Algebraic analysis for the Ore extension ring of differential time-varying delay operators

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***Abstract*— As far as we know, there is no algebraic (polynomial) approach for the study of linear differential time-delay systems in the case of a (sufficiently regular) time-varying delay. Based on the concept of skew polynomial rings developed by Ore in the 30s, the purpose of this paper is to construct the ring of differential time-delay operators as an Ore extension and to analyze its properties. Classical algebraic properties of this ring, such as noetherianity, its homological and Krull dimensions and the existence of Gröbner bases, are characterized in terms of the time-varying delay function. In conclusion, the algebraic analysis approach to linear systems theory allows us to study linear differential time-varying delay systems (*e.g.* existence of autonomous elements, controllability, parametrizability, flatness, behavioral approach) through methods coming from module theory, homological algebra and constructive algebra.**

I. INTRODUCTION

Differential time-delay systems have been extensively studied in the literature of control theory. Most of these previous works consider constant or distributed time-delays. Motivated by applications, such as in incompressible fluid flows in pipes, material or vehicular flows, metal-rolling processes, communication networks and so on [9], [12], [20], [22], the class of *differential time-varying delay (DTVD) systems* has been investigated mainly from a stability analysis viewpoint. See for instance [1], [14], [11], [21] and the references therein.

At the end of the 90s, following a mathematical theory developed by Malgrange, Bernstein, Sato, Kashiwara and others, the *algebraic analysis approach* to linear systems theory was initiated by Oberst, Fliess and Pommaret (see [6], [15], [16], [23] and the references therein). Within this approach, an intrinsic study of linear systems can be developed based on module theory, homological algebra and functional analysis. In particular, built-in properties of linear systems can be characterized by means of module properties independently of the system representation. Moreover, *Willems' behavioral approach* can also be realized and developed within this framework [15].

Based on the concepts of a *skew polynomial* [13] and an *Ore algebra* [4], computer algebra methods (*e.g. Gröbner* or *Janet bases* [2], [4]) and a constructive approach to module theory and homological algebra [19], an *effective algebraic analysis approach* to linear functional systems was initiated in [2], [5]. General classes of linear functional systems (*e.g.* differential systems, constant time-delay systems, discrete systems) can then be studied by means of common mathematical concepts, theorems and algorithms. This approach yields the development of the OREMODULES and OREMORPHISMS packages [3], [5] allowing the study of control concepts such as controllability, observability, parametrizability, flatness and equivalences thoroughly.

The goal of this paper is to develop an algebraic analysis approach to linear DTVD systems. With this aim in mind, firstly the ring of DTVD operators is shown to be an *Ore extension* D [13]. Important algebraic and homological properties of D are then characterized. Since D is also a *bijective skew PBW extension* [7], using [8], Gröbner basis techniques can be used to effectively test standard module properties and thus system properties. The corresponding algorithms will be implemented in the future.

General definitions of skew polynomial rings and Ore extensions are given in Section II as well as some examples. In Section III, we give the explicit construction of the ring D of DTVD operators as an Ore extension. Algebraic properties of this Ore extension are analyzed in Section IV, especially within a homological framework. In Section V, system properties are characterized in terms of module theory and homological algebra. Finally, in Section VI, we discuss some open questions concerning D.

II. SKEW POLYNOMIAL RINGS AND ORE EXTENSIONS

In this section, we briefly recall general concepts such as the definitions of *skew polynomial ring*, of an *Ore extension* and of an *Ore algebra*. We also show that the ring of differential time-varying delay operators is not an Ore algebra.

Definition 1 ([13]): Let A be a ring. An *Ore extension* $A[\partial;\sigma,\delta]$ of A is the noncommutative ring formed by elements of the form $\sum_{i=0}^{n} a_i\,\partial^i$ with $a_i \in A$, obeying the following commutation rule

$$\forall\, a \in A, \quad \partial\, a = \sigma(a)\,\partial + \delta(a), \tag{1}$$

where σ is a *ring endomorphism* of A (*i.e.* $\sigma : A \longrightarrow A$ satisfies $\sigma(1) = 1$, $\sigma(a+b) = \sigma(a) + \sigma(b)$ and $\sigma(a\,b) = \sigma(a)\,\sigma(b)$, for all $a, b \in A$) and δ is a *σ-derivation* of A, *i.e.* $\delta : A \longrightarrow A$ and it satisfies for all $a, b \in A$:

$$\delta(a+b) = \delta(a) + \delta(b), \tag{2}$$
$$\delta(a\,b) = \sigma(a)\,\delta(b) + \delta(a)\,b. \tag{3}$$

The Ore extension $A[\partial;\sigma,\delta]$ of A is also called a *skew polynomial ring* over A.

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Example 1: Let A be a ring and δ a *derivation* of A (*i.e.* $\delta : A \longrightarrow A$ satisfies (2) and (3) with $\sigma = \mathrm{id}_A$. Note that in this case, (2) is the standard Leibniz's rule). The ring $(A,\, \delta)$ is called a *a differential ring*. For instance, in the case $A = k[t]$ (resp. $A = k(t)$) denotes the commutative ring of polynomials (resp. rational functions) in t with coefficients in a field k (*e.g.* $k = \mathbb{Q}$, $\mathbb{R}$, or $\mathbb{C}$), or if A is the ring of analytic (or meromorphic) functions in t, then $\frac{d}{dt}$ is a derivation of A and $\left(A,\, \frac{d}{dt}\right)$ is a differential ring. In what follows, we will denote a derivation by $\frac{d}{dt}$. The Ore extension $O = A\left[\partial; \mathrm{id}_A, \frac{d}{dt}\right]$ of a ring A is then the ring of differential operators in ∂ with coefficients in A. An element of O is of the form $\sum_{i=0}^{n} a_i\, \partial^i$, where the product as to be understood as the composition of the following operators acting on A:

$$a_i : \begin{array}{ccc} A & \longrightarrow & A \\ a & \longmapsto & a_i\, a, \end{array} \qquad \partial : \begin{array}{ccc} A & \longrightarrow & A \\ a & \longmapsto & \frac{d}{dt}\, a. \end{array}$$

We may check that (1) is satisfied for $\sigma = \mathrm{id}_A$:

$$\begin{aligned} (\partial\, a_i)(a) &= \partial(a_i\, a) = \tfrac{d}{dt}\,(a_i\, a) = a_i\,\left(\tfrac{d}{dt}\, a\right) + \left(\tfrac{d}{dt}\, a_i\right)\, a \\ &= \left(a_i\, \partial + \tfrac{d}{dt}\, a_i\right)(a), \quad \forall\, a \in A. \end{aligned}$$

Example 2: Let $h \in \mathbb{R}_{>0} := \{t \in \mathbb{R} \mid t > 0\}$ and A be a ring of real-valued functions of t equipped with the endomorphism σ defined by $\sigma(a(t)) = a(t-h)$ for all $a \in A$. The Ore extension $O = A[S; \sigma, 0]$ of A is the *ring of time-delay (TD) operators with coefficients in A*. An element of O is of the form $\sum_{i=0}^{n} a_i\, S^i$, where the product as to be understood as the composition of the two operators:

$$a_i : \begin{array}{ccc} A & \longrightarrow & A \\ a & \longmapsto & a_i\, a, \end{array} \qquad S : \begin{array}{ccc} A & \longrightarrow & A \\ a(t) & \longmapsto & a(t-h). \end{array}$$

Similarly as above, if A is a ring of real-valued sequences on $\mathbb{Z}$, *i.e.* $A = \mathbb{R}^{\mathbb{Z}}$, then an element $a \in A$ can be written as $a = (a_i)_{i \in \mathbb{Z}}$. Let σ be the endomorphism defined by forward (resp. backward) shift $\sigma\,(a_i) = (a_{i+1})$ for (resp. $\sigma\,(a_i) = (a_{i-1})$) all $i \in \mathbb{Z}$. Then, $A\,[S; \sigma, 0]$ is the *skew polynomial ring of forward (resp. backward) shift operators*.

If our goal is to study linear systems of differential time-delay (DTD) equations, of differential difference equations, of partial differential equations and so on, *i.e.* rings of multivariate functional operators, then the Ore extension construction can be iterated. Hence, if $O = A[\partial; \sigma, \delta]$ is an Ore extension of A and σ_2 (resp. δ_2) is an endomorphism (resp. a σ_2-derivation) of O, then the Ore extension $B = O\,[\partial_2; \sigma_2, \delta_2] = A\,[\partial; \sigma, \delta]\,[\partial_2; \sigma_2, \delta_2]$ of O can be defined. Note that we then have:

$$\forall\, o \in O, \quad \partial_2\, o = \sigma_2(o)\, \partial_2 + \delta_2(o).$$

For $\partial \in O$, we have $\partial_2\, \partial = \sigma_2(\partial)\, \partial_2 + \delta_2(\partial)$. That implies that $\partial_2\, \partial$ is usually not equal to $\partial\, \partial_2$, *i.e.* ∂ and ∂_2 do not usually commute. In many standard applications though, functional operators commute with each other. This remark led to the introduction of the concept of an *Ore algebra*.

Definition 2 ([4]): If A is a k-algebra, then an Ore extension $A[\partial_1; \sigma_1, \delta_1] \cdots [\partial_n; \sigma_n, \delta_n]$ of A is called an *Ore algebra* if $\sigma_j(A) \subseteq A$ and $\delta_j(A) \subseteq A$ for $j = 1, \ldots, n$, and:

$$1 \le i < j \le n, \quad \sigma_j(\partial_i) = \partial_i, \quad \delta_j(\partial_i) = 0,$$

$$1 \le i, j \le n, \quad i \ne j, \quad \left\{ \begin{array}{l} (\sigma_j \circ \sigma_i)_{|A} = (\sigma_i \circ \sigma_j)_{|A}, \\ (\delta_j \circ \sigma_i)_{|A} = (\sigma_i \circ \delta_j)_{|A}, \\ (\delta_j \circ \delta_i)_{|A} = (\delta_i \circ \delta_j)_{|A}. \end{array} \right.$$

Example 3: Note that $\beta_1 = \frac{\partial}{\partial x_1}$ is a derivation of the polynomial ring $A = \mathbb{R}[x_1, x_2]$. Hence, we can define the Ore extension $O = A\,[\partial_1; \mathrm{id}_A, \beta_1]$ of A formed by the differential operators in ∂_1 with coefficients in A. Consider the following derivation of O:

$$\beta_2 = \frac{\partial}{\partial x_2} : \begin{array}{ccc} O & \longrightarrow & O \\ \sum_{i=0}^{r} a_i(x_1, x_2)\, \partial_1^i & \longmapsto & \sum_{i=0}^{r} \dfrac{\partial a_i(x_1, x_2)}{\partial x_1}\, \partial_1^i. \end{array}$$

Thus, the Ore extension $B = O\,[\partial_2; \mathrm{id}_O, \beta_2]$ of O can be formed. The conditions of an Ore algebra are clearly fulfilled ($\beta_2 \circ \beta_1 = \beta_1 \circ \beta_2$ is the Schwarz's theorem), which shows that B is an Ore algebra and $\partial_2\, \partial_1 = \partial_1\, \partial_2$, *i.e.* the operators ∂_1 and ∂_2 commute.

Example 4: Let $O = A\,[\partial; \mathrm{id}_A, \delta]$ be a ring of differential operators with coefficients in a differential $\mathbb{R}$-algebra A of functions of t with $\delta = \frac{d}{dt}$. Let h be a non-negative real and σ be the following map:

$$\sigma : \begin{array}{ccc} O & \longrightarrow & O \\ \sum_{i=0}^{r} a_i(t)\, \partial^i & \longmapsto & \sum_{i=0}^{r} a_i(t-h)\, \partial^i. \end{array}$$

Then, σ is an endomorphism of O since h is a constant. Thus, we can consider the Ore extension $B = O[S; \sigma, 0]$ of O. For $a \in A$, we have

$$(\delta \circ \sigma)\,(a(t)) = \frac{d}{dt}\, a(t-h) = \dot{a}(t-h) = (\sigma \circ \delta)\,(a(t))\,,$$

which shows that $(\delta \circ \sigma)_{|A} = (\sigma \circ \delta)_{|A}$. The remaining conditions for an Ore algebra can easily be checked and that shows that B is an Ore algebra. In particular, we have:

$$S\,(\partial) = \sigma(\partial)\, S = \partial\, S.$$

We now show that the ring of DTVD operators is not an Ore algebra. Let h be a smooth or an analytic function which satisfies:

$$\forall\, t \in \mathbb{R}_{\ge 0} := \{t \in \mathbb{R} \mid t \ge 0\}, \quad h(t) \ge 0. \qquad (4)$$

Let $\left(A, \delta = \frac{d}{dt}\right)$ be a differential ring of real-valued functions of t equipped with an endomorphism σ defined by $\sigma\,(a(t)) = a\,(t - h(t))$ for all $a \in A$. Then, we obtain

$$\begin{aligned} (\delta \circ \sigma)(a(t)) &= \delta(a(t - h(t))) = (1 - \dot{h}(t))\, \dot{a}(t - h(t)) \\ &= (1 - \dot{h}(t))\, (\sigma \circ \delta)(a(t)), \end{aligned}$$

which shows that $(\delta \circ \sigma)_{|A} = (1 - \dot{h})\, (\sigma \circ \delta)_{|A}$. Hence, we cannot construct an Ore algebra of DTVD operators.

III. An Ore extension construction

Let us develop an explicit Ore extension construction for the ring of DTVD operators. Let $\left(A, \delta = \frac{d}{dt}\right)$ be a differential ring of real-valued functions of t equipped with an endomorphism σ defined by $\sigma\left(a(t)\right) = a\left(t - h(t)\right)$ for all $a \in A$. We assume that $t \in A$ so that $h(t) = t - \sigma(t) \in A$. Let $O = A\left[\partial; \mathrm{id}_A, \frac{d}{dt}\right]$ be the skew polynomial ring of differential operators with coefficients in A.

We first note that the following map σ

$$\begin{array}{rcl} \sigma : O & \longrightarrow & O \\ \sum_{i=0}^r a_i(t)\,\partial^i & \longmapsto & \sum_{i=0}^r a_i(t-h(t))\,\partial^i, \end{array}$$

cannot be an endomorphism of O. Indeed, if so, then using the identity $\partial\, a(t) = a(t)\,\partial + \dot{a}(t)$ of O, we would get

$$\begin{aligned} \sigma(\partial\, a(t)) &= \sigma(\partial)\,\sigma(a(t)) = \partial\, a(t-h(t)) \\ &= a(t-h(t))\,\partial + (1-\dot{h}(t))\,a(t-h(t)), \\ \sigma(a(t)\,\partial + \dot{a}(t)) &= \sigma(a(t))\,\sigma(\partial) + \sigma(\dot{a}(t)) \\ &= a(t-h(t))\,\partial + \dot{a}(t-h(t)), \end{aligned} \tag{5}$$

which yields $\sigma(\partial\, a(t)) \neq \sigma(a(t)\,\partial + \dot{a}(t))$ apart from the case $\dot{h}(t) = 0$, *i.e.* $h(t) = h \in \mathbb{R}_{\geq 0}$ is a constant function.

In what follows, we suppose that h satisfies the conditions

$$\forall\, t \in \mathbb{R}_{\geq 0}, \quad \dot{h}(t) \neq 1,$$

and $1/(1-\dot{h}) \in A$. Then, we consider the following map:

$$\begin{array}{rcl} \sigma : O & \longrightarrow & O \\ \sum_{i=0}^r a_i(t)\,\partial^i & \longmapsto & \sum_{i=0}^r a_i(t-h(t))\left(\dfrac{1}{1-\dot{h}}\,\partial\right)^i. \end{array} \tag{6}$$

We claim that σ is an endomorphism of O.

Remark 1: If we note $x(t) := t - h(t)$, then we have:

$$\frac{d}{dt} = \frac{dx}{dt}\,\frac{d}{dx} = \left(1 - \dot{h}(t)\right)\frac{d}{dx} \;\Rightarrow\; \frac{d}{dx} = \frac{1}{1-\dot{h}(t)}\,\frac{d}{dt}.$$

Therefore, if we set $\partial_t := \partial = \dfrac{d}{dt}$ and $\partial_x := \dfrac{d}{dx}$, then the map σ can be understood as:

$$\sigma(a(t)) = a(x), \quad \sigma(\partial_t) = \partial_x.$$

Hence, σ corresponds to the *change of time-scale*, *i.e.* we pass from time t to time $x(t) := t - h(t)$.

Using the notation of Remark 1, we have:

$$\sigma\left(\sum_{i=0}^r a_i(t)\,\partial_t^i\right) = \sum_{i=0}^r a_i(x)\,\partial_x^i. \tag{7}$$

We clearly have $\sigma(1) = 1$. Moreover, σ is an additive map. Let us prove that σ is multiplicative. If we consider

$$d_1 := \sum_{i=0}^r a_i(t)\,\partial_t^i, \quad d_2 := \sum_{j=0}^s b_j(t)\,\partial^j \in O,$$

then we have $d_1\, d_2 = \sum_{k=0}^{r+s} c_k(t)\,\partial_t^k$, for some $c_k \in A$. Using (7), we have:

$$\sigma(d_1) = \sum_{i=0}^r a_i(x)\,\partial_x^i, \quad \sigma(d_2) = \sum_{j=0}^s b_j(x)\,\partial_x^j,$$

$$\sigma(d_1\, d_2) = \sum_{k=0}^{r+s} c_k(x)\,\partial_x^k.$$

Since $O = A\left[\partial_x; \mathrm{id}_A, \frac{d}{dx}\right]$ is a ring of differential operators in ∂_x with function coefficients in x, we also have:

$$\sum_{k=0}^{r+s} c_k(x)\,\partial_x^k = \left(\sum_{i=0}^r a_i(x)\,\partial_x^i\right)\left(\sum_{j=0}^s b_j(x)\,\partial_x^j\right).$$

That shows that $\sigma(d_1\, d_2) = \sigma(d_1)\,\sigma(d_2)$ and proves that σ is an endomorphism of O.

Since σ is an endomorphism of O, we can then introduce the Ore extension of O defined by:

$$D = O\,[S; \sigma, 0] = A\left[\partial; \mathrm{id}_A, \frac{d}{dt}\right][S; \sigma, 0].$$

In the rest of the text, this ring will be called a *ring of differential time-varying delay (DTVD) operators over A*. An element $d \in D$ is then of the form:

$$d = \sum_{0 \leq i+j \leq r} a_{ij}(t)\,\partial^i\, S^j, \quad a_{ij} \in A.$$

From the definition of an Ore extension, we have:

$$\forall\, o \in O, \quad S\, o = \sigma(o)\, S. \tag{8}$$

Considering $o := \partial$, we obtain:

$$S\,\partial = \frac{1}{1-\dot{h}}\,\partial\, S, \quad \textit{i.e.} \quad \partial\, S = (1-\dot{h})\, S\,\partial. \tag{9}$$

Note that (9) corresponds to the following identity:

$$\begin{aligned} (\partial\, S)(y(t)) &= \frac{d}{dt}\,y(t-h(t)) = (1-\dot{h}(t))\,\dot{y}(t-h(t)) \\ &= (1-\dot{h}(t))\,(S\,\partial)(y(t)). \end{aligned}$$

Remark 2: The ring D of DTVD operators can also be obtained by extending the ring $A[S; \sigma, 0]$ of time-varying delay operators, where σ is defined by $\sigma\left(a(t)\right) = a\left(t - h(t)\right)$. Indeed, if we define the following maps

$$\begin{array}{rcl} \sigma_2 : A[S;\sigma,0] & \longrightarrow & A[S;\sigma,0] \\ \sum_{i=0}^r a_i(t)\, S^i & \longmapsto & \sum_{i=0}^r a_i(t)\,((1-\dot{h}(t))\, S)^i, \end{array}$$

$$\begin{array}{rcl} \delta_2 : A[S;\sigma,0] & \longrightarrow & A[S;\sigma,0] \\ \sum_{i=0}^r a_i(t)\, S^i & \longmapsto & \sum_{i=0}^r \dot{a}_i(t)\, S^i, \end{array}$$

then we can check that σ_2 is an endomorphism of $A[S;\sigma,0]$ and β_2 a σ_2-derivation of $A[S;\sigma,0]$. Therefore, we can define the Ore extension $D' = A[S;\sigma,0][\partial;\sigma_2,\delta_2]$. Then, we have

$$\left\{\begin{array}{l} \partial\, a(t) = \sigma_2(a(t))\,\partial + \delta_2(a(t)) = a(t)\,\partial + \dot{a}(t), \\ \partial\, S = \sigma_2(S)\,\partial + \delta_2(S) = (1-\dot{h}(t))\, S\,\partial, \end{array}\right.$$

and we can check that $D' = D$.

IV. Algebraic properties of the Ore extension D

In this section, we study the properties of the ring D of DTVD operators defined in Section III.

We refer to [2], [13], [19] for basic algebraic definitions. Let us state again standard results on Ore extensions.

Theorem 1 ([13]): Let A be a noncommutative ring and $D = A[\partial; \sigma, \delta]$ a skew polynomial ring. Then, we have:

1) If A is a *domain* (*i.e.* A has no non-zero zero divisors) and σ is injective endomorphism of A, then D is a domain.
2) If A is a *left Ore domain* (*i.e.* A is a domain satisfying $A\,a_1 \cap A\,a_2 \neq 0$ for all $a_1,\, a_2 \in A \setminus \{0\}$) and σ is injective, then D is a left Ore domain.
3) If A is a left (right) noetherian ring (*i.e.* every left (right) ideal of A is finitely generated) and σ is an automorphism of A, then D is a left (right) noetherian ring. Moreover, if A is a domain, then D is a left (right) Ore domain.

We obtain the following important corollary of Theorem 1.

Corollary 1: Let $D = A\left[\partial; \mathrm{id}_A, \frac{d}{dt}\right][S; \sigma, 0]$ be the ring of DTVD operators defined in Section III. Then, we have:

1) If A is a domain, then so is D.
2) If A is a left Ore domain, then so is D.
3) If A is a left (right) noetherian ring and if $\sigma_{|A}$ is an automorphism of A, then D is a left (right) noetherian ring. Moreover, $\sigma_{|A}$ is an automorphism of A if and only if the function l defined by

$$\begin{array}{rcl} l : \mathbb{R}_{\geq 0} & \longrightarrow & \mathbb{R} \\ t & \longmapsto & t - h(t), \end{array} \tag{10}$$

admits an inverse $l^{-1} \in A$ which satisfies $a \circ l^{-1} \in A$ for all $a \in A$ (*e.g.* $h(t) := q\,t + h$, where $h \in \mathbb{R}_{\geq 0}$ and $0 < q < 1$). Finally, if A is a domain, then D is a left (right) Ore domain.

Proof: 1 and 2 are direct consequences of 1 and 2 of Theorem 1 and of the fact that id_A and σ, defined in Section III, are both injective endomorphisms of A.

3. Let us first prove that $\sigma_{|A}$ is an automorphism of A if and only if the function l defined by (10) admits an inverse in A. If σ is an automorphism of A, then for every $a \in A$, there exists a unique $b \in A$ such that $a(t) = \sigma(b(t)) = b(l(t))$. Taking $a = t \in A$, we get $t = (b \circ l)(t)$, i.e., $b \circ l = \mathrm{id}_{\mathbb{R}_{\geq 0}}$, which proves that l is injective and thus bijective on its image, and $b = l^{-1} \in A$. Moreover, for every $c \in A$, $\sigma^{-1}(c) \in A$, where $\sigma^{-1}(c(t)) = c(\sigma^{-1}(t)) = c(l^{-1}(t))$, which shows that $c \circ l^{-1} \in A$. Conversely, if (10) admits an inverse $l^{-1} \in A$ and $a \circ l^{-1} \in A$ for all $a \in A$, then we can define the endomorphism σ' of A by

$$\begin{array}{rcl} \sigma' : A & \longrightarrow & A \\ a(t) & \longmapsto & a(l^{-1}(t)), \end{array}$$

and we can easily check that we have $\sigma' \circ \sigma = \mathrm{id}_A$ and $\sigma \circ \sigma' = \mathrm{id}_A$, i.e., σ is an automorphism of A and $\sigma' = \sigma^{-1}$.

Let us now show that the automorphism σ of A extends to an automorphism of the ring $O = A\left[\partial; \mathrm{id}_A, \frac{d}{dt}\right]$ of differential operators by considering the action on O

$$\begin{array}{rcl} \sigma' : O & \longrightarrow & O \\ \displaystyle\sum_{i=0}^{r} a_i(t)\,\partial_t^i & \longmapsto & \displaystyle\sum_{i=0}^{r} a(l^{-1}(t)) \left(\frac{1}{\dot{(l^{-1})}(t)}\,\partial_t\right)^i = \sum_{i=0}^{r} a(y)\,\partial_y^i, \end{array}$$

with the notations $y := l^{-1}(t)$ and $\partial_y := \frac{1}{\dot{(l^{-1})}(t)}\,\partial_t$. Similarly as for (6), we can prove that σ' is an endomorphism of A. Using $l^{-1} \circ l = \mathrm{id}$ and $l \circ l^{-1} = \mathrm{id}$, we get:

$$\begin{array}{lcl} l(l^{-1}(t)) = t & \Rightarrow & \dot{l}(l^{-1}(t))\,\dot{(l^{-1})}(t) = 1, \\ l^{-1}(l(t)) = t & \Rightarrow & \dot{l^{-1}}(l(t))\,\dot{l}(t) = 1. \end{array}$$

Since $\dot{l} = 1 - \dot{h} \in A$, we obtain $1/\dot{(l^{-1})} = \dot{l}(l^{-1}) \in A$. Moreover, we have

$$\begin{array}{rl} & (\sigma' \circ \sigma)\left(\sum_{i=0}^{r} a_i(t)\,\partial_t^i\right) \\ = & \sigma'\left(\sum_{i=0}^{r} a_i(l(t))\left(\frac{1}{\dot{l}(t)}\,\partial_t\right)^i\right) \\ = & \sum_{i=0}^{r} a_i(t)\left(\frac{1}{\dot{l}(l^{-1}(t))\,\dot{(l^{-1})}(t)}\,\partial_t\right)^i \\ = & \sum_{i=0}^{r} a_i(t)\,\partial_t^i, \end{array}$$

$$\begin{array}{rl} & (\sigma \circ \sigma')\left(\sum_{i=0}^{r} a_i(t)\,\partial_t^i\right) \\ = & \sigma\left(\sum_{i=0}^{r} a_i(l^{-1}(t))\left(\frac{1}{\dot{(l^{-1})}(t)}\,\partial_t\right)^i\right) \\ = & \sum_{i=0}^{r} a_i(t)\left(\frac{1}{\dot{(l^{-1})}(l(t))\,\dot{l}(t)}\,\partial_t\right)^i \\ = & \sum_{i=0}^{r} a_i(t)\,\partial_t^i, \end{array}$$

and shows that σ is an automorphism of O and $\sigma' = \sigma^{-1}$. Finally, the results are consequences of 3 of Theorem 1. ■

Corollary 1 can be found again using the second construction of the ring of DTVD operators given in Remark 2.

Example 5: Let us consider $h_1(t) := 1/(1 + t^2)$ and $h_2(t) = 1 - h_1(t)$. We have $h_1(t),\, h_2(t) \in [0,\, 1[$ for all $t \in \mathbb{R}_{\geq 0}$. We can check that $1 - \dot{h}_1(t) = 1 + \frac{2\,t}{(1+t^2)^2} = 0$ and $1 - \dot{h}_2(t) = 1 - \frac{2\,t}{(1+t^2)^2} = 0$ have no real (positive) solutions. The functions $l_i : t \in \mathbb{R}_{\geq 0} \longmapsto t - h_i(t) \in [0,\, 1[$ are bijective for $i = 1,\, 2$. For $x \geq 0$, the equation $l_1(t) = t - h_1(t) = x$ admits a unique real positive solution defined by:

$$\begin{array}{rcl} l_1^{-1}(x) & := & \dfrac{x}{3} + \dfrac{\alpha_1}{6} - \dfrac{1}{3\,\alpha_1}\,(3 - x^2), \\ \alpha_1 & := & \big(8\,x^3 + 72\,x + 108 \\ & & +12\,\sqrt{3\,(4\,x^4 + 4\,x^3 + 8\,x^2 + 36\,x + 31)}\big)^{1/3}. \end{array}$$

Similarly, for $x \geq 0$, $l_2(t) = t - h_2(t) = x$ admits a unique real solution defined by:

$$\begin{array}{rcl} l_2^{-1}(x) & = & \dfrac{1}{3}\,(x + 1) + \dfrac{1}{6}\,\alpha_2 + \dfrac{(x^2 + 2\,x - 2)}{3\,\alpha_2}, \\ \alpha_2 & := & \big(8\,x^3 + 24\,x^2 + 96\,x + 188 \\ & & +\sqrt{3\,(4\,x^4 + 20\,x^3 + 44\,x^2 + 80\,x + 83)}\big)^{1/3}. \end{array}$$

For $i = 1, 2$, if A_i is a differential field such that $a \circ l_i \in A$ and $a \circ l_i^{-1} \in A$ for all $a \in A$, then using Corollary 1, $D_i = A_i \left[\partial; \mathrm{id}_{A_i}, \frac{d}{dt}\right] [S_i; \sigma_{h_i}, 0]$ are noetherian domains.

We have the following important result for the effective study of linear systems over the ring D of DTVD operators.

Theorem 2: Every left ideal of the Ore extension $D = A\left[\partial; \mathrm{id}_A, \frac{d}{dt}\right] [S; \sigma, 0]$, where A is a field and σ an automorphism, admits a *Gröbner basis* for an *admissible monomial order* [10] which can be computed by means of *Buchberger's algorithm* [10]. More generally, every left D-submodule of $D^{1\times p}$ admits a Gröbner basis for $p \in \mathbb{Z}_{\geq 0}$.

Proof: If A is a field, then the Ore extension $D = A\left[\partial; \mathrm{id}_A, \frac{d}{dt}\right] [S; \sigma, 0]$ of DTVD operators is a *bijective skew PBW extension* [7]. In [8], it is proved that Gröbner techniques [10] hold over a bijective skew PBW extension. ■

Within the algebraic analysis approach to linear systems theory, if D is a ring (of functional operators), $R \in D^{q\times p}$ a $q \times p$ matrix with entries in D and $\mathcal{F}$ a left D-module, then a *linear functional system* or a *behavior* is defined by:

$$\ker_{\mathcal{F}}(R.) := \{\eta \in \mathcal{F}^p := \mathcal{F}^{p\times 1} \mid R\,\eta = 0\}.$$

Now, if we consider the left D-submodule $D^{1\times q}\,R := \{\mu\, R \mid \mu \in D^{1\times q}\}$ of $D^{1\times r}$ defined by all the left D-linear combinations of the rows of R and the factor left D-module $M := D^{1\times p}/(D^{1\times q}\,R)$, then a standard homological algebra result (also called *Malgrange's remark*) states that we have

$$\ker_{\mathcal{F}}(R.) \cong \hom_D(M, \mathcal{F}), \tag{11}$$

where $\hom_D(M, \mathcal{F})$ denotes the abelian group (*i.e.* the $\mathbb{Z}$-module) of D-homomorphisms (*i.e.* left D-linear maps) from M to $\mathcal{F}$ [19]. For more details, see [2], [15], [17], [23]. A consequence of (11) is that a linear system/behavior can be studied by means of M (which encodes the system equations) and by $\mathcal{F}$ (which is the functional space where the solutions are sought, also called the *signal space* in the behavioral approach [15]). In particular, the module properties of M characterize built-in properties of the linear system $\ker_{\mathcal{F}}(R.)$ such as controllability, observability, flatness, *etc.* Let us state again basic module properties [19].

Definition 3: Let D be a domain and M a *finitely generated* left D-module, *i.e.* $M = \sum_{i=1}^{r} D\, m_i$, where $m_i \in M$.

1) M is *free* if there exists $r \in \mathbb{Z}_{\geq 0}$ such that $M \cong D^{1\times r}$, where $\cong$ stands for an *isomorphism*, *i.e.* a bijective homomorphism of left D-modules.
2) M is *stably free* if there exist $r, s \in \mathbb{Z}_{\geq 0}$ such that $M \oplus D^{1\times s} \cong D^{1\times r}$.
3) M is *projective* if there exist $r \in \mathbb{Z}_{\geq 0}$ and a left D-module N such that $M \oplus N \cong D^{1\times r}$.
4) M is *reflexive* if the canonical evaluation homomorphism $\varepsilon : M \longrightarrow M^{\star\star} := \hom_D(\hom_D(M, D), D)$, defined by $\varepsilon(m)(f) := f(m)$ for all $m \in M$ and for all $f \in M^\star := \hom_D(M, D)$, is an isomorphism.
5) The *torsion submodule* $t(M)$ of M is defined by:
$$t(M) := \{m \in M \mid \exists\, d \in D \setminus \{0\} : \; d\,m = 0\}.$$
6) M is *torsion-free* if $t(M) = 0$.
7) M is *torsion* if $t(M) = M$.

A free module is clearly stably free (take $s = 0$) and a stably free module is projective (take $N := D^{1\times s}$). Since a projective module is a direct summand of a finitely generated free module, we can easily check that $M \cong M^{\star\star}$, *i.e.* M is a reflexive module. Note that we have:

$$\ker\varepsilon = \{m \in M \mid \forall\, f \in M^\star : \; f(m) = 0\}.$$

Now, if $m \in t(M)$, then there exists $d \in D \setminus \{0\}$ such that $d\,m = 0$, which yields $d\,f(m) = f(d\,m) = f(0) = 0$ for all $f \in M^\star$, and thus $f(m) = 0$ since D is a domain, which shows that $t(M) \subseteq \ker\varepsilon$. Hence, if M is a reflexive, then M is torsion-free. The study of the converse of these results is an important issue in control theory since it is related, for instance, to recognizing when a controllable system is *parametrizable* or *flat* [2], [6], [17], [18]. Note that if D is a *principal ideal domain* (*i.e.* every left (resp. right) ideal of D are of the form $D\,d$ (resp. $d\,D$) for a certain $d \in D$) [19], as, *e.g.* $D := k[\partial; \mathrm{id}, 0]$ where k is a field, then a torsion-free module is free.

We now characterize *homological invariants* [13], [19] of the ring of DTVD operators which play important roles in the algebraic analysis approach [2], [17]. To do that, we first review basic definitions of homological algebra [19].

Definition 4: 1) A sequence of left (right) D-modules M_i and of $\delta_i \in \hom_D(M_i, M_{i-1})$ for $i \in \mathbb{Z}$ is a *complex* if $\delta_i \circ \delta_{i+1} = 0$, *i.e.* if $\operatorname{im}\delta_{i+1} \subseteq \ker\delta_i$ for all $i \in \mathbb{Z}$. This complex is denoted by:
$$M_\bullet : \; \ldots \xrightarrow{\delta_{i+2}} M_{i+1} \xrightarrow{\delta_{i+1}} M_i \xrightarrow{\delta_i} M_{i-1} \xrightarrow{\delta_{i-1}} \ldots$$
2) The *defect of exactness* of the complex $M_\bullet$ at M_i is the left (right) D-module defined by:
$$H_i(M_\bullet) := \ker\,\delta_i / \operatorname{im}\delta_{i+1}.$$
3) The complex $M_\bullet$ is said to be *exact at* M_i if we have $H_i(M_\bullet) = 0$, *i.e.* if $\ker\delta_i = \operatorname{im}\delta_{i+1}$, and is *exact* if $H_i(M_\bullet) = 0$ for all $i \in \mathbb{Z}$.
4) A *projective* (resp. *free*) *resolution* of a left D-module M is an exact sequence of the form
$$P_\bullet : \; \ldots \xrightarrow{\delta_3} P_2 \xrightarrow{\delta_2} P_1 \xrightarrow{\delta_1} P_0 \xrightarrow{\delta_1} M \longrightarrow 0,$$
where the P_i's are projective (resp. free) left D-modules. If $P_i = 0$ for $i \geq m + 1$, then the *length* of the resolution $P_\bullet$ is set to be m. Similar definitions hold for right D-modules.
5) The *left projective dimension* of a left D-module M, denoted by $\operatorname{lpd}_D(M)$, is the length of the shortest projective resolution of M (and similarly for the *right projective dimension* of a right D-module).
6) The *left* (resp. *right*) *global dimension* of D, denoted by $\operatorname{lgd} D$ (resp. $\operatorname{rgd} D$), is the supremum of $\operatorname{lpd}_D(M)$ (resp. $\operatorname{rpd}_D(M)$) over all the left (resp. right) D-modules M.

Theorem 3 (Auslander's theorem, Corollary 8.28 of [19]): If A is a *noetherian ring* (*i.e.* both left and right noetherian ring), then $\mathrm{lgld}\, A = \mathrm{rgld}\, A$. Then, we simply note $\mathrm{gld}\, A$.

Definition 5: A ring A for which every finitely generated left (resp. right) A-module has finite projective dimension is called a *left* (resp. *right*) *regular ring*.

Theorem 4 ([13]): Let A be a ring with finite left (resp. right) global dimension $\mathrm{lgld}\, A$ (resp. $\mathrm{rgld}\, A$) and σ an automorphism of A. Then, we have:

1) $\mathrm{lgld}\, A \leq \mathrm{lgld}\, A[\partial; \sigma, \delta] \leq \mathrm{lgld}\, A + 1$ (resp. $\mathrm{rgld}\, A \leq \mathrm{rgld}\, A[\partial; \sigma, \delta] \leq \mathrm{rgld}\, A + 1$).
2) If $\delta = 0$, then we have $\mathrm{lgld}\, A[\partial; \sigma, \delta] = \mathrm{lgld}\, A + 1$ (resp. $\mathrm{rgld}\, A[\partial; \sigma, \delta] = \mathrm{rgld}\, A + 1$).
3) If A is a semisimple Artinian ring (*e.g.* A is a field), then $\mathrm{lgld}\, A[\partial; \sigma, \delta] = 1$ (resp. $\mathrm{rgld}\, A[\partial; \sigma, \delta] = 1$).

Corollary 2: Let $D = A\left[\partial; \mathrm{id}_A, \frac{d}{dt}\right][S; \sigma, 0]$ be the ring of DTVD operators defined in Section III, where the function $l : t \longmapsto t - h(t)$ is bijective (*e.g.* $h(t) := q\,t + h$ where $h \in \mathbb{R}_{\geq 0}$ and $0 < q < 1$). If A has finite left (resp. right) global dimension $\mathrm{lgld}\, A$ (resp. $\mathrm{rgld}\, A$), then we have:

$$\mathrm{lgld}\, A + 1 \leq \mathrm{lgld}\, D \leq \mathrm{lgld}\, A + 2$$

(resp. $\mathrm{rgld}\, A + 1 \leq \mathrm{rgld}\, D \leq \mathrm{rgld}\, A + 2$). In particular, D is a left (resp. right) regular ring. If A a semisimple Artinian ring (*e.g.* A is a field), then we have $\mathrm{lgld}\, D = 2$ (resp. $\mathrm{rgld}\, D = 2$). Finally, if A is a noetherian ring, then we have $\mathrm{gld}\, D = \mathrm{lgld}\, D = \mathrm{lrgld}\, D$.

Example 6: Considering again Example 5, if $D = D_1$ or $D = D_2$, then we have $\mathrm{gld}\, D = 2$.

Definition 6: A ring A is said to be *projective stably free* if every finitely generated projective left/right A-module is stably free.

Theorem 5 (Serre's theorem): If A is a left regular noetherian ring and projective stably free (*e.g.* A is a field), then so is the ring $D = A\left[\partial; \mathrm{id}_A, \frac{d}{dt}\right][S; \sigma, 0]$.

Proof: It is a consequence of Corollary 12.3.3 of [13]. ∎

Corollary 3: Let $D = A\left[\partial; \mathrm{id}_A, \frac{d}{dt}\right][S; \sigma, 0]$ be the ring of DTVD operators defined in Section III, where the function $l : t \longmapsto t - h(t)$ is bijective (*e.g.* $h(t) := q\,t + h$, where $h \in \mathbb{R}_{\geq 0}$ and $0 < q < 1$), $R \in D^{q \times p}$, and $M = D^{1 \times p}/(D^{1 \times q}\, R)$ the left D-module finitely presented by R. Then, M admits a free resolution of length less than or equal to $\mathrm{lgld}\, D + 1$.

Proof: This can be proved as in Proposition 8 of [2]. ∎

For an explicit way to compute free resolutions of finitely generated left D-modules based on Gröbner basis techniques, see [2], [17] and the OREMODULES package [3].

We denote by $\mathrm{lKdim}\, A$ the *left Krull dimension* of the ring A. For more details, see [13] and the references therein.

Theorem 6 (Proposition 6.5.4 of [13]): If A is a left noetherian ring, σ an automorphism of A and δ a σ-derivation, then we have:

1) $\mathrm{lKdim}\, A \leq \mathrm{lKdim}\, A[\partial; \sigma, \delta] \leq \mathrm{lKdim}\, A + 1$.
2) $\mathrm{lKdim}\, A[S; \sigma, 0] = \mathrm{lKdim}\, A$.
3) If A is a left Artinian ring, then $\mathrm{lKdim}\, A[S; \sigma, 0] = 1$.

Corollary 4: Let $D = A\left[\partial; \mathrm{id}_A, \frac{d}{dt}\right][S; \sigma, 0]$ be the ring of DTVD operators defined in Section III, where the function $l : t \longmapsto t - h(t)$ is bijective (*e.g.* $h(t) := q\,t + h$, where $h \in \mathbb{R}_{\geq 0}$ and $0 < q < 1$), then we have:

$$\mathrm{lKdim} A \leq \mathrm{lKdim}\, D \leq \mathrm{lKdim}\, A + 1.$$

If D is a left noetherian domain, then we can define its *total quotient field* $K := \{d^{-1}\, n \mid 0 \neq d,\, n \in D\}$ and the *rank* of a left D-module M is then defined by

$$\mathrm{rank}_D(M) := \dim_K(K \otimes_D M),$$

where $K \otimes_D M$ is the left K-vector space obtained by extending the scalars of M from D to K [19]. In control theory, the rank of the left D-module $M = D^{1 \times p}/(D^{1 \times q}\, R)$ corresponds to the number of inputs of the linear system.

Theorem 7 (Theorems 11.1.14 and 11.1.17 of [13]): If D is a left noetherian domain, then any stably free left D-module M with $\mathrm{rank}_D(M) \geq \mathrm{lKdim}(D) + 1$ is free.

V. ALGEBRAIC ANALYSIS APPROACH

Let us introduce the concept of *extension modules* [19]. In this section, we suppose that D is a noetherian domain. Let $M = D^{1 \times p}/(D^{1 \times q}\, R)$ be the left D-module *finitely presented* by the system matrix $R \in D^{q \times p}$, which is associated with the linear system $\ker_{\mathcal{F}}(R.) = \{\eta \in \mathcal{F}^p \mid R\,\eta = 0\}$. Then, we have the following exact sequence

$$0 \longrightarrow \ker_D(.R) \xrightarrow{i} D^{1 \times q} \xrightarrow{.R} D^{1 \times p} \xrightarrow{\pi} M \longrightarrow 0,$$

where i is the canonical injection, $.R \in \hom_D(D^{1 \times q}, D^{1 \times p})$ is defined by $(.R)(\mu) = \mu\, R$ for all $\mu \in D^{1 \times q}$ and $\pi \in \hom_D(D^{1 \times p}, M)$ is the canonical projection which sends $\lambda \in D^{1 \times p}$ onto its *residue class* $\pi(\lambda) \in M$ (note that $\pi(\lambda) = \pi(\lambda')$ iff $\lambda - \lambda' \in D^{1 \times q}\, R$). Since D is a left noetherian ring, the finitely generated left D-module $D^{1 \times q}$ is *noetherian*, and thus $\ker_D(.R) = \{\mu \in D^{1 \times q} \mid \mu\, R = 0\}$ is a finitely generated left D-module. Thus, there exists a finite set of generators $\{\mu_j\}_{j=1,\ldots,r}$ of $\ker_D(.R)$. If we note $R_2 = (\mu_1^T \; \ldots \; \mu_r^T)^T \in D^{r \times q}$, then we have $\ker_D(.R) = \mathrm{im}_D(.R_2) = D^{1 \times r}\, R_2$. Thus, we get the following long exact sequence of left D-modules:

$$0 \longrightarrow \ker_D(.R_2) \xrightarrow{i_2} D^{1 \times r} \xrightarrow{.R_2} D^{1 \times q} \xrightarrow{.R} D^{1 \times p} \xrightarrow{\pi} M \longrightarrow 0.$$

Repeating the same arguments as above for $\ker_D(.R_2)$ and so on, we finally obtain a free resolution of M

$$\ldots \xrightarrow{.R_3} D^{1 \times p_2} \xrightarrow{.R_2} D^{1 \times p_1} \xrightarrow{.R_1} D^{1 \times p_0} \xrightarrow{\pi} M \longrightarrow 0,$$

where $p_0 = p$, $p_1 = q$, $p_2 = r$ and $R_1 = R$. Applying the *contravariant left exact functor* $\hom_D(\cdot, \mathcal{F})$ [19] to the *truncated free resolution* defined by

$$\ldots \xrightarrow{.R_3} D^{1\times p_2} \xrightarrow{.R_2} D^{1\times p_1} \xrightarrow{.R_1} D^{1\times p_0} \longrightarrow 0, \quad (12)$$

i.e. dualizing (12), we get the following complex

$$\mathcal{F}^\bullet : \ \ldots \xleftarrow{R_3.} \mathcal{F}^{p_2} \xleftarrow{R_2.} \mathcal{F}^{p_1} \xleftarrow{R_1.} \mathcal{F}^{p_0} \longleftarrow 0,$$

where $(R_i.)(\eta) = R_i\,\eta$ for all $\eta \in \mathcal{F}^{p_{i-1}}$ for $i \geq 1$. The defects of exactness of $\mathcal{F}^\bullet$, also *cohomologies* of $\mathcal{F}^\bullet$, are defined by:

$$\begin{cases} H^0(\mathcal{F}^\bullet) = \ker_{\mathcal{F}}(R_1.), \\ H^i(\mathcal{F}^\bullet) = \ker_{\mathcal{F}}(R_{i+1}.)/\mathrm{im}_{\mathcal{F}}(R_i.), \quad i \geq 1. \end{cases}$$

Using (11), we obtain $H^0(\mathcal{F}^\bullet) \cong \hom_D(M, \mathcal{F})$. More generally, an important result of homological algebra proves that the cohomologies $H^i(\mathcal{F}^\bullet)$'s do not depend on the choice of the free resolution of M, *i.e.* up to isomorphism, they depend only on M and $\mathcal{F}$ [19]. They are then denoted by

$$\begin{cases} \mathrm{ext}^i_D(M, \mathcal{F}) = \hom_D(M, \mathcal{F}), \\ \mathrm{ext}^i_D(M, \mathcal{F}) = H^i(\mathcal{F}^\bullet), \quad i \geq 1, \end{cases}$$

and are called *extension abelian groups*. In the case where $\mathcal{F} = D$, we can prove that the $\mathrm{ext}^i_D(M, D)$'s inherit a right D-module structure. Similarly, if M is a right D-module, then the $\mathrm{ext}^i_D(M, D)$'s inherit left D-module structures. For an implementation of the computation of the $\mathrm{ext}^i_D(M, D)$'s for certain Ore algebras, see the OREMODULES package [3].

Due to the homological nature of the main results of [2] (see also [17]), they can directly be applied to the ring of DTVD operators.

Theorem 8: Let A be a regular noetherian ring and $D = A\left[\partial; \mathrm{id}_A, \frac{d}{dt}\right][S; \sigma, 0]$ the ring of DTVD operators with coefficients in A defined in Section III, where the function $l : t \longmapsto t - h(t)$ is bijective (*e.g.* $h(t) := q\,t + h$, where $h \in \mathbb{R}_{\geq 0}$ and $0 < q < 1$), $R \in D^{q\times p}$, $M = D^{1\times p}/(D^{1\times q}\,R)$ and $N = D^q/(R\,D^p)$ the *Auslander transpose* of M. Then:

1) $t(M) \cong \mathrm{ext}^1_D(N, D)$.
2) M is torsion-free iff $\mathrm{ext}^1_D(N, D) = 0$.
3) M is reflexive iff $\mathrm{ext}^i_D(N, D) = 0$ for $i = 1, 2$.
4) M is projective iff $\mathrm{ext}^i_D(N, D) = 0$ for $i = 1, \ldots, \mathrm{gld}\, D$.
5) If A is a projective stably free ring (*e.g.* A is a field), then M is projective left D-module if and only if M is stably free left D-module. Moreover, M is free when $\mathrm{rank}_D(M) \geq \mathrm{lKdim}(D) + 1$.

Proof: 1 and 2 are direct consequences of Theorem 5 of [2]. 3 (resp. 4) is a consequence of Theorem 6 (resp. Theorem 7) of [2]. Finally, let us prove 5. A stably free left D-module is well-known to be a projective one. The converse is proved in Corollary 12.3.3 of [13]. ■

Within the algebraic analysis approach, the concept of an *injective cogenerator signal space* $\mathcal{F}$ plays a similar role as the one of an algebraic *closed field* in algebraic geometry (think about the solutions of $x^2 + 1 = 0$ in $\mathbb{R}$). A non-trivial module $M = D^{1\times p}/(D^{1\times q}\,R)$ then defines a non-zero linear system/behavior $\ker_{\mathcal{F}}(R.)$. Moreover, a complete duality exists between linear systems/behaviors and finitely presented left modules [2], [15], [17], [23].

Definition 7 ([19]):
1) A left D-module $\mathcal{F}$ is *cogenerator* if for every left D-module M and $m \in M \setminus \{0\}$, there exists $f \in \hom_D(M, \mathcal{F})$ such that $f(m) \neq 0$.
2) A left D-module $\mathcal{F}$ is *injective* if $\mathrm{ext}^i_D(M, \mathcal{F}) = 0$ for all $i \geq 1$ and for all left D-modules M.

For a given ring D, it can be shown that an injective cogenerator left D-module $\mathcal{F}$ always exists [19].

According to [2], [6], [15], [16], [17], [23], we can state the following general definitions.

Definition 8: Let D be a noetherian domain, $R \in D^{q\times p}$, $\mathcal{F}$ an injective cogenerator left D-module and $\ker_{\mathcal{F}}(R.)$.

1) An *observable* $\psi(\eta)$ of $\ker_{\mathcal{F}}(R.)$ is a D-linear combination of the system variables, *i.e.* $\psi(\eta) = \sum_{i=1}^p d_i\,\eta_i$, where $d_i \in D$ and $\eta = (\eta_1 \ \ldots \ \eta_p)^T \in \ker_{\mathcal{F}}(R.)$.
2) An observable $\psi(\eta)$ is called *autonomous* if it satisfies a D-linear relation by itself, *i.e.* $d\,\psi(\eta) = 0$ for some $d \in D \setminus \{0\}$. It is called *free* it is not autonomous.
3) The linear system is said to be *controllable* if every observable is free.
4) The linear system is said to be *parametrizable* if there exists a matrix $Q \in D^{p\times m}$ such that
$$\ker_{\mathcal{F}}(R.) = \mathrm{im}_{\mathcal{F}}(Q.) = Q\,\mathcal{F}^m,$$
i.e. if for every $\eta \in \ker_{\mathcal{F}}(R.)$, there exists $\xi \in \mathcal{F}^m$ such that $\eta = Q\,\xi$. Then, Q is called a *parametrization* and ξ a *potential*.
5) The linear system is said to be *flat* if there exists a parametrization $Q \in D^{p\times m}$ which admits a left inverse $T \in D^{m\times p}$, *i.e.* $T\,Q = I_p$. In other words, a flat system is a parametrizable system such that every component ξ_i of a potential ξ is an observable of the system. The potential ξ is then called a *flat output*.

The next theorem explicitly characterizes the above definitions in terms of properties of modules.

Theorem 9 ([2], [6]): Let $M = D^{1\times p}/(D^{1\times q}\,R)$ be the finitely presented left D-module associated with the linear system $\ker_{\mathcal{F}}(R.) = \{\eta \in \mathcal{F}^p \mid R\,\eta = 0\}$. With the hypotheses of Definition 8, we have:

1) The observables of the linear system are in one-to-one correspondence with the elements of M.
2) The autonomous elements of the linear system are in one-to-one correspondence with the torsion elements of M. The linear system is controllable iff M is torsion-free.
3) The linear system is parametrizable iff there exists a matrix $Q \in D^{p\times m}$ such that we have $M \cong D^{1\times p}\,Q$, *i.e.* iff M is a torsion-free left D-module. Then, the

matrix Q is a parametrization, *i.e.* $\ker_{\mathcal{F}}(R.) = Q\,\mathcal{F}^m$. A parametrization Q can be computed by checking that $\mathrm{ext}^1_D(N, D) = 0$, where $N = D^q/(R\,D^p)$.
4) The linear system is flat iff M is a free left D-module. Then, the bases of M are in one-to-one correspondence with the flat outputs of the linear system.

Combining Theorems 8 and 9, system properties listed in Definition 8 can be explicitly characterized in terms of module properties and in terms of the vanishing of the extension modules $\mathrm{ext}^i_D(N, D)$'s. Hence, the future implementation of Gröbner basis techniques for the ring D (see Theorem 2) in the OREMODULES package [3] will give us an effective way to check the system properties given in Definition 8.

Example 7: Let us consider the system $\dot{x}(t) = u(t-h(t))$, where $l : t \longmapsto t - h(t)$ is bijective (*e.g.* $h(t) := q\,t + h$, where $h \in \mathbb{R}_{\geq 0}$ and $0 < q < 1$). Let D be a ring of DTVD operators, $R = (\partial \quad - S) \in D^{1\times 2}$, $M = D^{1\times 2}/(D\,R)$ the left D-module finitely presented by R and $N = D/(R\,D^2)$ be the Auslander transpose of M. Let study M.

Using (9), *i.e.* $\partial\, S = (1 - \dot{h})\, S\, \partial = S\, \left(\sigma^{-1}\left(1 - \dot{h}\right)\, \partial\right)$, if $Q = \left(S \quad \sigma^{-1}\left(1 - \dot{h}\right)\, \partial\right)^T \in D^2$, then we obtain $\mathrm{im}_D(Q.) = \ker_D(R.)$, and thus we obtain the free resolution $0 \longleftarrow N \xleftarrow{\kappa} D \xleftarrow{R.} D^2 \xleftarrow{Q.} D \longleftarrow 0$. Dualizing this free resolution, we get the complex $0 \longrightarrow D \xrightarrow{.R} D^{1\times 2} \xrightarrow{.Q} D \longrightarrow 0$, which yields:

$$\begin{cases} \mathrm{ext}^1_D(N, D) = \ker_D(.Q)/\mathrm{im}_D(.R) \cong t(M), \\ \mathrm{ext}^2_D(N, D) = D/(D^{1\times 2}\, Q) = D/\left(S, \partial\right) \neq 0. \end{cases}$$

We obtain that M is a torsion-free but not a projective left D-module because $\ker_D(.Q) = \mathrm{im}_D(.R)$ and $\mathrm{gld}\, D = 2$.

If $\mathcal{F}$ is an injective left D-module, then applying the contravariant exact functor $\hom_D(\,\cdot\,, \mathcal{F})$ to the above complex, we obtain the following exact sequence of abelian groups:

$$0 \longleftarrow \mathcal{F} \xleftarrow{R.} \mathcal{F}^2 \xleftarrow{Q.} \mathcal{F} \longleftarrow \ker_{\mathcal{F}}(Q.) \longleftarrow 0.$$

In particular, we have $\ker_{\mathcal{F}}(R.) = \mathrm{im}_{\mathcal{F}}(Q.)$, which shows that Q is a parametrization of $\ker_{\mathcal{F}}(R.)$, *i.e.* we have:

$$\begin{cases} x(t) = S\,\xi(t) = \xi(t - h(t)), \\ u(t) = \sigma^{-1}\left(1 - \dot{h}\right)\, \partial\, \xi(t) = \left(1 - \dot{h}(l^{-1}(t))\right)\, \dot{\xi}(t). \end{cases}$$

Since σ is an automorphism of $A\left[\partial; \mathrm{id}_A, \frac{d}{dt}\right]$, then we can define *the skew Laurent polynomial ring* $E = A\left[\partial; \mathrm{id}_A, \frac{d}{dt}\right][S; \sigma, 0][S^{-1}; \sigma^{-1}, 0]$ [13]. Hence, $E \otimes_D M$ is a free left E-module of rank 1 and x is a basis of $E \otimes_D M$, *i.e.* the DTVD system is a *S-flat* system [6].

VI. Conclusion

In this paper, we propose an algebraic analysis approach for the ring of differential time-varying delay operators realized as an Ore extension D. The corresponding construction is explicit and provides a new algebraic approach for the study of linear differential varying time-delay systems. Moreover, homological algebraic properties of D were studied and its global and Krull dimensions were analyzed.

Nevertheless, some questions remain open. For instance, to determine whether or not D is an *Auslander regular ring* or *Cohen Macaulay* [13]. Another important point would be to define an *involution* of D so that Gröbner bases can be calculated for right D-modules. For more details, see [2]. Finally, an implementation of Gröbner basis techniques for D is an important issue for the constructive aspects.

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