Stafford's Reduction of Linear Partial Differential Systems

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Abstract: It is well-known that linear systems theory can been studied by means of module theory. In particular, to a linear ordinary/partial differential system corresponds a finitely presented left module over a ring of ordinary/partial differential operators. The structure of modules over rings of partial differential operators was investigated in Stafford's seminal work [18]. The purpose of this paper is to make some results obtained in [18] constructive. Our results are implemented in the Maple package STAFFORD. Finally, we give system-theoretic interpretations of Stafford's results within the behavioural approach (e.g., minimal representations, autonomous behaviours, direct decomposition of behaviours, differential flatness).

1. INTRODUCTION

It is well-known that linear systems theory can be studied by means of module theory (see, e.g., [2, 3, 4, 5, 12, 14, 15] and the references therein). The purpose of this paper is to develop constructive versions of important results obtained by Stafford in his seminal paper [18] on the module structure of rings of partial differential (PD) operators. Using the duality between linear systems (behaviours) and finitely presented left modules, we give system-theoretic interpretations of Stafford's theorems. Finally, based on Stafford's results, we obtain explicit conditions so that a linear PD system is equivalent to another one defined by fewer unknowns and fewer equations.

2. ALGEBRAIC ANALYSIS

In this section, we briefly review the *algebraic analysis approach* [7] to linear systems theory. For more details, see [2, 4, 11, 12, 13, 14]. In what follows, we shall assume that D is a *noetherian domain*, namely, a ring D without zero divisors and such that every left/right ideal of D is finitely generated as a left/right D-module [8, 17].

Let $R \in D^{q\times p}$ be a $(q \times p)$-matrix with entries in D and

$$.R\colon D^{1\times q} \longrightarrow D^{1\times p}, \quad \lambda \longmapsto \lambda R,$$

the left D-homomorphism (i.e., the left D-linear map) represented by R. Then, the cokernel of $.R$ is the factor left D-module $M := D^{1\times p}/(D^{1\times q} R)$, *finitely presented by* R. In order to describe M by means of *generators* and *relations*, let $\{f_j\}_{j=1,\dots,p}$ be the *standard basis* of $D^{1\times p}$, namely, f_j is the row vector of length p with 1 at position j and 0 elsewhere. Moreover, let $\pi\colon D^{1\times p} \longrightarrow M$ be the canonical projection onto M, i.e., the left D-homomorphism which maps $\lambda \in D^{1\times p}$ to its residue class $\pi(\lambda)$ in M. Then, π is surjective since by definition of M, every $m \in M$ is the class of certain λ's in $D^{1\times p}$, i.e., $m = \pi(\lambda) = \pi(\lambda + \nu R)$ for all $\nu \in D^{1\times q}$. If $y_j = \pi(f_j)$ for $j = 1,\dots,p$, then, for every $m \in M$, there exists $\lambda = (\lambda_1 \ \dots \ \lambda_p) \in D^{1\times p}$ such that

$$m = \pi(\lambda) = \pi\left(\sum_{j=1}^{p} \lambda_j f_j\right) = \sum_{j=1}^{p} \lambda_j \pi(f_j) = \sum_{j=1}^{p} \lambda_j y_j,$$

which shows that $\{y_j\}_{j=1,\dots,p}$ is a generating set for M. Let $R_{i\bullet}$ (resp., $R_{\bullet j}$) denote the i^{th} row (resp., j^{th} column) of R. Then $\{y_j\}_{j=1,\dots,p}$ satisfies the relations

$$\sum_{j=1}^{p} R_{ij}\, y_j = \sum_{j=1}^{p} R_{ij}\, \pi(f_j) = \pi\left(\sum_{j=1}^{p} R_{ij} f_j\right) = \pi(R_{i\bullet}) = 0 \tag{1}$$

for all $i = 1, \dots, q$, since $R_{i\bullet} \in D^{1\times q} R$ for $i = 1,\dots,q$.

Now, let $\mathcal{F}$ be a left D-module, $\mathcal{F}^p := \mathcal{F}^{p\times 1}$, and let

$$\ker_{\mathcal{F}}(R.) := \{\eta \in \mathcal{F}^p \mid R\,\eta = 0\}$$

be the *linear system* or *behaviour* defined by R and $\mathcal{F}$. A simple but fundamental remark due to Malgrange [10] is that $\ker_{\mathcal{F}}(R.)$ is isomorphic to the abelian group $\hom_D(M,\mathcal{F})$ of left D-homomorphisms from M to $\mathcal{F}$, i.e.,

$$\ker_{\mathcal{F}}(R.) \cong \hom_D(M,\mathcal{F}) \tag{2}$$

as abelian groups, where $\cong$ denotes an isomorphism (e.g., of abelian groups, left/right modules). This isomorphism can easily be described: if $\phi \in \hom_D(M,\mathcal{F})$, $\eta_j = \phi(y_j)$ for $j = 1,\dots,p$, and $\eta = (\eta_1 \ \dots \ \eta_p)^T \in \mathcal{F}^p$, then using (1), $R\,\eta = 0$ since for $i = 1,\dots,q$:

$$\sum_{j=1}^{p} R_{ij}\, \phi(y_j) = \phi\left(\sum_{j=1}^{p} R_{ij}\, y_j\right) = \phi\left(\pi\left(R_{i\bullet}\right)\right) = 0.$$

Moreover, for any $\eta = (\eta_1 \ \dots \ \eta_p)^T \in \ker_{\mathcal{F}}(R.)$, the map $\phi_\eta\colon M \longrightarrow \mathcal{F}$ defined by $\phi_\eta(y_j) = \eta_j$ for $j = 1,\dots,p$ is a well-defined left D-homomorphism from M to $\mathcal{F}$, i.e., we have $\phi_\eta \in \hom_D(M, \mathcal{F})$. Finally, the abelian group homomorphism $\chi\colon \ker_{\mathcal{F}}(R.) \longrightarrow \hom_D(M, \mathcal{F})$ defined by $\chi(\eta) = \phi_\eta$ is then bijective. For more details, see [2, 3, 14]. Hence, (2) shows that the linear system $\ker_{\mathcal{F}}(R.)$ can be studied in terms of $\hom_D(M, \mathcal{F})$, and thus, by means of the left D-modules M and $\mathcal{F}$. Since matrices R_1 and R_2 representing equivalent linear systems define isomorphic modules, $\hom_D(M, \mathcal{F})$ is a more intrinsic description of the linear system than $\ker_{\mathcal{F}}(R.)$ (e.g., it does not depend on the particular embedding of $\ker_{\mathcal{F}}(R.)$ into $\mathcal{F}^p$).

Example 1. Let A be a *differential ring*, namely, A is a ring equipped with *commuting derivations* δ_i for $i = 1, \ldots, n$, namely, maps $\delta_i : A \longrightarrow A$ satisfying

$$\forall\, a_1,\, a_2 \in A, \quad \begin{cases} \delta_i(a_1 + a_2) = \delta_i(a_1) + \delta_i(a_2), \\ \delta_i(a_1\, a_2) = \delta_i(a_1)\, a_2 + a_1\, \delta_i(a_2), \end{cases}$$

and $\delta_i \circ \delta_j = \delta_j \circ \delta_i$ for all $1 \leq i < j \leq n$. Moreover, let $D = A\langle \partial_1, \ldots, \partial_n \rangle$ be the (not necessarily commutative) polynomial ring of PD operators in $\partial_1, \ldots, \partial_n$ with coefficients in A, namely, every element $d \in D$ is of the form $d = \sum_{0 \leq |\mu| \leq r} a_\mu\, \partial^\mu$, where $r \in \mathbb{Z}_{\geq 0}$, $a_\mu \in A$, $\mu = (\mu_1\ \ldots\ \mu_n) \in (\mathbb{Z}_{\geq 0})^{1 \times n}$, $\partial^\mu = \partial_1^{\mu_1} \ldots \partial_n^{\mu_n}$ is a monomial in the commuting indeterminates $\partial_1, \ldots, \partial_n$, and:

$$\forall\, a \in A, \quad \partial_i\, a = a\, \partial_i + \delta_i(a).$$

For instance, if k is a field and $A = k[x_1, \ldots, x_n]$ (resp., $k(x_1, \ldots, x_n)$), then the so-called *Weyl algebra* $A\langle \partial_1, \ldots, \partial_n \rangle$ is simply denoted by $A_n(k)$ (resp., $B_n(k)$).

If $M = D^{1 \times p}/(D^{1 \times q}\, R)$ is the left D-module finitely presented by the matrix of PD operators $R \in D^{q \times p}$ and $\mathcal{F}$ a left D-module (e.g., $\mathcal{F} = A$), then the linear PD system $\ker_{\mathcal{F}}(R.)$ is intrinsically defined by $\hom_D(M, \mathcal{F})$.

If M, M' and M'' are three left/right D-modules and $f \in \hom_D(M', M)$ and $g \in \hom_D(M, M'')$ are such that $g \circ f = 0$, i.e., $\operatorname{im} f \subseteq \ker g$, then $M' \xrightarrow{f} M \xrightarrow{g} M''$ is called a *complex* (see, e.g., [17]). Moreover, if $\ker g = \operatorname{im} f$, then the complex is said to be *exact at* M (see, e.g., [17]).

By construction of the finitely presented left D-module $M = D^{1 \times p}/(D^{1 \times q}\, R)$, the following complex is exact:

$$D^{1 \times q} \xrightarrow{.R} D^{1 \times p} \xrightarrow{\pi} M \longrightarrow 0.$$

It is called a *finite presentation* of M [17].

The short exact sequence $0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$ *splits* if one of the following equivalent assertions holds:

(1) $\exists\, u \in \hom_D(M'', M)$: $g \circ u = \mathrm{id}_{M''}$.
(2) $\exists\, v \in \hom_D(M, M')$: $v \circ f = \mathrm{id}_{M'}$.
(3) $M \cong M' \oplus M''$, where $\oplus$ denotes the direct sum.

For more details, see, e.g., [17].

Within algebraic analysis, the module structure of rings of PD operators plays a fundamental role for the study of linear systems of PD equations [7]. In [2, 3, 13, 14], we have initiated the constructive study of module theory and homological algebra over *Ore algebras*, i.e., a certain class of noncommutative polynomial rings of *functional operators* such as rings of ordinary/partial differential operators, differential time-delay operators, or shift operators. Let us now recall a few classical definitions.

Definition 2. ([8, 17]). Let D be a noetherian domain and M a finitely generated left D-module.

- M is *free of rank* r if $M \cong D^{1 \times r}$.
- M is *stably free* if there exist r, $s \in \mathbb{Z}_{\geq 0}$ such that:
$$M \oplus D^{1 \times s} \cong D^{1 \times r}.$$
- M is *torsion-free* if its *torsion left D-submodule*
$$t(M) = \{m \in M \mid \exists\, d \in D \setminus \{0\} :\ d\, m = 0\}$$
is reduced to $\{0\}$, i.e., $t(M) = \{0\}$.
- M is *torsion* if $t(M) = M$.

Similar definitions hold for right D-modules.

See [2, 13, 14] for algorithms which test whether or not a finitely presented left D-module M is free, stably free, torsion-free, has torsion elements, or is torsion.

Since D is a noetherian domain, D has the *left* (and the *right*) *Ore property* [8], i.e., for all d_1, $d_2 \in D \setminus \{0\}$, there exist e_1, $e_2 \in D \setminus \{0\}$ such that $e_1\, d_1 = e_2\, d_2$ (resp., $d_1\, e_1 = d_2\, e_2$). This implies the existence of the *division ring of fractions* $Q(D) = S^{-1}\, D = D\, S^{-1}$ of D, where $S = D \setminus \{0\}$ [8]. If M is a finitely generated left/right D-module, then $Q(D) \otimes_D M$ (resp., $M \otimes_D Q(D)$) is a finitely generated left (resp., right) $Q(D)$-vector space and:

$$\begin{aligned} \operatorname{rank}_D(M) &:= \dim_{Q(D)}(Q(D) \otimes_D M) \\ &= \dim_{Q(D)}(M \otimes_D Q(D)). \end{aligned}$$

Proposition 3. ([2], Corollary 1). Let M be a finitely generated left D-module. Then, the assertions are equivalent:

(1) M is a torsion left D-module.
(2) $\operatorname{rank}_D(M) = 0$.
(3) $\hom_D(M, D) = 0$.

Theorem 4. (1) [8, 17] The following implications
free $\Rightarrow$ stably free $\Rightarrow$ torsion-free
hold for finitely generated left/right D-modules.
(2) [15] If k is a field of characteristic zero, $A = k[\![t]\!]$ the ring of formal power series with coefficients in k, or $A = k\{t\}$ the ring of locally convergent power series with coefficients in $k = \mathbb{R}$ or $\mathbb{C}$ (i.e., germs of real analytic/holomorphic functions), and $D = A\langle \partial \rangle$ the ring of ordinary differential (OD) operators with coefficients in A, then every stably free left D-module M with $\operatorname{rank}_D(M) \geq 2$ is free.
(3) [1, 18] If k is a field of characteristic zero, A is either the polynomial ring $k[x_1, \ldots, x_n]$, the field of rational functions $k(x_1, \ldots, x_n)$, the field of fractions $k((x_1, \ldots, x_n))$ of the domain $k[\![x_1, \ldots, x_n]\!]$ of formal power series with coefficients in k, or the field of fractions $k\{\{x_1, \ldots, x_n\}\}$ of the domain $k\{x_1, \ldots, x_n\}$ of locally convergent power series with coefficients in $k = \mathbb{R}$ or $\mathbb{C}$, and $D = A\langle \partial_1, \ldots, \partial_n \rangle$ the ring of PD operators with coefficients in A, then every stably free left D-module M with $\operatorname{rank}_D(M) \geq 2$ is free.

A constructive proof of *Stafford's theorem*, i.e., 3 of Theorem 4 for $D = A_n(k)$ and $B_n(k)$, was given in [14]. An implementation of computation of bases of finitely presented free left D-modules is available in the Stafford package [14] for $D = A_n(\mathbb{Q})$ and $B_n(\mathbb{Q})$.

Let us now consider the OD case (e.g., $D = \mathbb{R}\{t\}\langle \partial \rangle$). Since the inputs of a linear control system are generally considered as independent, then the number of inputs of a linear system defined by a finitely presented left D-module M is $\operatorname{rank}_D(M)$ [4]. Moreover, if a finitely presented left

D-module $M = D^{1\times p}/(D^{1\times q}\,R)$ is free, then the linear system $\ker_{\mathcal{F}}(R.)$ is called *differentially flat* [5]. Moreover, a torsion-free left D-module defines a controllable linear system $\ker_{\mathcal{F}}(R.)$ [4]. For more details, see [2, 15]. 2 of Theorem 4 asserts that every controllable linear control system with at least two inputs is differentially flat [15].

3. UNIMODULAR ELEMENTS

Let us introduce the concept of *unimodular elements.*

Definition 5. An element m of a left D-module M is called *unimodular* if there exists $\varphi \in \hom_D(M, D)$ such that:

$$\varphi(m) = 1.$$

The set of unimodular elements of M is denoted by $\mathrm{U}(M)$.

Let us show how unimodular elements of M can be used to decompose M into a direct sum. If $m \in \mathrm{U}(M)$, then there exists $\varphi \in \hom_D(M, D)$ such that $\varphi(m) = 1$. Thus, for any $d \in D$, $d = d\,\varphi(m) = \varphi(d\,m)$, which shows that φ is surjective, and we have the following short exact sequence:

$$0 \longrightarrow \ker\varphi \longrightarrow M \xrightarrow{\varphi} D \longrightarrow 0. \quad (3)$$

Since D is free, (3) splits (see, e.g., [17]). Therefore, we have $M \cong \ker\varphi \oplus D$ as left D-modules. More precisely, $\sigma \in \hom_D(D, M)$ defined by $\sigma(d) = d\,m$ for all $d \in D$ satisfies $\varphi\circ\sigma = \mathrm{id}_D$. Thus, $M = \ker\varphi \oplus \mathrm{im}\,\sigma = \ker\varphi \oplus D\,m$.

Remark 6. If m is a torsion element of M, then Proposition 3 shows that $\varphi(m) = 0$ for all $\varphi \in \hom_D(M, D)$. This implies $t(M) \cap U(M) = \emptyset$, i.e., $m \in \mathrm{U}(M) \Rightarrow m \notin t(M)$.

Let us now study the problem of computing unimodular elements of M. Let $M = D^{1\times p}/(D^{1\times q}\,R)$ be a left D-module finitely presented by $R \in D^{q\times p}$. Using Malgrange's remark (see Section 2), we obtain the following lemma.

Lemma 7. Let $M = D^{1\times p}/(D^{1\times q}\,R)$ be a finitely presented left D-module, $\pi\colon D^{1\times p} \longrightarrow M$ the canonical projection onto M, and

$$\begin{aligned} R.\colon D^p &\longrightarrow D^q \\ \eta &\longmapsto R\,\eta \end{aligned}$$

the right D-homomorphism represented by R. Then, we have $\hom_D(M, D) \cong \ker_D(R.) := \{\eta \in D^p \mid R\,\eta = 0\}$. In particular, for every $\varphi \in \hom_D(M, D)$, there exists $\mu \in \ker_D(R.)$, i.e., $R\,\mu = 0$, such that:

$$\forall\, \lambda \in D^{1\times p}, \quad \varphi(\pi(\lambda)) = \lambda\,\mu. \quad (4)$$

In what follows, φ defined by (4) will be denoted by φ_μ.

Remark 6 shows that if M is a torsion left D-module, then $\mathrm{U}(M) = \emptyset$. Hence, let us suppose that M is not torsion. Thus, by Proposition 3, $\ker_D(R.) \cong \hom_D(M, D) \neq 0$. Since D is a right noetherian ring, $\ker_D(R.)$ is a finitely generated right D-module, and thus there exists a matrix $Q \in D^{p\times m}$ such that $\ker_D(R.) = \mathrm{im}_D(Q.) := Q\,D^m$. Then we have the exact sequence $D^q \xleftarrow{R.} D^p \xleftarrow{Q.} D^m$ and, since $R\,Q = 0$, the following *complex* of left D-modules:

$$D^{1\times q} \xrightarrow{.R} D^{1\times p} \xrightarrow{.Q} D^{1\times m}.$$

Lemma 8. ([2], Theorem 5). With the above notations:

$$t(M) = \ker_D(.Q)/\mathrm{im}_D(.R), \;\; M/t(M) = D^{1\times p}/\ker_D(.Q).$$

In particular, $\pi(\lambda)$ is a torsion element of M iff $\lambda\,Q = 0$.

Remark 9. Combining Lemmas 7 and 8, we obtain that for every $\pi(\lambda) \in M \setminus t(M)$, i.e., $\lambda\,Q \neq 0$, there exists $\mu \in \ker_D(R.) = \mathrm{im}_D(Q.)$ such that $\varphi_\mu \in \hom_D(M, D)$ satisfies $\varphi_\mu(\pi(\lambda)) = \lambda\,\mu \neq 0$. Since $\mu = Q\,\xi$ for $\xi \in D^m$ and $\lambda\,Q \neq 0$, we only need to fix ξ such that $(\lambda\,Q)\,\xi \neq 0$.

By Lemma 7, every $\varphi \in \hom_D(M, D)$ is of the form φ_μ for a certain $\mu \in \ker_D(R.) = \mathrm{im}_D(Q.)$, i.e., for $\mu = Q\,\xi$ for some $\xi \in D^m$. Thus, the problem of finding a unimodular element $\pi(\lambda) \in M$ amounts to the following:

Problem 10. Find $\lambda^\star \in D^{1\times p}$ and $\xi^\star \in D^m$ such that:

$$\lambda^\star\,Q\,\xi^\star = 1.$$

We point out that Problem 10 corresponds to solving an inhomogeneous quadratic equation in the λ_i's and the ξ_j's. We also note that the problem of checking whether or not $\pi(\lambda)$ is a unimodular element of M is a linear problem: Check whether or not $\lambda\,Q \in D^{1\times m}$ admits a right inverse over D. For instance, this can be answered constructively for (not necessarily commutative) polynomial rings which admit Gröbner basis techniques (see, e.g., [2]).

If one entry of Q is invertible in D, then Problem 10 can be solved easily: if $Q_{ij} \in \mathrm{U}(D)$ and $\{f_i\}_{i=1,\ldots,p}$ (resp., $\{h_j\}_{j=1,\ldots,m}$) is the standard basis of $D^{1\times p}$ (resp., D^m), then $\lambda^\star := f_i$ and $\xi^\star := Q_{ij}^{-1}\,h_j$ are such that $\lambda^\star\,Q\,\xi^\star = 1$. Then, $m^\star := \pi(f_i) \in M$ is unimodular and $\varphi_{Q\,\xi^\star}(m^\star) = 1$.

More generally, if one row (resp., one column) of Q admits a right inverse (resp., a left inverse) over D, then Problem 10 can be solved easily. For instance, if the j^{th} column $Q_{\bullet j}$ of Q admits a left inverse $T \in D^{1\times p}$, then considering $\lambda^\star := T$ and $\xi^\star := h_j$, where $\{h_k\}_{k=1,\ldots,m}$ is the standard basis of D^m, and $\mu^\star := Q\,h_j$, we get $\lambda^\star\,\mu^\star = 1$, which proves that $m^\star := \pi(T) \in \mathrm{U}(M)$ and $\varphi_{\mu^\star}(m^\star) = 1$. Now, if the i^{th} row $Q_{i\bullet}$ of Q admits a right inverse $S \in D^m$, then considering $\lambda^\star := f_i$, where $\{f_k\}_{k=1,\ldots,p}$ is the standard basis of $D^{1\times p}$, $\xi^\star := S$, and $\mu^\star := Q\,S$, we then have $\lambda^\star\,\mu^\star = 1$, which shows that $m^\star := \pi(f_i) \in \mathrm{U}(M)$ and $\varphi_{\mu^\star}(m^\star) = 1$.

4. VERY SIMPLE RINGS

Let us introduce the concept of a *very simple domain.*

Definition 11. A domain D is called *very simple* if D is noetherian and satisfies:

$$\begin{aligned} &\forall\, a, b, c \in D, \; \forall\, d \in D\setminus\{0\}, \; \exists\, u, v \in D : \\ &D\,a + D\,b + D\,c = D\,(a + d\,u\,c) + D\,(b + d\,v\,c). \end{aligned} \quad (5)$$

Remark 12. If D is very simple, then considering $d = 1$ in (5), we obtain $D\,a + D\,b + D\,c = D\,(a + u\,c) + D\,(b + v\,c)$ for some $u, v \in D$, which shows that every left ideal of D generated by three elements, and thus, every finitely generated left ideal of D, can be generated by two elements.

If D is very simple, then choosing $a = b = 0$, $c = 1$, and $d \in D \setminus \{0\}$, there exist $u, v \in D$ such that

$$D = D\,d\,u + D\,d\,v,$$

which implies that there exist $s, t \in D$ such that:

$$s\,d\,u + t\,d\,v = 1.$$

Thus, $D\,d\,D = D$, which shows that every two-sided ideal of D is trivial, and thus, that D is a *simple* ring [8].

In fact, the following variant of (5) holds for D:

$$\begin{aligned} &\forall\, a, b, c \in D, \; \forall\, d_1, d_2 \in D\setminus\{0\}, \; \exists\, u, v \in D : \\ &D\,a + D\,b + D\,c = D\,(a + d_1\,u\,c) + D\,(b + d_2\,v\,c). \end{aligned}$$

Since D is a noetherian domain, it satisfies the right Ore condition (see Section 2). Hence, given $d_1, d_2 \in D \setminus \{0\}$, there exist $e_1,\, e_2 \in D \setminus \{0\}$ such that $d := d_1\, e_1 = d_2\, e_2$. If D is very simple, then there exist $u,\, v \in D$ such that:

$$\begin{aligned} D\,a + D\,b + D\,c &= D\,(a + d\,u\,c) + D\,(b + d\,v\,c) \\ &= D\,(a + d_1\,(e_1\,u)\,c) + D\,(b + d_2\,(e_2\,v)\,c). \end{aligned}$$

Theorem 13. ([18]). If k is a field of characteristic 0 (e.g., $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$), then the Weyl algebras $A_n(k)$ and $B_n(k)$ are very simple domains.

The computation of elements u and v defined in (5) is implemented in the Stafford package [14] based on algorithms developed in [6, 9] for the computation of two generators of left/right ideals generated by three elements.

Example 14. Let $D = A_2(\mathbb{Q})$, $a = \partial_1$, $b = \partial_2$, $c = x_1$, $d_1 \in D$ arbitrary, and $d_2 = x_1$. If we consider $u = 0$, $v = 1$, $a_2 := a + d_1\,u\,c = \partial_1$, and $b_2 := b + d_2\,v\,c = \partial_2 + x_1^2$, then (5) holds, i.e., $a = a_2$ and

$$\begin{cases} b = ((x_1\,(\partial_2 + x_1^2)\,a_2 + (-x_1\,\partial_1 + 2)\,b_2)/2, \\ c = -((\partial_2 + x_1^2)\,a_2 - \partial_1\,b_2)/2, \end{cases}$$

which shows that $D\,a + D\,b + D\,c = D\,a_2 + D\,b_2$.

Theorem 15. ([14]). The ring $D = A\langle \partial \rangle$ of OD operators with coefficients in the differential ring $A = k[\![t]\!]$ (resp., $k\{t\}$, where $k = \mathbb{R}, \mathbb{C}$) of formal power series (resp., locally convergent power series) is a very simple domain.

Theorem 16. ([1]). Let $A = k((x_1, \ldots, x_n))$ be the field of fractions of the domain of formal power series with coefficients in k. Then, the ring $A\langle \partial_1, \ldots, \partial_n \rangle$ of PD operators with coefficients in A is a very simple domain. The same result holds if A is the field of fractions $k\{\{x_1, \ldots, x_n\}\}$ of the domain of locally convergent power series with coefficients in $k = \mathbb{R}$ or $\mathbb{C}$.

Corollary 17. ([18]). Let D be a very simple domain and $d_1,\, d_2 \in D \setminus \{0\}$. Then, the following quadratic equation

$$y_1\, d_1\, z_1 + y_2\, d_2\, z_2 = 1 \qquad (6)$$

admits a solution $(y_1, y_2, z_1, z_2)^T \in D^4$.

Proof. This is the particular case $a = b = 0$, $c = 1$ of the condition given in 1 of Definition 11.

Elements y_1, y_2, z_1, and z_2 as in Corollary 17 can be computed by the Stafford package [14].

Since the *stable range* $\mathrm{sr}(D)$ [14] of a very simple domain D is 2, the following result holds.

Corollary 18. ([14]). Let D be a very simple domain and M a finitely generated stably free left D-module. If $\mathrm{rank}_D(M) \geq 2$, then M is free.

5. COMPUTATION OF UNIMODULAR ELEMENTS

Let us now show how to use Corollary 17 to solve Problem 10, and more generally, to give a constructive proof of the following theorem due to Stafford [18].

Theorem 19. ([18]). Let M and N be finitely generated left D-modules satisfying $M \subseteq N$ and $\mathrm{rank}_D(M) \geq 2$. Then, there exists $m \in M$ such that $m \in \mathrm{U}(N)$. Hence, $M = D\,m \oplus M' \subseteq N = D\,m \oplus N'$, where $M' = N' \cap M$.

Proof. We shall consider the slightly more general case of an injection $\iota : M \longrightarrow N$ rather than just an inclusion $M \subseteq N$. Since D is a noetherian ring and M and N are finitely generated, they are finitely presented (see, e.g., [17]). Let $R \in D^{q \times p}$ and $R' \in D^{q' \times p'}$ be two matrices such that $M := D^{1 \times p}/(D^{1 \times q}\, R)$ and $N := D^{1 \times p'}/(D^{1 \times q'}\, R')$. Let $\iota\colon M \longrightarrow N$ be an injection and $\pi\colon D^{1 \times p} \longrightarrow M$ (resp., $\pi'\colon D^{1 \times p'} \longrightarrow N$) the canonical projection. Then, the following diagram is commutative with exact rows

$$\begin{array}{ccccccc} D^{1\times q} & \xrightarrow{.R} & D^{1\times p} & \xrightarrow{\pi} & M & \longrightarrow 0 \\ \downarrow .P' & & \downarrow .P & & \downarrow \iota & \\ D^{1\times q'} & \xrightarrow{.R'} & D^{1\times p'} & \xrightarrow{\pi'} & N & \longrightarrow 0, \end{array}$$

i.e., $\iota(\pi(\eta)) = \pi'(\eta\, P)$ for all $\eta \in D^{1 \times p}$, where $P \in D^{p \times p'}$ is such that $R\, P = P'\, R'$ for some $P' \in D^{q \times q'}$. For more details, see [3]. The injectivity of ι is equivalent to the fact that for all $S \in D^{s \times p}$ and for all $T \in D^{s \times q'}$ satisfying $S\, P = T\, R'$, there exists $L \in D^{s \times q}$ such that $S = L\, R$. For more details, see [3]. Moreover, we have:

$$\iota(M) = \left(D^{1 \times (p+q')}\, (P^T \quad R'^T)^T\right)/(D^{1 \times p'}\, R') \subseteq N.$$

Since $\mathrm{rank}_D(M) \geq 2$, by Proposition 3, M is not torsion, i.e., there exists $m_1 := \pi(\eta_1) \in M \setminus t(M)$. Let $Q \in D^{p \times m}$ be such that $\ker_D(R.) = \mathrm{im}_D(Q.)$. Lemma 8 shows that we have to choose $\eta_1 \in D^{1 \times p}$ so that $\eta_1\, Q \neq 0$. Note that $m_1 \in t(M)$ if and only if $\iota(m_1) \in t(N)$ since ι is injective. Hence, if $Q' \in D^{p' \times m'}$ is such that $\ker_D(R'.) = \mathrm{im}_D(Q'.)$ and $\lambda_1 := \eta_1\, P \in D^{1 \times p'}$, then $\eta_1 \in D^{1 \times p}$ can equivalently be chosen such that it satisfies $\lambda_1\, Q' = \eta_1\, P\, Q' \neq 0$. By Remark 9, there exists $\mu_1 \in \ker_D(R'.) = \mathrm{im}_D(Q'.)$ such that $\varphi_1 := \varphi_{\mu_1} \in \hom_D(N, D)$ satisfies $\varphi_1(\iota(m_1)) \neq 0$, i.e., $\xi_1 \in D^{m'}$ can be chosen such that $\mu_1 := Q'\, \xi_1$ satisfies:

$$\varphi_1(\iota(m_1)) = \lambda_1\, \mu_1 = (\eta_1\, P\, Q')\, \xi_1 \neq 0.$$

The following diagram is commutative with exact rows:

$$\begin{array}{ccccccc} D^{1\times q} & \xrightarrow{.R} & D^{1\times p} & \xrightarrow{\pi} & M & \longrightarrow 0 \\ \downarrow & & \downarrow .(P\,\mu_1) & & \downarrow \varphi_1 \circ \iota & \\ 0 & \longrightarrow & D & \xrightarrow{\mathrm{id}} & D & \longrightarrow 0. \end{array}$$

Since $\mathrm{im}(\varphi_1 \circ \iota)$ is a left ideal of D containing the non-zero element $(\varphi_1 \circ \iota)(m_1)$, we have $\mathrm{rank}_D(\mathrm{im}(\varphi_1 \circ \iota)) = 1$. Then, using the following canonical short exact sequence

$$0 \longrightarrow \ker(\varphi_1 \circ \iota) \longrightarrow M \longrightarrow \mathrm{im}(\varphi_1 \circ \iota) \longrightarrow 0,$$

we get $\mathrm{rank}_D(\ker(\varphi_1 \circ \iota)) = \mathrm{rank}_D(M) - 1 \geq 1$ (see, e.g., [13]). Hence, $\ker(\varphi_1 \circ \iota)$ is not a torsion left D-module and there exists $m_2 \in \ker(\varphi_1 \circ \iota)$ such that $m_2 \notin t(M)$, or, equivalently, such that $\iota(m_2) \notin t(N)$. Let $S \in D^{r \times p}$ be such that $\ker_D(.(P\,\mu_1)) = \mathrm{im}_D(.S)$. Since we have

$$\ker(\varphi_1 \circ \iota) = \ker_D(.(P\,\mu_1))/(D^{1 \times q}\, R) = (D^{1 \times r}\, S)/(D^{1 \times q}\, R),$$

$\eta_2 := \nu\, S \in D^{1 \times p}$ and $\nu \in D^{1 \times r}$ defining $m_2 := \pi(\eta_2)$ have to be chosen so that $\nu\,(S\, Q) \neq 0$ or, equivalently, so that $\nu\,(S\, P\, Q') \neq 0$. Let $\lambda_2 := \eta_2\, P$ and consider $\xi_2 \in D^{m'}$ such that $(\lambda_2\, Q')\, \xi_2 = (\nu\, S\, P\, Q')\, \xi_2 \neq 0$ and $\mu_2 := Q'\, \xi_2 \in D^{p'}$. Then, $\varphi_2 := \varphi_{\mu_2} \in \hom_D(N, D)$ satisfies:

$$\varphi_2(\iota(m_2)) = \lambda_2\, \mu_2 = \nu\, S\, P\, Q'\, \xi_2 \neq 0.$$

By construction, we have $m_2 \in \ker(\varphi_1 \circ \iota)$, which yields:

$$\varphi_1(\iota(m_2)) = \lambda_2\, \mu_1 = 0.$$

If $\varphi_2(\iota(m_1)) = \lambda_1\, \mu_2 \neq 0$, then, by the right Ore property (see Section 2), there exist $r_1,\, r_2 \in D \setminus \{0\}$ such that:

$$(\lambda_1\, \mu_1)\, r_1 + (\lambda_1\, \mu_2)\, r_2 = 0. \qquad (7)$$

Let us then consider:

$$\begin{cases} \mu_2' := \mu_1\, r_1 + \mu_2\, r_2 \in \ker_D(R'.), \\ \varphi_2' := \varphi_{\mu_1}\, r_1 + \varphi_{\mu_2}\, r_2 = \varphi_{\mu_2'} \in \hom_D(N, D). \end{cases}$$

Then, using (7), we have:

$$\begin{cases} \varphi_2'(\iota(m_1)) = \lambda_1\, \mu_2' = \lambda_1\, (\mu_1\, r_1 + \mu_2\, r_2) = 0, \\ \varphi_2'(\iota(m_2)) = \lambda_2\, \mu_2' = \lambda_2\, (\mu_1\, r_1 + \mu_2\, r_2) = (\lambda_2\, \mu_2)\, r_2 \neq 0. \end{cases}$$

Therefore, without loss of generality, we may assume that

$$\varphi_2(\iota(m_1)) = \lambda_1\, \mu_2 = 0.$$

Let $d_1 := \lambda_1\, \mu_1 \neq 0$ and $d_2 := \lambda_2\, \mu_2 \neq 0$. Corollary 17 then shows that there exists $(y_1,\, y_2,\, z_1,\, z_2)^T \in D^4$ satisfying

$$y_1\, (\lambda_1\, \mu_1)\, z_1 + y_2\, (\lambda_2\, \mu_2)\, z_2 = 1.$$

If we now introduce

$$\begin{cases} \eta^\star := y_1\, \eta_1 + y_2\, \eta_2 \in D^{1\times p}, \\ m^\star := \pi(\eta^\star) \in M, \\ \mu^\star := \mu_1\, z_1 + \mu_2\, z_2 \in \ker_D(R'.), \\ \varphi := \varphi_{\mu^\star} \in \hom_D(N, D), \end{cases}$$

then we have

$$\begin{aligned} \varphi(\iota(m^\star)) &= \eta^\star\, P\, \mu^\star = (y_1\, \eta_1 + y_2\, \eta_2)\, P\, (\mu_1\, z_1 + \mu_2\, z_2) \\ &= (y_1\, \lambda_1 + y_2\, \lambda_2)\, (\mu_1\, z_1 + \mu_2\, z_2) \\ &= y_1\, (\lambda_1\, \mu_1)\, z_1 + y_2\, (\lambda_2\, \mu_2)\, z_2 = 1, \end{aligned}$$

which shows that $\iota(m^\star) \in \mathrm{U}(N)$ and yields:

$$N = D\, \iota(m^\star) \oplus \ker \varphi.$$

Moreover, $\psi := \varphi_{|\iota(M)} \in \hom_D(\iota(M), D)$ satisfies $\psi(\iota(m^\star)) = 1$. Thus, $\iota(m^\star)$ is a unimodular element of $\iota(M)$, which shows that

$$\iota(M) = D\, \iota(m^\star) \oplus \ker \psi,$$

and thus,

$$\iota(M) = D\, \iota(m^\star) \oplus M' \subseteq N = D\, \iota(m^\star) \oplus N',$$

where $N' := \ker \varphi$ and $M' := \ker \varphi_{|\iota(M)} = \ker \varphi \cap \iota(M)$.

For the precise description of the algorithm corresponding to Theorem 19 and examples, see [16]. Theorem 19 resembles the characterization of vector spaces over a *division ring* D [8] (e.g., a field) for which $\mathrm{rank}_D(M) \geq 1$.

Theorem 20. ([18]). Let D be a very simple domain and M a finitely generated left D-module. Then, there exist $r \in \mathbb{Z}_{\geq 0}$ and a left D-module M' with $\mathrm{rank}_D(M') \leq 1$ such that $M \cong D^{1\times r} \oplus M'$. Moreover, if M is torsion-free, then M' can be chosen as a left ideal of D, which can be generated by two elements.

Proof. If $\mathrm{rank}_D(M) \leq 1$, the first statement holds with $r = 0$ and $M' = M$. If $\mathrm{rank}_D(M) \geq 2$, then applying Theorem 19 to $N = M$ and $\iota = \mathrm{id}_M$, then there exists $m \in \mathrm{U}(M)$ such that $M = D\, m \oplus N \cong D \oplus N$, where $\mathrm{rank}_D(N) = \mathrm{rank}_D(M) - 1$. Repeating the same argument on $\mathrm{rank}_D(N)$ and so on, we obtain:

$$M \cong D^{1\times r} \oplus M', \quad \mathrm{rank}_D(M') \leq 1.$$

Now, if M is torsion-free, so is M'. Moreover, if $M' \neq 0$, then $\mathrm{rank}_D(M') = 1$ by Proposition 3, and thus, M' admits a *minimal parametrization* [2], namely, M' is isomorphic to a finitely generated left ideal I of D, which can be generated by two elements by Remark 12.

For the precise description of the algorithm corresponding to Theorem 20 and explicit examples, see [16].

A system-theoretic interpretation of Theorem 20 is that every linear system $\ker_{\mathcal{F}}(R.)$ defined by a matrix R with entries in a very simple domain D satisfies

$$\ker_{\mathcal{F}}(R.) \cong \mathcal{F}^r \oplus \ker_{\mathcal{F}}(R'.), \tag{8}$$

where $M' = D^{1\times p'}/(D^{1\times q'}\, R')$ and $\mathrm{rank}_D(M') \leq 1$. (8) states that a linear differential system $\ker_{\mathcal{F}}(R.)$ is isomorphic to the direct sum of a differentially flat system and a linear system $\ker_{\mathcal{F}}(R'.)$ with at most one input.

6. STAFFORD'S REDUCTION

We give an application of Theorem 19 by studying when a linear system $\ker_{\mathcal{F}}(Q.)$ is isomorphic to a linear system defined by fewer unknowns and fewer equations.

Let $Q \in D^{p\times m}$, $R \in D^{q\times p}$ be such that

$$\ker_D(.Q) = \mathrm{im}_D(.R) \tag{9}$$

(possibly $R = 0$ and $q = 0$), and let us consider the left D-modules $N := D^{1\times m}$ and $M := D^{1\times p}/(D^{1\times q}\, R)$. Using

$$\mathrm{coim}\, f := M/\ker f \cong \mathrm{im}\, f,$$

for all $f \in \hom_D(M, N)$ (see, e.g., [17]), and (9), we get:

$$\begin{aligned} M &= D^{1\times p}/\mathrm{im}_D(.R) = D^{1\times p}/\ker_D(.Q) \\ &= \mathrm{coim}_D(.Q) \cong \mathrm{im}_D(.Q). \end{aligned}$$

Hence, if we consider $\iota \in \hom_D(M, N)$ defined by

$$\forall\, \lambda \in D^{1\times p}, \quad \iota(\pi(\lambda)) := \lambda\, Q,$$

where $\pi\colon D^{1\times p} \longrightarrow M$ is the canonical projection onto M, then $L := \iota(M) = \mathrm{im}_D(.Q) \subseteq N = D^{1\times m}$. The following diagram is commutative with exact rows:

$$\begin{array}{ccccccc} D^{1\times q} & \xrightarrow{.R} & D^{1\times p} & \xrightarrow{\pi} & M & \longrightarrow 0 \\ \downarrow & & \downarrow .Q & & \downarrow \iota & \\ 0 & \longrightarrow & D^{1\times m} & \xrightarrow{\mathrm{id}} & D^{1\times m} & \longrightarrow 0. \end{array}$$

If $\mathrm{rank}_D(L) = \mathrm{rank}_D(M) \geq 2$, then we can apply Theorem 19 to find $\eta^\star \in D^{1\times p}$ and $\xi^\star \in D^m$ such that $m^\star := \pi(\eta^\star) \in M$ satisfies $\iota(m^\star) = \eta^\star\, Q \in \mathrm{U}(N)$ and $\varphi := \varphi_{\xi^\star}$ satisfies $\varphi(\iota(m^\star)) = \eta^\star\, Q\, \xi^\star = 1$. Then, we have

$$D^{1\times m} = D\, \iota(m^\star) \oplus \ker \varphi = D\, (\eta^\star\, Q) \oplus \ker \varphi.$$

Since $\iota(m^\star)$ is also a unimodular element of L, we get

$$D^{1\times p}\, Q = D\, \iota(m^\star) \oplus \ker \varphi_{|L} = D\, (\eta^\star\, Q) \oplus \ker \varphi_{|L},$$

where $\ker \varphi_{|L} = \ker \varphi \cap L$. Hence, we obtain:

$$\begin{aligned} P &:= D^{1\times m}/(D^{1\times p}\, Q) \\ &= (D\, (\eta^\star\, Q) \oplus \ker \varphi)/(D\, (\eta^\star\, Q) \oplus \ker \varphi_{|L}) \\ &\cong P' := \ker \varphi/\ker \varphi_{|L}. \end{aligned}$$

Now, $\ker \varphi = \ker_D(.\xi^\star)$ and $\ker \varphi_{|L} = \ker_D(.(Q\, \xi^\star))\, Q$, so that we have $P' = \ker_D(.\xi^\star)/(\ker_D(.(Q\, \xi^\star))\, Q)$, and:

$$\begin{cases} \mathrm{rank}_D(\ker \varphi) = \mathrm{rank}_D(N) - 1 = m - 1, \\ \mathrm{rank}_D(\ker \varphi_{|L}) = \mathrm{rank}_D(L) - 1. \end{cases}$$

Since $\eta^\star\, Q\, \xi^\star = 1$, the following left D-homomorphisms

$$.\eta^\star : D \longrightarrow D^{1\times p}, \quad .(\eta^\star\, Q) : D \longrightarrow D^{1\times m}$$

satisfy $.(Q\, \xi^\star) \circ .\eta^\star = \mathrm{id}_D$ and $.\xi^\star \circ (.(\eta^\star\, Q)) = \mathrm{id}_D$. Hence, we have the following split short exact sequences

$$0 \longrightarrow \ker_D(.\xi^\star) \longrightarrow D^{1\times m} \xrightarrow{.\xi^\star} D \longrightarrow 0,$$

$$0 \longrightarrow \ker_D(.(Q\, \xi^\star)) \longrightarrow D^{1\times p} \xrightarrow{.(Q\, \xi^\star)} D \longrightarrow 0,$$

i.e., $D^{1\times m} \cong D \oplus \ker_D(.\xi^\star)$ and $D^{1\times p} \cong D \oplus \ker_D(.(Q\, \xi^\star))$, which shows that $\ker_D(.\xi^\star)$ (resp., $\ker_D(.(Q\, \xi^\star))$) is a stably free left D-module of rank $m - 1$ (resp., $p - 1$).

Then, we have the following commutative exact diagram

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
0 & \longrightarrow & \ker_D(.(Q\,\xi^\star)) & \longrightarrow & D^{1\times p} & \xrightarrow{.(Q\,\xi^\star)} & D & \longrightarrow & 0\\
 & & \downarrow .Q & & \downarrow .Q & & \| & & \\
0 & \longrightarrow & \ker_D(.\xi^\star) & \longrightarrow & D^{1\times m} & \xrightarrow{.\xi^\star} & D & \longrightarrow & 0\\
 & & \downarrow \tau' & & \downarrow \tau & & \downarrow & & \\
0 & \longrightarrow & P' & \xrightarrow{\alpha} & P & \longrightarrow & 0, & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

where the left D-isomorphism $\alpha : P' \longrightarrow P$ is defined by $\forall\ \theta \in \ker_D(.\xi^\star), \alpha(\tau'(\theta)) = \tau(\theta)$, and τ (resp., τ') is the canonical projection onto P (resp., P').

Now, if $m \geq 3$, $\ker\varphi = \ker_D(.\xi^\star)$ is a free left D-module of rank $m-1$ by Corollary 18. Computing a basis of $\ker\varphi$, there exists a full row rank matrix $X \in D^{(m-1)\times m}$ such that $\ker\varphi = D^{1\times(m-1)}\,X$. Let $Y \in D^{s\times p}$ be such that $\ker_D(.(Q\,\xi^\star)) = D^{1\times s}\,Y$ and $Z = Y\,Q \in D^{s\times m}$. Thus:

$$P' = (D^{1\times(m-1)}\,X)/(D^{1\times s}\,Z).$$

Since $\ker_D(.X) = 0$, if $F \in D^{s\times(m-1)}$ is such that $Z = F\,X$, then Lemma 3.1 of [3] shows that

$$\begin{array}{rcl}\gamma : P'' := D^{1\times(m-1)}/(D^{1\times s}\,F) & \longrightarrow & P = D^{1\times m}/(D^{1\times p}\,Q)\\ \sigma(\nu) & \longmapsto & \tau(\nu\,X),\end{array}$$

is an isomorphism, where $\sigma : D^{1\times(m-1)} \longrightarrow P''$ is the canonical projection onto P'', which yields

$$P = D^{1\times m}/(D^{1\times p}\,Q) \cong P'' = D^{1\times(m-1)}/(D^{1\times s}\,F),$$

and shows that one generator of the left D-module P can be removed from the presentation given by the matrix Q.

Moreover, if $p \geq 3$, then $\ker_D(.(Q\,\xi^\star))$ is a free left D-module of rank $p-1$ by Corollary 18. Thus, there exists a full row rank matrix $G \in D^{(p-1)\times p}$ such that $\ker_D(.(Q\,\xi^\star)) = D^{1\times(p-1)}\,G$, i.e., $s = p-1$, and:

$$P = D^{1\times m}/(D^{1\times p}\,Q) \cong P'' = D^{1\times(m-1)}/(D^{1\times(p-1)}\,G).$$

Theorem 21. Let D be a very simple domain and P a left D-module given by $P = D^{1\times m}/(D^{1\times p}\,Q)$. Then, we have:

(1) If $\mathrm{rank}_D(D^{1\times p}\,Q) \geq 2$ and $m \geq 3$, then there exists $\overline{Q} \in D^{s\times(m-1)}$ such that
$$P \cong \overline{P} := D^{1\times(m-1)}/(D^{1\times s}\,\overline{Q}).$$

(2) Moreover, if $p \geq 3$, then $\overline{Q}$ can be chosen so that $s = p-1$, i.e., we have:
$$P \cong \overline{P} := D^{1\times(m-1)}/(D^{1\times(p-1)}\,\overline{Q}).$$

Strangely enough, Theorem 21 does not appear in [18]. Theorem 21 is implemented in the Stafford package [14]. We note that $\mathrm{rank}_D(D^{1\times p}\,Q) \geq 2$ means that at least two equations of $Q\,\zeta = 0$ are D-linearly independent.

Using (2) and $P \cong \overline{P}$, the following isomorphisms hold

$$\ker_{\mathcal{F}}(Q.) \cong \hom_D(P, \mathcal{F}) \cong \hom_D(\overline{P}, \mathcal{F}) \cong \ker_{\mathcal{F}}(\overline{Q}.),$$

which shows that the number of unknowns and equations of $\ker_{\mathcal{F}}(Q.)$ can be reduced according to Theorem 21.

Corollary 22. ([18]). Let D be a very simple domain and $P = D^{1\times m}/(D^{1\times p}\,Q)$ a torsion left D-module. Then, P can be generated by two elements.

Proof. If $m \leq 2$, then there is nothing to show. Let us suppose that $m \geq 3$. Since $\mathrm{rank}_D(P) = 0$ (see Proposition 3), $\mathrm{rank}_D(\mathrm{im}_D(.Q)) = m$. Applying $m-2$ times 1 of Theorem 21, we obtain $P \cong \overline{P} := D^{1\times 2}/(D^{1\times s}\,\overline{Q})$.

Since a torsion left D-module P defines an *autonomous linear system* (see, e.g., [2]), Corollary 22 shows that every autonomous linear differential system is equivalent to a linear differential system in two unknown functions. Moreover, any state space representation $\dot{x} - A\,x = 0$ is observable with respect to two outputs y_1, y_2 given by two generators of the corresponding torsion D-module.

For more results, see [16].

REFERENCES

[1] N. Caro, D. Levcovitz. On a Theorem of Stafford. *Cadernos de Mathemática*, 11, 63–70, 2010.

[2] F. Chyzak, A. Quadrat, D. Robertz. Effective algorithms for parametrizing linear control systems over Ore algebras. *Appl. Algebra Engrg. Comm. Comput.*, 16, 319–376, 2005.

[3] T. Cluzeau, A. Quadrat. Factoring and decomposing a class of linear functional systems. *Linear Algebra Appl.*, 428, 324–381, 2008.

[4] M. Fliess. Controllability revisited. In A. C. Antoulas (ed.), *The Influence of R. E. Kalman*, Springer, 463–474, 1991.

[5] M. Fliess, J. Lévine, P. Martin, P. Rouchon. Flatness and defect of nonlinear systems: introductory theory and examples. *Int. J. Control*, 61, 1327–1361, 1995.

[6] A. Hillebrand, W. Schmale. Towards an effective version of a theorem of Stafford. *J. Symbolic Comput.*, 32, 699–716, 2001.

[7] M. Kashiwara. *Algebraic Study of Systems of Partial Differential Equations*. Mémoires de la Société Mathématiques de France, 63, 1995.

[8] T. Y. Lam. *Lectures on Modules and Rings*. Graduate Texts in Mathematics 189, Springer, 1999.

[9] A. Leykin. Algorithmic proofs of two theorems of Stafford. *J. Symbolic Comput.*, 38, 1535–1550, 2004.

[10] B. Malgrange. Systèmes différentiels à coefficients constants. *Séminaire Bourbaki*, 1962/63, 1–11.

[11] U. Oberst. Multidimensional constant linear systems. *Acta Appl. Math.*, 20, 1–175, 1990.

[12] J.-F. Pommaret, A. Quadrat. Algebraic analysis of linear multidimensional control systems. *IMA J. Math. Control Inform.*, 16, 275–297, 1999.

[13] A. Quadrat. An introduction to constructive algebraic analysis and its applications. *Les cours du CIRM*, 1 no. 2: Journées Nationales de Calcul Formel, 281–471, 2010, Inria report 7354.

[14] A. Quadrat, D. Robertz. Computation of bases of free modules over the Weyl algebras. *J. Symbolic Comput.*, 42, 1113–1141, 2007, Stafford project (`http://wwwb.math.rwth-aachen.de/OreModules`).

[15] A. Quadrat, D. Robertz. Controllability and differential flatness of linear analytic ordinary differential systems. *Proceedings of MTNS 2010*, 05-07/07/10.

[16] A. Quadrat, D. Robertz. A constructive study of the module structure of rings of partial differential operators. Inria report, to appear, 2013.

[17] J. J. Rotman. *An Introduction to Homological Algebra*. Springer, 2nd edition, 2009.

[18] J. T. Stafford. Module structure of Weyl algebras. *J. London Math. Soc.*, 18, 429–442, 1978.