New perspectives in algebraic systems theory

Alban Quadrat[*]

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1 Introduction

Many systems coming from mathematical physics, applied mathematics and engineering sciences can be described by means of systems of ordinary or partial differential equations, difference equations, differential time-delay equations... If those systems are linear, then they can be defined by means of matrices with entries in (non-commutative) polynomial rings of functional operators such as the rings of differential operators, shift operators, time-delay operators... This idea can be traced back to J. Boole's two treatises [3, 4] on linear systems of differential and difference equations (1859, 1860). The rings of differential operators, shift operators and time-delay operators belong to an important class of non-commutative polynomial rings called *Ore algebras* ([32]) which have recently been studied in symbolic computation ([11]) and multidimensional systems theory ([9, 10]). Alternatively, infinite-dimensional linear systems can be described by means of matrices with entries in convolutional algebras ([28]) or Banach algebras such as the Hardy algebra $H^\infty(\mathbb{C}_+)$, the Callier-Desoer algebras $\mathcal{A}$ and $\hat{\mathcal{A}}$ and the Wiener algebra W_+ ([15, 64]). The underlying operational calculus was pioneered by O. Heaviside's work on linear differential equations (1880-1887).

Algebraic analysis is a mathematical theory which was created in the sixties by B. Malgrange ([31]), L. Ehrenpreis and V. Palamodov ([35]) for the study of linear systems of partial differential equations with constant coefficients. It was further developed by the Japanese school of M. Sato ([58]) and particularly by M. Kashiwara

*INRIA Sophia Antipolis, APICS project, 2004 Route des Lucioles, BP 93, 06902 Sophia Antipolis Cedex, France, `Alban.Quadrat@sophia.inria.fr`, `http://www-sop.inria.fr/members/Alban.Quadrat/index.html`.

([29]) for linear systems of partial differential equations with variable coefficients. Algebraic analysis was introduced in mathematical systems theory by U. Oberst ([34]), M. Fliess ([18, 19]) and J.-F. Pommaret ([38, 40]) in their study of differential linear control systems at the end of the eighties. Ideas of algebraic analysis have recently been extended to the case of Ore algebras and *coherent Banach algebras* in order to study within a unified mathematical framework the structural properties of different classes of multidimensional linear systems ([9, 12]) and the stabilization problems of infinite-dimensional linear systems ([45, 46]).

The purpose of these notes is to give a short introduction to the algebraic analysis approach to mathematical systems theory. Within this algebraic analysis framework, we shall explain how to study in a unified way (i.e., with common concepts, techniques, results, algorithms and implementations) time-invariant/time-varying continuous/discrete-time 1D linear systems (state-space, polynomial or behavioural representations) ([26, 37, 56]), multidimensional linear systems (e.g., systems over rings, differential time-delay systems, underdetermined systems of partial differential equations) ([5, 38, 62, 66]) and infinite-dimensional linear systems within an input-output approach (e.g., distributed-parameter systems such as partial differential, time-delay or convolutional equations) ([15, 64]). Finally, we shall show how to combine the previous approach with constructive algebra (e.g., *non-commutative Gröbner* or *Janet bases* ([11, 30, 55])) and symbolic computation to constructively study multidimensional linear systems and develop dedicated packages such as OreModules ([10]), OreMorphisms ([13]), JanetOre ([55]), QuillenSuslin ([17]) or Stafford ([53]).

2 Algebraic analysis approach to systems theory

A key idea in the algebraic analysis approach to mathematical systems theory is to associate the *finitely presented left* D*-module* $M = D^{1\times p}/(D^{1\times q}\, R)$ with a linear system defined by a matrix $R \in D^{q\times p}$ over a ring D (e.g., Ore algebras, convolution algebras, Banach algebras), where $D^{1\times p}$ denotes the set of row vectors of length p with entries in D and $D^{1\times q}\, R = \{\mu\, R \mid \mu \in D^{1\times q}\}$. We recall that the definition of a module is the same as the one of a vector space apart from the fact that the coefficients belong to a ring D (i.e., D admits non-trivial non-invertible elements for the multiplication operation) and not to a field (where every non-zero element is invertible for the multiplication operation). Moreover, if D is a non-commutative ring, i.e., $a\, b$ can be different from $b\, a$ for a and $b \in D$, then we need to specify if the elements of D act on the left or on the right on the elements of M. In particular, M is said to be a *left* (resp., *right*) D-module if, for all a_1 and $a_2 \in D$ and for all m_1 and $m_2 \in M$, we have $a_1\, m_1 + a_2\, m_2 \in M$ (resp., $m_1\, a_1 + m_2\, a_2 \in M$).

The idea of using module theory is natural as the structural properties of the linear system can be studied by means of algebraic manipulations on the system matrix R, i.e., by performing linear algebra over a ring which is called *module theory* in mathematics ([57]). One of the main interests for introducing the left D-module $M = D^{1\times p}/(D^{1\times q}\, R)$ is that it is intrinsically defined in the sense that different representations of the same linear system by means of different matrices R define the same left D-module M (up to *isomorphisms*). In terms of homological algebra, the different representations of the linear system define different *finite presentations* of M ([57]). Hence, even if the linear system can be described by means of different representations (e.g., state-space or input-output representations for 1D linear systems, Roesser or Fornasini-Marchesini models for multidimensional systems), the left D-module M intrinsically defines the underlying *linear system of equations*.

Another reason for introducing the left D-module M is that it allows us to give an intrinsic formulation of the concept of the *solutions space* of the linear system of equations defined by the left D-module M. Indeed, if $\mathcal{F}$ denotes a left D-module, then an $\mathcal{F}$*-solution* of the linear system of equations $R\,y = 0$, namely, $\eta \in \mathcal{F}^p$ satisfying $R\,\eta = 0$, is in a one-to-one correspondence with a *left D-homomorphism* (i.e., a D-linear application) from M to $\mathcal{F}$. For instance, we can consider $\mathcal{F}$ to be the space of smooth functions or distributions when D is the ring of differential or differential time-delay operators with constant coefficients, the ring of real analytic functions when D is the ring of differential operators with real analytic coefficients, the Hilbert space $H^2(\mathbb{C}_+)$ when $D = H^\infty(\mathbb{C}_+)$ or $\hat{\mathcal{A}}$, $L^p(\mathbb{R}_+)$ when $D = \mathcal{A}$. If we denote by $\hom_D(M, \mathcal{F})$ the *abelian group* (i.e., $\mathbb{Z}$-module) of D-homomorphisms from M to $\mathcal{F}$, then the following isomorphism of abelian groups holds:

$$\ker_{\mathcal{F}}(R.) = \{\eta \in \mathcal{F}^p \mid R\,\eta = 0\} \cong \hom_D(M, \mathcal{F}). \qquad (1)$$

That central observation, first due to B. Malgrange ([31]), was the starting point of the development of algebraic analysis. From (1), we see that the solution space $\hom_D(M, \mathcal{F})$ only depends on the left D-modules M (which intrinsically defines the linear system of equations) and $\mathcal{F}$ (which defines the functional space). In particular, as M does not depend on any particular representation of the linear system, i.e., does not depend on a particular matrix $R \in D^{q\times p}$ defining the linear system, the solution space $\hom_D(M, \mathcal{F})$ does not depend on the particular embedding of $\ker_{\mathcal{F}}(R.)$ into $\mathcal{F}^p$ (i.e., on the choice of specific "coordinate systems"). Similarly as in algebraic geometry where some curves or surfaces do not necessarily have points in certain fields or rings (e.g., no real solutions of a polynomial with real coefficients), the nature of the solution space $\hom_D(M, \mathcal{F})$ depends on $\mathcal{F}$. In algebraic geometric terms, an $\mathcal{F}$-solution of the linear system of equations defined by a finitely presented left D-module M corresponds to the so-called $\mathcal{F}$*-point*, where M plays the role of the underlying *scheme* in algebraic geometry ([16]).

We note that the right-hand side of (1) is called a *behaviour* within the *behavioural approach* pioneered by J. C. Willems ([37]). Hence, the behavioural approach can be studied within the algebraic analysis approach as it was first noticed by U. Oberst in 1990 ([34]). See also [43, 48]. In particular, we obtain an intrinsic characterization of the concept of behaviours as the algebraic analysis approach does not depend on any embedding of the solution space into a power $\mathcal{F}^p$ of the *signal space* $\mathcal{F}$. The idea of intrinsically defining the concept of an algebraic variety without using any particular embedding, as it is done in differential geometry by means of atlas and charts, was the beginning of A. Grothendieck's revolutionary revision of algebraic geometry based on the concept of schemes ([16]).

Another important remark concerning (1) is that the intrinsic reformulation of the solution space $\ker_{\mathcal{F}}(R.)$ in terms of $\hom_D(M, \mathcal{F})$, i.e., as the set of D-linear applications from a certain algebraic object M to $\mathcal{F}$, can be seen as the passage from classical mechanics to quantum mechanics where *observables* play a crucial role. In mathematics, it is a classical technique to define algebras of functions on (continuous, differentiable, algebraic, analytic) varieties. However, the recent adoption of the opposite point of view consisting in defining a space by means of an algebra of functions was extremely successful in mathematics and particularly in operator algebras due to I. M. Gel'fand's results ([21]), A. Grothendieck's revision of algebraic geometry ([16]) and A. Connes' non-commutative geometry ([14]). See also ([33]) for a reformulation of the differential geometry based on this idea. In particular, this new viewpoint has refreshed our conceptions of points, spaces and symmetries as it is well explained by P. Cartier in [8].

In the community of mathematical systems theory, the question of the mathematical status of the behavioural approach was raised and especially the question to know whether it belonged to algebra or analysis. The previous explanations show it should be considered as a geometrical theory. See also [59, 60] for interesting results concerning the behavioural approach based on algebraic geometric ideas. Another argument for this thesis is that, despite the important results in the theory of partial differential equations obtained by M. Malgrange, L. Ehrenpreis and V. Palamodov (e.g., fundamental principle, integral representation) and despite M. Sato's fundamental results (hyperfunctions, microfunctions, microdifferential or microlocal operators), algebraic analysis is nowadays considered a branch of algebraic geometry.

We can wonder what kind of geometry a behaviour $\ker_{\mathcal{F}}(R.)$ or, more generally, the solution space $\hom_D(M, \mathcal{F})$ can have. Following F. Klein's ideas (Erlangen program), a geometry can be defined by a group of symmetries. Hence, which kind of symmetries does the solution space $\hom_D(M, \mathcal{F})$ admit? One answer is to look at the *internal symmetries* of M, i.e., at the *D-endomorphism ring* of M, namely, the non-commutative ring $\mathrm{end}_D(M) = \hom_D(M, M)$ of the D-endomorphisms of M, and at its subgroup $\mathrm{aut}_D(M)$ formed by the *D-automorphisms* of M defined by invertible D-endomorphisms of M. Indeed, if $\phi \in \hom_D(M, \mathcal{F})$ corresponds to a certain $\mathcal{F}$-solution of the linear system of equations defined by M, then, for every $f \in \mathrm{end}_D(M)$, $\phi \circ f \in \hom_D(M, \mathcal{F})$ is another solution. Hence, the abelian group homomorphism $f^\star : \hom_D(M, \mathcal{F}) \longrightarrow \hom_D(M, \mathcal{F})$ defined by $f^\star(\phi) = \phi \circ f$, sends any solution of the linear system to another one, i.e., $f^\star \in \mathrm{end}_{\mathbb{Z}}(\hom_D(M, \mathcal{F}))$. In particular, if $f \in \mathrm{aut}_D(M)$, then $f^\star$ permutes the elements of the solution space $\hom_D(M, \mathcal{F})$, i.e., defines a *Galois-like transformation.* For more details, see [12].

For certain classes of (non-commutative) polynomial rings, M. Janet ([25]) and B. Buchberger ([7]) have developed two constructive algorithms which compute new sets of generators for a (left) ideal or module, called a *Janet* or a *Gröbner basis*. Algorithms rewriting any element of that (left) ideal or module in terms of the new generators were obtained. More generally, *normal forms* of general elements can be computed with respect to the Janet or Gröbner bases (see, e.g., [11, 30, 22, 55] and the references therein). For instance, the knowledge of a Janet or a Gröbner basis for the left D-submodule $D^{1\times q}\,R$ of $D^{1\times p}$ allows us to compute the normal form of any element $\lambda \in D^{1\times p}$ with respect to the basis, i.e., to compute a distinguished representative of the residue class of λ in the quotient left D-module $M = D^{1\times p}/(D^{1\times q}\,R)$. In particular, Gröbner or Janet basis techniques allow one to constructively work in the (left) ideals and modules (e.g., computation of kernels, images, factorizations or left/right/generalized inverses of multivariate polynomial matrices). They are nowadays implemented in different computer algebra systems (e.g., Maple, Mathematica, Singular, Macaulay 2, CoCoA) for different classes of commutative and non-commutative polynomial rings. We note that Gröbner bases over commutative polynomial rings were introduced by L. Habets in his study of differential time-delay systems ([23]). More recently, non-commutative Gröbner bases have played an important role in J. W. Helton's works and, particularly, in those concerning H^{∞}-control which use the package *NCAlgebra* ([24]).

Over certain (non-commutative) polynomial rings of functional operators for which the concept of Gröbner or Janet bases is well-defined, such as the rings of differential operators, shift operators, delay operators, we can constructively study the D-endomorphism ring $\mathrm{end}_D(M)$ of a finitely presented left D-module M as explained in [12] and implemented in the package OreMorphisms ([13]). Those results are useful in the study of the "geometries" of M and $\hom_D(M, \mathcal{F})$ and their invariants (e.g., quadratic first integrals, quadratic conservation laws) ([12]) as well as for the central issue in module theory consisting in recognizing whether or not two finitely presented left D-modules M and M' are isomorphic. If $D = k[s]$ is a univariate commutative polynomial ring over a field k (e.g., $k = \mathbb{Q}$, $\mathbb{R}$), then computing the *Smith canonical forms* ([26]) of two system matrices $R \in D^{q\times p}$ and $R' \in D^{q'\times p'}$ and comparing their *invariant factors*, we can decide whether or not the two left D-modules $M = D^{1\times p}/(D^{1\times q}\,R)$ and $M' = D^{1\times p'}/(D^{1\times q'}\,R')$ are isomorphic. However, no Smith canonical form exists over a multivariate polynomial ring ([6]). Hence, a way to study the equivalence problem is to try to decompose the modules into direct sums of *indecomposable modules* (i.e., into "atoms") and compare them. Indeed, in the case of $D = k[s]$, the Smith canonical form gives a decomposition of the module M into direct sums of *cyclic D-modules* defined by the invariant factors, i.e., into D-modules of the form $D/(D\,d_i)$, where d_i is an invariant factor of the Smith canonical form of R. Based on the computation of *idempotents* of D-endomorphisms of M, namely, $f \in \mathrm{end}_D(M)$ satisfying $f^2 = f$, the decomposition problem is constructively developed in [12]. More generally, the factorization, reduction and decomposition problems are studied in [12] and implemented in the package OreMorphisms ([13]). If $\mathrm{GL}_p(D) = \{U \in D^{p\times p} \mid \exists\, V \in D^{p\times p} :\ U\,V = V\,U = I_p\}$ denotes the group of invertible matrices of the non-commutative ring $D^{p\times p}$ and $R \in D^{q\times p}$, then those problems are respectively defined by:

1. Find two matrices $T \in D^{q\times r}$ and $S \in D^{r\times p}$ satisfying $R = T\,S$.
2. Find two matrices $V \in \mathrm{GL}_q(D)$ and $W \in \mathrm{GL}_p(D)$ such that the new matrix $\overline{R} = V\,R\,W$ has a block triangular form.
3. Find two matrices $V \in \mathrm{GL}_q(D)$ and $W \in \mathrm{GL}_p(D)$ such that the new matrix $\overline{R} = V\,R\,W$ has a block diagonal form.

A direct application of the first (resp., third) problem is that $\ker_{\mathcal{F}}(S.) \subseteq \ker_{\mathcal{F}}(R.)$ (resp., $\ker_{\mathcal{F}}(R.) \cong \ker_{\mathcal{F}}(R_1.) \oplus \ker_{\mathcal{F}}(R_2.)$, where R_1 and R_2 denote two complementary diagonal blocks). Concerning 2, if the matrix $\overline{R}$ is a 2×2 upper triangular matrix, i.e., $\overline{R}_{21} = 0$, then the integration of the linear system $R\,\eta = 0$ can be done "in cascade" as we first need to integrate $\overline{R}_{22}\,\overline{\eta}_2 = 0$ and then the inhomogeneous linear system $\overline{R}_{11}\,\overline{\eta}_1 = -\overline{R}_{12}\,\overline{\eta}_2$. Then, $\eta = W\,(\overline{\eta}_1^T \quad \overline{\eta}_2^T)^T$ is a solution of $R\,\eta = 0$. Hence, the constructive study of the internal symmetries of the left D-module M gives important information about the solution space $\hom_D(M, \mathcal{F})$. Finally, those previous results are used in [6, 12, 13] to simplify classical differential time-delay linear systems studied in the literature of control theory.

3 Dictionary between the structural properties of linear systems and modules

In Section 2, we explained that the philosophy of algebraic analysis was to first define the concept of equivalence of linear systems of equations in terms of isomorphic modules. Another interest of using algebraic analysis in mathematical systems theory is that the structural properties of a linear system can be studied by means of the structural properties of its underlying module as this module does not depend on any particular representations of the system (e.g., state-space or input-output representations). Hence, if we find a module characterization for a structural property of time-varying linear 1D systems (e.g., controllability, observability), then testing this module property for different representations of a linear system (e.g., state-space or input-output representations) must give again the different characterizations known within these different approaches (e.g., Kalman, Hautus and polynomial tests ([26, 56])). This intrinsic approach contrasts with control theory, which is historically divided into the state-space approach and the frequency-domain approach (which are in some extent proved to be equivalent) ([26, 56]). In philosophical terms, this algebraic analysis approach can be seen as a *structuralist approach* to mathematical systems theory. See [1] for a historical account of structuralism in mathematics (epitomized by the romantic images of N. Bourbaki and A. Grothendieck), social sciences, literature and arts.

A crucial point in module theory is that we cannot generally extract a basis from a finite family of generators of a left D-module M as we need to invert certain elements of D in the computation of an independent family of generators, which is generally impossible over a ring D in contrast with the case of a field. Hence, although its definition is similar to the one of a vector space, a left D-module M generally does not admit a basis. A direct consequence is that module theory is a richer algebraic theory than linear algebra in the sense that a finer classification of the structural properties of modules needs to be developed. We shall below recall a few of the module properties commonly used in module theory (see, e.g., [32, 57]). In order to do that, we first need to recall certain concepts in ring theory. A ring D is called a *domain* if $0 \neq a \in D$ and $0 \neq b \in D$ implies that $a\,b \neq 0$. A domain D is said to be *left noetherian* if every *left ideal* of D, namely, every left D-submodule of D, is *finitely generated*, i.e., admits a finite family of generators.

Definition 1 ([32, 57]). Let M be a finitely presented left D-module over either a left noetherian domain D or a commutative domain D. Then, we have:

1. M is *free* if there exists a non-negative integer r such that $M \cong D^{1\times r}$.
2. M is *stably free* if there exist two non-negative integers r and s such that $M \oplus D^{1\times s} \cong D^{1\times r}$, where $\oplus$ denotes the direct sum.
3. M is *projective* if there exist a left D-module N and a non-negative integer r such that $M \oplus N \cong D^{1\times r}$.
4. M is *reflexive* if the D-homomorphism $\varepsilon_M : M \longrightarrow \hom_D(\hom_D(M, D), D)$ defined by $\varepsilon_M(m)(f) = f(m)$, for all $f \in \hom_D(M, D)$ and for all $m \in M$, is an isomorphism (i.e., ε_M is both injective and surjective).
5. M is *torsion-free* if the *torsion left D-submodule* of M defined by
$$t(M) = \{m \in M \mid \exists\, 0 \neq P \in D : \; P\, m = 0\}$$
is reduced to 0. Elements of $t(M)$ are called the *torsion elements* of M.
6. M is *torsion* if $t(M) = M$.

From Definition 1, it is clear that a free left D-module M admits a basis. Moreover, we can prove that a free module is stably free, a stably free module is projective, a projective module is reflexive and a reflexive module is torsion-free ([57]). The converse implications only hold for special rings. For instance, if $D = k[s]$ is a univariate polynomial ring over a field k, then a torsion-free D-module is free. This result can be generalized to any *left ideal domain*, namely, a domain over which every left ideal of D can be generated by one element (e.g., the ring of ordinary differential operators with coefficients in a *differential field* ([18, 38]) such as $\mathbb{Q}(t)$ (the so-called *Weyl algebra* $B_1(\mathbb{Q})$) or in the field of meromorphic functions). However, there exist stably free but non free left modules over the Weyl algebra $D = A_1(\mathbb{Q})$ of ordinary differential operators with polynomial coefficients, but torsion-free left D-modules are stably free. In 1955, J.-P. Serre conjectured that projective modules over a commutative polynomial ring $D = k[x_1, \ldots, x_n]$, where k is a field, are free. This conjecture was proved to be true by D. Quillen and A. A. Suslin in 1976 only (see, e.g., [17, 57]). For an implementation of the Quillen-Suslin theorem in the package QuillenSuslin, see [17]. A theorem of J. T. Stafford ([32]) asserts that a stably free left module of at least rank 2 over the Weyl algebra $A_n(k)$ (resp., $B_n(k)$)[1], where k is a field of characteristic 0 (e.g., $k = \mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$), is free. See [53] for a constructive algorithm of this result and its implementation in the package Stafford. Finally, we can prove that a reflexive finitely presented $H^\infty(\mathbb{C}_+)$-module is free ([45]).

Over the years, a large dictionary between the structural properties of linear systems and modules has been developed in the study of analysis and synthesis

[1] $A_n(k)$ (resp., $B_n(k)$) is the ring of partial differential operators in $\partial/\partial_1, \ldots, \partial/\partial_n$ with coefficients in $k[x_1, \ldots, x_n]$ (resp., $k(x_1, \ldots, x_n)$).

problems such as the stabilization problems of infinite-dimensional linear systems ([15, 64]) and the variational and optimal control problems of multidimensional linear systems ([44]). Some of those equivalences are summed up in Table 1. For more results and details, see [9, 18, 19, 34, 36, 38, 39, 40, 41, 44, 45, 46, 47, 48, 49, 50, 61, 63, 65, 67] and the references therein. Certain results in Table 1 are only valid for an *injective cogenerator* left D-module $\mathcal{F}$ ([34, 48, 57, 65]). This concept somehow plays a similar role as the one of algebraically closed fields in algebraic geometry (e.g., $\mathbb{C}$). A classical result in homological algebra asserts that every non-commutative ring D admits an injective cogenerator left D-module $\mathcal{F}$ ([57]). For more results not based on the use of injective cogenerator modules, see [48, 59, 60]. We point out that certain system properties cannot fit into Table 1 because, for instance, they are characterized in terms of properties of different modules (e.g., observability).

Finally, we note that module theory was first introduced by R. Kalman in realization theory ([27]). But his approach is rather different from the one developed in algebraic analysis. We also refer the reader to the interesting works of E. W. Kamen ([28]), E. D. Sontag ([62]) and P. A. Fuhrmann ([20]) where module theory plays an important role (see the references therein).

4 Constructive homological algebra & packages

The results developed in Section 3 raises the natural but important question of recognizing whether or not a finitely presented left D-module M has non-trivial torsion-elements or is torsion-free, reflexive, projective, stably free or free. Within algebraic analysis, the powerful techniques of homological algebra are used to study the structural properties of modules within a unified mathematical framework. More precisely, as those techniques do not depend on particular representations of the linear system, i.e., on particular *free/projective resolutions* of the left D-module M defining the linear system of equations ([57]), they naturally form powerful mathematical tools to answer the previous question. Moreover, they are generic methods which do not depend on the underlying ring D. The particular properties of the ring D are only used to simplify certain situations (e.g., the vanishing of certain abelian groups defining *homological invariants* of the left D-module M). Hence, homological algebra allows us to develop a unified treatment of linear systems defined over different kind of rings.

Methods of homological algebra and category theory are sometimes called *abstract non-sense* in the literature of mathematics with the pernicious effect of scaring non-specialists (whereas it was first a self-mockery within the mathematical community). Contrary to this idea, homological algebra studies in full generality simple mathematical objects which appear in different theories. As in abstract painting where real forms are depicted in a simplified and rather reduced way (lines, circles, squares), homological algebra only focuses on the simple but fundamental features of certain complex mathematical objects. It is nowadays the backbone of many important mathematical theories (e.g., algebraic topology, group theory, algebraic number theory, algebraic geometry, analytic geometry, non-commutative geometry) and it

Table 1. *Classification of structural properties*

Module M	Structural properties	Stabilization problems Optimal control
Torsion	Autonomous system Poles/zeros classifications	
With torsion	Existence of autonomous elements	
Torsion-free	No autonomous elements, Controllability, Parametrizability, π-freeness	Variational problem without constraints (Euler-Lagrange equations)
Reflexive	Filter identification	
Projective	Internal stabilizability, Bézout identities	Existence of a parametrization all stabilizing controllers Computation of Lagrange parameters without integration
Free	Flatness, Poles placement, Doubly coprime factorization	Youla-Kučera parametrization Optimal controller

daily plays a more and more important role in mathematical physics. In particular, it develops powerful mathematical tools based on abstract algebraic objects such as complexes, exact sequences, homology and cohomology groups, chain complexes, extension or torsion functors, spectral sequences ([57]). In recent years, using the developments and implementations of Gröbner and Janet bases over certain classes of Ore algebras ([11, 30, 55]), it was shown in [9, 10, 12, 13, 40, 43, 54] how to make some of those important tools constructive. The corresponding constructive algorithms have been implemented in the packages OreModules ([10]) and OreMorphisms ([13]) and have been demonstrated in different examples appearing in the control theory literature such as differential time-delay systems (see the libraries of examples of those packages). Hence, combining constructive homological algebraic techniques with the dictionary developed in Section 3 (see Table 1), we can constructively determine the structural properties of the corresponding linear system. The advantage of describing the structural properties within the language of homological algebra is transferred to the implementation of the algorithms in OreModules and OreMorphisms: all algorithms are stated and implemented in sufficient generality such that the different non-commutative polynomial rings of functional operators implemented in the package *Ore_algebra* (available in the current releases of Maple) are covered at the same time.

Let us shortly explain how to test whether or not the finitely presented left D-module $M = D^{1\times p}/(D^{1\times q}\, R)$ has some non-trivial torsion elements, or is torsion-free, reflexive or projective (projective modules being stably free for the important Ore algebras). As explained in [9, 40] and implemented in OreModules ([10]), if we introduce the right D-module $N = D^q/(R\, D^p)$, where D^p denotes the set of column vectors of length p with entries in D, then, based on computations of left and right kernels of matrices, we can compute the so-called *extension* left D-modules $\mathrm{ext}^i_D(N, D)$ for $i = 1, \ldots, \mathrm{gldim}(D)$, where $\mathrm{gldim}(D)$ denotes the so-called *global dimension* of D (e.g., $\mathrm{gldim}(D) = n$ if $D = k[x_1, \ldots, x_n]$ is a commutative polynomial ring over a field k) ([32, 53, 57]). For instance, the left D-module $\mathrm{ext}^1_D(N, D)$ is defined by

$$\mathrm{ext}^1_D(N, D) = \ker_D(.Q)/(D^{1\times q}\, R),$$

where $Q \in D^{p\times m}$ is any matrix satisfying $\ker_D(R.) \triangleq \{\nu \in D^p \mid R\,\nu = 0\} = Q\, D^m$ and $\ker_D(.Q) \triangleq \{\lambda \in D^{1\times p} \mid \lambda\, Q = 0\}$. Using homological algebraic techniques, we can prove that $\mathrm{ext}^1_D(N, D)$ only depends on M ([42]). Then, the vanishing of the $\mathrm{ext}^i_D(N, D)$'s classifies the structural properties of the left D-module M as shown in Table 2.

In the particular case of a *full row rank matrix* $R \in D^{q\times p}$ over the commutative polynomial ring $D = \mathbb{R}[x_1, \ldots, x_n]$, namely, the rows of R are D-linearly independent (i.e., $\ker_D(.R) = 0$), the previous result allows us to find again the different concepts of *primeness* (see the fourth column of Table 2), which play an important role in multidimensional systems theory and are recalled hereafter.

Definition 2 ([5, 66]). Let $D = \mathbb{R}[x_1, \ldots, x_n]$ be a commutative polynomial ring, $R \in D^{q\times p}$ a full row rank matrix, J the ideal generated by the $q \times q$ minors of R and $V(J)$ the algebraic variety defined by $V(J) = \{\xi \in \mathbb{C}^n \mid P(\xi) = 0,\ \forall\, P \in J\}$.

1. R is called *minor left-prime* if $\dim_{\mathbb{C}} V(J) \leq n - 2$, i.e., the greatest common divisor of the $q \times q$ minors of R is 1.

2. R is called *weakly zero left-prime* if $\dim_{\mathbb{C}} V(J) \leq 0$, i.e., the $q \times q$ minors of R may only vanish simultaneously in a finite number of points of $\mathbb{C}^n$.

3. R is called *zero left-prime* if $\dim_{\mathbb{C}} V(J) = -1$, i.e., the $q \times q$ minors of R do not vanish simultaneously in $\mathbb{C}^n$.

Table 2. *Classification of structural properties*

Module M	$\mathbf{ext}_D^i(N, D)$	$\dim_D(N)$	**Primeness**
With torsion	$t(M) \cong \mathrm{ext}_D^1(N, D)$	$n-1$	$\emptyset$
Torsion-free	$\mathrm{ext}_D^1(N, D) = 0$	$n-2$	Minor left-prime
Reflexive	$\mathrm{ext}_D^i(N, D) = 0$, $i = 1, 2$	$n-3$	
...	...	...	...
...	$\mathrm{ext}_D^i(N, D) = 0$, $1 \leq i \leq n-1$	0	Weakly zero left-prime
Projective	$\mathrm{ext}_D^i(N, D) = 0$, $1 \leq i \leq n$	-1	Zero left-prime

The fourth column of Table 2 can be generalized to the case of the ring D of differential operators over a differential field K (e.g., $\mathbb{Q}(x_1, \ldots, x_n)$, the ring of meromorphic functions). See the third column of Table 2, where $\dim_D(N)$ denotes the *dimension* of the right D-module N, i.e., the dimension of the *characteristic variety* of the linear system of partial differential equations defined by the *formal adjoint* $\widetilde{R}$ of R in the sense of the theory of distributions ([40]).

If M is a torsion-free left D-module, then the vanishing of $\mathrm{ext}_D^1(N, D)$ gives a matrix $Q \in D^{p \times m}$ satisfying that $M \cong D^{1 \times p}\, Q$. Hence, if $\mathcal{F}$ denotes an injective left D-module, we then obtain $\ker_{\mathcal{F}}(R.) = Q\,\mathcal{F}^m$, i.e., the behaviour $\ker_{\mathcal{F}}(R.)$ admits the *parametrization* defined by Q ([9, 10, 38, 40]). Within the behavioural approach, this parametrization is called an *image representation*. Moreover, if M is a reflexive left D-module, then $\mathrm{ext}_D^i(N, D) = 0$, for $i = 1$ and 2, which proves that the new behaviour $\ker_{\mathcal{F}}(Q.)$ is also *parametrizable*, i.e., there exists $P \in D^{m \times l}$ satisfying $\ker_{\mathcal{F}}(Q.) = P\,\mathcal{F}^l$. If M is a projective left D-module, then we obtain a *chain of successive parametrizations*, namely, we have $\ker_{\mathcal{F}}(R.) = Q_0\,\mathcal{F}^{m_1}$ and $\ker_{\mathcal{F}}(Q_i.) = Q_{i+1}\,\mathcal{F}^{m_{i+2}}$, $0 \leq i \leq n-2$. For instance, the divergence operator in $\mathbb{R}^3$ is parametrizable by the curl operator and the curl operator is parametrized by the gradient operator. We can prove that if M is a free left D-module of rank m, then there exists an *injective parametrization* of $\ker_{\mathcal{F}}(R.)$, namely, there exist $Q \in D^{p \times m}$

and $T \in D^{m \times p}$ satisfying that $\ker_{\mathcal{F}}(R.) = Q\,\mathcal{F}^m$ and $T\,Q = I_m$ ([9, 17, 53]). Hence, for every element $\eta \in \ker_{\mathcal{F}}(R.)$, there exists a unique $\xi \in \mathcal{F}^m$ satisfying that $\eta = Q\,\xi$ and $\xi = T\,\eta$. For more details, see [17, 53] and the packages QuillenSuslin ([17]) and Stafford ([53]). The behaviour $\ker_{\mathcal{F}}(R.)$ is then said to be *flat* ([19]) and ξ is called a *flat output* of the system. Finally, we refer the reader to [51, 52, 54] for *Monge parametrizations* of a solution space $\hom_D(M, \mathcal{F})$ admitting non-trivial *autonomous elements*, i.e., whose underlying left D-module M admits a non-trivial torsion left D-submodule $t(M)$. The Monge parametrization of the solution space $\hom_D(M, \mathcal{F})$ is obtained by gluing the integration of the autonomous elements of $\hom_D(M, \mathcal{F})$, i.e., $\hom_D(t(M), \mathcal{F})$, to a parametrization of the parametrizable solution space $\hom_D(M/t(M), \mathcal{F})$ defined by the left D-module $M/t(M)$.

The increasing role of homological algebra in mathematical systems theory, mathematical physics and other fields has recently motivated the development of packages based on more and more powerful homological algebraic techniques like, for instance, OreModules ([10]), OreMorphisms ([13]) and `homalg` ([2]). We are convinced that this phenomenon is a precursory sign of a new era when computer algebra and symbolic computation will play the equivalent role for pure mathematics as the one played by numerical analysis in engineering sciences.

5 Conclusion

We hope to have convinced the reader that the algebraic analysis approach to linear systems gives an aesthetic unification of different existing theories (e.g., state-space and input-output representation approaches, systems over rings, discrete-time or continuous-time systems, multidimensional or infinite-dimensional systems, analysis or synthesis problems). Within this new algebraic approach to mathematical systems theory, open questions and conjectures have been solved (e.g., intrinsic study of the structural properties of the linear systems, generalization of the concepts of primeness to non-full row rank matrices with coefficients in some Ore algebras ([9, 40]), constructive studies of the computation of (injective/minimal/successive) parametrizations of multidimensional linear systems ([9, 17, 38, 39, 41, 53]), Lin-Bose's conjectures on a generalization of Serre's conjecture ([17]), Lin's conjecture on the relations between internal stabilizability and the existence of doubly coprime factorizations for multidimensional systems ([49]) and for general linear systems (the so-called Vidyasagar-Schneider-Francis' question) ([45, 47, 49]), generalization of the Youla-Kučera parametrization of all stabilizing controllers for internally stabilizable plants which do not necessarily admit doubly coprime factorizations ([47, 50])). In particular, using constructive homological algebra techniques, the different results have been implemented for some classes of linear systems into symbolic computation packages such as OreModules ([10]), OreMorphisms ([13]), JanetOre ([55]), QuillenSuslin ([17]) or Stafford ([53]). However, even if some open problems have been solved, there remain many unanswered questions which need to be studied. Thus, our hope for the near future is that what have been achieved (and demonstrated therein) will entice young scientists to further develop the algebraic analysis approach to mathematical systems theory.

Bibliography

[1] A. D. Aczel. *The Artist and the Mathematician. The story of Nicolas Bourbaki, the genius mathematician who never existed.* High Stakes, 2006.

[2] M. Barakat and D. Robertz. `homalg`: A meta-package for homological algebra. To appear in *Journal of Algebra and Applications*, 2008. `http://wwwb.math.rwth-aachen.de/homalg/`.

[3] J. Boole. *Treatise on Differential Equations.* Macmillan, 1859.

[4] J. Boole. *Treatise on the Calculus of Finite Differences.* Macmillan, 1860.

[5] N. K. Bose. *Multidimensional Systems Theory and Applications.* Kluwer, 2 edition, 2003.

[6] M. S. Boudellioua and A. Quadrat. Reduction of linear systems based on Serre's theorem. In *Proceedings of MTNS 2008, Virginia (USA)*, 2008.

[7] B. Buchberger. *An algorithm for finding a basis for the residue class rings of a zero-dimensional polynomial ideal.* PhD thesis, University of Innsbruck (Austria), 1965.

[8] P. Cartier. A mad day's work: from Grothendieck to Connes and Kontsevich. The evolution of concepts of space and symmetry. *Bulletin of the American Mathematical Society*, 38:389–408, 2001.

[9] F. Chyzak, A. Quadrat, and D. Robertz. Effective algorithms for parametrizing linear control systems over Ore algebras. *Applicable Algebra in Engineering, Communications and Computing*, 16(5):319–376, 2005.

[10] F. Chyzak, A. Quadrat, and D. Robertz. OreModules: A symbolic package for the study of multidimensional linear systems. In J. Chiasson and J.-J. Loiseau, editors, *Applications of Time-Delay Systems*, LNCIS 352, pages 233–264. Springer, 2007. `http://wwwb.math.rwth-aachen.de/OreModules/`.

[11] F. Chyzak and B. Salvy. Non-commutative elimination in Ore algebras proves multivariate identities. *Journal of Symbolic Computation*, 26:187–227, 1998.

[12] T. Cluzeau and A. Quadrat. Factoring and decomposing a class of linear functional systems. *Linear Algebra and Its Applications*, 428(1):324–381, 2008.

[13] T. Cluzeau and A. Quadrat. OreMorphisms: A homological algebraic package for factoring, reducing and decomposing linear functional systems. In J.-J. Loiseau, W. Michiels, S.-I. Niculescu, and R. Sipahi, editors, *Topics in Time-Delay Systems: Analysis, Algorithms and Control*, LNCIS. Springer, 2008. http://www-sop.inria.fr/members/Alban.Quadrat/OreMorphisms/index.html.

[14] A. Connes. *Noncommutative Geometry*. Academic Press, 1994.

[15] R. F. Curtain and H. Zwart. *An Introduction to Infinite-Dimensional Linear Systems Theory*, volume 21 of *Texts in Applied Mathematics*. Springer, 1991.

[16] D. Eisenbud and J. Harris. *The Geometry of Schemes*, volume 197 of *Graduate Texts in Mathematics*. Springer, 2000.

[17] A. Fabiańska and A. Quadrat. Applications of the Quillen-Suslin theorem in multidimensional systems theory. In H. Park and G. Regensburger, editors, *Gröbner Bases in Control Theory and Signal Processing*, Radon Series on Computation and Applied Mathematics 3, pages 23–106. de Gruyter publisher, 2007. http://wwwb.math.rwth-aachen.de/QuillenSuslin/index.html.

[18] M. Fliess. Some basic structural properties of generalized linear systems. *Systems & Control Letters*, 15:391–396, 1990.

[19] M. Fliess and H. Mounier. Controllability and observability of linear delay systems: an algebraic approach. *ESAIM: Control, Optimisation and Calculus of Variations*, 3:301–314, 1998.

[20] P. A. Fuhrmann. Algebraic systems theory: An analyst's point of view. *Journal of Franklin Institute*, 305:521–540, 1976.

[21] I. Gel'fand, R. Raikov, and G. Shilov. *Commutative normed rings*. Chelsea, 1964.

[22] G.-M. Greuel and G. Pfister. *A Singular Introduction to Commutative Algebra*. Springer, 2002.

[23] L. Habets. *Algebraic and computational aspects of time-delay systems*. PhD thesis, University of Eindhoven (The Netherlands), 1994.

[24] J. W. Helton, M. Stankus, and J. J. Wavrik. Computer simplification of formulas in linear systems theory. *IEEE Transactions on Automatic Control*, 43(3):302–314, 1998. http://www.math.ucsd.edu/~ncalg/.

[25] M. Janet. *Leçons sur les systèmes d'équations aux dérivées partielles*. Cahiers scientifiques IV. Gauthier-Villars, 1929.

[26] T. Kailath. *Linear Systems*. Prentice-Hall, 1980.

[27] R. E. Kalman, P. L. Falb, and M. A. Arbib. *Topics in Mathematical System Theory*. McGraw-Hill, 1969.

[28] E. W. Kamen. On an algebraic theory of systems defined by convolution operators. *Mathematical Systems Theory*, 9(1):57–74, 1975.

[29] M. Kashiwara. *D-modules and Microlocal Calculus*, volume 217 of *Translations of Mathematical Monographs*. American Mathematical Society, 2003.

[30] V. Levandovsky. *Non-commutative Computer Algebra for polynomial algebras: Gröbner bases, applications and implementation*. PhD thesis, University of Kaiserslautern (Germany), 2005.

[31] B. Malgrange. Systèmes à coefficients constants. *Séminaire Bourbaki*, 246:79–89, 1962-63.

[32] J. C. McConnell and J. C. Robson. *Noncommutative Noetherian Rings*, volume 30 of *Graduate Studies in Mathematics*. American Mathematical Society, 2000.

[33] J. Nestruev. *Smooth Manifolds and Observables*, volume 220 of *Graduate Texts in Mathematics*. Springer, 2003.

[34] U. Oberst. Multidimensional constant linear systems. *Acta Applicandæ Mathematicæ*, 20:1–175, 1990.

[35] V. P. Palamodov. *Linear Differential Operators with Constant Coefficients*. Springer, 1970.

[36] H. K. Pillai and S. Shankar. A behavioural approach to control of distributed systems. *SIAM Journal of Control and Optimization*, 37:388–408, 1999.

[37] J. W. Polderman and J. C. Willems. *Introduction to Mathematical Systems Theory: A Behavioral Approach*, volume 26 of *Texts in Applied Mathematics*. Springer-Verlag, 1998.

[38] J.-F. Pommaret. *Partial Differential Control Theory*, volume 530 of *Mathematics and Its Applications*. Kluwer, 2001.

[39] J.-F. Pommaret and A. Quadrat. Generalized Bezout identity. *Applicable Algebra in Engineering, Communications and Computing*, 9:91–116, 1998.

[40] J.-F. Pommaret and A. Quadrat. Algebraic analysis of linear multidimensional control systems. *IMA Journal of Control and Information*, 16(3):275–297, 1999.

[41] J.-F. Pommaret and A. Quadrat. Localization and parametrization of linear multidimensional systems. *Systems & Control Letters*, 37:247–260, 1999.

[42] J.-F. Pommaret and A. Quadrat. Equivalences of linear control systems. In *Proceedings of MTNS 2000, Perpignan (France)*, 2000.

[43] J.-F. Pommaret and A. Quadrat. A functorial approach to the behaviour of multidimensional control systems. *Applied Mathematics and Computer Science*, 13(1):7–13, 2003.

[44] J.-F. Pommaret and A. Quadrat. A differential operator approach to multidimensional optimal control. *International Journal of Control*, 77:821–836, 2004.

[45] A. Quadrat. The fractional representation approach to synthesis problems: an algebraic analysis viewpoint. Part I: (Weakly) doubly coprime factorizations. *SIAM Journal of Control and Optimization*, 42:266–299, 2003.

[46] A. Quadrat. The fractional representation approach to synthesis problems: an algebraic analysis viewpoint. Part II: Internal stabilization. *SIAM Journal of Control and Optimization*, 42:300–320, 2003.

[47] A. Quadrat. On a generalization of the Youla-Kučera parametrization. Part I: the fractional ideal approach to SISO systems. *Systems & Control Letters*, 50:135–148, 2003.

[48] A. Quadrat. An algebraic interpretation to the operator-theoretic approach to stabilizability. Part I: SISO systems. *Acta Applicandæ Mathematicæ*, 88:1–45, 2005.

[49] A. Quadrat. A lattice approach to analysis and synthesis problems. *Mathematics of Control, Signals, and Systems*, 18:147–186, 2006.

[50] A. Quadrat. On a generalization of the Youla-Kučera parametrization. Part II: The lattice approach to MIMO systems. *Mathematics of Control, Signals, and Systems*, 18:199–235, 2006.

[51] A. Quadrat and D. Robertz. Parametrizing all solutions of uncontrollable multidimensional linear systems. In *Proceedings of the* 16^{th} *IFAC World Congress, Prague (Czech Republic)*, 2005.

[52] A. Quadrat and D. Robertz. On the Monge problem and multidimensional optimal control. In *Proceedings of MTNS 2006, Kyoto (Japan)*, 2006.

[53] A. Quadrat and D. Robertz. Computation of bases of free modules over the Weyl algebras. *Journal of Symbolic Computation*, 42(11-12):1113–1141, 2007. `wwwb.math.rwth-aachen.de/OreModules/stafford.html`.

[54] A. Quadrat and D. Robertz. Baer's extension problem for multidimensional linear systems. In *Proceedings of MTNS 2008, Virginia (USA)*, 2008.

[55] D. Robertz. Janet bases and applications. In M. Rosenkranz and D. Wang, editors, *Groebner Bases in Symbolic Analysis*, Radon Series on Computation and Applied Mathematics 2, pages 139–168. de Gruyter publisher, 2007.

[56] H. H. Rosenbrock. *State space and Multivariable Theory*. Nelson, 1970.

[57] J. J. Rotman. *An Introduction to Homological Algebra*. Academic Press, 1979.

[58] M. Sato, T. Kawai, and M. Kashiwara. Microfunctions and pseudo-differential equations. In Komatsu, editor, *Hyperfunctions and pseudo-differential equations*, Lecture Notes in Mathematics, pages 265–529. Springer-Verlag, 1973.

[59] S. Shankar. The Nullstellensatz for systems of PDE. *Advances in Applied Mathematics*, 23:360–374, 1999.

[60] S. Shankar. The lattice structure of behaviours. *SIAM Journal of Control and Optimization*, 39:1817–1832, 2001.

[61] S. Shankar and V. R. Sule. Algebraic geometric aspects of feedback stabilization. *SIAM Journal of Control and Optimization*, 30:11–30, 1992.

[62] E. D. Sontag. Linear systems over commutative rings: A survey. *Ricerche di Automatica*, 7:1–34, 1978.

[63] V. R. Sule. Feedback stabilization over commutative rings: the matrix case. *SIAM Journal of Control and Optimization*, 32:1675–1695, 1994.

[64] M. Vidyasagar. *Control System Synthesis. A Factorization Approach.* MIT Press, 1985.

[65] J. Wood. Modules and behaviours in nD systems theory. *Multidimensional Systems and Signal Processing*, 11(1-2):11–48, 2000.

[66] E. Zerz. *Topics in Multidimensional Linear Systems Theory*, volume 256 of *Lecture Notes in Control and Information Sciences.* Springer, 2000.

[67] E. Zerz. An algebraic analysis approach to linear time-varying systems. *IMA Journal of Mathematical Control and Information*, 23:113–126, 2006.