Reduction of linear systems based on Serre's theorem

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Abstract. Within a module-theoretic approach, we study when a (multidimensional) linear system can be defined by one equation. When this reduction is possible, we show how to compute the corresponding equation. Based on a theorem by J.-P. Serre ([22]), our results use the algebraic concept of Baer extensions. Hence, we simplify and generalize different results on the reduction problem for different classes of (multidimensional) linear systems.

Keywords. Linear systems, reduction problem, Serre's theorem, module theory.

1 Introduction

Over the years, the Smith canonical form has played an important role in the study of linear systems defined over an univariate commutative polynomial ring $k[x]$, where k is a field (see, e.g., [11, 20]). The concept of the Smith canonical form can be extended to a multivariate commutative polynomial ring $k[x_1, \ldots, x_n]$ when defined as being the diagonal matrix formed by the invariant polynomials $\gamma_1, \ldots, \gamma_r$ defined as the successive quotients $\gamma_i = \alpha_i/\alpha_{i-1}$ of the greatest common divisors α_i of the $i \times i$-minors of the matrix ($\alpha_0 = 1$, r is the rank of the matrix). Despite its interest in multidimensional systems theory, the problem of reducing a multivariate polynomial matrix to its Smith form by means of unimodular transformations has only been sparsely studied in the control literature. See the few exceptions [1, 8, 9, 10, 12] and the references therein. An important result in this direction is the following one.

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Theorem 1 ([1]). *Let $D = \mathbb{R}[x_1, \ldots, x_n]$ be the commutative polynomial ring with coefficients in $\mathbb{R}$, $R \in D^{p\times p}$ a full row rank matrix and $\mathbb{R}^\star = \mathbb{R}\backslash\{0\}$. Then, there exist $V \in D^{p\times p}$ and $W \in D^{p\times p}$ satisfying* $\det V \in \mathbb{R}^\star$, $\det W \in \mathbb{R}^\star$ *and*

$$V\,R\,W = \left(\begin{array}{cc} I_{p-1} & 0 \\ 0 & u\,\det R \end{array}\right), \quad u \in \mathbb{R}^\star,$$

iff there exists a column vector $\Lambda \in D^p$ admitting a left-inverse over D such that the matrix $(R \quad -\Lambda) \in D^{p\times(p+1)}$ admits a right-inverse over D.

The purpose of this paper is to show that Theorem 1 has deep connections with a result due to J.-P. Serre ([22]) based on the algebraic concept of *Baer extensions* ([18, 19, 21]). Using this connection, we simplify and generalize known results on the reduction problem for different classes of (multidimensional) linear systems. We refer the reader to [2] for more results, examples and applications.

2 A pedestrian approach to Baer extensions

In what follows, we shall denote by D a non-commutative left and right *noetherian domain* ([21]), $D^{1\times p}$ (resp., D^q) the left (resp., right) D-module formed by row (resp., column) vectors of length p (resp., q) with entries in D and $R \in D^{q\times p}$ a $q \times p$ matrix with entries in D. Moreover, we shall denote by:

$$\begin{array}{rcl} .R : D^{1\times q} & \longrightarrow & D^{1\times p} \\ \mu & \longmapsto & \mu\,R, \end{array} \qquad \begin{array}{rcl} R. : D^{p} & \longrightarrow & D^{q} \\ \eta & \longmapsto & R\,\eta. \end{array}$$

In the rest of the paper, for reasons of simplicity, we shall suppose that $p \geq q$ and R has *full row rank*, i.e., $\ker_D(.R) = \{\mu \in D^{1\times q} \mid \mu\,R = 0\} = 0$.

Let us define the factor left D-module $M = D^{1\times p}/(D^{1\times q}\,R)$ *finitely presented* by R and $\pi : D^{1\times p} \longrightarrow M$ the D-morphism (i.e., D-linear application) sending any element $\lambda \in D^{1\times p}$ to its residue class $\pi(\lambda)$ in M. We have the *exact sequence*

$$0 \longrightarrow D^{1\times q} \xrightarrow{.R} D^{1\times p} \xrightarrow{\pi} M \longrightarrow 0, \tag{1}$$

namely, $.R$ is an injective D-morphism as $\ker_D(.R) = 0$, $\ker_D \pi = D^{1\times q}\,R$ and π is a surjective D-morphism. The D-morphism π is surjective as, by definition of M, every element $m \in M$ has the form $m = \pi(\lambda)$, for a certain $\lambda \in D^{1\times p}$.

Let us consider the matrices $\Lambda \in D^q$ and $P = (R \quad -\Lambda) \in D^{q\times(p+1)}$, the finitely presented left D-module $E = D^{1\times(p+1)}/(D^{1\times q}\,P)$ and the exact sequence:

$$0 \longrightarrow D^{1\times q} \xrightarrow{.P} D^{1\times(p+1)} \xrightarrow{\varrho} E \longrightarrow 0. \tag{2}$$

Let us study the connections between the left D-modules M and E. If we denote by $X = (I_p^T \quad 0^T)^T \in D^{(p+1)\times p}$, then the identity $R = P\,X$ induces the following *commutative exact diagram* of left D-modules ([21])

$$\begin{array}{ccccccccc} 0 \longrightarrow & D^{1\times q} & \xrightarrow{.P} & D^{1\times(p+1)} & \xrightarrow{\varrho} & E & \longrightarrow 0 \\ & \| & & \downarrow .X & & & \\ 0 \longrightarrow & D^{1\times q} & \xrightarrow{.R} & D^{1\times p} & \xrightarrow{\pi} & M & \longrightarrow 0, \end{array}$$

and the well-defined D-morphism $\beta : E \longrightarrow M$ defined by:

$$\forall\, \mu_1 \in D^{1\times p}, \quad \forall\, \mu_2 \in D, \quad \beta(\varrho((\mu_1 \quad \mu_2))) = \pi((\mu_1 \quad \mu_2)\, X) = \pi(\mu_1).$$

For all $m \in M$, we know that there exists $\mu_1 \in D^{1\times p}$ such that $m = \pi(\mu_1)$, and thus, $m = \beta(\varrho((\mu_1 \quad 0)))$, which proves that β is surjective, i.e., $\operatorname{im}\beta = M$.

Now, an element $\varrho((\mu_1 \quad \mu_2)) \in \ker\beta$ satisfies $\pi(\mu_1) = 0$, i.e., $\mu_1 = \nu_1\, R$ for a certain $\nu_1 \in D^{1\times q}$. Hence, we obtain:

$$\ker\beta = \{\varrho((\nu_1\, R \quad \mu_2)) = \varrho((0 \quad \mu_2 + \nu_1\, \Lambda)) \mid \nu_1 \in D^{1\times q},\ \mu_2 \in D\} = \{\varrho((0 \quad \nu)) \mid \nu \in D\}.$$

Let us denote by $\alpha : D \longrightarrow \ker\beta$ the D-isomorphism defined by $\alpha(\nu) = \varrho((0 \quad \nu))$ for all $\nu \in D$. The exact sequence $0 \longrightarrow \ker\beta \xrightarrow{i} E \xrightarrow{\beta} \operatorname{im}\beta \longrightarrow 0$ becomes:

$$0 \longrightarrow D \xrightarrow{\alpha} E \xrightarrow{\beta} M \longrightarrow 0. \tag{3}$$

The exact sequence (3) is called a *(Baer) extension of D by M* (see, e.g., [19, 21]).

Let us now consider the matrices $\Theta \in D^p$, $\overline{\Lambda} = \Lambda + R\,\Theta$, $\overline{P} = (R \quad -\overline{\Lambda})$ and the left D-module $\overline{E} = D^{1\times(p+1)}/(D^{1\times q}\,\overline{P})$. Let us denote by $\overline{\varrho} : D^{1\times(p+1)} \longrightarrow \overline{E}$ the projection onto $\overline{E}$. Doing as previously with $\overline{E}$, we obtain the extension of D by M defined by $0 \longrightarrow D \xrightarrow{\overline{\alpha}} \overline{E} \xrightarrow{\overline{\beta}} M \longrightarrow 0$, with the following notations:

$$\forall\, \nu \in D, \quad \overline{\alpha}(\nu) = \overline{\varrho}((0 \quad \nu))), \quad \forall\, \mu_1 \in D^{1\times p},\ \forall\, \mu_2 \in D, \quad \overline{\beta}(\overline{\varrho}((\mu_1 \quad \mu_2)) = \pi(\mu_1).$$

If we denote by $\mathrm{GL}_p(D) = \{U \in D^{p\times p} \mid \exists\, V \in D^{p\times p} :\ U\,V = V\,U = I_p\}$ and

$$V = \begin{pmatrix} I_p & \Theta \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_{p+1}(D),$$

then we have $P = \overline{P}\,V$, which induces the following commutative exact diagram:

$$\begin{array}{ccccccc}
0 \longrightarrow & D^{1\times q} & \xrightarrow{.\overline{P}} & D^{1\times(p+1)} & \xrightarrow{\overline{\varrho}} & \overline{E} & \longrightarrow 0 \\
 & \| & & \downarrow .V & & & \\
0 \longrightarrow & D^{1\times q} & \xrightarrow{.P} & D^{1\times(p+1)} & \xrightarrow{\varrho} & E & \longrightarrow 0.
\end{array}$$

Hence, we obtain the D-isomorphism $\phi : \overline{E} \longrightarrow E$ defined by:

$$\forall\, \mu_1 \in D^{1\times p},\ \forall\, \mu_2 \in D, \quad \phi(\overline{\varrho}((\mu_1 \quad \mu_2))) = \varrho((\mu_1 \quad \mu_2)\, V) = \varrho((\mu_1 \quad \mu_2 + \mu_1\, \Theta)).$$

Then, for all $\nu \in D$, we have $(\phi \circ \overline{\alpha})(\nu) = \phi(\overline{\varrho}((0 \quad \nu))) = \varrho((0 \quad \nu)) = \alpha(\nu)$, which proves that $\alpha = \phi \circ \overline{\alpha}$. Moreover, for all $\mu_1 \in D^{1\times p}$ and $\mu_2 \in D$, we have

$$(\beta \circ \phi)(\overline{\varrho}((\mu_1 \quad \mu_2))) = \beta(\varrho((\mu_1 \quad \mu_2 + \mu_1\, \Theta))) = \pi_1(\mu_1) = \overline{\beta}(\overline{\varrho}((\mu_1 \quad \mu_2))),$$

which shows that $\overline{\beta} = \beta \circ \phi$. Therefore, we obtain the commutative exact diagram:

$$\begin{array}{ccccccc}
0 \longrightarrow & D & \xrightarrow{\overline{\alpha}} & \overline{E} & \xrightarrow{\overline{\beta}} & M & \longrightarrow 0 \\
 & \| & & \downarrow \phi & & \| & \\
0 \longrightarrow & D & \xrightarrow{\alpha} & E & \xrightarrow{\beta} & M & \longrightarrow 0.
\end{array} \tag{4}$$

We are led to the following definition of *equivalent extensions*.

Definition 2 ([19, 21]). Two extensions $e:\, 0 \longrightarrow D \xrightarrow{\alpha} E \xrightarrow{\beta} M \longrightarrow 0$ and $\overline{e}:\, 0 \longrightarrow D \xrightarrow{\overline{\alpha}} \overline{E} \xrightarrow{\overline{\beta}} M \longrightarrow 0$ of D by M are said to be *(Baer) equivalent* if there exists a D-morphism $\phi: \overline{E} \longrightarrow E$ satisying $\alpha = \phi \circ \overline{\alpha}$ and $\overline{\beta} = \beta \circ \phi$, i.e., such that (4) is a commutative exact diagram.

If e and $\overline{e}$ are equivalent, then we can easily prove that ϕ is necessarily a D-isomorphism (e.g., apply the *snake lemma* ([21]) to (4)). Hence, we can easily check that Definition 2 is an equivalence relation $\sim$ on the set of extensions of D by M ([21]). We denote by $\mathrm{e}_D(M, D)$ the set of all equivalence classes of extensions of D by M and $[e]$ the equivalence class of the extension e in $\mathrm{e}_D(M, D)$.

The previous results show that the extensions of D by M defined by the left D-modules E and $\overline{E}$, i.e., by means of Λ and $\overline{\Lambda} = \Lambda + R\,\Theta$, $\Theta \in D^p$, are equivalent.

Let us now develop the relations between the matrices Λ and $\overline{\Lambda} = \Lambda + R\,\Theta$ and $\mathrm{e}_D(M, D)$. In order to do that, we need to introduce the right D-module

$$\mathrm{ext}^1_D(M, D) = D^q/(R\, D^p), \tag{5}$$

called *the first extension right D-module of M with value in D*. The notation $\mathrm{ext}^1_D(M, D)$ is explained by the fact that, using techniques of homological algebra, we can prove that the right D-module $D^q/(R\, D^p)$ only depends on M and not on the choice of the matrix R which presents M (see, e.g., [21]). Moreover, as R has full row rank, the higher extension right D-modules $\mathrm{ext}^i_D(M, D)$, $i \geq 2$, are reduced to 0. We recall that the extension left D-modules $\mathrm{ext}^i_D(D^q/(R\, D^p), D)$, $i \geq 1$, play important roles in the classification of algebraic properties of the left D-module $M = D^{1\times p}/(D^{1\times q}\, R)$, and thus, of the structural properties of the *behaviour*

$$\ker_{\mathcal{F}}(R.) = \{\eta \in \mathcal{F}^p \mid R\,\eta = 0\},$$

where $\mathcal{F}$ denotes a left D-module usually called the *signal space* (see [3, 15, 16, 23]).

If we denote by $\rho: D^q \longrightarrow \mathrm{ext}^1_D(M, D) = D^q/(R\, D^p)$ the projection onto $\mathrm{ext}^1_D(M, D)$, then, for all $\Theta \in D^p$, we have $\rho(\overline{\Lambda}) = \rho(\Lambda + R\,\Theta) = \rho(\Lambda)$, i.e., the matrices Λ and $\overline{\Lambda} = \Lambda + R\,\Theta$ belong to the same residue class in $\mathrm{ext}^1_D(M, D)$.

We have just proved that every element $\rho(\Lambda) \in \mathrm{ext}^1_D(M, D)$ defines a unique equivalence class $[e]$ of extensions of D by M defined by the exact sequence

$$e:\, 0 \longrightarrow D \xrightarrow{\alpha} E \xrightarrow{\beta} M \longrightarrow 0,$$

where the left D-module E is defined by $E = D^{1\times(p+1)}/(D^{1\times q}\,(R \quad -\Lambda))$.

Let us study the converse of this result. To do that, we consider an extension $0 \longrightarrow D \xrightarrow{\varepsilon} F \xrightarrow{\delta} M \longrightarrow 0$ of D by M. Let us denote by $\{f_i\}_{i=1,\ldots,p}$ the *standard basis* of $D^{1\times p}$, namely, f_i is the vector with 1 at the i^{th} position and 0 elsewhere.

Using the fact that the D-morphism δ is surjective, for all $i = 1, \ldots, p$, there exists $\zeta_i \in F$ satisfying that $\delta(\zeta_i) = \pi(f_i) \in M$. For all $j = 1, \ldots, q$, we get:

$$\delta\left(\sum_{k=1}^{p} R_{jk}\,\zeta_k\right) = \sum_{k=1}^{p} R_{jk}\,\delta(\zeta_k) = \sum_{k=1}^{p} R_{jk}\,\pi(f_k) = \pi\left(\sum_{k=1}^{p} R_{jk}\,f_k\right) = 0.$$

As $\ker\delta = \operatorname{im}\varepsilon$ and ε is injective, there exists a unique element $\lambda_j \in D$ satisfying $\varepsilon(\lambda_j) = \sum_{k=1}^{p} R_{jk}\,\zeta_k$. If we denote by $\Lambda = (\lambda_1 \ \ldots \ \lambda_q)^T \in D^q$, then we have $\rho(\Lambda) \in \operatorname{ext}^1_D(M, D)$. We can check that $\rho(\Lambda)$ is well-defined as if we consider other pre-images $\overline{\zeta}_i$'s of the $\pi(f_i)$'s, i.e., $\delta(\overline{\zeta}_i) = \pi(f_i)$, $i = 1, \ldots, p$, and the corresponding $\overline{\Lambda}$, then there exists $\Theta \in D^p$ satisfying $\overline{\Lambda} = \Lambda + R\,\Theta$, i.e., $\rho(\overline{\Lambda}) = \rho(\Lambda)$ ([18, 19]).

We can prove that there is a one-to-one correspondence between the elements of $\operatorname{ext}^1_D(M, D)$ and the equivalence classes of extensions of D by M (see, e.g., [18, 19, 21]). An important consequence of this result is that every equivalence class of extensions of D by M contains a representative defined by means of a left D-module $E(\Lambda) = D^{1\times(p+1)}/(D^{1\times q}\,(R \quad -\Lambda))$ for a certain $\Lambda \in D^q$. Then, the *Baer sum* $[e_1] + [e_2]$ of two equivalent classes $[e_1]$ and $[e_2]$ of extensions of D by M, respectively defined by representatives formed by $E(\Lambda_1)$ and $E(\Lambda_2)$, is the equivalence class of the extension defined by $E(\Lambda_1 + \Lambda_2)$. Endowed with the Baer sum and the neutral element defined by the equivalence class of the extension defined by $E(0) = D \oplus M$, $\mathrm{e}_D(M, D)$ becomes an abelian group (see, e.g., [18, 19, 21]).

The next result plays an important role in homological algebra (see, e.g., [21]).

Theorem 3. *We have the abelian group isomorphism* $\operatorname{ext}^1_D(M, D) \cong \mathrm{e}_D(M, D)$.

3 Serre's theorem

A natural question is whether or not there exists an element $\rho(\Lambda) \in \operatorname{ext}^1_D(M, D)$ such that the left D-module $E = D^{1\times(p+1)}/(D^{1\times q}\,(R \quad -\Lambda))$, defining an extension of D by M, is respectively *torsion-free*, *reflexive*, *projective* or *free* ([21]). In [22], J.-P. Serre studied that problem for projective and free modules of *projective dimensions* equal to 1 ([21]) over a commutative ring D. If the ring D is *regular* in the sense that every finitely generated left D-module admits a *finite free resolution* ([21]), then the previous hypothesis is equivalent to the existence of a full row rank matrix $R \in D^{q\times p}$ satisfying $M = D^{1\times p}/(D^{1\times q}\,R)$ (see [17] for a constructive algorithm computing the matrix R and its implementation in OreModules ([4])).

Let us recall a few definitions that will play important roles in what follows.

Definition 4 ([21]). Let D be a left noetherian domain, $R \in D^{q\times p}$ a matrix and $M = D^{1\times p}/(D^{1\times q}\,R)$ the left D-module finitely presented by R.

1. M is said to be *free of rank* l if $M \cong D^{1\times l}$, where $L \cong M$ denotes that the left D-modules L and M are isomorphic as left D-modules.

2. M is said to be *stably free* if there exist two non-negative integers m and l such that $M \oplus D^{1\times m} \cong D^{1\times l}$, where $\oplus$ denotes the direct sum of left D-modules.

3. M is said to be *projective* if there exist a non-negative integer l and a left D-module L such that $M \oplus L \cong D^{1\times l}$.

We can easily check that a free module is stably free and a stably free module is projective but the converses are generally not true. If a left D-module M is presented by a full row rank matrix R, then M is stably free iff M is projective (see, e.g., [7, 17]). The following proposition explicitly characterizes stably free and free modules.

Proposition 5 ([3, 7, 17]). *Let D be a left noetherian domain, $R \in D^{q\times p}$ a full row rank matrix and the left D-module $M = D^{1\times p}/(D^{1\times q}\, R)$. Then, we have:*

1. *M is a stably free left D-module of rank $p-q$ iff R admits a right-inverse over D, namely, a matrix $S \in D^{p\times q}$ satisfying $R\,S = I_q$.*
2. *M is a free left D-module of rank $p-q$ iff there exists $U \in \mathrm{GL}_p(D)$ such that:*

$$R\,U = (I_q \quad 0).$$

If we denote by $U = (S \quad Q)$, where $S \in D^{p\times q}$ and $Q \in D^{p\times(p-q)}$, then the D-morphism $\varphi : M \longrightarrow D^{1\times(p-q)}$ defined by $\varphi(\pi(\lambda)) = \lambda\, Q$, for all $\lambda \in D^{1\times p}$, is an isomorphism whose inverse is defined by $\varphi^{-1}(\mu) = \pi(\mu\, T)$, for all $\mu \in D^{1\times(p-q)}$, where $T \in D^{(p-q)\times p}$ is the submatrix of $U^{-1} = (R^T \quad T^T)^T$. Finally, $\{\pi(T_i)\}_{i=1,\ldots,p-q}$ defines a basis of the free left D-module M of rank $p-q$, where T_i denotes the i^{th} row of the matrix T.

We recall that, by definition of the extension right D-modules, we have:

$$\mathrm{ext}^1_D(M,D) = D^q/(R\,D^p), \quad \mathrm{ext}^1_D(E,D) = D^q/(P\,D^{p+1}).$$

Using the inclusions $R\,D^p \subseteq P\,D^{p+1} \subseteq D^q$ and the classical *third isomorphism theorem* (see, e.g., [21]), we obtain the exact sequence of left D-modules:

$$0 \longrightarrow (P\,D^{p+1})/(R\,D^p) \xrightarrow{j} \mathrm{ext}^1_D(M,D) \xrightarrow{\sigma} \mathrm{ext}^1_D(E,D) \longrightarrow 0. \tag{6}$$

Hence, $\mathrm{ext}^1_D(E,D) = 0$ is equivalent to $\mathrm{ext}^1_D(M,D) = (P\,D^{p+1})/(R\,D^p)$. Using the definition of $P = (R \quad -\Lambda)$, we note that the left D-module $(P\,D^{p+1})/(R\,D^p)$ is the right D-module generated by the residue class $\rho(\Lambda)$ of $\Lambda \in D^q$ in $\mathrm{ext}^1_D(M,D)$.

Lemma 6. *With the previous notations, $\mathrm{ext}^1_D(E,D) = 0$ iff $\mathrm{ext}^1_D(M,D)$ is the cyclic right D-module generated by $\rho(\Lambda)$.*

Let us understand the algebraic condition $\mathrm{ext}^1_D(E,D) = 0$. By definition, $\mathrm{ext}^1_D(E,D) = 0$ is equivalent to $D^q = P\,D^{p+1}$. If we now denote by $\{g_j\}_{j=1,\ldots,q}$ the standard basis of D^q, then the last module equality is equivalent to, for $j = 1,\ldots,q$, there exists $S_j \in D^{p+1}$ satisfying $g_j = P\,S_j$, i.e., to the existence of a matrix

$S = (S_1^T \; \ldots \; S_q^T)^T \in D^{(p+1)\times q}$ satisfying $P\,S = I_q$, which, by 2 of Proposition 5, is equivalent to E is stably free. Hence, we obtain the following simple result.

Lemma 7. *With the previous notations, $\mathrm{ext}_D^1(E, D) = 0$ iff the left D-module E is stably free.*

Similarly, $\mathrm{ext}_D^1(M, D) = 0$ is equivalent to the existence of a right-inverse of R over D, i.e., to the fact that M is a stably free left D-module.

Combining Lemmas 6 and 7, we get the following non-commutative generalization of J.-P. Serre's result ([22]).

Theorem 8. *Let $R \in D^{q\times p}$ be a full row rank matrix, i.e., $\ker_D(.R) = 0$, $\Lambda \in D^q$, $P = (R \quad -\Lambda)$ and $M = D^{1\times p}/(D^{1\times q}\,R)$ (resp., $E = D^{1\times (p+1)}/(D^{1\times q}\,P)$) the left D-module finitely presented by R (resp., P) defining an extension of D by M:*

$$0 \longrightarrow D \xrightarrow{\alpha} E \xrightarrow{\beta} M \longrightarrow 0.$$

Then, the following results are equivalent:

1. *The left D-module E is stably free.*
2. *The matrix $P = (R \quad -\Lambda) \in D^{q\times(p+1)}$ admits a right-inverse over D.*
3. $\mathrm{ext}_D^1(E, D) = D^q/(P\,D^{p+1}) = 0.$
4. $\mathrm{ext}_D^1(M, D) = D^q/(R\,D^p)$ *is the cyclic right D-module generated by $\rho(\Lambda)$, where $\rho : D^q \longrightarrow \mathrm{ext}_D^1(M, D)$ denotes the projection onto $\mathrm{ext}_D^1(M, D)$.*

Finally, the previous equivalences only depend on the residue class $\rho(\Lambda)$ of $\Lambda \in D^q$ in the right D-module $\mathrm{ext}_D^1(M, D) = D^q/(R\,D^p)$.

In particular, Theorem 8 is fulfilled if $\mathrm{ext}_D^1(M, D) = 0$, i.e., if M is a stably free left D-module or, equivalently, if R admits a right-inverse over D. Indeed, $\mathrm{ext}_D^1(M, D) = 0$ is then the trivial cyclic left D-module or, equivalently, using the exact sequence (6), we obtain that $\mathrm{ext}_D^1(E, D) = 0$ or we can take $\Lambda = 0$ in P.

In [3, 4], we showed how to compute the right D-modules $\mathrm{ext}_D^1(M, D)$ and $\mathrm{ext}_D^1(E, D)$ when D is a non-commutative polynomial ring over which Gröbner bases exist for any term order (e.g., certain *Ore algebras* ([3])). Using a right Gröbner basis computation, we can check whether or not $\mathrm{ext}_D^1(E, D) = 0$ or the existence of a right inverse of P over D. However, apart from particular situations, we do not know yet how to recognize the existence of $\Lambda \in D^q$ satisfying 2 of Theorem 8.

On simple examples over a commutative polynomial ring D with coefficients in a computable field k, we can use a generic vector $\Lambda \in D^q$ with a fixed degree and compute the D-module $\mathrm{ext}_D^1(E, D) = D^{1\times q}/(D^{1\times(p+1)}\,P^T)$ by means of a Gröbner basis computation and check whether or not $\mathrm{ext}_D^1(E, D)$ vanishes on a *constellation* of semi-algebraic sets in the k-parameters of Λ ([13]). See [13] for an interesting survey where these results are explained and implemented in SINGULAR. They are particularly interesting when $\mathrm{ext}_D^1(M, D) = D^{1\times q}/(D^{1\times p}\,R^T)$ is 0-dimensional.

4 Applications of Serre's theorem

We are now in position to give applications of Theorem 8 to the reduction problem.

Theorem 9. *Let $R \in D^{q\times p}$ be a full row rank matrix and $\Lambda \in D^q$ such that there exists $U \in \mathrm{GL}_{p+1}(D)$ satisfying that $(R \quad -\Lambda)\, U = (I_q \quad 0)$. If we denote by*

$$U = \begin{pmatrix} S_1 & Q_1 \\ S_2 & Q_2 \end{pmatrix},$$

where $S_1 \in D^{p\times q}$, $S_2 \in D^{1\times q}$, $Q_1 \in D^{p\times(p+1-q)}$, $Q_2 \in D^{1\times(p+1-q)}$, and the left D-module $L = D^{1\times(p+1-q)}/(D\,Q_2)$ defined by the following exact sequence

$$0 \longrightarrow D \xrightarrow{.Q_2} D^{1\times(p+1-q)} \xrightarrow{\kappa} L \longrightarrow 0, \tag{7}$$

then we have:

$$M = D^{1\times p}/(D^{1\times q}\,R) \;\cong\; L = D^{1\times(p+1-q)}/(D\,Q_2). \tag{8}$$

Conversely, if M is isomorphic to a left D-module L defined by the exact sequence (7) for a certain matrix $Q_2 \in D^{1\times(p+1-q)}$, then there exist two matrices $\Lambda \in D^q$ and $U \in \mathrm{GL}_{p+1}(D)$ satisfying $(R \quad -\Lambda)\, U = (I_q \quad 0)$.

Proof. $\Rightarrow$ By hypothesis, we have $(R \quad -\Lambda)\, S = I_q$, where $S = (S_1^T \quad S_2^T)^T$, which shows that $P = (R \quad -\Lambda)$ admits a right-inverse over D. By Theorem 8, the extension (3) of D by M is defined by a stably free left D-module E, and thus, free of rank $p-q+1$ by 2 of Proposition 5 applied to E. Moreover, by 2 of Proposition 5, the D-morphism $\varphi : E \longrightarrow D^{1\times(p+1-q)}$ defined by $\varphi(\varrho((\mu_1 \quad \mu_2))) = \mu_1\, Q_1 + \mu_2\, Q_2$, for all $\mu_1 \in D^{1\times p}$ and $\mu_2 \in D$, is an isomorphism, which shows that we have the following equivalence of extensions of D by M:

$$\begin{array}{ccccccccc}
0 \longrightarrow & D & \xrightarrow{\alpha} & E & \xrightarrow{\beta} & M & \longrightarrow 0 \\
 & \| & & \downarrow \varphi & & \| & \\
0 \longrightarrow & D & \xrightarrow{\varphi\circ\alpha} & D^{1\times(p+1-q)} & \xrightarrow{\beta\circ\varphi^{-1}} & M & \longrightarrow 0.
\end{array}$$

In the standard bases of D and $D^{1\times(p+1-q)}$, we have

$$(\varphi\circ\alpha)(1) = \varphi(\alpha(1)) = \varphi(\varrho((0 \quad 1)) = Q_2,$$

which shows that $\varphi\circ\alpha : D \longrightarrow D^{1\times(p+1-q)}$ is defined by $(\varphi\circ\alpha)(\nu) = \nu\, Q_2$, for all $\nu \in D$. If we denote by $L = D^{1\times(p+1-q)}/(D\,Q_2)$, $\kappa : D^{1\times(p+1-q)} \longrightarrow L$ the projection onto L, then we obtain (7) and $L = \mathrm{coker}_D(.Q_2) \cong \mathrm{im}\,(\beta\circ\varphi^{-1}) = M$.

$\Leftarrow$ Let us suppose that there exist $Q_2 \in D^{1\times(p+1-q)}$ such that (7) holds and a D-isomorphism $\gamma : L \longrightarrow M$. Then, we have the following extension of D by M:

$$0 \longrightarrow D \xrightarrow{.Q_2} D^{1\times(p+1-q)} \xrightarrow{\gamma\circ\kappa} M \longrightarrow 0. \tag{9}$$

By Theorem 3, there exists $\Lambda \in D^q$ such that the equivalence class of (9) corresponds to the element $\rho(\Lambda) \in \mathrm{ext}^1_D(M, D)$. Then, $\rho(\Lambda)$ defines an extension of D by M of the form (3), where $E = D^{1\times(p+1)}/(D^{1\times q}\,(R \quad -\Lambda))$, which belongs to the same equivalence class as (9). Using the fact that extensions of D by M belonging to the same equivalence class are defined by D-isomorphic central left D-modules, we obtain that E is a free left D-module of rank $p+1-q$, which, by 2 of Proposition 5, implies that there exists $U \in \mathrm{GL}_{p+1}(D)$ satisfying that $(R \quad -\Lambda)\,U = (I_q \quad 0)$. □

Theorem 9 is a module-theoretic generalization of Theorem 1 (e.g., extension to non-square matrices over non-commutative domains, generalization of unimodular equivalence by isomorphism equivalence). The condition that Λ admits a left-inverse is not used in Theorem 9. For results using it (e.g., the computation of $V \in \mathrm{GL}_q(D)$ and $W \in \mathrm{GL}_p(D)$ satisfying $V\,R\,W = \mathrm{diag}(I_{q-1}, Q_2)$), see [2].

We now give an explicit description of the isomorphism obtained in Theorem 9.

Corollary 10 ([2]). *The D-isomorphism (8) obtained in Theorem 9 is defined by:*

$$\begin{array}{rcl} M = D^{1\times p}/(D^{1\times q}\,R) & \longrightarrow & L = D^{1\times(p+1-q)}/(D\,Q_2) \\ \pi(\lambda) & \longmapsto & \kappa(\lambda\,Q_1). \end{array}$$

Moreover, its inverse $\psi^{-1} : L \longrightarrow M$ is defined by $\psi^{-1}(\kappa(\mu)) = \pi(\mu\,T_1)$, where:

$$U^{-1} = \begin{pmatrix} R & -\Lambda \\ T_1 & T_2 \end{pmatrix} \in \mathrm{GL}_{p+1}(D), \quad T_1 \in D^{(p+1-q)\times p}, \quad T_2 \in D^{(p+1-q)}.$$

A straightforward consequence of Corollary 10 is the following result.

Corollary 11. *Let $\mathcal{F}$ be a left D-module and the following two linear systems $\ker_{\mathcal{F}}(R.) = \{\eta \in \mathcal{F}^p \mid R\,\eta = 0\}$ and $\ker_{\mathcal{F}}(Q_2.) = \{\zeta \in \mathcal{F}^{p+1-q} \mid Q_2\,\zeta = 0\}$. Then, we have the abelian group isomorphism $\ker_{\mathcal{F}}(R.) \cong \ker_{\mathcal{F}}(Q_2.)$ and:*

$$\ker_{\mathcal{F}}(R.) = Q_1\,\ker_{\mathcal{F}}(Q_2.), \quad \ker_{\mathcal{F}}(Q_2.) = T_1\,\ker_{\mathcal{F}}(R.).$$

The conditions 1, 2 and 3 stated in the next corollary of Theorem 9 are known in module theory to imply that a stably free left D-module is free respectively due to a nice property of left principal ideal domains (PID), the Quillen-Suslin theorem and Stafford's theorem. For more details and references, see [5, 7, 17, 21].

Corollary 12. *Let $R \in D^{q\times p}$ be a full row rank matrix and $\Lambda \in D^q$ a column vector such that the matrix $P = (R \quad -\Lambda) \in D^{q\times(p+1)}$ admits a right-inverse over D. Then, Theorem 9 holds when D satisfies one of the following properties:*

1. *D is a (left) PID, i.e., every (left) ideal of D can be generated by one element,*
2. *$D = A[x_1, \ldots, x_n]$ is a commutative polynomial ring and A a field or a PID,*

3. *D is one of the two* Weyl algebras $A_n(k)$ *or* $B_n(k)$ *(namely, algebras of the differential operators in* $d_1 = \frac{\partial}{\partial x_1}, \ldots, d_n = \frac{\partial}{\partial x_n}$ *with coefficients in the polynomial ring* $k[x_1, \ldots, x_n]$ *or in the field of rational functions* $k(x_1, \ldots, x_n)$*), k a field of characteristic 0 (e.g.,* $k = \mathbb{Q}, \mathbb{R}, \mathbb{C}$*) and* $p - q \geq 1$.

Example 13. We consider the model of a string with an interior mass defined by

$$\begin{cases} \phi_1(t) + \psi_1(t) - \phi_2(t) - \psi_2(t) = 0, \\ \dot{\phi}_1(t) + \dot{\psi}_1(t) + \eta_1\, \phi_1(t) - \eta_1\, \psi_1(t) - \eta_2\, \phi_2(t) + \eta_2\, \psi_2(t) = 0, \\ \phi_1(t - 2\, h_1) + \psi_1(t) - u(t - h_1) = 0, \\ \phi_2(t) + \psi_2(t - 2\, h_2) - v(t - h_2) = 0, \end{cases} \tag{10}$$

and studied in [14], where η_1 and η_2 are two constant parameters and $h_1, h_2 \in \mathbb{R}_+$ are such that $\mathbb{Q}\, h_1 + \mathbb{Q}\, h_2$ is a 2-dimensional $\mathbb{Q}$-vector space. Let us denote by $D = \mathbb{Q}(\eta_1, \eta_2)\, [\partial, \sigma_1, \sigma_2]$ the commutative polynomial algebra of differential incommensurable time-delay operators in ∂, σ_1 and σ_2, where $\partial\, f(t) = \dot{f}(t)$, $\sigma_1\, f(t) = f(t - h_1)$ and $\sigma_2 f(t) = f(t - h_2)$. The system matrix $R \in D^{4 \times 6}$ of (10) is then defined by:

$$R = \begin{pmatrix} 1 & 1 & -1 & -1 & 0 & 0 \\ \partial + \eta_1 & \partial - \eta_1 & -\eta_2 & \eta_2 & 0 & 0 \\ \sigma_1^2 & 1 & 0 & 0 & -\sigma_1 & 0 \\ 0 & 0 & 1 & \sigma_2^2 & 0 & -\sigma_2 \end{pmatrix} \in D^{4 \times 6}.$$

Let us consider $\Lambda = (0 \quad -1 \quad 0 \quad 0)^T \in D^4$ and $P = (R \quad -\Lambda) \in D^{4 \times 7}$. Using the package OreModules ([4]), we can check that P admits a right-inverse over D and, by 2 of Corollary 12, we get that $M = D^{1 \times 6}/(D^{1 \times 4}\, R) \cong L = D^{1 \times 3}/(D\, \overline{R})$. Using OreModules, let us compute $\overline{R} \in D^{1 \times 3}$. We obtain that the matrix

$$U = \begin{pmatrix} 0 & 0 & -1 & 0 & -1 & -\sigma_1 & 0 \\ 0 & 0 & 1 & 0 & 0 & \sigma_1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & \sigma_2 \\ -1 & 0 & 0 & -1 & -1 & 0 & -\sigma_2 \\ 0 & 0 & -\sigma_1 & 0 & -\sigma_1 & 1 - \sigma_1^2 & 0 \\ -\sigma_2 & 0 & 0 & -\sigma_2 & -\sigma_2 & 0 & 1 - \sigma_2^2 \\ \eta_2 & 1 & 2\, \eta_1 & 2\, \eta_2 & \partial + \eta_1 + \eta_2 & 2\, \eta_1\, \sigma_1 & 2\, \eta_2\, \sigma_2 \end{pmatrix} \in \mathrm{GL}_7(D),$$

satisfies $(R \quad -\Lambda)\, U = (I_4 \quad 0)$, and thus, $\overline{R} = (\partial + \eta_1 + \eta_2 \quad 2\, \eta_1\, \sigma_1 \quad 2\, \eta_2\, \sigma_2)$. Hence, (10) is equivalent to the following differential time-delay equation:

$$\dot{x}_1(t) + (\eta_1 + \eta_2)\, x_1(t) + 2\, \eta_1\, x_2(t - h_1) + 2\, \eta_2\, x_3(t - h_2) = 0.$$

That result was also obtained in [6] using D-morphism computations and resolutions of algebraic Riccati equations of the form $X\, R\, X = X$ as explained in [5, 6].

For more results on the reduction problem (e.g., on the problem of reducing the matrix R to $\mathrm{diag}(I_m, \star)$, where $1 \leq m \leq q - 1$, based on $\mathrm{ext}_D^1(M, D^{1 \times (q-m)}) \cong \mathrm{ext}_D^1(M, D) \otimes_D D^{1 \times (q-m)}$), applications in systems theory and examples, see [2].

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