Further results on Serre's reduction of multidimensional linear systems

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***Abstract*— Serre's reduction aims at reducing the number of unknowns and equations of a linear functional system (e.g., system of ordinary or partial differential equations, system of differential time-delay equations, system of difference equations). Finding an equivalent representation of a linear functional system containing fewer equations and fewer unknowns generally simplifies the study of its structural properties, its closed-form integration and different numerical issues. The purpose of this paper is to present a constructive approach to Serre's reduction for linear functional systems.**

I. An algebraic analysis approach to linear systems theory

In what follows, D will denote a *noncommutative noetherian domain*, namely, a unital ring satisfying that $d\,d'$ is not necessarily equal to $d'\,d$ for $d,\, d' \in D$, containing no nontrivial *zero-divisors*, i.e., $d\,d' = 0$ yields $d = 0$ or $d' = 0$, and every left (resp., right) ideal of D is *finitely generated*, i.e., can be generated by a finite family of elements of D as a left (resp., right) D-module ([9], [16]). Moreover, we shall denote by $D^{1\times p}$ (resp., D^q) the left (resp., right) D-module formed by row (resp., column) vectors of length p (resp., q) with entries in D and by $R \in D^{q\times p}$ a $q \times p$ matrix R with entries in D. Moreover, we shall use the following notations:

$$\begin{array}{rcl} .R : D^{1\times q} & \longrightarrow & D^{1\times p} \\ \mu & \longmapsto & \mu\, R, \end{array} \qquad \begin{array}{rcl} R. : D^{p} & \longrightarrow & D^{q} \\ \eta & \longmapsto & R\,\eta. \end{array} \tag{1}$$

Since the image $\mathrm{im}_D(.R) = D^{1\times q}\,R$ of the *left D-homomorphism* $.R : D^{1\times q} \longrightarrow D^{1\times p}$ defined by (1), i.e., $\mathrm{im}_D(.R) = \{\lambda \in D^{1\times p} \mid \exists\, \mu \in D^{1\times q} : \lambda = \mu\, R\}$, is a left D-submodule of $D^{1\times p}$, we can introduce the *quotient left D-module* $M = D^{1\times p}/(D^{1\times q}\,R)$ and the left D-homomorphism $\pi : D^{1\times p} \longrightarrow M$ which sends $\lambda \in D^{1\times p}$ to its residue class $\pi(\lambda)$ in M. In particular, $\pi(\lambda) = \pi(\lambda')$ iff there exists $\mu \in D^{1\times q}$ such that $\lambda - \lambda' = \mu\, R$. The left D-module $M = D^{1\times p}/(D^{1\times q}\,R)$ is then said to be *finitely presented* by R ([16]). Let us describe the left D-module $M = D^{1\times p}/(D^{1\times q}\,R)$ in terms of *generators* and *relations*. Let $\{f_j\}_{j=1,\ldots,p}$ be the *standard basis* of the left D-module $D^{1\times p}$, namely, f_j is the row vector of length p with 1 at the j^{th} position and 0 elsewhere, and $y_j \triangleq \pi(f_j) \in M$ for $j = 1, \ldots, p$. Since every $m \in M$ has the form $m = \pi(\lambda)$ for a certain row vector $\lambda = (\lambda_1 \ \ldots \ \lambda_p) \in D^{1\times p}$,

$$m = \pi\left(\sum_{j=1}^{p} \lambda_j\, f_j\right) = \sum_{j=1}^{p} \lambda_j\, \pi(f_j) = \sum_{j=1}^{p} \lambda_j\, y_j,$$

which shows that every element m of M can be written as a left D-linear combination of the y_j's, i.e., $\{y_j\}_{j=1,\ldots,p}$ is a *family of generators* of M. M is said to be *finitely generated* ([16]). If $R_{i\bullet}$ denotes the i^{th} row of the matrix $R \in D^{q\times p}$, then $R_{i\bullet} \in D^{1\times q}\,R$ which yields $\pi(R_{i\bullet}) = 0$, and thus

$$\pi\left(\sum_{j=1}^{p} R_{ij}\, f_j\right) = \sum_{j=1}^{p} R_{ij}\, \pi(f_j) = \sum_{j=1}^{p} R_{ij}\, y_j = 0, \tag{2}$$

for $i = 1, \ldots, q$, and shows that the generators $\{y_j\}_{j=1,\ldots,p}$ of M satisfy the *left D-linear relations* (2), or, in other words, $y = (y_1 \ \ldots \ y_p)^T \in M^p$ satisfies $R\,y = 0$.

If $\mathcal{F}$ is a left D-module and $\hom_D(M, \mathcal{F})$ is the abelian group (i.e., $\mathbb{Z}$-module) of the left D-homomorphisms from M to $\mathcal{F}$, then Malgrange's remark ([8]) asserts that

$$\ker_{\mathcal{F}}(R.) \triangleq \{\eta \in \mathcal{F}^p \mid R\,\eta = 0\} \cong \hom_D(M, \mathcal{F}), \tag{3}$$

where $\cong$ is an *isomorphism*, i.e., a bijective homomorphism. The linear system $\ker_{\mathcal{F}}(R.)$ is also called a *behaviour*. The above isomorphism $\chi : \ker_{\mathcal{F}}(R.) \longrightarrow \hom_D(M, \mathcal{F})$ can be easily defined: for all $\eta \in \ker_{\mathcal{F}}(R.)$, we can defined $\chi(\phi) = \phi_\eta \in \hom_D(M, \mathcal{F})$ by $\phi_\eta(\pi(\lambda)) = \lambda\,\eta$ for all $\lambda \in D^{1\times p}$. It is well-defined since if $\lambda \in D^{1\times q}\,R$, then there exists $\mu \in D^{1\times q}$ such that $\lambda = \mu\,R$, and thus $\pi(\lambda) = 0$, which, on the one hand, yields $\phi_\eta(\pi(\lambda)) = \phi_\eta(0) = 0$ and, on the other hand, $\lambda\,\eta = \mu\,(R\,\eta) = 0$. The inverse χ^{-1} is then defined by $\chi^{-1}(\phi) = (\phi(y_1) \ \ldots \ \phi(y_p))^T \in \mathcal{F}^p$, where $\{y_j = \pi(f_j)\}_{j=1,\ldots,p}$ is a family of generators of M as explained above. Indeed, if $\eta = (\phi(y_1) \ \ldots \ \phi(y_p))^T$, then

$$\sum_{j=1}^{p} R_{ij}\,\eta_j = \sum_{j=1}^{p} R_{ij}\,\phi(y_j) = \phi\left(\sum_{j=1}^{p} R_{ij}\,y_j\right) = \phi(0) = 0,$$

i.e., $\eta \in \ker_{\mathcal{F}}(R.)$, and $(\chi^{-1} \circ \chi)(\phi) = \chi^{-1}(\phi_\eta) = \eta$.

The algebraic analysis approach to linear systems theory aims at intrinsically studying the linear system $\ker_{\mathcal{F}}(R.)$ by means of $\hom_D(M, \mathcal{F})$, i.e., by means of the left D-modules $M = D^{1\times p}/(D^{1\times q}\,R)$ and $\mathcal{F}$ ([3], [8], [10], [11]).

Definition 1 ([6], [9], [16]): Let D be a left noetherian domain and $M = D^{1\times p}/(D^{1\times q}\,R)$ the left D-module finitely presented by the matrix $R \in D^{q\times p}$.

1) M is *free of rank* $r \in \mathbb{N} = \{0, 1, \ldots\}$ if $M \cong D^{1\times r}$.

2) M is *stably free of rank* $r-s$ if there exist $r,\, s \in \mathbb{N}$ such that $M \oplus D^{1\times s} \cong D^{1\times r}$, where $\oplus$ denotes the direct sum of left D-modules.
3) M is *projective* if there exist $r \in \mathbb{N}$ and a left D-module P such that $M \oplus P \cong D^{1\times r}$.
4) M is *torsion-free* if the torsion left D-submodule

$$t(M) = \{m \in M \mid \exists\, d \in D \setminus \{0\}:\ d\,m = 0\}$$

of M is reduced to 0, i.e., $t(M) = 0$.
5) M is *torsion* if $t(M) = M$, i.e., every $m \in M$ is a *torsion element* of M, namely, $m \in t(M)$.
6) M is *cyclic* if M is generated by one element $m \in M$, i.e., $M = D\,m \triangleq \{d\,m \mid d \in D\}$.

A free module is clearly stably free (take $s = 0$ in 2 of Definition 1) and a stably free module is projective (take $P = D^{1\times s}$ in 3 of Definition 1) and a projective module is torsion-free (since it can be embedded into a free, and thus, into a torsion-free module) but the converse of these results are generally not true for a general left noetherian domain.

Theorem 1 ([6], [9], [15], [16]): 1) If D is a *principal left ideal domain*, namely, every left ideal of D can be generated by one element of D (e.g., the ring of ordinary differential operators with coefficients in a differential field such that $K = \mathbb{R}$ or $\mathbb{R}(t)$), then every finitely generated torsion-free left D-module is free.
2) If $D = k[x_1, \ldots, x_n]$ is a commutative polynomial ring over a field k, then every finitely generated projective D-module is free (Quillen-Suslin theorem).
3) If k is a field of characteristic 0 (e.g., $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$) and $D = A_n(k)$ (resp., $B_n(k)$) is the first (resp., second) Weyl algebra of partial differential operators in $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}$ with coefficients in $k[x_1, \ldots, x_n]$ (resp., $k(x_1, \ldots, x_n)$), then every finitely generated projective left D-module is stably free and every stably free left D-module of rank at least 2 is free (Stafford's theorem).
4) If D is the ring of ordinary differential operators with coefficients in the ring of formal power series $k[\![t]\!]$, where k is a field of characteristic 0, or in the ring of convergent power series $k\{t\}$ with coefficients in $k = \mathbb{R}$ or $\mathbb{C}$, then every finitely generated projective left D-module is stably free and every stably free left D-module of rank at least 2 is free.

Let us characterize stably free and free modules.

Proposition 1 ([5], [13]): Let D be a noetherian domain, $R \in D^{q\times p}$ a *full row rank matrix*, i.e., $\ker_D(.R) = 0$, and $M = D^{1\times p}/(D^{1\times q}\,R)$.
1) M is a projective left D-module iff M is a stably free left D-module.
2) M is a stably free left D-module of rank $p-q$ iff R admits a *right-inverse* over D, namely, iff there exists a matrix $S \in D^{p\times q}$ satisfying $R\,S = I_q$.
3) M is a free left D-module of rank $p-q$ iff there exists a matrix $U \in \mathrm{GL}_p(D)$, where

$$\mathrm{GL}_p(D) = \{V \in D^{p\times p} \mid \exists\, W \in D^{p\times p}:\ V\,W = W\,V = I_p\},$$

such that $R\,U = (I_q \quad 0)$. If we write $U = (S \quad Q)$, where $S \in D^{p\times q}$ and $Q \in D^{p\times (p-q)}$, then

$$\begin{array}{rcl} \psi : M & \longrightarrow & D^{1\times (p-q)} \\ \pi(\lambda) & \longmapsto & \lambda\, Q, \end{array}$$

is a left D-isomorphism and ψ^{-1} is defined by:

$$\begin{array}{rcl} \psi^{-1} : D^{1\times (p-q)} & \longrightarrow & M \\ \mu & \longmapsto & \pi(\mu\, T), \end{array}$$

where the matrix $T \in D^{(p-q)\times p}$ is defined by:

$$U^{-1} = \begin{pmatrix} R \\ T \end{pmatrix} \in D^{p\times p}.$$

Then, $M \cong D^{1\times p}\,Q = D^{1\times (p-q)}$ and the matrix Q is called an *injective parametrization* of M. Finally, $\{\pi(T_{i\bullet})\}_{i=1,\ldots,p-q}$ defines a basis of the free left D-module M of rank $p-q$.

Let D be a left noetherian domain and $R \in D^{q\times p}$. Then, the left D-submodule $\ker_D(.R) = \{\mu \in D^{1\times q} \mid \mu\, R = 0\}$ of $D^{1\times q}$ is finitely generated (see, e.g., [16]). Therefore, there exists a finite family of generators $\{\mu_k\}_{k=1,\ldots,r}$ of $\ker_D(.R)$ and defining $R_2 = (\mu_1^T \ \ldots \ \mu_r^T)^T \in D^{r\times p}$, we get $\ker_D(.R) = D^{1\times r}\, R_2$. Similarly, we can find a matrix $R_3 \in D^{s\times r}$ such that $\ker_D(.R_2) = D^{1\times s}\, R_3$ and so on. We are led to the *concept of a finite free resolution* of M.

Definition 2: 1) A *complex* of left (resp., right) D-modules, denoted by

$$M_\bullet \ \ldots \xrightarrow{d_{i+2}} M_{i+1} \xrightarrow{d_{i+1}} M_i \xrightarrow{d_i} M_{i-1} \xrightarrow{d_{i-1}} \ldots, \quad (4)$$

is a sequence of left (resp., right) D-homomorphisms $d_i : M_i \longrightarrow M_{i-1}$ between left (resp., right) D-modules which satisfy $\mathrm{im}\, d_{i+1} \subseteq \ker d_i$, i.e.,

$$\forall\, i \in \mathbb{Z}, \quad d_i \circ d_{i+1} = 0.$$

2) The *defect of exactness* of (4) at M_i is defined by:

$$H_i(M_\bullet) \triangleq \ker d_i / \mathrm{im}\, d_{i+1}.$$

3) The complex (4) is *exact at* M_i if $H_i(M_\bullet) = 0$, i.e., $\ker\, d_i = \mathrm{im}\, d_{i+1}$, and *exact* if $\ker\, d_i = \mathrm{im}\, d_{i+1}$ for all $i \in \mathbb{Z}$. An exact complex is called an *exact sequence*.
4) A *finite free resolution* of the left D-module M is an exact sequence of the form

$$\ldots \xrightarrow{.R_3} D^{1\times p_2} \xrightarrow{.R_2} D^{1\times p_1} \xrightarrow{.R_1} D^{1\times p_0} \xrightarrow{\pi} M \longrightarrow 0, \quad (5)$$

where $R_i \in D^{p_i \times p_{i-1}}$ and $.R_i : D^{1\times p_i} \longrightarrow D^{1\times p_{i-1}}$ is defined by $(.R_i)(\lambda) = \lambda\, R_i$ for all $\lambda \in D^{1\times p_i}$.

If D is a left noetherian domain, then the above comment shows that the left D-module $M = D^{1\times p}/(D^{1\times q}\, R)$ admits a finite free resolution of the form (5), where $R_1 = R$, $p_0 = p$

and $p_1 = p$. If $\mathcal{F}$ is a left D-module, then a necessary condition for the *solvability* of the inhomogeneous linear system $R_1\,\eta = \zeta$ for a fixed $\zeta \in \mathcal{F}^{p_1}$ is $R_2\,\zeta = 0$, where $R_2 \in D^{p_2 \times p_1}$ is such that $\ker_D(.R_1) = D^{1\times p_2} R_2$. Indeed, for every $\mu \in \ker_D(.R_1)$, $R_1\,\eta = \zeta$ yields $\mu\,\zeta = \mu\,R_1\,\eta = 0$. Let us study when the necessary condition $R_2\,\zeta = 0$ is also sufficient. We need to investigate the defect of exactness $\ker_{\mathcal{F}}(R_2.)/\mathrm{im}_{\mathcal{F}}(R_1.)$ of the following complex at $\mathcal{F}^{p_1}$

$$\mathcal{F}^{p_2} \xleftarrow{R_2.} \mathcal{F}^{p_1} \xleftarrow{R_1.} \mathcal{F}^{p_0}, \tag{6}$$

where $R_i. : \mathcal{F}^{p_{i-1}} \longrightarrow \mathcal{F}^{p_i}$ is defined by $(R_i.)(\eta) = R_i\,\eta$ for all $\eta \in \mathcal{F}^{p_{i-1}}$ and $i = 1, 2$. Indeed, for a fixed $\zeta \in \mathcal{F}^{p_1}$, there exists $\eta \in \mathcal{F}^{p_0}$ satisfying $R_1\,\eta = \zeta$ iff $\zeta \in \mathrm{im}_{\mathcal{F}}(R_1.) = R_1\,\mathcal{F}^{p_0}$ and the necessary condition $R_2\,\zeta = 0$ (since $R_2\,R_1 = 0$) means that $\zeta \in \ker_{\mathcal{F}}(R_2.)$. Therefore, there exists $\eta \in \mathcal{F}^{p_1}$ satisfying $R_1\,\eta = \zeta$ iff the residue class of ζ in $\ker_{\mathcal{F}}(R_2.)/\mathrm{im}_{\mathcal{F}}(R_1.)$ is reduced to 0. A key result in homological algebra proves that the defect of exactness of (6) at $\mathcal{F}^{p_1}$ depends only on M and $\mathcal{F}$ and not on the choice of the beginning of the finite free resolution (5) of the left D-module M (see [16]). Hence, up to isomorphism, the defect of exactness of (6) at $\mathcal{F}^{p_1}$ is denoted by:

$$\mathrm{ext}^1_D(M, \mathcal{F}) \cong \ker_{\mathcal{F}}(R_2.)/\mathrm{im}_{\mathcal{F}}(R_1.). \tag{7}$$

If the complex (6) is exact at $\mathcal{F}^{p_1}$, i.e., $\mathrm{ext}^1_D(M, \mathcal{F}) = 0$, then the necessary condition $R_2\,\zeta = 0$ for the solvability of the inhomogeneous linear system $R_1\,\eta = \zeta$ is also sufficient. This fact explains why the *extension abelian group* $\mathrm{ext}^1_D(M, \mathcal{F})$ plays an important role in linear systems theory.

II. Baer's extensions

In this section, we extend the results obtained in [2]. Let D be a noetherian domain and $R \in D^{q\times p}$ a full row rank matrix, i.e., $\ker_D(.R) = 0$. Then, we have the following *short exact sequence* of left D-modules

$$0 \longrightarrow D^{1\times q} \xrightarrow{.R} D^{1\times p} \xrightarrow{\pi} M \longrightarrow 0, \tag{8}$$

i.e., $.R$ is an injective left D-homomorphism (since $\ker_D(.R) = 0$), $\ker_D \pi = D^{1\times q} R$ and π is a surjective left D-homomorphism (since, by definition of M, every element $m \in M$ has the form $m = \pi(\lambda)$ for a certain $\lambda \in D^{1\times p}$).

Let $0 \leq r \leq q-1$ and let us now consider the matrices

$$\Lambda \in D^{q\times(q-r)}, \quad P = (R \quad -\Lambda) \in D^{q\times(p+q-r)},$$

the left D-module $E = D^{1\times(p+q-r)}/(D^{1\times q} P)$ finitely presented by the full row rank matrix P. Then, the following short exact sequence of left D-modules holds

$$0 \longrightarrow D^{1\times q} \xrightarrow{.P} D^{1\times(p+q-r)} \xrightarrow{\varrho} E \longrightarrow 0, \tag{9}$$

where $\varrho : D^{1\times(p+q-r)} \longrightarrow E$ is the canonical projection onto E, i.e., the left D-homomorphism which sends an element $\zeta \in D^{1\times(p+q-r)}$ to its residue class $\varrho(\zeta)$ in E.

Let us study the connections between the left D-modules M and E. If $X = (I_p^T \quad 0^T)^T \in D^{(p+q-r)\times p}$, then the identity $R = P\,X$ induces the commutative exact diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & D^{1\times q} & \xrightarrow{.P} & D^{1\times(p+q-r)} & \xrightarrow{\varrho} & E & \longrightarrow & 0 \\ & & \| & & \downarrow .X & & & & \\ 0 & \longrightarrow & D^{1\times q} & \xrightarrow{.R} & D^{1\times p} & \xrightarrow{\pi} & M & \longrightarrow & 0, \end{array}$$

and the left D-homomorphism $\beta : E \longrightarrow M$ defined by

$$\beta(\varrho((\mu_1 \quad \mu_2))) = \pi((\mu_1 \quad \mu_2)\,X) = \pi(\mu_1),$$

for all $\mu_1 \in D^{1\times p}$ and all $\mu_2 \in D^{1\times(q-r)}$. For every $m \in M$, there exists $\mu_1 \in D^{1\times p}$ such that $m = \pi(\mu_1)$ and thus $m = \beta(\varrho((\mu_1 \quad 0)))$, which proves that β is surjective.

Let us study $\ker\beta$. An element $\varrho((\mu_1 \quad \mu_2)) \in \ker\beta$ satisfies $\pi(\mu_1) = 0$, i.e., $\mu_1 = \nu\,R$ for a certain $\nu \in D^{1\times q}$. Since $\varrho((\nu\,R \quad -\nu\,\Lambda)) = 0$, we get $\varrho((\nu\,R \quad 0)) = \varrho((0 \quad \nu\,\Lambda))$

$$\begin{aligned} \Rightarrow\ \ker\beta &= \{\varrho((\nu\,R \quad \mu_2)) = \varrho((0 \quad \mu_2 + \nu\,\Lambda)) \\ &\qquad\qquad \mid \nu \in D^{1\times q},\ \mu_2 \in D^{1\times(q-r)}\Big\} \\ &= \Big\{\varrho((0 \quad \xi)) \mid \xi \in D^{1\times(q-r)}\Big\}. \end{aligned}$$

Let $\gamma : D^{1\times(q-r)} \longrightarrow \ker\beta$ be the left D-isomorphism defined by $\gamma(\xi) = \varrho((0 \quad \xi))$ for all $\xi \in D^{1\times(q-r)}$ (i.e., γ is injective and surjective). The canonical short exact sequence $0 \longrightarrow \ker\beta \xrightarrow{i} E \xrightarrow{\beta} \mathrm{im}\,\beta \longrightarrow 0$ then yields

$$0 \longrightarrow D^{1\times(q-r)} \xrightarrow{\alpha} E \xrightarrow{\beta} M \longrightarrow 0, \tag{10}$$

where $\alpha = i \circ \gamma$. The short exact sequence (10) is called a *Baer extension of* $D^{1\times(q-r)}$ *by* M (see, e.g., [16]) and we shall simply say an *extension of* $D^{1\times(q-r)}$ *by* M.

Let us now introduce the matrices $\Theta \in D^{p\times(q-r)}$,

$$\overline{\Lambda} = \Lambda + R\,\Theta \in D^{q\times(q-r)}, \ \overline{P} = (R \quad -\overline{\Lambda}) \in D^{q\times(p+q-r)},$$

and the left D-module $\overline{E} = D^{1\times(p+q-r)}/(D^{1\times q}\,\overline{P})$ finitely presented by $\overline{P}$. Let $\overline{\varrho} : D^{1\times(p+q-r)} \longrightarrow \overline{E}$ be the canonical projection onto $\overline{E}$. As previously with the left D-module $\overline{E}$, we obtain the extension of $D^{1\times(q-r)}$ by M defined by

$$0 \longrightarrow D^{1\times(q-r)} \xrightarrow{\overline{\alpha}} \overline{E} \xrightarrow{\overline{\beta}} M \longrightarrow 0,$$

where $\overline{\alpha}(\xi) = \overline{\varrho}((0 \quad \xi))$ and $\overline{\beta}(\overline{\varrho}((\mu_1 \quad \mu_2))) = \pi(\mu_1)$ for all $\xi \in D^{1\times(q-r)}$, all $\mu_1 \in D^{1\times p}$ and all $\mu_2 \in D^{1\times(q-r)}$.

If we introduce the matrix V defined by

$$V = \begin{pmatrix} I_p & \Theta \\ 0 & I_{q-r} \end{pmatrix} \in \mathrm{GL}_{p+q-r}(D),$$

then $P = \overline{P}\,V$ induces the commutative exact diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & D^{1\times q} & \xrightarrow{.\overline{P}} & D^{1\times(p+q-r)} & \xrightarrow{\overline{\varrho}} & \overline{E} & \longrightarrow & 0 \\ & & \| & & \downarrow .V & & & & \\ 0 & \longrightarrow & D^{1\times q} & \xrightarrow{.P} & D^{1\times(p+q-r)} & \xrightarrow{\varrho} & E & \longrightarrow & 0. \end{array}$$

Since $V \in \mathrm{GL}_{p+q-r}(D)$, we get the left D-isomorphism $\psi : \overline{E} \longrightarrow E$ defined by

$$\psi(\overline{\varrho}((\mu_1 \quad \mu_2))) = \varrho((\mu_1 \quad \mu_2)\,V) = \varrho((\mu_1 \quad \mu_1\,\Theta + \mu_2)),$$

for all $\mu_1 \in D^{1\times p}$ and all $\mu_2 \in D^{1\times(q-r)}$. Then, we have

$$(\psi \circ \overline{\alpha})(\xi) = \psi(\overline{\varrho}((0 \quad \xi))) = \varrho((0 \quad \xi)) = \alpha(\xi),$$

for all $\xi \in D^{1\times(q-r)}$, which proves $\alpha = \psi \circ \overline{\alpha}$. Now,

$$\begin{aligned}(\beta \circ \psi)(\overline{\varrho}((\mu_1 \quad \mu_2))) &= \beta(\varrho((\mu_1 \quad \mu_2 + \mu_1 \Theta))) \\ &= \pi_1(\mu_1) = \overline{\beta}(\overline{\varrho}((\mu_1 \quad \mu_2))),\end{aligned}$$

for all $\mu_1 \in D^{1\times p}$ and all $\mu_2 \in D^{1\times(q-r)}$, which proves $\overline{\beta} = \beta \circ \psi$. Thus, we get the commutative exact diagram:

$$\begin{array}{ccccccccc} 0 \longrightarrow & D^{1\times(q-r)} & \xrightarrow{\overline{\alpha}} & \overline{E} & \xrightarrow{\overline{\beta}} & M & \longrightarrow 0 \\ & \| & & \downarrow \psi & & \| & \\ 0 \longrightarrow & D^{1\times(q-r)} & \xrightarrow{\alpha} & E & \xrightarrow{\beta} & M & \longrightarrow 0. \end{array} \tag{11}$$

We are then led to the definition of *equivalent extensions*.

Definition 3 ([16]): Two extensions of $D^{1\times(q-r)}$ by M

$$\begin{aligned} e : 0 \longrightarrow D^{1\times(q-r)} \xrightarrow{\alpha} E \xrightarrow{\beta} M \longrightarrow 0, \\ \overline{e} : 0 \longrightarrow D^{1\times(q-r)} \xrightarrow{\overline{\alpha}} \overline{E} \xrightarrow{\overline{\beta}} M \longrightarrow 0, \end{aligned}$$

are said to be *equivalent* if there exists a left D-homomorphism $\psi : \overline{E} \longrightarrow E$ satisfying $\alpha = \psi \circ \overline{\alpha}$ and $\overline{\beta} = \beta \circ \psi$, i.e., if (11) is a commutative exact diagram.

If e and $\overline{e}$ are equivalent extensions, then we can easily check that ψ is necessarily a left D-isomorphism (e.g., apply the *snake lemma* ([16]) to (11)). Hence, $\sim$ is an equivalence relation on the set of extensions of $D^{1\times(q-r)}$ by M ([16]). We denote by $\mathrm{e}_D\left(M, D^{1\times(q-r)}\right)$ the set of all equivalence classes of extensions of $D^{1\times(q-r)}$ by M and $[e]$ the equivalence class of the extension e of $D^{1\times(q-r)}$ by M.

The previous results show that the extensions of $D^{1\times(q-r)}$ by M defined by E and $\overline{E}$, i.e., by means of the matrices Λ and $\overline{\Lambda} = \Lambda + R\,\Theta$ for $\Theta \in D^{p\times(q-r)}$, are equivalent, and thus they define the same equivalence class in $\mathrm{e}_D\left(M, D^{1\times(q-r)}\right)$.

Let us now explain another relation between $\mathrm{e}_D\left(M, D^{1\times(q-r)}\right)$ and the matrices Λ and $\overline{\Lambda} = \Lambda + R\,\Theta$. Using (8), i.e., $R_2 = 0$, and $\mathcal{F} = D^{1\times(q-r)}$, we get $\ker_{\mathcal{F}}(R_2.) = D^{q\times(q-r)}$ and (7) yields:

$$\mathrm{ext}^1_D\left(M, D^{1\times(q-r)}\right) \cong D^{q\times(q-r)}/\left(R\,D^{p\times(q-r)}\right). \tag{12}$$

If $\rho : D^{q\times(q-r)} \longrightarrow D^{q\times(q-r)}/\left(R\,D^{p\times(q-r)}\right)$ is the canonical projection, then we have

$$\forall\, \Theta \in D^{p\times(q-r)}, \quad \rho(\overline{\Lambda}) = \rho(\Lambda + R\,\Theta) = \rho(\Lambda),$$

i.e., Λ and $\overline{\Lambda} = \Lambda + R\,\Theta$ define the same residue class in $D^{q\times(q-r)}/\left(R\,D^{p\times(q-r)}\right)$. We have just proved that every element $\rho(\Lambda) \in D^{q\times(q-r)}/\left(R\,D^{p\times(q-r)}\right)$ defines the equivalence class $[e]$ of extensions of $D^{1\times(q-r)}$ by M, where

$$e : 0 \longrightarrow D^{1\times(q-r)} \xrightarrow{\alpha} E \xrightarrow{\beta} M \longrightarrow 0,$$

and the left D-module E is finitely presented by the matrix $P = (R \quad -\Lambda)$, i.e., $E = D^{1\times(p+q-r)}/(D^{1\times q}\,P)$.

Let us now study the converse of this result. We first consider the following extension of $D^{1\times(q-r)}$ by M:

$$0 \longrightarrow D^{1\times(q-r)} \xrightarrow{\varepsilon} F \xrightarrow{\delta} M \longrightarrow 0. \tag{13}$$

Let $\{f_i\}_{i=1,\ldots,p}$ be the standard basis of $D^{1\times p}$, namely, f_i is the row vector with 1 at the i^{th} position and 0 elsewhere. Since the left D-homomorphism δ is surjective, there exists $\zeta_i \in F$ such that $\delta(\zeta_i) = \pi(f_i) \in M$ for $i = 1, \ldots, p$. Then,

$$\begin{aligned}\delta\left(\sum_{k=1}^{p} R_{jk}\,\zeta_k\right) &= \sum_{k=1}^{p} R_{jk}\,\delta(\zeta_k) = \sum_{k=1}^{p} R_{jk}\,\pi(f_k) \\ &= \pi\left(\sum_{k=1}^{p} R_{jk}\,f_k\right) = \pi(R_{j\bullet}) = 0,\end{aligned}$$

for $j = 1, \ldots, q$. Since $\ker\delta = \operatorname{im}\varepsilon$ and ε is injective, there exists a unique element $\lambda_j \in D^{1\times(q-r)}$ such that $\sum_{k=1}^{p} R_{jk}\,\zeta_k = \varepsilon(\lambda_j)$. If $\Lambda = (\lambda_1^T \ \ldots \ \lambda_q^T)^T \in D^{q\times(q-r)}$, then we get $\rho(\Lambda) \in D^{q\times(q-r)}/\left(R\,D^{p\times(q-r)}\right)$. Let us check that the residue class $\rho(\Lambda)$ of Λ is well-defined, i.e., it does not depend on the choice of the pre-images ζ_i's of the $\pi(f_i)$'s. Let us consider other pre-images $\overline{\zeta}_i$'s of the $\pi(f_i)$, i.e., $\delta(\overline{\zeta}_i) = \pi(f_i)$ for $i = 1, \ldots, p$. Using the same arguments, there exists $\overline{\lambda}_j \in D^{1\times(q-r)}$ such that $\sum_{k=1}^{p} R_{jk}\,\overline{\zeta}_k = \varepsilon(\overline{\lambda}_j)$ for $j = 1, \ldots, q$. But, $\delta(\overline{\zeta}_i) = \delta(\zeta_i)$ yields $\delta(\overline{\zeta}_i - \zeta_i) = 0$, i.e., $\overline{\zeta}_i - \zeta_i \in \ker\delta = \operatorname{im}\varepsilon$ and thus there exists $\theta_i \in D^{1\times(q-r)}$ such that $\overline{\zeta}_i = \zeta_i + \varepsilon(\theta_i)$

$$\begin{aligned}\Rightarrow\ \varepsilon(\overline{\lambda}_j) &= \sum_{k=1}^{p} R_{jk}\,\overline{\zeta}_k = \varepsilon(\lambda_j) + \sum_{k=1}^{p} R_{jk}\,\varepsilon(\theta_k) \\ &= \varepsilon\left(\lambda_j + \sum_{k=1}^{p} R_{jk}\,\theta_k\right).\end{aligned} \tag{14}$$

If we introduce the following two matrices

$$\overline{\Lambda} = \begin{pmatrix} \overline{\lambda}_1 \\ \vdots \\ \overline{\lambda}_q \end{pmatrix} \in D^{q\times(q-r)}, \quad \Theta = \begin{pmatrix} \theta_1 \\ \vdots \\ \theta_p \end{pmatrix} \in D^{p\times(q-r)},$$

then, since ε is injective, (14) yields $\overline{\lambda}_j = \lambda_j + \sum_{k=1}^{p} R_{jk}\,\theta_k$ for $j = 1, \ldots, q$, i.e., $\overline{\Lambda} = \Lambda + R\,\Theta$, and thus $\rho(\overline{\Lambda}) = \rho(\Lambda + R\,\Theta) = \rho(\Lambda)$, which proves that every extension (13) of $D^{1\times(q-r)}$ by M defines a unique element $\rho(\Lambda)$ of the right D-module $D^{q\times(q-r)}/\left(R\,D^{p\times(q-r)}\right)$. Finally, let us show that every extension in the same equivalence class of (13) in $\mathrm{e}_D\left(M, D^{1\times(q-r)}\right)$ defines the same element $\rho(\Lambda)$. Let us consider an extension of $D^{1\times(q-r)}$ by M in the same equivalence class of (13), i.e., the commutative exact diagram

$$\begin{array}{ccccccc} 0 \longrightarrow & D^{1\times(q-r)} & \xrightarrow{\varepsilon} & F & \xrightarrow{\delta} & M & \longrightarrow 0 \\ & \| & & \downarrow \psi & & \| & \\ 0 \longrightarrow & D^{1\times(q-r)} & \xrightarrow{\varepsilon'} & F' & \xrightarrow{\delta'} & M & \longrightarrow 0, \end{array}$$

holds for a certain left D-isomorphism $\psi \in \hom_D(F, F')$. Using $\delta' \circ \psi = \delta$, we obtain that $\delta'(\psi(\zeta_i)) = \delta(\zeta_i) = \pi(f_i)$ for $i = 1, \ldots, p$, and applying ψ to $\sum_{k=1}^{p} R_{jk}\,\zeta_k = \varepsilon(\lambda_j)$ and using $\varepsilon' = \psi \circ \varepsilon$, we get $\sum_{k=1}^{p} R_{jk}\,\psi(\zeta_k) = \varepsilon'(\lambda_j)$

for $j = 1, \ldots, q$, which yields the same matrix $\Lambda = (\lambda_1^T \ \ldots \ \lambda_q^T)$ as previously, and thus the same $\rho(\Lambda)$.

Hence, there is a one-to-one correspondence between the elements of the right D-module $D^{q\times(q-r)}/\left(R\, D^{p\times(q-r)}\right) \cong \mathrm{ext}_D^1\left(M, D^{1\times(q-r)}\right)$ and the equivalence classes of extensions of $D^{1\times(q-r)}$ by M. This result is attributed to Baer. An important consequence of this result is that every equivalence class of extensions of $D^{1\times(q-r)}$ by M contains an extension

$$0 \longrightarrow D^{1\times(q-r)} \xrightarrow{\alpha} E_{\rho(\Lambda)} \xrightarrow{\beta} M \longrightarrow 0,$$

where $E_{\rho(\Lambda)} = D^{1\times(p+q-r)}/(D^{1\times q}\,(R \quad -\Lambda))$ for a certain $\Lambda \in D^{q\times(q-r)}$. The *Baer sum* $[e_1]+[e_2]$ of two equivalence classes $[e_1]$ and $[e_2]$ of extensions of $D^{1\times(q-r)}$ by M, respectively defined by representatives formed by $E_{\rho(\Lambda_1)}$ and $E_{\rho(\Lambda_2)}$, is the equivalence class of the extension defined by $E_{\rho(\Lambda_1+\Lambda_2)}$. See [14], [16] for proofs. Endowed with the Baer sum and the neutral element defined by the equivalence class of the extension of $D^{1\times(q-r)}$ by M defined by

$$E_{\rho(0)} = D^{1\times(p+q-r)}/(D^{1\times q}\,(R \quad 0)) \cong D^{1\times(q-r)} \oplus M,$$

i.e., the equivalence class of the *split short exact sequence*

$$0 \longrightarrow D^{1\times(q-r)} \xrightarrow{\alpha} D^{1\times(q-r)} \oplus M \xrightarrow{\beta} M \longrightarrow 0,$$

we can prove that $\mathrm{e}_D\left(M, D^{1\times(q-r)}\right)$ inherits an abelian group structure and $\mathrm{e}_D\left(M, D^{1\times(q-r)}\right)$ is isomorphic to the abelian group $\mathrm{ext}_D^1\left(M, D^{1\times(q-r)}\right)$ (see, e.g., [14], [16]).

Theorem 2 ([14], [16]): We have:

$$\mathrm{ext}_D^1\left(M, D^{1\times(q-r)}\right) \cong \mathrm{e}_D\left(M, D^{1\times(q-r)}\right).$$

Substituting $r = q-1$ in (12), we obtain the isomorphism $\mathrm{ext}_D^1(M, D) \cong D^q/(R\,D^p)$. A classical result in homological algebra asserts that

$$\mathrm{ext}_D^1\left(M, D^{1\times(q-r)}\right) \cong \mathrm{ext}_D^1(M, D)^{1\times(q-r)},$$

for all left D-modules M. If $\tau : D^q \longrightarrow D^q/(R\,D^p)$ is the canonical projection, then an element $\rho(\Lambda)$ can be interpreted as a row vector of length $q-r$ formed by the elements $\tau(\Lambda_{\bullet i}) \in D^q/(R\,D^p)$, where $\Lambda_{\bullet i}$ is the i^{th} column of the matrix $\Lambda \in D^{q\times(q-r)}$, i.e.:

$$\rho(\Lambda) = (\tau(\Lambda_{\bullet 1}) \ \ldots \ \tau(\Lambda_{\bullet (q-r)})) \in (D^q/(R\,D^p))^{1\times(q-r)}.$$

III. Serre's reduction

In what follows, we shall assume that M is finitely presented by a full row rank matrix $R \in D^{q\times p}$, i.e., $\ker_D(.R) = 0$ and $M = D^{1\times p}/(D^{1\times q}\,R)$. A natural question is whether or not there exists $\rho(\Lambda)$ such that the left D-module $E_{\rho(\Lambda)} = D^{1\times(p+q-r)}/(D^{1\times q}\,P)$ – finitely presented by $P = (R \quad -\Lambda)$ and defining an extension of $D^{1\times(q-r)}$ by M – is projective, stably free or free. In [17], J.-P. Serre studied this problem for the commutative polynomial ring $D = k[x_1, \ldots, x_n]$, where k is a field.

By definition of the extension right D-module, we have:

$$\begin{cases} \mathrm{ext}_D^1(M, D) \cong D^q/(R\,D^p), \\ \mathrm{ext}_D^1(E, D) \cong D^q/\left(P\,D^{(p+q-r)}\right). \end{cases}$$

Now, using the following inclusions of right D-modules

$$R\,D^p \subseteq P\,D^{(p+q-r)} = R\,D^p + \Lambda\,D^{(q-r)} \subseteq D^q,$$

if $N = \left(P\,D^{(p+q-r)}\right)/(R\,D^p)$, then the following short exact sequence of right D-modules holds

$$0 \longrightarrow N \xrightarrow{j} \mathrm{ext}_D^1(M, D) \xrightarrow{\sigma} \mathrm{ext}_D^1(E, D) \longrightarrow 0, \quad (15)$$

where j is the canonical injection. Hence, (15) shows that

$$\begin{aligned} & \mathrm{ext}_D^1(E, D) = 0 \\ \Leftrightarrow \quad & \mathrm{ext}_D^1(M, D) \cong N = \left(R\,D^p + \Lambda\,D^{(q-r)}\right)/(R\,D^p) \\ \Leftrightarrow \quad & \mathrm{ext}_D^1(M, D) \cong \left(R\,D^p + \sum_{i=1}^{q-r} \Lambda_{\bullet i}\,D\right)/(R\,D^p) \\ \Leftrightarrow \quad & \mathrm{ext}_D^1(M, D) \cong \sum_{i=1}^{q-r} \tau(\Lambda_{\bullet i})\,D \end{aligned}$$

where $\tau : D^p \longrightarrow D^p/(R\,D^q)$ is the canonical projection. Hence, $\mathrm{ext}_D^1(E, D) = 0$ iff the right D-module $D^p/(R\,D^q)$ is generated by $\{\tau(\Lambda_{\bullet i})\}_{i=1,\ldots,q-r}$ of $q-r$ elements.

Lemma 1: $\mathrm{ext}_D^1(E, D) = 0$ iff the right D-module $D^p/(R\,D^q)$ is generated by $\{\tau(\Lambda_{\bullet i})\}_{i=1,\ldots,q-r}$, i.e., iff $\mathrm{ext}_D^1(M, D)$ can be generated by $q-r$ elements.

$\mathrm{ext}_D^1(E, D) = 0$ is equivalent to $D^q = P\,D^{(p+q-r)}$. If $\{g_k\}_{k=1,\ldots,q}$ is the standard basis of D^q, then the above equality is equivalent to the existence of $S_k \in D^{(p+q-r)}$ satisfying $g_k = P\,S_k$ for $k = 1, \ldots, q$, i.e., to the existence of $S = (S_1 \ \ldots \ S_q) \in D^{(p+q-r)\times q}$ satisfying $P\,S = I_q$, i.e., a right-inverse of P over D, which, by 2 of Proposition 1, is equivalent to E is a stably free left D-module.

Lemma 2: $\mathrm{ext}_D^1(E, D) = 0$ iff the left D-module E is stably free of rank $p-r$.

Combining Lemmas 1 and 2, we get the following result.

Theorem 3: Let D be a noetherian domain, $R \in D^{q\times p}$ a full row rank matrix, namely, $\ker_D(.R) = 0$, $\Lambda \in D^{q\times(q-r)}$, $P = (R \quad -\Lambda) \in D^{q\times(p+q-r)}$ and $M = D^{1\times p}/(D^{1\times q}\,R)$ (resp., $E = D^{1\times(p+q-r)}/(D^{1\times q}\,P)$) the left D-module finitely presented by R (resp., P) which defines the following extension of $D^{1\times(q-r)}$ by M:

$$0 \longrightarrow D^{1\times(q-r)} \xrightarrow{\alpha} E \xrightarrow{\beta} M \longrightarrow 0.$$

Then, the following results are equivalent:

1) The left D-module E is stably free of rank $p-r$.
2) The matrix $P = (R \quad -\Lambda) \in D^{q\times(p+q-r)}$ admits a right-inverse with entries in D.
3) $\mathrm{ext}_D^1(E, D) \cong D^q/\left(P\,D^{(p+q-r)}\right) = 0$.
4) The right D-module $D^q/(R\,D^p) \cong \mathrm{ext}_D^1(M, D)$ finitely presented by R is generated by the family $\{\tau(\Lambda_{\bullet i})\}_{i=1,\ldots,q-r}$, where $\tau : D^q \longrightarrow D^q/(R\,D^p)$ is the canonical projection.

Finally, the previous equivalences depend only on the residue class $\rho(\Lambda)$ of $\Lambda \in D^{q\times(q-r)}$ in

$D^{q\times(q-r)}/\left(R\,D^{p\times(q-r)}\right)$, i.e., they depend only on the row vector $(\tau(\Lambda_{\bullet 1})\ \ldots\ \tau(\Lambda_{\bullet(q-r)}))\in(D^q/(R\,D^p))^{1\times(q-r)}$.

Remark 1: Theorem 3 was first obtained by J.-P. Serre in [17] for a commutative ring D and $r=q-1$. In this case, $\mathrm{ext}^1_D(M,D)$ is the right D-module generated by $\tau(\Lambda)$, i.e., $\mathrm{ext}^1_D(M,D)$ is the cyclic right D-module generated by $\tau(\Lambda)$.

On simple examples over a commutative polynomial ring $D=k[x_1,\ldots,x_n]$ with coefficients in a computable field k (e.g., $k=\mathbb{Q}$ or $\mathbb{F}_p$ where p is a prime number), we can take a generic matrix $\Lambda\in D^{q\times(q-r)}$ with a fixed total degree in the x_i's and study the D-module $\mathrm{ext}^1_D(E,D)\cong D^{1\times q}/\left(D^{1\times(p+q-r)}\,P^T\right)$ by means of a Gröbner basis computation and check whether or not the D-module $\mathrm{ext}^1_D(E,D)$ vanishes on certain branches of the corresponding *tree of integrability conditions* ([12]) or on certain parts of the underlying *constellation* of semi-algebraic sets in the k-parameters of Λ ([7]). In particular, we can test whether or not a non-zero constant belongs to the *annihilator* $\mathrm{ann}_D(\mathrm{ext}^1_D(E,D)$ of the D-module $\mathrm{ext}^1_D(E,D)$, namely,

$$\{d\in D\mid \forall\, n\in\mathrm{ext}^1_D(E,D),\ d\,n=0\},$$

i.e., whether or not $\mathrm{ann}_D(\mathrm{ext}^1_D(E,D)=D$. Since, $\hom_D(\mathrm{ext}^1_D(E,D),D)\cong\ker_D(.R)=0$ by a right D-module analogue of (3), $\mathrm{ext}^1_D(E,D)$ is a torsion right D-module (see Corollary 1 of [3]), and thus we obtain $\mathrm{ext}^1_D(E,D)=0$ iff $\mathrm{ann}_D(\mathrm{ext}^1_D(E,D))=D$.

The constellation technique is particularly interesting when the finitely presented $D=k[x_1,\ldots,x_n]$-module $D^q/(R\,D^q)$ is *0-dimensional*, i.e., when the ring $A=D/I$ is a finite k-vector space, where $I=\mathrm{ann}_D(D^q/(R\,D^q))$. Indeed, a Gröbner basis computation of the D-module $R\,D^p$ then gives a set of row vectors $\{\lambda_k\}_{k=1,\ldots,s}$, where $\lambda_k\in D^q$ and $s=\dim_k(A)$, such that $D^q/(R\,D^q)=\bigoplus_{k=1}^s k\,\tau(\lambda_k)$. Then, we can consider a generic matrix of the form

$$\Lambda=\left(\sum_{k=1}^s a_{1k}\,\lambda_k\quad\ldots\quad\sum_{k=1}^s a_{(q-r)k}\,\lambda_k\right)\in D^{q\times(q-r)},$$

where the a_{lk}'s are arbitrary elements of the field k for $l=1,\ldots,(q-r)$ and $k=1,\ldots,s$, and compute the constellation of semi-algebraic sets corresponding to the possible vanishing of the D-module $\mathrm{ext}^1_D(E,D)$.

Apart from the previous 0-dimensional case, we do not know yet how to recognize the existence of $\Lambda\in D^{q\times(q-r)}$ satisfying 2 of Theorem 3. However, using an ansatz, we can give the sketch of an algorithm in the case of the second Weyl algebra $B_n(k)$. This case encapsulates the cases of a commutative polynomial ring and the first Weyl algebra $A_n(k)$ since $k[x_1,\ldots,x_n]\subset A_n(k)\subset B_n(k)$.

Algorithm 1:
- **Input:** Let k be an algebraically closed computational field, $D=B_n(k)$, $R\in D^{q\times p}$ a full row rank matrix and three non-negative integers α, β and γ.
- **Output:** A set (possibly empty) of $\{\Lambda_i\}_{i\in I}$ such that the matrix $(R\quad-\Lambda_i)$ admits a right-inverse over D.

1) Consider an ansatz $\Lambda\in D^{q\times(q-r)}$ whose entries have a fixed total order α in the ∂_i's and a fixed total degree β (resp., γ) for the polynomial numerators (resp., denominators) in the x_j's of the arbitrary coefficients of the ansatz Λ.
2) Compute a Gröbner basis of the right D-module $R\,D^p$.
3) Compute the normal form $\overline{\Lambda}_{\bullet i}$ of the i^{th} column $\Lambda_{\bullet i}$ of Λ in $D^q/(R\,D^p)$ for $i=1,\ldots,q-r$.
4) Compute the obstructions for projectivity of the left D-module $\overline{E}=D^{1\times(p+q-r)}/(D^{1\times q}\,(R\quad-\overline{\Lambda}))$ (e.g., computation of a Gröbner basis of the right D-module $(R\quad-\overline{\Lambda})\,D^{(p+q-r)}$ or computation of the π-polynomials of the left D-module $\overline{E}$ ([3])).
5) Solve the systems in the arbitrary coefficients of the ansatz Λ obtained by making the obstructions vanish.
6) Return the set of solutions for Λ.

For examples, we refer the reader to [2].

IV. Serre's reduction problem

Theorem 4: Let D be a noetherian domain, $R\in D^{q\times p}$ a full row rank matrix, $0\le r\le q-1$ and $\Lambda\in D^{q\times(q-r)}$ such that there exists $U\in\mathrm{GL}_{p+q-r}(D)$ satisfying:

$$(R\quad-\Lambda)\,U=(I_q\quad 0). \tag{16}$$

If we decompose the matrix U as follows

$$U=\left(\begin{array}{cc} S_1 & Q_1\\ S_2 & Q_2\end{array}\right), \tag{17}$$

where $S_1\in D^{p\times q}$, $S_2\in D^{(q-r)\times q}$, $Q_1\in D^{p\times(p-r)}$ and $Q_2\in D^{(q-r)\times(p-r)}$, and if we introduce the left D-module $L=D^{1\times(p-r)}/(D^{1\times(q-r)}\,Q_2)$ finitely presented by the full row rank matrix Q_2, i.e., defined by the short exact sequence

$$0\longrightarrow D^{1\times(q-r)}\xrightarrow{.Q_2}D^{1\times(p-r)}\xrightarrow{\kappa}L\longrightarrow 0, \tag{18}$$

then we have:

$$M=D^{1\times p}/(D^{1\times q}\,R)\ \cong\ L=D^{1\times(p-r)}/(D^{1\times(q-r)}\,Q_2). \tag{19}$$

Conversely, if M is isomorphic to a left D-module L defined by the short exact sequence (18) for a certain matrix $Q_2\in D^{(q-r)\times(p-r)}$, then there exist $\Lambda\in D^{q\times(q-r)}$ and $U\in\mathrm{GL}_{p+q-r}(D)$ such that $(R\quad-\Lambda)\,U=(I_q\quad 0)$.

Proof: $\Rightarrow$ By hypothesis, we have $(R\quad-\Lambda)\,S=I_q$, where $S=(S_1^T\quad S_2^T)^T$, which shows that $P=(R\quad-\Lambda)$ admits a right-inverse over D. By Theorem 3, the extension (10) of $D^{1\times(q-r)}$ by M is then defined by a stably free left D-module E, and thus, free of rank $p-r$ by 3 of Proposition 1 applied to E. Moreover, by 3 of Proposition 1, the left D-homomorphism $\psi:E\longrightarrow D^{1\times(p-r)}$ defined by $\psi(\varrho((\mu_1\quad\mu_2)))=\mu_1\,Q_1+\mu_2\,Q_2$ for all $\mu_1\in D^{1\times p}$ and all $\mu_2\in D^{1\times(q-r)}$, is a left D-isomorphism, which yields the equivalence of extensions of $D^{1\times(q-r)}$ by M:

$$\begin{array}{ccccccccc}
0\longrightarrow & D^{1\times(q-r)} & \xrightarrow{\alpha} & E & \xrightarrow{\beta} & M & \longrightarrow 0\\
 & \| & & \downarrow\psi & & \| & \\
0\longrightarrow & D^{1\times(q-r)} & \xrightarrow{\psi\circ\alpha} & D^{1\times(p-r)} & \xrightarrow{\beta\circ\psi^{-1}} & M & \longrightarrow 0.
\end{array}$$

Using the standard basis $\{e_i\}_{i=1,\ldots,q-r}$ of $D^{1\times(q-r)}$, we get

$$(\psi\circ\alpha)(e_i)=\psi(\alpha(e_i))=\psi(\varrho((0 \quad e_i))=e_i\,Q_2,$$

for $i=1,\ldots,q-r$, i.e., $\psi\circ\alpha: D^{1\times(q-r)}\longrightarrow D^{1\times(p-r)}$ is defined by $(\psi\circ\alpha)(\nu)=\nu\,Q_2$ for $\nu\in D^{1\times(q-r)}$. The matrix Q_2 has full row rank since $\psi\circ\alpha$ is injective as the composition of two injective left D-homomorphisms. If $L=D^{1\times(p-r)}/(D^{1\times(q-r)}\,Q_2)$ is the left D-module finitely presented by $Q_2\in D^{(q-r)\times(p-r)}$ and $\kappa: D^{1\times(p-r)}\longrightarrow L$ the canonical projection onto L, then we get (18) and:

$$L=\mathrm{coker}_D(.Q_2)\cong\mathrm{im}\,(\beta\circ\psi^{-1})=M.$$

$\Leftarrow$ Let us suppose that there exists a left D-isomorphism $\gamma: L\longrightarrow M$, where L is defined by (18). Then, we have the following extension of $D^{1\times(q-r)}$ by M:

$$0\longrightarrow D^{1\times(q-r)}\xrightarrow{.Q_2}D^{1\times(p-r)}\xrightarrow{\gamma\circ\kappa}M\longrightarrow 0. \qquad (20)$$

By Theorem 2, the equivalence class of extension (20) defines a unique element $\rho(\Lambda)$ of the right D-module $D^{q\times(q-r)}/\left(R\,D^{p\times(q-r)}\right)$, where $\Lambda\in D^{q\times(q-r)}$. Then, the left D-module $E=D^{1\times(p+q-r)}/(D^{1\times q}\,(R \quad -\Lambda))$ defines the extension (10) of $D^{1\times(q-r)}$ by M which belongs to the same equivalence class as (20). Since extensions of $D^{1\times(q-r)}$ by M belonging to the same equivalence class are defined by isomorphic central left D-modules (see the comment after Definition 3), we obtain $E\cong D^{1\times(p-r)}$. Hence, E is a free left D-module of rank $p-r$, which, by 2 of Proposition 1, implies the existence $U\in\mathrm{GL}_{p+q-r}(D)$ such that (16). ■

Corollary 1: With the notations of Theorem 4, the left D-isomorphism (19) obtained in Theorem 4 is defined by:

$$\begin{array}{rcl} M=D^{1\times p}/(D^{1\times q}\,R) & \xrightarrow{\varphi} & L=D^{1\times(p-r)}/(D^{1\times(q-r)}\,Q_2)\\ \pi(\lambda) & \longmapsto & \kappa(\lambda\,Q_1).\end{array}$$

Moreover, its inverse $\varphi^{-1}: L\longrightarrow M$ is defined by $\varphi^{-1}(\kappa(\mu))=\pi(\mu\,T_1)$, where:

$$U^{-1}=\left(\begin{array}{cc} R & -\Lambda\\ T_1 & -T_2\end{array}\right)\in\mathrm{GL}_{p+q-r}(D),$$

where $T_1\in D^{(p-r)\times p}$ and $T_2\in D^{(p-r)\times(q-r)}$. These results depend only on the residue class $\rho(\Lambda)$ of $\Lambda\in D^{q\times(q-r)}$ in:

$$D^{q\times(q-r)}/\left(R\,D^{p\times(q-r)}\right)\cong\mathrm{ext}^1_D(M,D)^{1\times(q-r)}.$$

Proof: Let us first check that φ is well-defined: if $\lambda,\,\lambda'\in D^{1\times p}$ are such that $\pi(\lambda)=\pi(\lambda')$, then there exists $\nu\in D^{1\times q}$ such that $\lambda=\lambda'+\nu\,R$ and using (16), where $U\in\mathrm{GL}_{p+q-r}(D)$ is defined by (17), we get $R\,Q_1=\Lambda\,Q_2$:

$$\begin{array}{rl}\Rightarrow\ \varphi(\pi(\lambda)) & =\kappa(\lambda\,Q_1)=\kappa(\lambda'\,Q_1)+\kappa((\nu\,\Lambda)\,Q_2)\\ & =\kappa(\lambda'\,Q_1)=\varphi(\pi(\lambda')).\end{array}$$

Similarly, let us prove that the left D-homomorphism

$$\begin{array}{rcl}\phi: L & \longrightarrow & M\\ \kappa(\mu) & \longmapsto & \pi(\mu\,T_1),\end{array}$$

is well-defined: if $\mu,\,\mu'\in D^{1\times(p-r)}$ satisfy $\kappa(\mu)=\kappa(\mu')$, then there exists $\theta\in D^{1\times(q-r)}$ such that $\mu=\mu'+\theta\,Q_2$ and using the identity $U\,U^{-1}=I_{p+q-r}$, we get $Q_2\,T_1=-S_2\,R$

$$\begin{array}{rl}\Rightarrow\ \phi(\kappa(\mu)) & =\pi(\mu\,T_1))=\pi(\mu'\,T_1)-\pi((\theta\,S_2)\,R)\\ & =\pi(\mu'\,T_1)=\phi(\kappa(\mu')).\end{array}$$

The identity $U\,U^{-1}=I_{p+q-r}$ yields $S_1\,R+Q_1\,T_1=I_p$ and

$$\begin{array}{rl}(\phi\circ\varphi)(\pi(\lambda)) & =\phi(\kappa(\lambda\,Q_1))=\pi(\lambda\,Q_1\,T_1)\\ & =\pi(\lambda)-\pi((\lambda\,S_1)\,R)=\pi(\lambda),\end{array}$$

i.e., $\phi\circ\varphi=\mathrm{id}_M$. Using $U^{-1}\,U=I_{p+q-r}$, we get

$$T_1\,Q_1-T_2\,Q_2=I_{p-r},$$

$$\begin{array}{rl}\Rightarrow\ (\varphi\circ\phi)(\kappa(\mu)) & =\varphi(\pi(\mu\,T_1))=\kappa(\mu\,T_1\,Q_1)\\ & =\kappa(\mu)+\kappa((\mu\,T_2)\,Q_2)=\kappa(\mu),\end{array}$$

i.e., $\varphi\circ\phi=\mathrm{id}_L$, and thus φ is an isomorphism and $\varphi^{-1}=\phi$. ■

Corollary 2: Let $\mathcal{F}$ be a left D-module and:

$$\left\{\begin{array}{l}\ker_{\mathcal{F}}(R.)=\{\eta\in\mathcal{F}^p\mid R\,\eta=0\},\\ \ker_{\mathcal{F}}(Q_2.)=\{\zeta\in\mathcal{F}^{(p-r)}\mid Q_2\,\zeta=0\}.\end{array}\right.$$

Then, we have $\ker_{\mathcal{F}}(R.)\cong\ker_{\mathcal{F}}(Q_2.)$ and:

$$\ker_{\mathcal{F}}(R.)=Q_1\,\ker_{\mathcal{F}}(Q_2.),\quad \ker_{\mathcal{F}}(Q_2.)=T_1\,\ker_{\mathcal{F}}(R.).$$

Corollary 3: Let $R\in D^{q\times p}$ be a full row rank matrix and $\Lambda\in D^{q\times(q-r)}$ such that $P=(R \quad -\Lambda)\in D^{q\times(p+q-r)}$ admits a right-inverse over D. Then, Theorem 4 holds when D satisfies one of the following properties:

1) D is a left principal ideal domain (e.g., the ring of ordinary differential operators with coefficients in a differential field such that $\mathbb{R}$, $\mathbb{R}(t)$ or $\mathbb{R}\{t\}[t^{-1}]$),
2) $D=k[x_1,\ldots,x_n]$ is a commutative polynomial ring over a field k,
3) D is one of the two *Weyl algebras* $A_n(k)$ or $B_n(k)$, where k a field of characteristic 0 and $p-r\geq 2$.
4) D is the ring of ordinary differential operators with coefficients in $k[\![t]\!]$, where k is a field of characteristic 0, or in $k\{t\}$, where $k=\mathbb{R}$ or $\mathbb{C}$, and $p-r\geq 2$.

Proof: If D satisfy one of the four conditions, then the stably free left D-module E finitely presented by $P=(R \quad -\Lambda)\in D^{q\times(p+q-r)}$, is free of rank $p-r$ by Theorem 1. The result is then a consequence of Theorem 4. ■

If Corollary 3 holds, then it is enough to search for a matrix $\Lambda\in D^{q\times(q-r)}$ such that $P=(R \quad -\Lambda)$ admits a right-inverse over D by Proposition 1 (see Algorithm 1).

The next corollary generalizes a result of [1].

Corollary 4: With the notations of Theorem 4 and Corollary 1, if the matrix $\Lambda\in D^{q\times(q-r)}$ admits a left-inverse $\Gamma\in D^{(q-r)\times q}$, i.e., $\Gamma\,\Lambda=I_{q-r}$, then the matrix Q_1 admits the left-inverse $T_1-T_2\,\Gamma\,R\in D^{(p-r)\times p}$ and the left D-module $\ker_D(.Q_1)$ is stably free of rank r.

If the left D-module $\ker_D(.Q_1)$ is free of rank r, then there exists a matrix $Q_3 \in D^{p\times r}$ such that:

$$W \triangleq (Q_3 \quad Q_1) \in \mathrm{GL}_p(D).$$

If we write $W^{-1} = (Y_3^T \quad Y_1^T)^T$, where $Y_3 \in D^{r\times p}$ and $Y_1 \in D^{(p-r)\times p}$, then $X \triangleq (R\,Q_3 \quad \Lambda) \in \mathrm{GL}_q(D)$ and:

$$V \triangleq X^{-1} = \left(\begin{array}{c} Y_3\,S_1 \\ Q_2\,Y_1\,S_1 - S_2 \end{array}\right).$$

The matrix R is then equivalent to the matrix $X\,\mathrm{diag}(I_r, Q_2)\,W^{-1}$ or equivalently:

$$V\,R\,W = \left(\begin{array}{cc} I_r & 0 \\ 0 & Q_2 \end{array}\right).$$

Finally, the left D-module $\ker_D(.Q_1)$ is free when D satisfies 1 or 2 of Corollary 3 or if D is $A_n(k)$ or $B_n(k)$ over a field k of characteristic 0 and $r \geq 2$ or if D is the ring of ordinary differential operators with coefficients in $k[[t]]$, where k a field of characteristic 0, or with coefficients in $k\{t\}$, where $k = \mathbb{R}$ or $\mathbb{C}$, and $r \geq 2$.

Proof: Using (16) and (17), we get the identities:

$$\left\{\begin{array}{l} R\,S_1 - \Lambda\,S_2 = I_q, \\ R\,Q_1 = \Lambda\,Q_2, \\ T_1\,S_1 = T_2\,S_2, \\ T_1\,Q_1 - T_2\,Q_2 = I_{p-r}. \end{array}\right. \quad (21)$$

Moreover, we know that there exists $\Gamma \in D^{(q-r)\times q}$ such that $\Gamma\,\Lambda = I_{q-r}$. Pre-multiplying the second equation of (21) by Γ, we get $Q_2 = \Gamma\,R\,Q_1$, which, combined with the last equation of (21), yields $(T_1 - T_2\,\Gamma\,R)\,Q_1 = I_{p-r}$ and proves that Q_1 admits a left-inverse over D. Then, the following short exact sequence

$$0 \longrightarrow \ker_D(.Q_1) \xrightarrow{\;i\;} D^{1\times p} \xrightarrow{\;.Q_1\;} D^{1\times(p-r)} \longrightarrow 0 \quad (22)$$

ends with the free left D-module $D^{1\times(p-r)}$, and thus splits, namely, we have $D^{1\times p} \cong \ker_D(.Q_1) \oplus D^{1\times(p-r)}$ (see, e.g., [16]), which proves that $\ker_D(.Q_1)$ is a stably free left D-module of rank $p-(p-r) = r$.

Now, let us suppose that $\ker_D(.Q_1)$ is a free left D-module of rank r and let denote by $\psi : D^{1\times r} \longrightarrow \ker_D(.Q_1)$ a left D-isomorphism. The split short exact sequence (22) yields

$$0 \longrightarrow D^{1\times r} \underset{\xleftarrow{\;.Q_3\;}}{\xrightarrow{\;.Y_3\;}} D^{1\times p} \underset{\xleftarrow{\;.Y_1\;}}{\xrightarrow{\;.Q_1\;}} D^{1\times(p-r)} \longrightarrow 0, \quad (23)$$

where $Y_3 \in D^{r\times p}$ is a matrix such that $(i\circ\psi)(\nu) = \nu\,Y_3$ for all $\nu \in D^{1\times r}$. Hence, if we write $W = (Q_3 \quad Q_1) \in D^{p\times p}$, then the previous split short exact sequence yields

$$\begin{array}{l} (Q_3 \quad Q_1)\left(\begin{array}{c} Y_3 \\ Y_1 \end{array}\right) = Q_3\,Y_3 + Q_1\,Y_1 = I_p, \\ \left(\begin{array}{c} Y_3 \\ Y_1 \end{array}\right)(Q_3 \quad Q_1) = \left(\begin{array}{cc} I_r & 0 \\ 0 & I_{p-r} \end{array}\right) = I_p, \end{array} \quad (24)$$

i.e., $W \in \mathrm{GL}_p(D)$ and $W^{-1} = (Y_3^T \quad Y_1^T)^T$. The second identity of (21) yields:

$$R\,W = (R\,Q_3 \quad \Lambda\,Q_2) = (R\,Q_3 \quad \Lambda)\left(\begin{array}{cc} I_r & 0 \\ 0 & Q_2 \end{array}\right). \quad (25)$$

Using the identities of (21) and (24), we obtain

$$\begin{array}{l} (R\,Q_3 \quad \Lambda)\left(\begin{array}{c} Y_3\,S_1 \\ Q_2\,Y_1\,S_1 - S_2 \end{array}\right) \\ = R\,S_1 - R\,Q_1\,Y_1\,S_1 + \Lambda\,Q_2\,Y_1\,S_1 - \Lambda\,S_2 \\ = I_q - (R\,Q_1 - \Lambda\,Q_2)\,Y_1\,S_1 = I_q, \end{array}$$

and thus $X \triangleq (R\,Q_3 \quad \Lambda) \in \mathrm{GL}_q(D)$ since D is a noetherian ring and thus a *stably finite ring* (i.e., a ring over which every left or right invertible matrix is invertible ([6])) and:

$$V \triangleq X^{-1} = \left(\begin{array}{c} Y_3\,S_1 \\ Q_2\,Y_1\,S_1 - S_2 \end{array}\right).$$

Using (25), we finally obtain $V\,R\,W = \mathrm{diag}(I_r, Q_2)$. ■

For more results on Serre's reduction of linear systems of partial differential equations, see [4].

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