$(\delta^2+1)\,\mu_1 + 2\,\delta\,\mu_2 - \frac{d}{dt}(1-\delta^2)\,\mu_3 = 0. \qquad (3)$$

$$\begin{cases} y_1(t) = \xi(t) + \xi(t-2), \\ y_2(t) = 2\,\xi(t-1), \\ u(t) = \dot{\xi}(t) - \dot{\xi}(t-2). \end{cases} \qquad (4)$$

$$\begin{cases} \dot{y}_1(t) - \dot{y}_2(t-1) - u(t) = 0, \\ 2\,y_1(t-1) - y_2(t) - y_2(t-2) = 0, \\ \dot{y}_1(t-1) - \dot{y}_2(t-1) = 0, \end{cases} \qquad (5)$$

$\Rightarrow t(M) \neq 0 \Rightarrow$ **the system is not controllable**:

$\theta(t) = 2\,y_1(t-1) - y_2(t) - y_2(t-1), \quad \dot{\theta}(t) = 0.$

$\Rightarrow (-c/2, -c, 0)^T$ **is not parametrized by** (4).

Controllability

• **Kalman system**: $\dot{x} = A(t)\,x + B(t)\,u.$ (1)

• We consider the $D = K\left[\frac{d}{dt}\right]$-module defined by:

$$M = D^{1\times(n+m)}/(D^{1\times n}\,R),$$

where $R = \left(\frac{d}{dt}\,I_n - A(t) \quad -B(t)\right)$.

• $\widetilde{N} = D^{1\times n}/(D^{1\times(n+m)}\,\widetilde{R})$ is defined by:

$$\begin{cases} -(\dot{\lambda} + A(t)^T\,\lambda) = 0, \\ -B(t)^T\,\lambda = 0, \end{cases} \Rightarrow \begin{cases} \dot{\lambda} = -A(t)^T\,\lambda, \\ B(t)^T\,\lambda = 0. \end{cases} \qquad (2)$$

• (1) is **controllable** iff M is **torsion-free**, i.e., iff we have (R has full row rank and $n = 1$)

$$t(M) \cong \text{ext}^1_D(\widetilde{N}, D) = 0 \Leftrightarrow d(\widetilde{N}) = -1 \Leftrightarrow \widetilde{N} = 0.$$

• The **formal integrability** of (2) gives:

$$B(t)^T\,\dot{\lambda} + \dot{B}(t)^T\,\lambda = 0 \Rightarrow (-B^T\,A^T + \dot{B}^T)\,\lambda = 0$$

$$\Rightarrow (B^T\,(A^2)^T - B^T\,\dot{A}^T - 2\dot{B}^T A^T + \ddot{B}^T)\,\lambda = 0\ldots$$

• System (1) is **controllable** iff:

$$\text{rk}_K\,(B \mid AB - \dot{B} \mid A^2 B + \ldots \mid A^{n-1} B + \ldots \mid \ldots) = n.$$

Maple package OREMODULES

OREMODULES

- OREMODULES is a tool-box developed in *Maple*.

- OREMODULES uses *Mgfun* developed by F. Chyzak

`http://algo.inria.fr/chyzak/mgfun.html.`

- OREMODULES is developed by Chyzak-Q.-Robertz.

- OREMODULES can handle linear systems of ODEs, PDEs, differential time-delay systems, multidimensional discrete systems. . .

- OREMODULES computes:

1. autonomous elements, non-controllable elements,

2. parametrizations of under-determined systems,

3. left-/right-/generalized inverses,

4. flat outputs, π-polynomials,

5. first integrals of motion,

6. Euler-Lagrange equations. . .

- A **second release is available** on the web page:

`http://wwwb.math.rwth-aachen.de/OreModules.`

List of the functions

Main functions

`Parametrization`
`MinimalParametrization(s)`
`AutonomousElements`
`LeftInverse(Rat)`
`LocalLeftInverse`
`RightInverse(Rat)`
`GeneralizedInverse(Rat)`
`PiPolynomial`
`FirstIntegral`
`LQEquations`

Module theory

`TorsionElements`
`Exti(Rat)`
`Extn(Rat)`
`Quotient(Rat)`
`SyzygyModule(Rat)`
`Resolution(Rat)`
`FreeResolution(Rat)`
`OreRank`

Some low-level functions

`DefineOreAlgebra`
`Involution`
`Factorize`
`Mult`
`ApplyMatrix`

- **Library of examples**: More than 30 examples.

Applications

Patching problem

- **Two pendula mounted on a car**:

$$\left\{\begin{array}{l} m_1\, L_1\, \ddot{w}_1(t) + m_2\, L_2\, \ddot{w}_2(t) - w_3(t) + (M + m_1 + m_2)\, \ddot{w}_4(t) = 0, \\ m_1\, L_1^2\, \ddot{w}_1(t) - m_1\, L_1\, g\, w_1(t) + m_1\, L_1\, \ddot{w}_4(t) = 0, \\ m_2\, L_2^2\, \ddot{w}_2(t) - m_2\, L_2\, g\, w_2(t) + m_2\, L_2\, \ddot{w}_4(t) = 0. \end{array}\right. \qquad (\star)$$

- $(\star)$ is **parametrizable** iff $L_1 \neq L_2$.

- A **parametrization** of $(\star)$ is given by:

$$\left\{\begin{array}{ll} w_1(t) = & -L_2\, \xi^{(4)}(t) + g\, \ddot{\xi}(t), \\ w_2(t) = & -L_1\, \xi^{(4)}(t) + g\, \ddot{\xi}(t), \\ w_3(t) = & L_1\, L_2\, M\, \xi^{(6)}(t) \\ & -(L_1\, m_2 + L_2\, m_1 + g\, (L_1 + L_2)\, M)\, \xi^{(4)}(t) \\ & +g^2\, (m_1 + m_2 + M)\, \xi^{(2)(t)} \\ w_4(t) = & L_1\, L_2\, \xi^{(4)}(t) - g\, (L_1 + L_2)\, \ddot{\xi}(t) + g^2\, \xi(t). \end{array}\right.$$

$$\xi(t) = \frac{1}{g^2\, (L_1 - L_2)}\, (L_1^2\, w_1(t) - L_2^2\, w_2(t) + (L_1 - L_2)\, w_4(t)).$$

- **Patching problem** $\Leftrightarrow$ **controllability**: $\quad T > 0$.

$w^p = (w_1^p, w_2^p, w_3^p, w_4^p)$ a **past trajectory** of $(\star)$ on $]-\infty, 0[$.

$w^f = (w_1^f, w_2^f, w_3^f, w_4^f)$ a **future trajectory** of $(\star)$ on $]T, +\infty[$.

$\Rightarrow \exists\, w = (w_1, w_2, w_3, w_4) \in C^\infty(\mathbb{R})^4$ trajectory of $(\star)$:

$$\left\{\begin{array}{l} w_{]-\infty,0[} = w^p, \\ w_{]T,+\infty[} = w^f. \end{array}\right.$$

It is **enough to find** $\xi \in C^\infty(\mathbb{R})$ such that:

$$\xi_{]-\infty,0[} = \xi^p \quad \& \quad \xi_{]T,+\infty[} = \xi^f.$$

Motion planning

- **Flexible rod with a torque**:

$$\begin{cases} \sigma^2 \frac{\partial^2 q(\tau,x)}{\partial \tau^2} - \frac{\partial^2 q(\tau,x)}{\partial x^2} = 0, \\ \frac{\partial q}{\partial x}(\tau,0) = -u(\tau), \\ \frac{\partial q}{\partial x}(\tau,L) = -J\,\frac{\partial^2 q}{\partial \tau^2}(\tau,L), \\ y(\tau) = q(\tau,L). \end{cases} \qquad (\star)$$

- Using **d'Alembert formula**

$$\begin{cases} q(\tau,x) = \phi(\tau + \sigma\, x) + \psi(\tau - \sigma\, x), \\ t = (\sigma/J)\,\tau, \quad v = (2\,J/\sigma^2)\,u, \end{cases}$$

$$(\star) \Rightarrow \ddot{y}(t+1)+\ddot{y}(t-1)+\dot{y}(t+1)-\dot{y}(t-1) = v(t)$$

$$\Leftrightarrow \begin{cases} y(t) = \xi(t-1), \\ v(t) = \ddot{\xi}(t) + \ddot{\xi}(t-2) + \dot{\xi}(t) - \dot{\xi}(t-2). \end{cases}$$

- If y_r is a **desired trajectory**

$$\xi_r(t-1) = y_r(t) \Rightarrow \xi_r(t) = y_r(t+1),$$

thus, we obtain the **open-loop control law**:

$$v_r(t) = \ddot{\xi}_r(t) + \ddot{\xi}_r(t-2) + \dot{\xi}_r(t) - \dot{\xi}_r(t-2)$$

$$= \ddot{y}_r(t+1) + \ddot{y}_r(t-1) + \dot{y}_r(t+1) - \dot{y}_r(t-1).$$

Optimal control

• **Problem**: Let us minimize the **cost function**

$$\frac{1}{2}\int_0^T (x(t)^2 + u(t)^2)\,dt$$

where $\dot{x}(t) + x(t) - u(t) = 0$, $x(0) = x_0$.

• $\dot{x}(t) + x(t) - u(t) = 0$ is **parametrized** by:

$$\begin{cases} \xi(t) = x(t), \\ \dot{\xi}(t) + \xi(t) = u(t). \end{cases} \qquad (\star)$$

• By substitution of $(\star)$ in the cost, we are led to:

Minimize $\frac{1}{2}\int_0^T (\xi(t)^2 + (\dot{\xi}(t) + \xi(t))^2)\,dt$.

Using Euler-Lagrange equations, we then obtain

$$\begin{cases} \xi(t) = x(t), \\ \dot{\xi}(t) + \xi(t) = u(t), \\ \ddot{\xi}(t) - 2\,\xi(t) = 0, \\ \dot{\xi}(T) + \xi(T) = 0, \\ \xi(0) = x_0, \end{cases}$$

which, by integration, gives the **optimal controller**:

$$u(t) = \frac{-e^{\sqrt{2}(t-T)} + e^{-\sqrt{2}(t-T)}}{(1-\sqrt{2})\,e^{\sqrt{2}(t-T)} - (1+\sqrt{2})\,e^{-\sqrt{2}(t-T)}}\,x(t).$$

Variational problem

• We extremize **the electromagnetism Lagragian**

$$\int \left(\frac{1}{2\,\mu_0} \parallel \vec{B} \parallel^2 - \frac{\epsilon_0}{2} \parallel \vec{E} \parallel^2 \right) dx_1\, dx_2\, dx_3\, dt \quad (1)$$

where $\vec{B}$ and $\vec{E}$ satisfy:

$$\begin{cases} \vec{\nabla}.\vec{B} = 0, \\ \vec{\nabla} \wedge \vec{E} + \dfrac{\partial \vec{B}}{\partial t} = 0. \end{cases} \quad (2)$$

• System (2) is **parametrizable**:

$$\begin{cases} \frac{\partial \vec{B}}{\partial t} + \vec{\nabla} \wedge \vec{E} = \vec{0}, \\ \vec{\nabla}\,.\,\vec{B} = 0, \end{cases} \Leftrightarrow \begin{cases} \vec{E} = -\vec{\nabla}\,V - \frac{\partial \vec{A}}{\partial t}, \\ \vec{B} = \vec{\nabla} \wedge \vec{A}. \end{cases} \quad (3)$$

• Substituting (3) in (1), we obtain a **variational problem in $\vec{A}$ and V without differential constraint**.

The variation of this problem gives ($c^2 = 1/(\epsilon_0\,\mu_0)$)

$$\begin{cases} \dfrac{1}{c^2}\dfrac{\partial^2 \vec{A}}{\partial t^2} - \Delta\,\vec{A} = 0, \\ \dfrac{1}{c^2}\dfrac{\partial^2 V}{\partial t^2} - \Delta\,V = 0, \\ \vec{\nabla} \wedge \vec{A} = \vec{B}, \\ -\vec{\nabla} V - \dfrac{\partial \vec{A}}{\partial t} = \vec{E}, \end{cases} \quad \text{(electromagnetic waves)}$$

with the **gauge condition** $\vec{\nabla}.\vec{A} + \frac{1}{c^2}\frac{\partial V}{\partial t} = 0$.

Poles placement

- Let us consider the system:

$$D\left(\frac{d}{dt},\delta\right)\,y(t)=N\left(\frac{d}{dt},\delta\right)\,u(t)\quad (1).$$

- Let us consider the feedback law:

$$A\left(\frac{d}{dt},\delta\right)\,u(t)=B\left(\frac{d}{dt},\delta\right)\,y(t)\quad (2)$$

- If (1) is **parametrizable**, then we have:

$$(1)\Leftrightarrow\left(\begin{array}{c} y(t)\\ u(t)\end{array}\right)=\left(\begin{array}{c}\widetilde{N}\left(\frac{d}{dt},\delta\right)\\ \widetilde{D}\left(\frac{d}{dt},\delta\right)\end{array}\right)\xi(t),\quad \forall\,\xi.$$

- The **closed-loop dynamic** is then given by:

$$(B\,\widetilde{N}-A\,\widetilde{D})\,\xi(t)=0.$$

- **Let us consider a closed-loop dynamic** S.

Then, there exists a feedback law (2) satisfying

$$(B\quad -A)\left(\begin{array}{c}\widetilde{N}\\ \widetilde{D}\end{array}\right)=S=\left(\begin{array}{c} S_1\\ \vdots\\ S_r\end{array}\right)\quad (3)$$

iff $S_i\in k\left[\frac{d}{dt},\delta\right]^{1\times p}\,(\widetilde{N}^T\quad \widetilde{D}^T)^T$, $i=1,\ldots,r$.

(2) can be computed by means of **Gröbner bases**.

- If (1) is a **flat system**, then (3) is always feasible:

$$(B\quad -A)=S\,(\widetilde{Y}\quad \widetilde{X})+Q\,(D\quad -N),\ \forall\,Q,$$

$$\text{where}\quad \xi(t)=\widetilde{Y}\,y(t)+\widetilde{X}\,u(t).$$

Conclusion

• The **algebraic analysis approach** allowed us to:

1. Develop **general concepts** and **results**.

2. Obtain **generic algorithms which constructively check the structural properties** of linear systems.

• An **implementation** of these algorithms have been done in OREMODULES:

`http://wwwb.math.rwth-aachen.de/OreModules.`

• A **large library of examples** is available.

• **Question**: Is it possible to use Ore algebras for the study of hybrid systems?

• An algebraic analysis approach to **synthesis problems** has been developed in:

A. Q., The fractional representation approach to synthesis problems: an algebraic analysis viewpoint.
Part I: (weakly) doubly coprime factorizations,
Part II: Internal stabilization,
SIAM J. Control & Optimization, 42 (2003), 266-299, 300-320.

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