On a general robust stability test based on the homological perturbation lemma

Alban Quadrat

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On a general robust stability test based on the homological perturbation lemma

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Abstract: Within the lattice approach to synthesis problems, we show how a general unstructured robust stability test can be obtained directly by applying the homological perturbation lemma, a standard method developed in algebraic topology and homological algebra. This robust stability test generalizes and unifies various results from the robust control literature.

Key-words: Robust control, H_∞ control, model uncertainties, unstructured robust stability tests, homological perturbation lemma, lattice approach, fractional representation approach

* Sorbonne Université and Université de Paris, CNRS, Inria Paris, IMJ-PRG, F-75005 Paris, France (alban.quadrat@inria.fr, `https://who.rocq.inria.fr/Alban.Quadrat/`)

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We dedicate this paper to the memory of my dear Aunt Martine, with my deepest love and gratitude.

RESEARCH CENTRE PARIS

2 rue Simone Iff - CS 42112
75589 Paris Cedex 12

Sur un test général de stabilité robuste basé sur le lemme de perturbation homologique

Résumé : Dans le cadre de l'approche des problèmes de synthèse par réseaux algébriques, nous montrons comment un test général de stabilité robuste non structurée peut être obtenu grâce à l'application du lemme de perturbation homologique, une méthode classique développée en topologie algébrique et algèbre homologique. Ce test de stabilité robuste généralise et unifie de nombreux résultats de la littérature de la commande robuste.

Mots-clés : Commande robuste, commande H_∞, incertitudes de modèle, tests de stabilité robuste non structurée, lemme de perturbation homologique, réseaux algébriques, approche par représentations fractionnaires

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1 Introduction

Robust control aims to control uncertain systems – subject to disturbances – by controller design. Since Zames introduced the Hardy algebra $H^\infty(\mathbb{C}_+)$ into the study of synthesis problems of uncertain systems – defined in the frequency domain – (i.e., the sensitivity minimization problem) [39], H_∞ robust control has been actively developed, especially in the 1990s. For example, see [9, 11, 37, 38, 40] and references therein. In particular, H_∞ control has provided a mathematical framework to study the concepts of bounded model uncertainties and robustness, developed robust stability tests, characterized robust controllers, etc. It is a rich mathematical interplay between control theory, complex analysis, and operator theory. Finally, engineers in companies now widely use H_∞ methods, showing that robust control has been a major success of control theory in recent years.

In the robust control literature, unstructured robust stability tests have been obtained for finite- and infinite-dimensional systems [9, 11, 14, 15, 37, 38, 40]. Given a stabilizing controller C of P, these tests characterize the *stability robustness* for a given model uncertainty P_Δ of a plant P, namely, they characterize Δ which ensures that C stabilizes all systems of P_Δ. Eight different model uncertainties P_Δ have been considered in the literature (additive uncertainty, multiplicative input or output uncertainty, multiplicative uncertainty, inverse multiplicative input or output uncertainty, and coprime factor uncertainty). For each of them, sharp bounds have been obtained for the maximum H_∞-norm on Δ in terms of the inverse of the H_∞-norm of a given transfer matrix of the closed-loop system defined by P and C.

This paper presents a general unstructured robust stability test (Theorem 5). It is stated within the *fractional representation approach* to analysis and synthesis problems, developed by Vidyasagar, Desoer, Francis, etc. [10, 37]. Thus, it holds for linear systems that can be defined by transfer matrices with entries in the quotient field $\mathcal{K}$ of an integral domain $\mathcal{A}$ of single-input single-output (SISO) proper and stable systems. In particular, it can be applied to several standard classes of linear systems (e.g., continuous/discrete finite-dimensional/infinite-dimensional systems, multidimensional systems). Moreover, it is only assumed that the plant P is *internally stabilizable*, i.e., that there exists a stabilizing controller C of P. Thus, we do not (necessarily) assume that P has a *doubly coprime factorization* as classically done in the literature. Recall that the existence of a doubly coprime factorization is only a sufficient condition for internal stabilizability (see Theorem 1 and Example 1), and is only necessary for certain (but important) classes of systems [36, 37, 29]. Moreover, the computation of doubly coprime factorizations is usually a hard problem while, in many practical cases, an explicit stabilizing controller is known. The set of uncertainties Δ considered in this paper combines the eight model uncertainties mentioned above. As a consequence, we find again the standard unstructured robust stability tests appearing in the literature (see Example 6, Table 1, and Section 7.3). If P has a doubly coprime factorization, then the general unstructured robust stability test reduces to the standard robust stability test for coprime factor uncertainty [37] (see Section 7.3). Thus, the unstructured robust stability test proposed here compactly encapsulates − with minimal assumptions − various results from the robust control literature.

It is worth noting that this general unstructured robust stability test naturally introduces the two projectors Π_P and Π_C of the closed-loop system onto P and C respectively [12]. Under the assumption that P has a doubly coprime factorization, these projectors were used in [37, 12, 14, 15] to obtain the robust stability test for coprime factor uncertainty for finite and infinite-dimensional linear systems (i.e., for $\mathcal{A} = RH_\infty$ and $H^\infty(\mathbb{C}_+)$). They are also at the core of the *loop-shaping* method [38] and of the definition of the *stability margin* and *optimal robust radius* [12, 14, 15].

To obtain this general unstructured robust stability test (Theorem 5), we combine two different area of mathematics, namely, the *algebraic lattice approach* to analysis and synthesis problems, initiated in [26, 28, 29] (see also [1] for an overview), and the so-called *homological perturbation lemma* [2, 4, 7, 16, 17, 18, 34, 13] developed in *algebraic topology* [19] and *homological algebra* [33].

The lattice approach provides a mathematical approach to the study of linear systems defined by transfer matrices. Within this approach, two *lattices* [3] can be naturally associated with a linear system defined by a transfer matrix P with entries in $\mathcal{K}$. System properties for P can then be translated into algebraic properties of the lattices. Using algebra, module theory, and homological algebra [33], one can then obtain general characterizations for the existence of (weakly) coprime factorizations, of internal stabilizability, of the set of all the stabilizing controllers of a stabilizable system (which does not necessarily have a doubly coprime factorization), of the internal model principle, etc. [22, 26, 28, 29]. Some of these results are recalled in this paper. Finally, the *operator-theoretic approach* to linear systems [14, 15, 37], based on graphs of linear operators, can be proved to be dual to the lattice approach [27].

The study of the so-called *deformation retracts* in topology motivated the development of the *homological perturbation lemma* in algebraic topology [19] and homological algebra [33]. See [4, 7, 16] and the references therein. The homological perturbation lemma (also called the *basic perturbation lemma*) provides explicit formulas in homological algebra and algebraic topology. It can be traced back to Shih's doctoral thesis (supervised by H. Cartan) on the computation of the homology of fibre spaces [35]. Its abstract modern formulation was found by Brown in [4] and is at the core of the so-called *homological perturbation theory* [2, 17, 18]. For an overview, we refer the reader to Crainic's classic notes [7]. Finally, methods of homological algebra can be made algorithmic in certain contexts, and thus, they can be implemented in computer algebra systems and used to develop effective aspects of algebraic topology (e.g., *effective homology*, *reductions* between two *chain complexes*) [34, 13]. In this context, the homological perturbation lemma plays an important role [34, 13].

The control theory community does not pay much attention to algebraic topology. The main exception is Jonckheere's pioneering work on algebraic and differential topology methods for robust stability [20].

Within the lattice approach to linear systems theory, we show that the general unstructured robust stability test (Theorem 5) is a straightforward consequence of a basic version of the homological perturbation lemma. Since this version can be easily proved by elementary algebraic arguments and its proof does not appear in the literature, we provide proof in the Appendix. The idea of applying the homological perturbation lemma to the two canonical *split short exact sequences* defining the two lattices associated with an internally stabilizable plant P comes from the fact that stabilizable plants (and only them) define an algebraic version of deformation retracts. This remark is also at the core of the development of a *noncommutative geometric approach* to robust control, initiated in [30, 32], following Connes' ideas and methods [6]. We note that the main interest of the lattice approach is to formulate general results and draw deep connections with other mathematical theories. As is often the case in algebra, these results, once precisely formulated, can be checked again by direct verification. An elementary computational verification of Theorem 5 is given in Corollary 2. In this paper, we have chosen to present the general method following the adage "Give a man a fish and you feed him for a day; teach a man to fish and you feed him for a lifetime". A preliminary version of this work appeared in the conference paper [31].

Plan. This section gives the general framework and the main notations. In Section 2, we briefly recall the fractional representation approach. Within this approach, we recall a general necessary and sufficient condition for internal stabilizability (Theorem 1), explain its connections with left/right coprime factorizations (Example 1), and briefly emphasize the deep connections that exist between stabilizability, algebra, topology, and differential geometry (paragraph after Lemma 1). In Section 3, the main results of the lattice approach to analysis and synthesis problems are recalled (Theorem 2). In Section 4, a brief introduction to basic homological algebra is given in connection with the lattice approach. In Section 5, we explain how the concepts of split short exact sequences and retracts are related to internal stabilizability and the parametrization of all the stabilizing controllers. A simple version of the homological perturbation lemma (Theorem 4) is given in Section 6. To keep this section short and accessible to a wide audience, the proofs are given in the appendix. The main result of the paper is stated in Theorem 5 and proved in the appendix. An elementary computational verification of this result is given in Corollary 2. Thus, the main result is accessible and can be verified by any reader without special knowledge of algebra. The rest of the section is devoted to showing how standard results on robust stability tests are applications of this general test. Finally, in the appendix, we list the proofs of the version of the homological perturbation lemma used in this paper and its application to the lattice approach yielding a detailed proof of the main result of the paper using homological algebra arguments.

Notations. In the following, $\mathcal{A}$ denotes an integral domain (namely, a commutative ring in which $a_1\, a_2 = 0$ and $a_1 \neq 0$ yield $a_2 = 0$) and $Q(\mathcal{A}) = \{n/d \mid 0 \neq d,\, n \in \mathcal{A}\}$ the field of fractions of $\mathcal{A}$. The $\mathcal{A}$-module formed by all the $q \times p$ matrices with entries in $\mathcal{A}$ is denoted by $\mathcal{A}^{q\times p}$ and the identity matrix of $\mathcal{A}^{p\times p}$ by I_p. If $R \in \mathcal{A}^{q\times p}$ and $\mathcal{B}$ is an $\mathcal{A}$-module (e.g., $\mathcal{B} = \mathcal{A}$ or $\mathcal{K}$), then we will use the notation

$$\ker_{\mathcal{B}}(.R) := \{\lambda \in \mathcal{B}^{1\times q} \mid \lambda\, R = 0\}, \quad \mathrm{im}_{\mathcal{B}}(.R) = \mathcal{B}^{1\times q}\, R := \{\lambda\, R \in \mathcal{B}^{1\times p} \mid \lambda \in \mathcal{B}^{1\times q}\},$$
$$\ker_{\mathcal{B}}(R.) := \{\eta \in \mathcal{B}^{p\times 1} \mid R\, \eta = 0\}, \quad \mathrm{im}_{\mathcal{B}}(R.) = R\, \mathcal{B}^{p\times 1} := \{R\, \eta \in \mathcal{B}^{p\times 1} \mid \eta \in \mathcal{B}^{p\times 1}\}.$$

We denote by $R_{i\bullet}$ (resp., $R_{\bullet j}$) the i^{th} row (resp., the j^{th} column) of the matrix R.

Let $\mathrm{GL}_q(\mathcal{A}) = \{A \in \mathcal{A}^{q\times q} \mid \exists\, B \in \mathcal{A}^{q\times q} :\; A\, B = B\, A = I_q\}$ be the *general linear group of $\mathcal{A}$ of index* q, i.e., the group of invertible matrices of $\mathcal{A}^{q\times q}$. Thus, $\mathrm{GL}_1(\mathcal{A})$ is the group of the invertible elements of $\mathcal{A}$, also denoted by $\mathrm{U}(\mathcal{A})$. Using the commutativity of $\mathcal{A}$, note that $\mathrm{GL}_q(\mathcal{A}) = \{A \in \mathcal{A}^{q\times q} \mid \det(A) \in \mathrm{U}(\mathcal{A})\}$.

If $\mathcal{M}$ is an $\mathcal{A}$-module, then $\mathrm{id}_{\mathcal{M}}$ is the identity endomorphism of $\mathcal{M}$, i.e., $\mathrm{id}_{\mathcal{M}}(m) = m$ for all $m \in \mathcal{M}$.

If $\mathcal{L}$ and $\mathcal{M}$ are two $\mathcal{A}$-modules, an $\mathcal{A}$-linear map $f : \mathcal{L} \longrightarrow \mathcal{M}$, i.e., $f(a_1\, l_1 + a_2\, l_2) = a_1\, f(l_1) + a_2\, f(l_2)$ for all $a_1,\, a_2 \in \mathcal{A}$ and $l_1,\, l_2 \in \mathcal{L}$, is called an $\mathcal{A}$-*homomorphism* and $\mathrm{Hom}_{\mathcal{A}}(\mathcal{L}, \mathcal{M})$ denotes the $\mathcal{A}$-module formed by all the $\mathcal{A}$-homomorphisms from $\mathcal{L}$ to $\mathcal{M}$. Finally, we also denote $\mathrm{Hom}_{\mathcal{A}}(\mathcal{M}, \mathcal{M})$ by $\mathrm{End}_{\mathcal{A}}(\mathcal{M})$.

2 The fractional representation approach

In this paper, we will study analysis and synthesis problems for shift-invariant linear systems within the *fractional representation approach* initiated in the eighties by Desoer, Vidyasagar, etc. See [10, 37] and the references therein. In particular, this approach has played a major role in the successful development of the H_∞ control in the nineties (see, e.g., [9, 11, 37, 40]).

The main idea at the core of the fractional representation approach is to view a stability test as a *membership problem*: given an integral domain $\mathcal{A}$ and P an element of the field of fractions $\mathcal{K} := Q(\mathcal{A})$ of $\mathcal{A}$, then the single-input single-output (SISO) linear system defined by the transfer function $P \in \mathcal{K}$ is $\mathcal{A}$-*stable* if $P \in \mathcal{A}$, or *unstable* if $P \in \mathcal{K} \backslash \mathcal{A}$. In other words, $\mathcal{A}$ defines an integral domain of SISO $\mathcal{A}$-stable systems. More generally, a multi-input multi-output (MIMO) linear system defined by a transfer matrix $P \in \mathcal{K}^{q\times r}$ is $\mathcal{A}$-*stable* if $P \in \mathcal{A}^{q\times r}$, *unstable* otherwise. Different rings $\mathcal{A}$ can be considered for different classes of linear systems (e.g., continuous or discrete, finite or infinite-dimensional or multidimensional systems) and stability concepts under consideration. See [9, 10, 37] and the examples below.

Let us now give some examples of integral domains of SISO stable systems. For finite-dimensional systems, one can consider the ring RH_∞ of proper and stable real rational functions, whose elements correspond to asymptotically stable finite-dimensional linear time-invariant systems [37, 40], for infinite-dimensional systems, the Hardy algebra $H^\infty(\mathbb{C}_+)$ of bounded holomorphic functions in the open right half-plane $\mathbb{C}_+ := \{s \in \mathbb{C} \mid \Re(s) > 0\}$ of $\mathbb{C}$, whose elements correspond to $L^2(\mathbb{R}_+) - L^2(\mathbb{R}_+)$-stability in the time-domain [9], or the Wiener algebra defined by

$$\mathcal{W} = \left\{ \widehat{f} + \sum_{i=0}^{+\infty} a_i\, e^{-h_i s} \mid f \in L^1(\mathbb{R}_+),\; (a_i)_{i\geq 0} \in l_1(\mathbb{N}),\; 0 = h_0 < h_1 < h_2 < \ldots \right\}, \tag{1}$$

whose elements are $L^\infty(\mathbb{R}_+) - L^\infty(\mathbb{R}_+)$-stability in the time-domain [10, 9], and the ring

$$\mathcal{R}_n = \{n/d \mid 0 \neq d,\, n \in \mathbb{R}[z_1, \ldots, z_n],\, \mathcal{V}(d) \cap \overline{\mathbb{D}}^n = \emptyset\}$$

of structurally stable multidimensional systems − where $\mathcal{V}(d) = \{z = (z_1, \ldots, z_n) \in \mathbb{C}^n \mid d(z) = 0\}$ is the set of complex solutions of d and $\overline{\mathbb{D}}^n = \{z = (z_1, \ldots, z_n) \in \mathbb{C}^n \mid |z_i| \leq 1, \ i = 1, \ldots, n\}$ the closed unit polydisc of $\mathbb{C}^n$ − whose elements are $l^\infty(\mathbb{Z}^n) - l^\infty(\mathbb{Z}^n)$ stable [28, 29]. For more details and examples, see [9, 10, 37, 28, 29, 21] and the references therein.

Let us give a few standard definitions of the fractional representation approach.

Definition 1 ([10, 28, 37])**.** Let $\mathcal{A}$ be an integral domain of SISO stable systems, $\mathcal{K} = Q(\mathcal{A})$ its field of fractions, $P \in \mathcal{K}^{q \times p}$, and $p = q + r$.

1. A *left fractional representation* of P is given by a pair of matrices (D, N), where $D \in \mathcal{A}^{q \times q}$, $\det D \neq 0$, and $N \in \mathcal{A}^{q \times p}$, satisfying $P = D^{-1} N$.
2. A *right fractional representation* of P is given by a pair of matrices $(\widetilde{D}, \widetilde{N})$, where $\widetilde{D} \in \mathcal{A}^{r \times r}$, $\det \widetilde{D} \neq 0$, and $\widetilde{N} \in \mathcal{A}^{p \times r}$, satisfying $P = \widetilde{N} \widetilde{D}^{-1}$.
3. A left fractional representation $P = D^{-1} N$ of P is said to be *weakly left coprime* if we have $\mathcal{K}^{1 \times q} R \cap \mathcal{A}^{1 \times p} = \mathcal{A}^{1 \times q} R$, where $R = (D \quad -N) \in \mathcal{A}^{q \times p}$.
4. A right fractional representation $P = \widetilde{N} \widetilde{D}^{-1}$ of P is said to be *weakly right coprime* if we have $\widetilde{R} \mathcal{K}^{r \times 1} \cap \mathcal{A}^{p \times 1} = \widetilde{R} \mathcal{A}^{r \times 1}$, where $\widetilde{R} = (\widetilde{N}^T \quad \widetilde{D}^T)^T \in \mathcal{A}^{p \times r}$.
5. P has a doubly weakly coprime factorization if both a weakly left and a weakly right coprime factorization exist for P.
6. A left fractional representation $P = D^{-1} N$ of P is said to be *left coprime* if $R = (D \quad -N) \in \mathcal{A}^{q \times p}$ has a right inverse $S = (X^T \quad Y^T)^T$, where $X \in \mathcal{A}^{q \times q}$ and $Y \in \mathcal{A}^{r \times q}$, i.e., $D X - N Y = I_q$.
7. A right fractional representation $P = \widetilde{N} \widetilde{D}^{-1}$ of P is said to be *right coprime* if $\widetilde{R} = (\widetilde{N}^T \quad \widetilde{D}^T)^T \in \mathcal{A}^{p \times r}$ has a left inverse $\widetilde{S} = (-\widetilde{Y} \quad \widetilde{X})$, where $\widetilde{X} \in \mathcal{A}^{r \times r}$ and $\widetilde{Y} \in \mathcal{A}^{r \times q}$, i.e., $-\widetilde{Y} \widetilde{N} + \widetilde{X} \widetilde{D} = I_r$.
8. P has a doubly coprime factorization if both a left and a right coprime factorization exist for P.

Remark 1. $P \in \mathcal{K}^{q \times p}$ always has a left (resp., right) fractional representation $P = D^{-1} N$ (resp., $P = \widetilde{N} \widetilde{D}^{-1}$): writing the entries P_{ij} of P as $P_{ij} = n_{ij}/d_{ij}$, where $0 \neq d_{ij}$, $n_{ij} \in \mathcal{A}$, for $i = 1, \ldots, r$ and $j = 1, \ldots, q$, then consider $d = \prod_{1 \leq i \leq r, 1 \leq j \leq q} d_{ij}$, $D = d \, I_p$, and $N := D \, P \in \mathcal{A}^{q \times r}$ (resp., $\widetilde{D} = d \, I_r$ and $\widetilde{N} := P \, \widetilde{D} \in \mathcal{A}^{q \times r}$).

Remark 2. A left coprime factorization is a weakly left coprime factorization. Indeed, if $\lambda \in \mathcal{K}^{1 \times q}$ is such that $\lambda R \in \mathcal{A}^{1 \times p}$, then $\lambda R S \in \mathcal{A}^{1 \times q}$ and using $R S = I_q$, we get $\lambda \in \mathcal{A}^{1 \times q}$. Similarly, a right coprime factorization is weakly right coprime. Finally, note that the converse (i.e., a weakly left/right coprime factorization is a left/right coprime factorization) is usually not true. See Theorem 2 below.

Remark 3. Suppose that P has both a left and a right coprime factorization, i.e., $P = D^{-1} N = \widetilde{N} \widetilde{D}^{-1}$, where $D X - N Y = I_q$ and $-\widetilde{Y}' \widetilde{N} + \widetilde{X}' \widetilde{D} = I_r$. If we set $O = -\widetilde{Y}' X + \widetilde{X}' Y \in \mathcal{A}^{r \times q}$, $\widetilde{Y} = \widetilde{Y}' + O \, D$, and $\widetilde{X} = \widetilde{X}' + O \, N$, then we have $-\widetilde{Y} \widetilde{N} + \widetilde{X} \widetilde{D} = I_r$ and $-\widetilde{Y} X + \widetilde{X} Y = 0$, which gives the Bézout identity

$$\begin{pmatrix} D & -N \\ -\widetilde{Y} & \widetilde{X} \end{pmatrix} \begin{pmatrix} X & \widetilde{N} \\ Y & \widetilde{D} \end{pmatrix} = \begin{pmatrix} I_q & 0 \\ 0 & I_r \end{pmatrix} = I_p. \tag{2}$$

Now consider a plant $P \in \mathcal{K}^{q \times r}$, a controller $C \in \mathcal{K}^{r \times q}$, and consider the closed-loop system defined by Figure 1. Writing the equations at the nodes, we have

$$\begin{cases} e_1 = u_1 - y_2 = u_1 - P \, e_2, \\ e_2 = u_2 - y_1 = u_2 - C \, e_2, \end{cases} \Leftrightarrow \underbrace{\begin{pmatrix} I_q & P \\ C & I_r \end{pmatrix}}_{:= G(P,C)} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}. \tag{3}$$

Definition 2 ([10, 37])**.** Let $P \in \mathcal{K}^{q \times r}$ be a plant and $C \in \mathcal{K}^{r \times q}$ a controller.

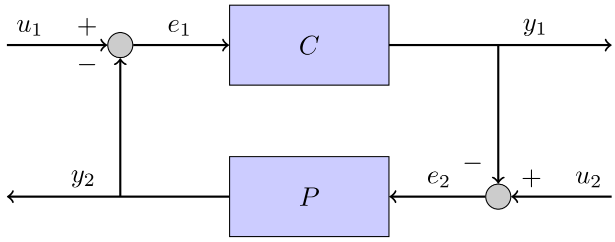

Figure 1: Closed-loop system

1. The closed-loop system defined by Figure 1 is said to be *well-posed* if the matrix $G(P, C)$ is non-singular, i.e., $\det(I_q - P\,C) \neq 0$ or $\det(I_r - C\,P) \neq 0$.
2. $C \in \mathcal{K}^{r\times q}$ *internally stabilizes* (or just *stabilizes*) $P \in \mathcal{K}^{q\times r}$ if the closed-loop system is well-posed and $H(P,\,C) := G(P, C)^{-1} \in \mathcal{A}^{p\times p}$, where

$$\begin{aligned} H(P,\,C) &= \begin{pmatrix} (I_q - P\,C)^{-1} & -(I_q - P\,C)^{-1}\,P \\ -C\,(I_q - P\,C)^{-1} & I_r + C\,(I_q - P\,C)^{-1}\,P \end{pmatrix} \\ &= \begin{pmatrix} I_q + P\,(I_r - C\,P)^{-1}\,C & -P\,(I_r - C\,P)^{-1} \\ -(I_r - C\,P)^{-1}\,C & (I_r - C\,P)^{-1} \end{pmatrix}. \end{aligned} \tag{4}$$

   Then, P is *internally stabilizable* and C is a *stabilizing controller* of P.
3. The set of all the stabilizing controllers of P is denoted by $\mathrm{Stab}(P)$.

If the closed-loop system defined by Figure 1 is *well-posed*, then we have

$$\begin{aligned} \begin{pmatrix} e_1 \\ y_1 \end{pmatrix} &= \Pi_C \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \ \Pi_C = \begin{pmatrix} (I_q - P\,C)^{-1} & -(I_q - P\,C)^{-1}\,P \\ C\,(I_q - P\,C)^{-1} & -C\,(I_q - P\,C)^{-1}\,P \end{pmatrix} \in \mathcal{K}^{p\times p}, \\ \begin{pmatrix} y_2 \\ e_2 \end{pmatrix} &= \Pi_P \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \ \Pi_P = \begin{pmatrix} -P\,(I_r - C\,P)^{-1}\,C & P\,(I_r - C\,P)^{-1} \\ -(I_r - C\,P)^{-1}\,C & (I_r - C\,P)^{-1} \end{pmatrix} \in \mathcal{K}^{p\times p}. \end{aligned} \tag{5}$$

Writing

$$\Pi_C = \begin{pmatrix} I_q \\ C \end{pmatrix} (I_q - P\,C)^{-1}\,(I_q \quad -P), \quad \Pi_P = \begin{pmatrix} P \\ I_r \end{pmatrix} (I_r - C\,P)^{-1}\,(-C \quad I_r), \tag{6}$$

we can then easily check that $\Pi_C^2 = \Pi_C$ and $\Pi_P^2 = \Pi_P$, i.e., Π_C and Π_P are two *idempotents* of the noncommutative ring $\mathcal{K}^{p\times p}$. Note that (5) shows that Π_C (resp., Π_P) is the projection of the closed-loop system onto the controller $y_1 = C\,e_1$ (resp., the system $y_2 = P\,e_2$). Furthermore, using the node equations (3), we have

$$\Pi_C \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \Pi_P \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} e_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} y_2 \\ e_2 \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \ \Rightarrow \ \Pi_C + \Pi_P = I_p. \tag{7}$$

Using (6), we have $\Pi_C\,\Pi_P = \Pi_P\,\Pi_C = 0$, which proves that Π_C and Π_P are two *orthogonal idempotents*.
Note that $H(P, C) \in \mathcal{A}^{p\times p}$ if and only if $\Pi_C \in \mathcal{A}^{p\times p}$ if and only if $\Pi_P \in \mathcal{A}^{p\times p}$. .

Lemma 1. *The following statements are equivalent:*

- *The controller $C \in \mathcal{K}^{r\times q}$ internally stabilizes $P \in \mathcal{K}^{q\times r}$.*
- *The closed-loop system defined by Figure 1 is well-posed and Π_C is an idempotent of $\mathcal{A}^{p\times p}$, namely, $\Pi_C^2 = \Pi_C$, i.e., Π_C is a stable projector.*

- *The closed-loop system defined by Figure 1 is well-posed and* Π_P *is an idempotent of* $\mathcal{A}^{p\times p}$*, namely,* $\Pi_P^2 = \Pi_P$*, i.e.,* Π_P *is a stable projector.*

Finally, we have $\Pi_C + \Pi_P = I_p$ *and* $\Pi_C\,\Pi_P = \Pi_P\,\Pi_C = 0$*, i.e.,* $\{\Pi_C, \Pi_p\}$ *is a complete set of orthogonal idempotents of the ring* $\mathcal{A}^{p\times p}$*.*

Lemma 1 was first obtained in [12] within the *operator-theoretic approach* to stabilization problems (see [14, 15] and the references therein). For $\mathcal{A} = RH_\infty$ and $H^\infty(\mathbb{C}_+)$, Π_C and Π_P play fundamental role in robust stabilization (*stability margin*, *loop-shaping*). One can prove that $\|\Pi_C\|_\infty = \|\Pi_P\|_\infty$ [12]. Finally, the *optimal robust radius* corresponds to the inverse of $\inf_{C\in\mathrm{Stab}(P)} \|\Pi_C\|_\infty$ [14, 15, 38].

The next theorem is a characterization of the internal stabilizability.

Theorem 1 ([28])**.** *The plant* $P \in \mathcal{K}^{q\times p}$ *is internally stabilizable if and only if one of the following equivalent statements is satisfied:*

1. *There exists* $F = (U^T \quad V^T)^T \in \mathcal{A}^{p\times q}$*, where* $U \in \mathcal{A}^{q\times q}$ *and* $V \in \mathcal{A}^{r\times q}$*, such that*

$$F\,P \in \mathcal{A}^{p\times r}, \quad (I_q \quad -P)\,F = I_q. \tag{8}$$

If $\det U \neq 0$*, then* $C = V\,U^{-1}$ *internally stabilizes* P *and*

$$U = (I_q - P\,C)^{-1}, \quad V = C\,(I_q - P\,C)^{-1}, \quad \Pi_P = F\,(I_q \quad -P).$$

2. *There exists* $G = (-\widetilde{V} \quad \widetilde{U}) \in \mathcal{A}^{r\times p}$*, where* $\widetilde{V} \in \mathcal{A}^{r\times q}$ *and* $\widetilde{U} \in \mathcal{A}^{r\times r}$*, such that*

$$P\,G \in \mathcal{A}^{q\times p}, \quad G\,\left(P^T \quad I_r^T\right)^T = I_r, \tag{9}$$

If $\det \widetilde{U} \neq 0$*, then* $C = \widetilde{U}^{-1}\,\widetilde{V}$ *internally stabilizes* C *and*

$$\widetilde{U} = (I_r - C\,P)^{-1}, \quad \widetilde{V} = C\,(I_r - C\,P)^{-1}, \quad \Pi_C = \left(P^T \quad I_r^T\right)^T\,G.$$

Algebraic and geometric interpretations of Theorem 1 are given in Section 5 (see Remark 12).

Remark 4. Note that $U = (I_q - P\,C)^{-1}$ is the *input sensitivity matrix*, also denoted by S_o in the literature, and $\widetilde{U} = (I_r - C\,P)^{-1}$ the *input sensitivity matrix*, also denoted by S_i. The transfer matrix $V = C\,(I_q - P\,C)^{-1} = C\,U = C\,(I_r - C\,P)^{-1} = \widetilde{V} = C\,\widetilde{U}$ does not seem to have a specific name in the literature (despite Remark 5 below). Finally, $T_i = -V\,P$ is the *complementary input sensitivity transfer matrix* and $T_o = -P\,V$ is the *complementary output sensitivity transfer matrix*.

Remark 5. Internal stabilizability can be expressed using only V: P is internally stabilizable if and only if there exists $V \in \mathcal{A}^{r\times q}$ satisfying $V\,P \in \mathcal{A}^{r\times r}$, $U = I_q + P\,V \in \mathcal{A}^{q\times q}$, i.e., $P\,V \in \mathcal{A}^{q\times q}$, and $U\,P = (I_q + P\,V)\,P \in \mathcal{A}^{q\times r}$.

Example 1. Suppose that P has a left coprime factorization $P = D^{-1}\,N$, where $D\,X - N\,Y = I_q$, $D \in \mathcal{A}^{q\times q}$, $N \in \mathcal{A}^{q\times r}$, $X \in \mathcal{A}^{q\times q}$, and $Y \in \mathcal{A}^{r\times q}$. Then, we have $X - P\,Y = D^{-1}$, and thus, $X\,D - P\,Y\,D = I_q$. So, if we consider the matrices $U = X\,D \in \mathcal{A}^{q\times q}$ and $V = Y\,D \in \mathcal{A}^{r\times q}$, then $F = (U^T \quad V^T)^T \in \mathcal{A}^{p\times q}$ then satisfies

$$F\,P = \begin{pmatrix} X\,N \\ Y\,N \end{pmatrix} \in \mathcal{A}^{p\times r}, \quad (I_q \quad -P)\,F = X\,D - P\,Y\,D = I_q.$$

Then, 1 of Theorem 1 shows that $C = V\,U^{-1} = Y\,X^{-1}$ internally stabilizes P and we find again a standard result of the fractional representation approach [10, 37]. Finally, we have $\Pi_C = (X^T \quad Y^T)^T\,(D \quad -N)$.

Similarly, if P has right coprime factorization $P = \widetilde{N}\,\widetilde{D}^{-1}$, $-\widetilde{Y}\,\widetilde{N} + \widetilde{X}\,\widetilde{D} = I_r$, then the matrix $G = (-\widetilde{D}\,\widetilde{Y} \quad \widetilde{D}\,\widetilde{X}) \in \mathcal{A}^{r\times p}$ satisfies $P\;G = (-\widetilde{N}\,\widetilde{Y} \quad \widetilde{N}\,\widetilde{X}) \in \mathcal{A}^{q\times p}$. Also, $-\widetilde{Y}\,\widetilde{N} + \widetilde{X}\,\widetilde{D} = I_r$ results $-\widetilde{Y}\,P + \widetilde{X} = \widetilde{D}^{-1}$, and thus, $-\widetilde{D}\,\widetilde{Y}\,P + \widetilde{D}\,\widetilde{X} = I_r$, i.e., $G\,M = I_r$. Then, 2 of Theorem 1 shows that $C = (\widetilde{D}\,\widetilde{X})^{-1}\,(\widetilde{D}\,\widetilde{Y}) = \widetilde{X}^{-1}\,\widetilde{Y}$ stabilizes P [10, 37]. Finally, we have $\Pi_P = (\widetilde{N}^T \quad \widetilde{D}^T)^T\,\left(-\widetilde{Y} \quad \widetilde{X}\right)$.

The existence of a left/right coprime factorization for P is thus a sufficient condition for internal stabilizability. If P has a weakly left/right coprime factorization, then it is also a necessary condition (see Corollary 8 of [28]). Furthermore, it was shown that every transfer matrix with entries in $\mathcal{K}$ has a weakly left/right coprime factorization if and only if $\mathcal{A}$ is a so-called *coherent Sylvester domain* (e.g., RH_∞, $H^\infty(\mathbb{C}_+)$, $\mathcal{R}_2$) [24, 29]. Thus, over a coherent Sylvester domain $\mathcal{A}$, internal stabilizability is equivalent to the existence of a left/right coprime factorization. For a direct proof of this result for RH_∞ and $H^\infty(\mathbb{C}_+)$ see [37, 36]. But the question is open for some algebras used in control theory [10, 21, 37].

3 The lattice approach to analysis and synthesis problems

Within the fractional representation approach, analysis and synthesis problems can be reformulated using the stability set-theoretic approach mentioned in Section 2 and characterized by algebraic conditions. Using the ring structure of $\mathcal{A}$ − which allows to consider sums and compositions of systems [39] − as well as the linearity property of linear systems, the fractional representation approach is naturally rooted in *module theory*, i.e., roughly speaking, in linear algebra over the ring $\mathcal{A}$. See [24, 25]. We recall that the definition of an $\mathcal{A}$*-module* is the same as that of a vector space except that the scalars belong to a ring $\mathcal{A}$, but not a field (in which every nonzero element has an inverse for the multiplication) like, e.g., $\mathcal{K}$ [33]. In [28, 29], it was shown that the theory of *algebraic lattices*, developed in algebraic geometry and number theory, is a natural mathematical framework for studying linear systems defined by transfer matrices. The algebraic lattice approach is a module theory that mimics linear algebra over the field of fractions $\mathcal{K}$ of the ring $\mathcal{A}$. This section briefly explains this approach.

Furthermore, in robust control, one wants to characterize close systems for a topology or a metric related to the robustness in stabilizability [39, 37, 14, 15, 38]. This requires that the ring $\mathcal{A}$ is a *normed ring* [23], namely, a ring equipped with a norm $\|\cdot\|_{\mathcal{A}}$ for which the multiplication defines a continuous function in the two arguments (e.g., $\mathcal{A} = RH_\infty$ for the norms $\|\cdot\|_\infty$ or $\|\cdot\|_{\mathcal{W}}$; see below) or, even better, a *Banach algebra*, namely, a *complete* norm ring [23]. Thus, a natural mathematical framework for robust control is module theory over a Banach algebra $\mathcal{A}$ (more precisely, the *category of finitely presented* $\mathcal{A}$*-modules* over a *coherent* Banach algebra $\mathcal{A}$ such as, e.g., $\mathcal{A} = H^\infty(\mathbb{C}_+)$ [24, 25]). Finally, we note that the *operator-theoretic approach* using *graphs* [37, 14, 15] is dual to the algebraic lattice approach [27].

Let us first review some standard definitions of module theory.

Definition 3 ([33])**.** *Let $\mathcal{A}$ be an integral domain, $\mathcal{K} = Q(\mathcal{A})$ its quotient fields, and $\mathcal{P}$ a* finitely generated *$\mathcal{A}$-module, namely, there exists a finite set I and $g_i \in \mathcal{P}$ for $i \in I$ such that*

$$\mathcal{P} = \sum_{i \in I} \mathcal{A}\, g_i := \left\{ \sum_{i \in I} a_i\, g_i \mid a_i \in \mathcal{A},\ i \in I \right\}.$$

1. *The* rank *of $\mathcal{P}$ is the dimension of the $\mathcal{K}$-vector space generated by $\mathcal{P}$, namely, $\mathcal{K} \otimes_{\mathcal{A}} \mathcal{P}$, where $\otimes_{\mathcal{A}}$ stands for the tensor product of $\mathcal{A}$-modules, namely, the dimension of the $\mathcal{K}$-vector space obtained by extending the coefficients of $\mathcal{P}$ from $\mathcal{A}$ to $\mathcal{K}$, i.e.,* $\operatorname{rank}_{\mathcal{A}}(\mathcal{P}) = \dim_{\mathcal{K}}(\mathcal{K} \otimes_{\mathcal{A}} \mathcal{P})$.

2. *The $\mathcal{A}$-module $\mathcal{P}$ is said to be* free *if $\mathcal{M}$ has a basis or equivalently if $\mathcal{M}$ is isomorphic to a finite direct copy of $\mathcal{A}$, i.e., $\mathcal{P} \cong \mathcal{A}^l$, where $\cong$ stands for an* isomorphism *(namely, an injective and surjective $\mathcal{A}$-homomorphism) and $l \in \mathbb{N}$.*

3. *The $\mathcal{A}$-module $\mathcal{P}$ is said to be* projective *if there exist an $\mathcal{A}$-module $\mathcal{Q}$ and $m \in \mathbb{N}$ such that $\mathcal{P} \oplus \mathcal{Q} \cong \mathcal{A}^m$, where $\oplus$ denotes the* direct sum *of $\mathcal{A}$-modules.*

Remark 6. Using Definition 3, a free $\mathcal{A}$-module $\mathcal{P}$ is projective with $\mathcal{Q} = 0$.

Let $\mathcal{A}$ be an integral domain of SISO stable plants, $\mathcal{K} = Q(\mathcal{A})$ its field of fractions, $P \in \mathcal{K}^{q \times r}$ a transfer matrix of the linear system $y = P\, u$, and $p = r + q$. Setting $L = (I_q \quad -P) \in \mathcal{K}^{q \times p}$, the linear system $y = P\, u$ can be rewritten as $L\, \begin{pmatrix} y^T & u^T \end{pmatrix} = 0$ and we can introduce the following $\mathcal{A}$-module

$$\mathcal{L} := L\, \mathcal{A}^{p \times 1} = \left\{ l_1 - P\, l_2 \mid l_1 \in \mathcal{A}^{q \times 1},\ l_2 \in \mathcal{A}^{r \times 1} \right\}$$

formed by the $\mathcal{A}$-linear combinations of the columns of L. The p columns of L form a set of generators of $\mathcal{L}$, and thus, $\mathcal{L}$ is a *finitely generated* $\mathcal{A}$-module. We have $\mathcal{L} \subseteq \mathcal{K}^{q\times 1}$, i.e., $\mathcal{L}$ is an $\mathcal{A}$-submodule of $\mathcal{K}^{q\times 1}$.

We can also consider the *descriptor form* of the linear system, namely, $(y^T \quad u^T)^T = M\,\xi$ for all $\xi \in \mathcal{A}^{r\times 1}$, where $M = (P^T \quad I_q^T)^T \in \mathcal{K}^{p\times r}$. Hence, we can define the following $\mathcal{A}$-module

$$\mathcal{M} := \mathcal{A}^{1\times p}\, M = \left\{m_1\, P + m_2 \mid m_1 \in \mathcal{A}^{1\times q},\ m_2 \in \mathcal{A}^{1\times r}\right\}$$

formed by the $\mathcal{A}$-linear combinations of the rows of M. The p rows of M form a set of generators of $\mathcal{M}$, and thus, $\mathcal{M}$ is a finitely generated $\mathcal{A}$-module. We have $\mathcal{M} \subseteq \mathcal{K}^{1\times r}$, i.e., $\mathcal{M}$ is an $\mathcal{A}$-submodule of $\mathcal{K}^{1\times r}$.

The $\mathcal{A}$-modules $\mathcal{L}$ and $\mathcal{M}$ belong to a special class of $\mathcal{A}$-modules called *algebraic lattices* or *lattices* in what follows for simplicity.

Definition 4 ([3])**.** *Let $\mathcal{V}$ be a finite-dimensional $\mathcal{K}$-vector space. Then, an $\mathcal{A}$-submodule $\mathcal{M}$ of $\mathcal{V}$ is a* lattice of $\mathcal{V}$ *if there exist two free $\mathcal{A}$-submodules $\mathcal{F}_1$ and $\mathcal{F}_2$ of $\mathcal{V}$ such that $\mathcal{F}_1 \subseteq \mathcal{M} \subseteq \mathcal{F}_2$ and*

$$\operatorname{rank}_{\mathcal{A}}(\mathcal{F}_1) = \dim_{\mathcal{K}}(\mathcal{V}).$$

Example 2. Let us show that $\mathcal{L}$ is a lattice of $\mathcal{K}^{q\times 1}$. Writing $P = D^{-1}\, N$, where $D \in \mathcal{A}^{q\times q}$, $\det D \neq 0$, and $N \in \mathcal{A}^{q\times r}$ (see Remark 1), then $\mathcal{F}_1 = \mathcal{A}^{q\times 1} \subseteq \mathcal{L} \subseteq \mathcal{F}_2 = D^{-1}\,\mathcal{A}^{q}$. The first inclusion is clear. For the second, if $R = (D \quad -N) \in \mathcal{A}^{q\times p}$, then $\mathcal{L} = (D^{-1}\, R)\,\mathcal{A}^{p\times 1} \subseteq D^{-1}\,\mathcal{A}^{q\times 1}$ since $R\,\mathcal{A}^{p\times 1} \subseteq \mathcal{A}^{q\times 1}$.

Now let us prove that $\mathcal{M}$ is a lattice of $\mathcal{K}^{1\times r}$. If we write $P = \widetilde{N}\,\widetilde{D}^{-1}$, where $\widetilde{D} \in \mathcal{A}^{r\times r}$, $\det \widetilde{D} \neq 0$, $\widetilde{N} \in \mathcal{A}^{q\times r}$ (see Remark 1), and $\widetilde{R} = (\widetilde{N}^T \quad \widetilde{D}^T)^T$, then $\mathcal{F}_1 = \mathcal{A}^{1\times r} \subseteq \mathcal{M} \subseteq \mathcal{F}_2 = \mathcal{A}^{1\times r}\,\widetilde{D}^{-1}$. Indeed, the first inclusion is clear and $\mathcal{M} = \mathcal{A}^{1\times p}\,\widetilde{R}\,\widetilde{D}^{-1} \subseteq \mathcal{A}^{1\times r}\,\widetilde{D}^{-1}$ since $\mathcal{A}^{1\times p}\,\widetilde{R} \subseteq \mathcal{A}^{1\times r}$.

Let us now give definitions concerning *lattice homomorphisms.*

Definition 5 ([3])**.** *Let $\mathcal{V}$ and $\mathcal{W}$ be two finitely generated $\mathcal{K}$-vector spaces, $\mathcal{L}$ a lattice of $\mathcal{V}$, $\mathcal{M}$ a lattice of $\mathcal{W}$, and $\operatorname{Hom}_{\mathcal{K}}(\mathcal{V}, \mathcal{W})$ the $\mathcal{K}$-vector space of the $\mathcal{K}$-linear maps from $\mathcal{V}$ to $\mathcal{W}$.*

1. $\mathcal{M} : \mathcal{L} := \{f \in \operatorname{Hom}_{\mathcal{K}}(\mathcal{V}, \mathcal{W}) \mid f(\mathcal{L}) \subseteq \mathcal{M}\}$.
2. *The* dual lattice *of $\mathcal{L}$ is defined by $\mathcal{A} : \mathcal{L}$.*

Remark 7. One can prove that $\mathcal{M} : \mathcal{L}$ is a lattice of the finite-dimensional $\mathcal{K}$-vector space $\operatorname{Hom}_{\mathcal{K}}(\mathcal{V}, \mathcal{W})$ [3]. Furthermore, it can be shown that every element of $\operatorname{Hom}_{\mathcal{A}}(\mathcal{L}, \mathcal{M})$ corresponds to the restriction of a unique element of $\mathcal{M} : \mathcal{L}$ to $\mathcal{M}$ [3].

Example 3. Let us characterize the lattice $\mathcal{A} : \mathcal{L}$ of $\operatorname{Hom}_{\mathcal{K}}(\mathcal{K}^{q\times 1}, \mathcal{K}) \cong \mathcal{K}^{1\times q}$ and the lattice $\mathcal{A} : \mathcal{M}$ of $\operatorname{Hom}_{\mathcal{K}}(\mathcal{K}^{1\times r}, \mathcal{K}) \cong \mathcal{K}^{r\times 1}$.

Considering $\mathcal{V} = \mathcal{K}^{q\times 1}$ and $\mathcal{W} = \mathcal{K}$, and using the isomorphism of $\mathcal{K}$-vector spaces $\operatorname{Hom}_{\mathcal{K}}(\mathcal{K}^{q\times 1}, \mathcal{K}) \cong \mathcal{K}^{1\times q}$ obtained by associating $f \in \operatorname{Hom}_{\mathcal{K}}(\mathcal{K}^{q\times}, \mathcal{K})$ with $(f(e_1) \ \ldots \ f(e_q)) \in \mathcal{K}^{1\times q}$, where $\{e_i\}_{i=1,\ldots,q}$ denotes the *standard basis* of $\mathcal{K}^{q\times 1}$ (i.e., e_i is a vector with 1 at the i^{th} position and 0 elsewhere), we get

$$\mathcal{A} : \mathcal{L} = \left\{\lambda \in \mathcal{K}^{1\times q} \mid \lambda\,(I_q \quad -P)\,\mathcal{A}^{p\times 1} \subseteq \mathcal{A}\right\} = \left\{\lambda \in \mathcal{A}^{1\times q} \mid \lambda\, P \in \mathcal{A}^{1\times r}\right\}.$$

Considering $\mathcal{V} = \mathcal{K}^{1\times r}$, $\mathcal{W} = \mathcal{K}$, and using $\operatorname{Hom}_{\mathcal{K}}(\mathcal{K}^{1\times r}, \mathcal{K}) \cong \mathcal{K}^{r\times 1}$, we have

$$\mathcal{A} : \mathcal{M} = \left\{\mu \in \mathcal{K}^{r\times 1} \mid \mathcal{A}^{1\times p}\,(P^T \quad I_r^T)^T\,\mu \subseteq \mathcal{A}\right\} = \left\{\mu \in \mathcal{A}^{r\times 1} \mid P\,\mu \in \mathcal{A}^{q\times 1}\right\}.$$

In the SISO case, $\mathcal{A} : \mathcal{L} = \mathcal{A} : \mathcal{M}$ is the ideal of $\mathcal{A}$ formed by all the possible $\mathcal{A}$-denominators of P.

Remark 8. Given $P \in \mathcal{K}^{q\times r}$ and considering the notations of Example 2, we can define the lattice $\mathcal{P} = R\,\mathcal{A}^{p\times 1}$ of $\mathcal{K}^{q\times 1}$ and the lattice $\mathcal{Q} = \mathcal{A}^{1\times p}\,\widetilde{R}$ of $\mathcal{K}^{1\times r}$. Moreover, we have [28]

$$\mathcal{A} : \mathcal{P} = \left\{\nu \in \mathcal{K}^{1\times q} \mid \nu\, R \in \mathcal{A}^{1\times p}\right\}, \quad \mathcal{A} : \mathcal{Q} = \left\{\xi \in \mathcal{K}^{r\times 1} \mid \widetilde{R}\,\xi \in \mathcal{A}^{p\times 1}\right\}.$$

As shown in [28, 29], properties of the linear system $y = P\,u$ can be characterized by algebraic properties of the lattices $\mathcal{L}$, $\mathcal{M}$, $\mathcal{P}$, and $\mathcal{Q}$. Let us state some of them.

Theorem 2. *[28, 29] We have the following results:*

1. *P has a weakly left coprime factorization if and only if $\mathcal{A} : \mathcal{L}$ is a free lattice of $\mathcal{K}^{1\times q}$, i.e., if and only if there exists $D \in \mathcal{A}^{q\times q}$ with $\det D \neq 0$ such that $\mathcal{A} : \mathcal{L} = \mathcal{A}^{1\times q}\, D$. Then, $N = D\, P \in \mathcal{A}^{q\times r}$ and $P = D^{-1}\, N$ is a weakly left coprime factorization.*

2. *P has a weakly right coprime factorization if and only if $\mathcal{A} : \mathcal{M}$ is a free lattice of $\mathcal{K}^{r\times 1}$, i.e., if and only if there exists $\widetilde{D} \in \mathcal{A}^{r\times r}$ with $\det \widetilde{D} \neq 0$ such that $\mathcal{A} : \mathcal{M} = \widetilde{D}\, \mathcal{A}^{r\times 1}$. Then, $\widetilde{N} = P\, \widetilde{D} \in \mathcal{A}^{r\times r}$ and $P = \widetilde{N}\, \widetilde{D}^{-1}$ is a weakly right coprime factorization.*

3. *P has a left coprime factorization if and only if $\mathcal{L}$ is a free lattice of $\mathcal{K}^{q\times 1}$, i.e., if and only if there exists $D \in \mathcal{A}^{q\times q}$ with $\det D \neq 0$ such that $\mathcal{L} = D^{-1}\, \mathcal{A}^{q\times 1}$. Then, $N = D\, P \in \mathcal{A}^{q\times r}$ and $P = D^{-1}\, N$ is a left coprime factorization.*

4. *P has a right coprime factorization if and only if $\mathcal{M}$ is a free lattice of $\mathcal{K}^{1\times r}$, i.e., if and only if there exists a matrix $\widetilde{D} \in \mathcal{A}^{r\times r}$ with $\det \widetilde{D} \neq 0$ such that $\mathcal{M} = \mathcal{A}^{1\times r}\, \widetilde{D}$. Then, $\widetilde{N} = P\, \widetilde{D} \in \mathcal{A}^{r\times r}$ and $P = \widetilde{N}\, \widetilde{D}^{-1}$ is a right coprime factorization.*

Let $P = D^{-1}\, N$ be a left coprime factorization. By 3 of Theorem 2, we have $\mathcal{L} = D^{-1}\, \mathcal{A}^{q\times 1}$, and thus, $\mathcal{A} : \mathcal{L} = \{\lambda \in \mathcal{A}^{1\times q} \mid \lambda\, D^{-1}\, \mathcal{A}^{q\times 1} \subseteq \mathcal{A}\} = \mathcal{A}^{1\times q}\, D$. Using 1 of Theorem 2, $P = D^{-1}\, N$ is then a weakly left coprime factorization. Similarly, right coprime factorizations are weakly right coprime factorizations.

In Section 5, we will characterize stabilizability in terms of the properties of the lattices $\mathcal{L}$ and $\mathcal{M}$.

4 A quick look at homological algebra

Let us start with some standard definitions of homological algebra.

Definition 6 ([33]). *Let $\mathcal{P} = (\mathcal{P}_i)_{i\in\mathbb{Z}}$ be a sequence of $\mathcal{A}$-modules and $d = (d_i)_{i\in\mathbb{Z}}$ a sequence of $\mathcal{A}$-homomorphisms, where $d_i \in \hom_{\mathcal{A}}(\mathcal{P}_i, \mathcal{P}_{i-1})$ for $i \in \mathbb{Z}$.*

1. *The sequence $(\mathcal{P}, d)$ is called a* complex *if $d_i \circ d_{i+1} = 0$ for all $i \in \mathbb{Z}$, i.e., if $\operatorname{im} d_{i+1} \subseteq \ker d_i$ for all $i \in \mathbb{Z}$. The complex $(\mathcal{P}, d)$ is simply denoted by*

$$(\mathcal{P}, d) : \ldots \xrightarrow{d_{i+2}} \mathcal{P}_{i+1} \xrightarrow{d_{i+1}} \mathcal{P}_i \xrightarrow{d_i} \mathcal{P}_{i-1} \xrightarrow{d_{i-1}} \ldots \tag{10}$$

A complex with only a finite of non-zero $\mathcal{A}$-modules $\mathcal{P}_i$ is simply denoted by

$$0 \longrightarrow \mathcal{M}_s \xrightarrow{d_s} \ldots \xrightarrow{d_{r+2}} \mathcal{P}_{r+1} \xrightarrow{d_{r+1}} \mathcal{P}_{r+1} \xrightarrow{d_r} \mathcal{M}_r \longrightarrow 0.$$

2. *A complex $(\mathcal{P}, d)$ is said to* exact at $\mathcal{P}_i$ *if $\ker d_i = \operatorname{im} d_{i+1}$, and is an* exact sequence of $\mathcal{A}$-modules *if it is exact at all the $\mathcal{P}_i$'s, i.e., if $\ker d_i = \operatorname{im} d_{i+1}$ for all $i \in \mathbb{Z}$.*

3. *A* short exact sequence *is an exact sequence of the form*

$$0 \longrightarrow \mathcal{P}_2 \xrightarrow{d_2} \mathcal{P}_1 \xrightarrow{d_1} \mathcal{P}_0 \longrightarrow 0, \tag{11}$$

i.e., d_2 is injective, d_1 is surjective, and $\ker d_1 = \operatorname{im} d_2$.

Let us illustrate the above concepts within the lattice approach of Section 3.

Let $P \in \mathcal{K}^{q\times r}$, $p = q + r$, $L = (I_q \quad -P) \in \mathcal{K}^{q\times p}$, $M = (P^T \quad I_q^T)^T \in \mathcal{K}^{p\times r}$, $\mathcal{L} = L\, \mathcal{A}^{p\times 1}$, and $\mathcal{M} = \mathcal{A}^{1\times r}\, M$. Also note

$$\operatorname{Syz}(\mathcal{L}) := \ker_{\mathcal{A}^{p\times 1}}(L.) = \left\{\eta = (\eta_1^T \quad \eta_2^T)^T \in \mathcal{A}^{(q+r)\times 1} \mid \eta_1 = P\, \eta_2\right\},$$

$$\operatorname{Syz}(\mathcal{M}) := \ker_{\mathcal{A}^{1\times p}}(.M) = \left\{\gamma = (\gamma_1 \quad \gamma_2) \in \mathcal{A}^{1\times(q+r)} \mid \gamma_2 = -\gamma_1\, P\right\}.$$

Then, we have the following short exact sequences of $\mathcal{A}$-modules

$$0 \longrightarrow \mathrm{Syz}(\mathcal{L}) \xrightarrow{i} \mathcal{A}^{p\times 1} \xrightarrow{L.} \mathcal{L} \longrightarrow 0,$$

$$0 \longleftarrow \mathcal{M} \xleftarrow{.M} \mathcal{A}^{1\times p} \xleftarrow{j} \mathrm{Syz}(\mathcal{M}) \longleftarrow 0,$$

where i (resp., j) denotes the canonical inclusion. Indeed, $L.$ (resp., $.M$) is a surjective $\mathcal{A}$-homomorphism and $\ker_{\mathcal{A}^{p\times 1}}(L.) = \mathrm{Syz}(\mathcal{L})$ (resp., $\ker_{\mathcal{A}^{1\times p}}(.M) = \mathrm{Syz}(\mathcal{M})$).

If $\eta = (\eta_1^T \quad \eta_2^T)^T \in \mathrm{Syz}(\mathcal{L})$, then $P\,\eta_2 = \eta_1 \in \mathcal{A}^{q\times 1}$, and thus, $\eta_2 \in \mathcal{A} : \mathcal{M}$ and $\eta = M\,\eta_2$, which shows that $\mathrm{Syz}(\mathcal{L}) = M\,(\mathcal{A} : \mathcal{M})$. Thus, we have the following short exact sequence of $\mathcal{A}$-modules

$$0 \longrightarrow \mathcal{A} : \mathcal{M} \xrightarrow{M.} \mathcal{A}^{p\times 1} \xrightarrow{L.} \mathcal{L} \longrightarrow 0. \tag{12}$$

Similarly, if $\gamma = (\gamma_1 \quad \gamma_2) \in \mathrm{Syz}(\mathcal{M})$, then $\gamma_1\,P = -\gamma_2 \in \mathcal{A}^{1\times r}$, which yields $\gamma_1 \in \mathcal{A} : \mathcal{L}$, $\gamma = (\gamma_1 \quad \gamma_2) = (\gamma_1 \quad -\gamma_1\,P) = \gamma_1\,L$, and shows that $\mathrm{Syz}(\mathcal{M}) = (\mathcal{A} : \mathcal{L})\,L$. Thus, we have the following short exact sequence of $\mathcal{A}$-modules

$$0 \longleftarrow \mathcal{M} \xleftarrow{.M} \mathcal{A}^{1\times p} \xleftarrow{.L} \mathcal{A} : \mathcal{L} \longleftarrow 0. \tag{13}$$

We can now state the standard so-called *splitting lemma* for a short exact sequence (11).

Proposition 1 (Proposition 2.28 & Exercise 2.8 of [33], [19], p. 47)**.** *Let (11) be a short exact sequence of $\mathcal{A}$-modules. The following statements are then equivalent:*

1. *There exists $h_0 \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{P}_0, \mathcal{P}_1)$ such that $d_1 \circ h_0 = \mathrm{id}_{\mathcal{P}_0}$.*
2. *There exist $h_0 \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{P}_0, \mathcal{P}_1)$ and $h_1 \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{P}_1, \mathcal{P}_2)$ such that*
$$d_2 \circ h_1 + h_0 \circ d_1 = \mathrm{id}_{\mathcal{P}_1}. \tag{14}$$
3. *There exists $h_1 \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{P}_1, \mathcal{P}_2)$ such that $h_1 \circ d_2 = \mathrm{id}_{\mathcal{P}_2}$.*
4. *There exists an isomorphism from $\mathcal{P}_0 \oplus \mathcal{P}_2$ to $\mathcal{P}_1$, i.e., $\mathcal{P}_1 \cong \mathcal{P}_0 \oplus \mathcal{P}_2$.*

Then, (11) is called a short split exact sequence *and it is denoted by*

$$0 \longrightarrow \mathcal{P}_2 \underset{h_1}{\overset{d_2}{\rightleftarrows}} \mathcal{P}_1 \underset{h_0}{\overset{d_1}{\rightleftarrows}} \mathcal{P}_0 \longrightarrow 0. \tag{15}$$

In particular, we have $d_1 \circ h_0 = \mathrm{id}_{\mathcal{P}_0}$, $d_2 \circ h_1 + h_0 \circ d_1 = \mathrm{id}_{\mathcal{P}_1}$, and $h_1 \circ d_2 = \mathrm{id}_{\mathcal{P}_2}$.

Remark 9. The $\mathcal{A}$-homomorphism h_0 of Proposition 1 is called a *section* of d_1, d_1 a *retraction* of h_0, and $\mathcal{P}_0$ a *retract* of $\mathcal{P}_1$. Similarly, d_2 is a section of h_1, h_1 a retraction of d_2, and $\mathcal{P}_2$ a retract of $\mathcal{P}_1$. $(h_0,\, h_1)$ is also called a *homotopy* of (11) and (11) satisfying Proposition 1 is called a *contractible complex.* Finally, (d_1, h_0) (resp., (h_1, d_2)) is called a *splitting* of the idempotent $e_1 = h_0 \circ d_1 \in \mathrm{End}_{\mathcal{A}}(\mathcal{P}_1)$ (resp., $e_2 = d_2 \circ h_1 \in \mathrm{End}_{\mathcal{A}}(\mathcal{P}_1)$). For more details, see, e.g., [33].

In Section 5, we will relate Proposition 1 to the stabilizability condition.

The sequence formed by the $\mathcal{A}$-homomorphisms h_0 and h_1 in (15) is not necessarily a complex, i.e., $\mathrm{im}\, h_0$ is not necessarily an $\mathcal{A}$-submodule of $\ker h_1$. The next corollary − which gives parametrizations of all the right (resp., left) inverses of d_1 (resp., d_2) − also shows that, up to a change of notation, it is always possible to assume that the homomorphisms h_0 and h_1 also define a split short exact sequence.

Corollary 1. *Using the notations of Proposition 1, we have:*

1. *All the right inverses of d_1 are of the form*

$$\forall\, q \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{P}_0, \mathcal{P}_2), \quad h_0(q) = h_0 + d_2 \circ q.$$

2. All the left inverse of d_2 are of the form

$$\forall\, q \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{P}_0, \mathcal{P}_2), \quad h_1(q) = h_1 - q \circ d_1.$$

3. If $h_1 \circ h_0 \neq 0$, then setting $q_\star = -h_1 \circ h_0 \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{P}_0, \mathcal{P}_2)$ then

$$h_1(-q_\star) \circ h_0 = 0, \quad h_1 \circ h_0(q_\star) = 0.$$

4. If (15) is a split short exact sequence and $h_1 \circ h_0 = 0$, then

$$0 \longrightarrow \mathcal{P}_0 \xrightarrow{h_0} \mathcal{P}_1 \xrightarrow{h_1} \mathcal{P}_2 \longrightarrow 0 \tag{16}$$

is also a split short exact sequence of $\mathcal{A}$-modules. Then, we will say that

$$0 \longrightarrow \mathcal{P}_2 \underset{h_1}{\overset{d_2}{\rightleftarrows}} \mathcal{P}_1 \underset{h_0}{\overset{d_1}{\rightleftarrows}} \mathcal{P}_0 \longrightarrow 0$$

is a bi-split short exact sequence *of $\mathcal{A}$-modules.*

5. Up to a change of notation for h_0 or h_1, (16) can always assumed to be a split short exact sequence. Then, we have the following bi-split short exact sequences

$$\forall\, q \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{P}_0, \mathcal{P}_2), \quad 0 \longrightarrow \mathcal{P}_2 \underset{h_1(q)}{\overset{d_2}{\rightleftarrows}} \mathcal{P}_1 \underset{h_0(q)}{\overset{d_1}{\rightleftarrows}} \mathcal{P}_0 \longrightarrow 0. \tag{17}$$

Proof. 1. We have $d_1 \circ h_0(q) = \mathrm{id}_{\mathcal{P}_0}$ for all $q \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{P}_0, \mathcal{P}_2)$. Now suppose that h_0' is a right inverse of d_1, i.e., $d_1 \circ h_0 = \mathrm{id}_{\mathcal{P}_0}$. Then, $d_1 \circ (h_0' - h_0) = 0$, which shows that $(h_0' - h_0)(p) \in \ker d_1 = \mathrm{im}\, d_2$ for all $p \in \mathcal{P}_0$, and thus, using the injectivity of d_2, there exists a unique $p' \in \mathcal{P}_2$ such that $(h_0' - h_0)(p) = d_2(p')$. Consider the map $q : \mathcal{P}_0 \longrightarrow \mathcal{P}_2$ be defined by mapping p to p'. Using the $\mathcal{A}$-linearity of h_0', h_0 and d_2, we get $q \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{P}_0, \mathcal{P}_2)$. Finally, we have $h_0' = h_0 + d_2 \circ q$.

2. We have $h_1(q) \circ d_2 = \mathrm{id}_{\mathcal{P}_2}$ for all $q \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{P}_0, \mathcal{P}_2)$. Now suppose that h_1' is a left inverse of d_2, i.e., $h_1' \circ d_2 = \mathrm{id}_{\mathcal{P}_2}$. Then, $(h_1 - h_1') \circ d_2 = 0$, which shows that $h_1 - h_1'$ reduces to 0 onto $\mathrm{im}\, d_2$. Using $\mathcal{P}_0 = \mathrm{im}\, d_1$, we can consider the map $q : \mathcal{P}_0 \longrightarrow \mathcal{P}_2$ defined by mapping $d_1(p)$ to $(h_1 - h_1')(p)$ for all $p \in \mathcal{P}_1$. It is well-defined because if $d_1(p) = d_1(p')$, then $p - p' \in \ker d_1 = \mathrm{im}\, d_2$, and thus, $p - p' = d_2(p'')$ for some $p'' \in \mathcal{P}_2$, which gives $(h_1 - h_1')(p) = (h_1 - h_1')(p)$. Using the $\mathcal{A}$-linearity of d_1, h_1', and h_1, we get $q \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{P}_0, \mathcal{P}_2)$. Finally, we have $q(d_1(p)) = (h_1 - h_1')(p)$ for all $p \in \mathcal{P}_1$, which gives $h_1' = h_1 - q \circ d_1$.

3. If we set $q_\star = -h_1 \circ h_0 \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{P}_0, \mathcal{P}_2)$ then $h_1(-q_\star) \circ h_0 = 0$ and $h_1 \circ h_0(q_\star) = 0$. Using Remark 9, note that $h_1(-q_\star) = h_1 \circ (\mathrm{id}_{\mathcal{P}_1} - e_1) = h_1 \circ e_2$ and $h_0(q_\star) = (\mathrm{id}_{\mathcal{P}_1} - e_2) \circ h_0 = e_1 \circ h_0$.

4. Let us assume that $h_1 \circ h_0 = 0$, i.e., $\mathrm{im}\, h_0 \subseteq \ker h_1$. Then, $h_1 \circ d_2 = \mathrm{id}_{\mathcal{P}_2}$ (resp., $d_1 \circ h_0 = \mathrm{id}_{\mathcal{P}_0}$) shows that h_1 is surjective (resp., h_0 is injective), and if $p \in \ker h_1$, then $d_2 \circ h_1 + h_0 \circ d_1 = \mathrm{id}_{\mathcal{P}_1}$ yields $p = h_0(d_1(p)) \in \mathrm{im}\, h_0$, i.e., $\ker h_1 = \mathrm{im}\, h_0$, which proves the exactness at $\mathcal{P}_1$. So (16) is a short exact sequence. Finally, the identities of Proposition 1 show that (16) is also a split short exact sequence.

5. If we consider either $h_0' = h_0$ and $h_1' = h_1(-q_\star) = h_1 - h_1 \circ h_0 \circ d_1$, or $h_0' = h_0(q_\star) = h_0 - d_2 \circ h_1 \circ h_0$ and $h_1' = h_1$, then the short exact (16) splits, where h_0' and h_1' replace h_0 and h_1 respectively. Thus, up to a change of notation, we can always assume that (16) is a split short exact of $\mathcal{A}$-modules. Finally, we have $h_1(q_1) \circ h_0(q_0) = (h_1 - q_1 \circ d_1) \circ (h_0 + d_2 \circ q_0) = q_0 - q_1$. Thus, $h_1(q_1) \circ h_0(q_0) = 0$ if and only if $q_1 = q_0$, which proves the exactness of the last complex by 4. □

In Section 5, we will show how Corollary 1 can be used to obtain a parametrization of all the stabilizing controllers of an internally stabilizable plant (which does not necessarily have coprime factorizations).

5 Further characterizations of internal stabilizability

In this section, we illustrate the concept of a split exact sequence within the lattice approach developed in Section 3, i.e., for the short exact sequences (12) and (13).

Introducing the following matrices

$$F = \begin{pmatrix} I_q \\ C \end{pmatrix} (I_q - P\,C)^{-1} \in \mathcal{K}^{p \times q}, \quad G = (I_r - C\,P)^{-1}\,(-C \quad I_r) \in \mathcal{K}^{r \times p},$$

and recalling the notations $L = (I_q \quad -P) \in \mathcal{K}^{q \times p}$ and $M = (P^T \quad I_q^T)^T \in \mathcal{K}^{p \times r}$, using (6) and (7), we have the following identities

$$F\,L = \Pi_C, \quad M\,G = \Pi_P, \quad L\,F = I_q, \quad G\,M = I_r, \quad F\,L + M\,G = I_p. \tag{18}$$

We can consider the following $\mathcal{A}$-homomorphisms

$$\begin{array}{rcl} F. : \; \mathcal{L} = L\,\mathcal{A}^{p \times 1} & \longrightarrow & \mathcal{K}^{p \times 1} \\ L\,\eta & \longmapsto & F\,L\,\eta = \Pi_C\,\eta, \end{array} \qquad \begin{array}{rcl} .G : \; \mathcal{M} = \mathcal{A}^{1 \times p}\,M & \longrightarrow & \mathcal{K}^{p \times 1} \\ \gamma\,M & \longmapsto & \gamma\,M\,G = \gamma\,\Pi_P. \end{array} \tag{19}$$

We can now state a theorem that connects the stabilizability characterizations given in Theorem 1, the splitting of the short exact sequences (12) and (13), and the concept of a projective lattice.

Theorem 3 (Thm. 3 of [28]). *The following statements are equivalent:*

1. *The controller $C \in \mathcal{K}^{r \times q}$ internally stabilizes the plant $P \in \mathcal{K}^{q \times r}$.*
2. *The short exact sequence (12) splits, i.e., there exists $F \in \mathcal{A}^{p \times q}$ such that*

$$F\,P \in \mathcal{A}^{p \times r}, \quad L\,F = I_q, \tag{20}$$

or equivalently, $\mathcal{L} \oplus (\mathcal{A} : \mathcal{M}) \cong \mathcal{A}^{p \times 1}$, i.e., $\mathcal{L}$ is a projective lattice of $\mathcal{K}^{q \times 1}$.

3. *The short exact sequence (13) splits, i.e., there exists $G \in \mathcal{A}^{r \times p}$ such that*

$$P\,G \in \mathcal{A}^{q \times p}, \quad G\,M = I_r, \tag{21}$$

or equivalently, $\mathcal{M} \oplus (\mathcal{A} : \mathcal{L}) \cong \mathcal{A}^{1 \times p}$, i.e., $\mathcal{M}$ is a projective lattice of $\mathcal{K}^{1 \times r}$.

Since we will need arguments from the proof of Theorem 3, we will detail it below.

Proof. Let us assume that 1 holds, i.e., P is internally stabilizable and C stabilizes P. Then, $F \in \mathcal{A}^{p \times q}$, $G \in \mathcal{A}^{r \times p}$, and $\Pi_1, \Pi_2 \in \mathcal{A}^{p \times p}$ (see Lemma 1), so that the target spaces of the $\mathcal{A}$-homomorphisms (19) are $\mathcal{A}^{p \times 1}$ and $\mathcal{A}^{p \times 1}$ respectively, i.e.,

$$\begin{array}{l} 0 \longrightarrow \mathcal{A} : \mathcal{M} \xrightarrow{M.} \mathcal{A}^{p \times 1} \underset{F.}{\overset{L.}{\rightleftarrows}} \mathcal{L} \longrightarrow 0, \\ 0 \longleftarrow \mathcal{M} \underset{.M}{\overset{.G}{\rightleftarrows}} \mathcal{A}^{1 \times p} \xleftarrow{.L} \mathcal{A} : \mathcal{L} \longleftarrow 0. \end{array}$$

Using (18), $(L. \circ F.)(l) = L\,F\,l = l$ for all $l \in \mathcal{L}$, i.e., $L. \circ F. = \mathrm{id}_{\mathcal{L}}$, which shows that (12) is a split short exact sequence by 1 of Proposition 1. By 4 of Proposition 1, we have $\mathcal{L} \oplus (\mathcal{A} : \mathcal{M}) \cong \mathcal{A}^{p \times 1}$, and thus, $\mathcal{L}$ is a projective lattice of $\mathcal{K}^{q \times 1}$ by 3 of Definition 4, which proves 2.

Using (18), $(.M \circ .G)(m) = m\,G\,M = m$ for all $m \in \mathcal{M}$, i.e., $.M \circ .G = \mathrm{id}_{\mathcal{M}}$, which shows that (13) is a split short exact sequence by 2 of Proposition 1. By 4 of Proposition 1, we have $\mathcal{M} \oplus (\mathcal{A} : \mathcal{L}) \cong \mathcal{A}^{1 \times p}$, and thus, $\mathcal{M}$ is a projective lattice of $\mathcal{K}^{1 \times r}$ by 3 of Definition 4, which proves 3.

Suppose that 2 holds. By Proposition 1, there exists $\sigma \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{L}, \mathcal{A}^{p \times 1})$ such that $L. \circ \sigma = \mathrm{id}_{\mathcal{L}}$. Using Remark 7, σ corresponds to a unique element of

$$\mathcal{A}^{p \times 1} : \mathcal{L} = \left\{ T \in \mathcal{K}^{p \times q} \mid T\,(I_q \quad -P)\,\mathcal{A}^{p \times 1} \subseteq \mathcal{A}^{p \times 1} \right\} = \left\{ T \in \mathcal{A}^{p \times q} \mid T\,P \in \mathcal{A}^{p \times r} \right\}.$$

Therefore, there exists $F \in \mathcal{A}^{p\times q}$ satisfying $F\,P \in \mathcal{A}^{p\times r}$ such that $\sigma(l) = F\,l$ for all $l \in \mathcal{L}$. Then, $L.\circ\sigma = \mathrm{id}_{\mathcal{L}}$ yields $L\,F\,l = l$ for all $l \in \mathcal{L} = (I_q \quad -P)\,\mathcal{A}^{p\times 1}$, and thus, in particular for all $l \in \mathcal{A}^q$, which yields $L\,F = I_q$. Finally, $\Pi_1 = F\,L = (F \quad -F\,P) \in \mathcal{A}^{p\times p}$, which shows that C internally stabilizes P by Lemma 1 and proves 1.

Suppose that 3 holds. By Proposition 1, there exists $\kappa \in \mathrm{Hom}(\mathcal{M}, \mathcal{A}^{1\times p})$ such that $.M\circ\kappa = \mathrm{id}_{\mathcal{M}}$. Using Remark 7, κ corresponds to a unique element of

$$\mathcal{A}^{1\times p} : \mathcal{M} = \left\{T \in \mathcal{K}^{r\times p} \mid \mathcal{A}^{1\times p}\,(P^T \quad I_r^T)^T\,T \subseteq \mathcal{A}^{1\times p}\right\} = \left\{T \in \mathcal{A}^{r\times p} \mid T\,G \in \mathcal{A}^{q\times p}\right\}.$$

Therefore, there exists $G \in \mathcal{A}^{p\times r}$ satisfying $P\,G \in \mathcal{A}^{q\times p}$ such that $\kappa(m) = m\,G$ for all $m \in \mathcal{M}$. Then, $.M\circ\kappa = \mathrm{id}_{\mathcal{M}}$ gives $m\,G\,M = m$ for all $m \in \mathcal{M} = \mathcal{A}^{1\times p}\,(P^T \quad I_r^T)^T$, and thus, in particular, for all $m \in \mathcal{A}^{1\times r}$, which yields $G\,M = I_r$. Finally, $\Pi_2 = M\,G = ((P\,G)^T \quad G^T) \in \mathcal{A}^{p\times p}$, which shows that C internally stabilizes P by Lemma 1 and proves 1. □

If we write $L = (U^T \quad V^T)^T$, where $U \in \mathcal{A}^{q\times q}$ and $V \in \mathcal{A}^{r\times q}$, and $\det U \neq 0$, then Theorem 1 shows that $C = V\,U^{-1} \in \mathrm{Stab}(P)$. Similarly, if we write $G = (-\widetilde{V} \quad \widetilde{U})$, where $\widetilde{V} \in \mathcal{A}^{r\times q}$ and $\widetilde{U} \in \mathcal{A}^{r\times r}$, then Theorem 1 shows that $C' = \widetilde{U}^{-1}\,\widetilde{V} \in \mathrm{Stab}(P)$.

Remark 10. Theorem 3 (resp., Theorem 2) shows that internal stabilizability (resp., the existence of a coprime factorization) corresponds to the projectivity (resp., freeness) of the lattices $\mathcal{L}$ and $\mathcal{M}$. By Remark 6, projective modules are free but the opposite is usually not true, except for the so-called *projective-free rings* (e.g., RH_∞, $H^\infty(\mathbb{C}_+)$, $\mathcal{W}$, $\mathcal{R}_n$) [37, 26, 5, 29].

Remark 11. Note $P\,G = P\,(I_r - C\,P)^{-1}\,(-C \quad I_r) \in \mathcal{A}^{q\times p}$, which shows that the columns of G belong to $\mathcal{A} : \mathcal{M}$ (see the characterization of $\mathcal{A} : \mathcal{M}$ in Section 3). The $\mathcal{A}$-homomorphism $G. : \mathcal{A}^{p\times 1} \longrightarrow \mathcal{A} : \mathcal{M}$ defined by $G(\xi) = G\,\xi$ for all $\xi \in \mathcal{A}^{p\times 1}$ then satisfies the identity $(M.)\circ(G.) + (F.)\circ(L.) = \Pi_P. + \Pi_C. = I_p.$ (see (18)), and we have the following short split exact sequence

$$0 \longrightarrow \mathcal{A} : \mathcal{M} \underset{G.}{\overset{M.}{\rightleftarrows}} \mathcal{A}^{p\times 1} \underset{F.}{\overset{L.}{\rightleftarrows}} \mathcal{L} \longrightarrow 0. \tag{22}$$

Using $G\,F = 0$ and 4 of Corollary 1, the lower sequence formed by the $\mathcal{A}$-homomorphisms $F.$ and $G.$ is exact, and thus, (22) is a bi-split short exact sequence.

Note that $F\,P \in \mathcal{A}^{p\times r}$, which shows that the rows of F belong to $\mathcal{A} : \mathcal{L}$ (see the characterization of $\mathcal{A} : \mathcal{L}$ in Section 3) and the $\mathcal{A}$-homomorphism $.L : \mathcal{A}^{1\times p} \longrightarrow \mathcal{A} : \mathcal{L}$ defined by $(.L)(\theta) = \theta\,L$ for all $\theta \in \mathcal{A}^{1\times p}$ satisfies $(.G)\circ(.M) + (.L)\circ(.F) = .\Pi_P + .\Pi_C = .I_p$ (see (18)) and we have the following split short exact sequence

$$0 \longleftarrow \mathcal{M} \underset{.M}{\overset{.G}{\rightleftarrows}} \mathcal{A}^{1\times p} \underset{.L}{\overset{.F}{\rightleftarrows}} \mathcal{A} : \mathcal{L} \longleftarrow 0. \tag{23}$$

Using $G\,F = 0$ and 4 of Corollary 1, the upper sequence formed by $.G$ and $.F$ is exact, and thus, (23) is a bi-split short exact sequence of $\mathcal{A}$-modules.

The bi-split short exact sequences (22) and (23) can be proved to be *dual* [28].

Remark 12. Theorem 3 shows that P is internally stabilizable if and only if the lattice $\mathcal{L} = L\,\mathcal{A}^{p\times 1}$of $\mathcal{K}^{q\times 1}$ is a retract of $\mathcal{A}^{p\times 1}$. Geometrically, $\mathcal{L}$ can be thought of as an abstract manifold of the affine space $\mathcal{K}^{q\times 1}$ and $\mathcal{L}$ can be embedded into the subspace $\mathcal{A}^{p\times 1}$ if and only if P is internally stabilizable. This embedding is obtained using a stabilizing controller C of P and the closed-loop system defined in Figure 1. The coordinates of $\mathcal{A}^{p\times 1}$ are the inputs $u = (u_1^T \quad u_2^T)^T$ of the closed-loop system and $\mathcal{L} \cong F(\mathcal{L}) = \Pi_C\,\mathcal{A}^{p\times 1}$. The stabilizing controller C defines a *coordinate system* on the abstract manifold $\mathcal{L}$. As explained in Section 3, $\{L_{\bullet k}\}_{k=1,\ldots,p}$ is a set of generators of $\mathcal{L}$. Setting $U = (I_q - P\,C)^{-1} \in \mathcal{A}^{q\times q}$, $V = C\,(I_q - P\,C)^{-1} \in \mathcal{A}^{r\times q}$, and $F = \left(U^T \quad V^T\right)^T$, then $\{F_{k\bullet}\}_{k=1,\ldots,p}$ defines a set of $\mathcal{A}$-*forms* of $\mathcal{L}$, i.e., $F_{k\bullet}. \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{L}, \mathcal{A})$ for $k = 1, \ldots, p$. Then, we have $l = L\,F\,l = \sum_{k=1}^{p} L_{\bullet k}\,F_{k\bullet}(l)$ for all $l \in \mathcal{L}$, which shows that every "point" $l \in \mathcal{L}$ can be located in "the coordinate system" $\{L_{\bullet k}\}_{k=1,\ldots,p}$ of $\mathcal{L}$ by the "coordinates" $F(l) = (F_{1\bullet}(l), \ldots, F_{p\bullet}(l))^T \in \mathcal{A}^{p\times 1}$. Finally, suppose there is a *differential calculus* on $\mathcal{A}$ [6, 30, 32]. Then, it can be extended from $\mathcal{A}$ to $\mathcal{A}^{p\times 1}$ and finally to $\mathcal{L}$ using the concept of *connections* or *covariant derivative* (*parallel transport*) and the retraction of $\mathcal{L}$ of $\mathcal{A}^{p\times 1}$ [30, 32]. Similarly for $\mathcal{M}$.

Finally, a Quillen-Cuntz theorem [8] asserts that connections only exist on finitely generated projective modules, and thus, differential geometric concepts such as connections, *Christoffel symbols*, *curvatures*, *characteristic classes*, etc., exist only internally stabilizable plants [30, 32].

To finish the section, let P be an internally stabilizable plant, C a stabilizing controller of P, and let

$$\begin{cases} U = (I_q - P\,C)^{-1} \in \mathcal{A}^{q\times q}, \\ V = C\,(I_q - P\,C)^{-1} \in \mathcal{A}^{r\times q}, \\ F = \begin{pmatrix} U^T & V^T \end{pmatrix}^T \in \mathcal{A}^{p\times q}, \end{cases} \quad \begin{cases} \widetilde{U} = (I_r - C\,P)^{-1} \in \mathcal{A}^{r\times r}, \\ \widetilde{V} = (I_r - C\,P)^{-1}\,C \in \mathcal{A}^{r\times q}, \\ G = \begin{pmatrix} -\widetilde{V} & \widetilde{U} \end{pmatrix} \in \mathcal{A}^{r\times p}. \end{cases} \tag{24}$$

We can apply Corollary 1 to the split short exact sequences (22) and (23). First, using Remark 7, we have $\hom_{\mathcal{A}}(\mathcal{L}, \mathcal{A} : \mathcal{M}) \cong (\mathcal{A} : \mathcal{M}) : \mathcal{L}$, where

$$\Omega := (\mathcal{A} : \mathcal{M}) : \mathcal{L} = \left\{ Q \in \mathcal{A}^{r\times q} \mid Q\,P \in \mathcal{A}^{r\times r},\, P\,Q \in \mathcal{A}^{q\times q},\, P\,Q\,P \in \mathcal{A}^{q\times r} \right\} \tag{25}$$

(see Example 3 of [29] for the explicit computations). Applying 1 of Corollary 1 to (22) yields

$$\forall\, Q \in \Omega, \quad F(Q) = F + M\,Q = \begin{pmatrix} U + P\,Q \\ V + Q \end{pmatrix},$$

is such that $F(Q) \in \mathcal{A}^{p\times q}$, $F(Q)\,P \in \mathcal{A}^{p\times r}$, and $L\,F(Q) = I_q$, and thus, $C(Q) = (V + Q)\,(U + P\,Q)^{-1}$ is a stabilizing controller of P for all $Q \in \Omega$.

Similarly, we have $\hom_{\mathcal{A}}(\mathcal{M}, \mathcal{A} : \mathcal{L}) \cong (\mathcal{A} : \mathcal{L}) : \mathcal{M}$. We can check that $(\mathcal{A} : \mathcal{L}) : \mathcal{M} = (\mathcal{A} : \mathcal{M}) : \mathcal{L}$ (see [28]). Applying 2 of Corollary 1 to (22) yields

$$\forall\, Q' \in \Omega, \quad G(Q') = G - Q'\,L = \begin{pmatrix} -\widetilde{V} - Q' & \widetilde{U} + Q'\,P \end{pmatrix}$$

is such that $G(Q') \in \mathcal{A}^{r\times p}$, $P\,G(Q') \in \mathcal{A}^{q\times p}$, and $G('Q)\,M = I_q$, and thus, $C'(Q') = (\widetilde{U}+Q'\,P)^{-1}\,(\widetilde{V}+Q')$ is a stabilizing controller of P for all $Q' \in \Omega$.

Using $G\,F = 0$, $L\,M = 0$, $L\,F = I_q$, and $G\,M = I_r$, we have $G(Q')\,F(Q) = Q - Q'$, and thus, $G(Q')\,F(Q) = 0$ if and only if $Q' = Q$ and, by 5 of Corollary 1,

$$\forall\, Q \in \Omega, \quad C(Q) = (V + Q)\,(U + P\,Q)^{-1} = C'(Q) = (\widetilde{U} + Q\,P)^{-1}\,(\widetilde{V} + Q) \tag{26}$$

parametrizes all the stabilizing controllers of P [29] and we have the bi-split short exact sequences

$$\forall\, Q \in \Omega, \quad 0 \longrightarrow \mathcal{A} : \mathcal{M} \underset{G(Q).}{\overset{M.}{\rightleftarrows}} \mathcal{A}^{p\times 1} \underset{F(Q).}{\overset{L.}{\rightleftarrows}} \mathcal{L} \longrightarrow 0.$$

We can show that Ω is a finitely generated projective $\mathcal{A}$-module and $\Omega = G\,\mathcal{A}^{p\times p}\,F = \sum_{i,j=1}^{p} \mathcal{A}\,(G_{i\bullet}\,F_{\bullet j})$ [29], i.e., we have $Q = G\,T\,F$ for all $T \in \mathcal{A}^{p\times p}$. We emphasize that the parametrization (26) of $\mathrm{Stab}(P)$ assumes only that P is stabilizable, i.e., that there exists a stabilizing controller $C_\star$. Finally, if P has a doubly coprime factorization $P = D^{-1}\,N = \widetilde{N}\,\widetilde{D}^{-1}$, then using 2 and 3 of Theorem 2, we first have

$$\begin{aligned} \Omega &= \left(\widetilde{D}\,\mathcal{A}^{r\times 1}\right) : \left(D^{-1}\,\mathcal{A}^{q\times 1}\right) = \{T \in \mathcal{K}^{r\times q} \mid T\,D^{-1}\,\mathcal{A}^{q\times 1} \subseteq \widetilde{D}\,\mathcal{A}^{r\times 1}\} \\ &= \left\{T \in \mathcal{K}^{r\times q} \mid \exists\, \Lambda \in \mathcal{A}^{r\times q} :\; T\,D^{-1} = \widetilde{D}\,\Lambda\right\} = \widetilde{D}\,\mathcal{A}^{r\times q}\,D, \end{aligned}$$

which shows that Ω is a free $\mathcal{A}$-module of rank $q\,r$ and we can consider $Q = \widetilde{D}\,\Lambda\,D$ for all $\Lambda \in \mathcal{A}^{r\times q}$. Finally, Example 1 shows that $F = (D^T\,(X^T \quad Y^T))^T$ and $G = \widetilde{D}\,(-\widetilde{Y} \quad \widetilde{X})$, which, by substitution into (26), reduces (26) to the well-known Youla-Kučera parametrization [10, 37].

Remark 13. Continuing the geometric interpretation of stabilizability developed in Remark 12, the parametrization of all the stabilizing controllers (26) of P can be understood as the parametrization of all the possible embeddings of $\mathcal{L}$ in $\mathcal{A}^{p\times 1}$ and

$$\forall\, Q \in \Omega, \quad \begin{cases} \Pi_C(Q) = (F + M\,Q)\,L = \Pi_C + M\,Q\,L = \Pi_C + \begin{pmatrix} P \\ I_r \end{pmatrix} Q\,(I_q \quad -P), \\ \Pi_P(Q) = M\,(G - Q\,L) = \Pi_P - M\,Q\,L = \Pi_P - \begin{pmatrix} P \\ I_r \end{pmatrix} Q\,(I_q \quad -P). \end{cases} \tag{27}$$

Let us note $\varepsilon(Q) = \begin{pmatrix} P^T & I_r^T \end{pmatrix}^T Q\,(I_q \quad -P) \in \mathcal{A}^{p\times p}$ for all $Q \in \Omega$. We have $\varepsilon(Q)^2 = 0$. Furthermore, using the identities $L\,\varepsilon(Q) = 0$ and $\varepsilon(Q)\,M = 0$, we can check again that $\Pi_C(Q)^2 = \Pi_C(Q) \in \mathcal{A}^{p\times p}$, $\Pi_P(Q)^2 = \Pi_P(Q) \in \mathcal{A}^{p\times p}$, and $\Pi_C(Q) + \Pi_P(Q) = I_p$ for all $Q \in \Omega$. Therefore, $\{\Pi_C(Q),\, \Pi_P(Q)\}_{Q\in\Omega}$ forms a family of complete sets of orthogonal idempotents of $\mathcal{A}^{p\times p}$ parametrized by the elements Q of the finitely generated projective $\mathcal{A}$-module Ω. Finally, if P has a doubly coprime factorization $P = D^{-1}\,N = \widetilde{N}\,\widetilde{D}^{-1}$, then $\Omega = \widetilde{D}\,\mathcal{A}^{r\times q}\,D$ and $\varepsilon(Q) = \varepsilon\left(\widetilde{D}\,\Lambda\,D\right) = \begin{pmatrix} \widetilde{N}^T & \widetilde{D}^T \end{pmatrix}^T \Lambda\,(D \quad -N)$ for all $\Lambda \in \mathcal{A}^{r\times q}$.

6 The homological perturbation lemma

Let us introduce the concept of a *perturbation of a complex*. This definition will play a central role.

Definition 7. *Let $(\mathcal{P}, d)$ be the complex defined by (10). Then, a* perturbation $\delta = (\delta_i)_{i\in\mathbb{Z}}$ *of $(\mathcal{P}, d)$ is a sequence of $\mathcal{A}$–homomorphisms $\delta_i \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{P}_i, \mathcal{P}_{i-1})$ for $i \in \mathbb{Z}$ which is such that*

$$\ldots \xrightarrow{f_{i+2}+\delta_{i+2}} \mathcal{P}_{i+1} \xrightarrow{f_{i+1}+\delta_{i+1}} \mathcal{P}_i \xrightarrow{f_i+\delta_i} \mathcal{P}_{i-1} \xrightarrow{f_{i-1}+\delta_{i-1}} \ldots$$

is a complex, i.e., satisfies the following conditions

$$\forall\, i \in \mathbb{Z}, \quad (f_{i-1} + \delta_{i-1}) \circ (f_i + \delta_i) = \delta_{i-1} \circ \delta_i + \delta_{i-1} \circ f_i + f_{i-1} \circ \delta_i = 0.$$

Let us first state a useful technical lemma (proof is given in the appendix).

Lemma 2. *Let us consider $\mathcal{M}$ and $\mathcal{P}$ be two $\mathcal{A}$-modules, $\delta \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{P}, \mathcal{M})$, and $h \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{M}, \mathcal{P})$. Then, the following statements are equivalent*

1. $\mathrm{id}_{\mathcal{P}} + h \circ \delta \in \mathrm{End}_{\mathcal{A}}(\mathcal{P})$ *is invertible.*
2. $\mathrm{id}_{\mathcal{M}} + \delta \circ h \in \mathrm{End}_{\mathcal{A}}(\mathcal{M})$ *is invertible.*

Furthermore, we have

$$(\mathrm{id}_{\mathcal{P}} + h \circ \delta)^{-1} = \mathrm{id}_{\mathcal{P}} - h \circ (\mathrm{id}_{\mathcal{M}} + \delta \circ h)^{-1} \circ \delta,$$
$$(\mathrm{id}_{\mathcal{M}} + \delta \circ h)^{-1} = \mathrm{id}_{\mathcal{M}} - \delta \circ (\mathrm{id}_{\mathcal{P}} + h \circ \delta)^{-1} \circ h.$$

Finally, if $\mathrm{id}_{\mathcal{M}} + \delta \circ h$ *or* $\mathrm{id}_{\mathcal{P}} + h \circ \delta$ *is invertible, then we have the following identity*

$$h \circ (\mathrm{id}_{\mathcal{M}} + \delta \circ h)^{-1} = (\mathrm{id}_{\mathcal{P}} + h \circ \delta)^{-1} \circ h.$$

We can now state the so-called *homological perturbation lemma* in a simple case, namely, the case of a contractible complex defined by a split short exact sequence.

Theorem 4. *Let us consider the following bi-split short exact sequence of $\mathcal{A}$-modules*

$$0 \longrightarrow \mathcal{M}_2 \underset{h_1}{\overset{d_2}{\rightleftarrows}} \mathcal{M}_1 \underset{h_0}{\overset{d_1}{\rightleftarrows}} \mathcal{M}_0 \longrightarrow 0,$$

and a perturbation $\mathcal{M}_2 \xrightarrow{\delta_2} \mathcal{M}_1 \xrightarrow{\delta_1} \mathcal{M}_0$ *of the following short exact sequence*

$$0 \longrightarrow \mathcal{M}_2 \xrightarrow{d_2} \mathcal{M}_1 \xrightarrow{d_1} \mathcal{M}_0 \longrightarrow 0,$$

i.e., $\delta_1 \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{M}_1, \mathcal{M}_0)$ *and* $\delta_2 \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{M}_2, \mathcal{M}_1)$ *satisfy the following identity*

$$(d_1 + \delta_1) \circ (d_2 + \delta_2) = d_1 \circ \delta_2 + \delta_1 \circ d_2 + \delta_1 \circ \delta_2 = 0. \tag{28}$$

If the following two conditions

1. $(\mathrm{id}_{\mathcal{M}_0} + \delta_1 \circ h_0) \in \mathrm{End}_{\mathcal{A}}(\mathcal{M}_0)$ *is invertible, i.e., is an automorphism of* $\mathcal{M}_0$,
2. $(\mathrm{id}_{\mathcal{M}_1} + \delta_2 \circ h_1) \in \mathrm{End}_{\mathcal{A}}(\mathcal{M}_1)$ *is invertible, i.e., is an automorphism of* $\mathcal{M}_1$,

are satisfied, setting the following notations

$$\begin{cases} H_0 = h_0 \circ (\mathrm{id}_{\mathcal{M}_0} + \delta_1 \circ h_0)^{-1} = (\mathrm{id}_{\mathcal{M}_1} + h_0 \circ \delta_1)^{-1} \circ h_0 \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{M}_0, \mathcal{M}_1), \\ H_1 = h_1 \circ (\mathrm{id}_{\mathcal{M}_1} + \delta_2 \circ h_1)^{-1} = (\mathrm{id}_{\mathcal{M}_2} + h_1 \circ \delta_2)^{-1} \circ h_1 \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{M}_1, \mathcal{M}_2), \end{cases} \tag{29}$$

then we have the following bi-split short exact sequence of A-modules

$$0 \longrightarrow \mathcal{M}_2 \underset{H_1}{\overset{d_2+\delta_2}{\rightleftarrows}} \mathcal{M}_1 \underset{H_0}{\overset{d_1+\delta_1}{\rightleftarrows}} \mathcal{M}_0 \longrightarrow 0. \tag{30}$$

The proof of Theorem 4 is a straightforward verification given in the appendix. It does not require difficult homological algebra machinery [33] as for the general case [4, 7, 16, 34, 13] (which generalizes Theorem 4 to *morphisms of chain complexes* [33]). See Crainic's notes [7] for a standard survey.

7 A general unstructured robust stability test

7.1 Main result

We can now state the main result of the paper. It is a direct application of Theorem 4.

Theorem 5. *Let* $\mathcal{A}$ *be an integral domain of SISO stable plants and* $K = Q(\mathcal{A})$ *its field of fractions. Moreover, let* $P \in \mathcal{K}^{q\times r}$ *be a stabilizable plant,* $C \in \mathcal{K}^{r\times q}$ *a stabilizing controller of* P*, and set* $p = q + r$,

$$\begin{cases} U = (I_q - P\,C)^{-1} \in \mathcal{A}^{q\times q}, \\ V = C\,(I_q - P\,C)^{-1} \in \mathcal{A}^{r\times q}, \\ F = \begin{pmatrix} U^T & V^T \end{pmatrix}^T \in \mathcal{A}^{p\times q}, \end{cases} \quad \begin{cases} \widetilde{U} = (I_r - C\,P)^{-1} \in \mathcal{A}^{r\times r}, \\ \widetilde{V} = (I_r - C\,P)^{-1}\,C = V \in \mathcal{A}^{r\times q}, \\ G = \begin{pmatrix} -\widetilde{V} & \widetilde{U} \end{pmatrix} \in \mathcal{A}^{r\times p}, \end{cases} \tag{31}$$

$\Pi_C = F\,(I_q \quad -P) \in \mathcal{A}^{r\times p}$, *and* $\Pi_P = (P^T \quad I_r^T)^T\,M \in \mathcal{A}^{r\times p}$. *Finally, let*

$$\Theta = \begin{pmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{pmatrix} \in \mathcal{A}^{p\times p}, \quad \Xi = \begin{pmatrix} \Xi_{11} & \Xi_{12} \\ \Xi_{21} & \Xi_{22} \end{pmatrix} \in \mathcal{A}^{p\times p},$$

where $\Theta_{11}, \Xi_{11} \in \mathcal{A}^{q\times q}$, $\Theta_{12}, \Xi_{12} \in \mathcal{A}^{q\times r}$, $\Theta_{21}, \Xi_{21} \in \mathcal{A}^{r\times q}$, $\Theta_{22}, \Xi_{22} \in \mathcal{A}^{r\times r}$, *and*

$$\begin{aligned} \Delta = (\Delta_1 \quad -\Delta_2) &:= (I_q \quad -P)\,\Theta = (\Theta_{11} - P\,\Theta_{21} \quad \Theta_{12} - P\,\Theta_{22}), \\ \nabla = \begin{pmatrix} \nabla_1 \\ \nabla_2 \end{pmatrix} &:= \Xi \begin{pmatrix} P \\ I_r \end{pmatrix} = \begin{pmatrix} \Xi_{11}\,P + \Xi_{12} \\ \Xi_{21}\,P + \Xi_{22} \end{pmatrix}. \end{aligned} \tag{32}$$

Then, C *internally stabilizable the plants of the form*

$$\begin{aligned} P_\delta &:= (I_q + \Delta_1)^{-1}\,(P + \Delta_2) = (I_q + \Theta_{11} - P\,\Theta_{21})^{-1}\,(-\Theta_{12} + P\,(I_r + \Theta_{22})) \\ &= (P + \nabla_1)\,(I_r + \nabla_2)^{-1} = (\Xi_{12} + (I_q + \Xi_{11})\,P)\,(I_r + \Xi_{22} + \Xi_{21}\,P)^{-1} \end{aligned} \tag{33}$$

for all matrices $\Theta, \Xi \in \mathcal{A}^{p\times p}$ *satisfying the following conditions*

$$\begin{cases} I_p + \Pi_C\,\Theta \in \mathrm{GL}_p(\mathcal{A}), \\ I_p + \Xi\,\Pi_P \in \mathrm{GL}_p(\mathcal{A}), \\ \det(I_q + \Delta_1) = \det(I_q + \Theta_{11} - P\,\Theta_{21}) \neq 0, \\ \det(I_r + \nabla_2) = \det(I_r + \Xi_{22} + \Xi_{21}\,P) \neq 0. \end{cases} \tag{34}$$

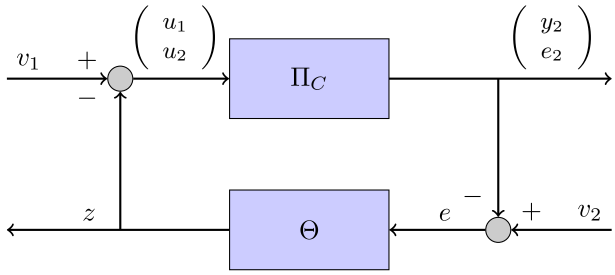

Figure 2: Interpretation of the stability condition $I_p + \Xi\,\Pi_P \in \mathrm{GL}_p(\mathcal{A})$

Finally, the first two conditions of (34) are equivalent to

$$\left\{\begin{array}{l} I_q + \Delta\, F = I_q + \Delta_1\, U + \Delta_2\, V \in \mathrm{GL}_q(\mathcal{A}), \\ I_r + G\,\nabla = I_r - \widetilde{V}\,\nabla_1 + \widetilde{U}\,\nabla_2 \in \mathrm{GL}_r(\mathcal{A}), \end{array}\right. \Leftrightarrow \left\{\begin{array}{l} \det(I_q + \Delta\, F) \in \mathrm{U}(\mathcal{A}), \\ \det(I_r + G\,\nabla) \in \mathrm{U}(\mathcal{A}). \end{array}\right. \tag{35}$$

The proof of Theorem 5 is given in the appendix.

Remark 14. We emphasize that Theorem 5 does not require the existence of a doubly coprime factorization for P. It only requires that P is stabilizable, i.e., P is stabilized by a controller C.

If P has a doubly coprime factorization, in Section 7.3, we will show that Theorem 5 gives again the standard robust stability test for coprime factor uncertainty.

Remark 15. The first condition of (34), i.e., $I_p + \Pi_C\,\Theta \in \mathrm{GL}_p(\mathcal{A})$, is equivalent to the fact that the stable transfer matrix $\Theta \in \mathcal{A}^{p\times p}$ internally stabilizes the stable system $\Pi_C \in \mathcal{A}^{p\times p}$ (see Figure 2). Similarly the second condition of (34), i.e., $I_p + \Xi\,\Pi_P \in \mathrm{GL}_p(\mathcal{A})$, asserts that $\Xi \in \mathcal{A}^{p\times p}$ internally stabilizes $\Pi_P \in \mathcal{A}^{p\times p}$.

Example 4. We illustrate Theorem 5 for SISO plants. Let $p \in \mathcal{K}$ be a stabilizable plant, c a stabilizing controller of p, $u = 1/(1 - p\,c) \in \mathcal{A}$, and $v = c/(1 - p\,c) \in \mathcal{A}$. Set $\delta_1 = \theta_{11} - \theta_{12}\,p$ and $\delta_2 = -\theta_{12} + \theta_{22}\,p$ for $\theta_{11}, \theta_{12}, \theta_{21}, \theta_{22} \in \mathcal{A}$. Then, we have

$$I_2 + \Pi_c\,\Theta = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} u & -u\,p \\ v & -v\,p \end{pmatrix} \begin{pmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{pmatrix} = \begin{pmatrix} 1 + u\,\delta_1 & -u\,\delta_2 \\ v\,\delta_1 & 1 - v\,\delta_2 \end{pmatrix}.$$

Theorem 5 then shows that c stabilizes all the plants of the form

$$p_\delta = \frac{p + \delta_2}{1 + \delta_1} = \frac{-\theta_{12} + (1 + \theta_{22})\,p}{1 + \theta_{11} - \theta_{21}\,p}$$

for all $\theta_{11}, \theta_{12}, \theta_{21}, \theta_{22} \in \mathcal{A}$ such that $1 + \theta_{11} - \theta_{21}\,p \neq 0$ and satisfying the following condition

$$\begin{aligned} I_2 + \Pi_c\,\Theta \in \mathrm{GL}_2(\mathcal{A}) \;\Leftrightarrow\; & \det(I_2 + \Pi_c\,\Theta) = 1 + u\,\delta_1 - v\,\delta_2 \\ & = 1 + \theta_{11}\,u + \theta_{12}\,v - \theta_{21}\,(u\,p) - \theta_{22}\,(v\,p) \in \mathrm{U}(\mathcal{A}). \end{aligned}$$

For SISO systems, a factorization representation is both left and right. So, if $\partial_1 = \xi_{11}\,p + \xi_{12}$ and $\partial_2 = \xi_{21}\,p + \xi_{22}$ for $\xi_{11}, \xi_{12}, \xi_{21}, \xi_{22} \in \mathcal{A}$, then the robust stability test for

$$p_\delta = \frac{p + \partial_1}{1 + \partial_2} = \frac{\xi_{12} + (1 + \xi_{11})\,p}{1 + \xi_{22} + \xi_{21}\,p}$$

is the same as above after changing the notations $\xi_{11} = \theta_{22}$, $\xi_{12} = -\theta_{12}$, $\xi_{21} = -\theta_{21}$, and $\xi_{22} = \theta_{11}$.

Remark 16. Let $\mathcal{F}_\ell(\Lambda, P) = \Lambda_{11} + \Lambda_{12}\,P\,(I_r - \Lambda_{22}\,P)^{-1}\,\Lambda_{21}$ be the *lower linear fractional transformation* (LFT) (see, e.g., [40]), where $P \in \mathcal{K}^{q\times r}$ and $\Lambda \in \mathcal{K}^{(q+r)\times(r+q)}$ is the matrix defined by

$$\Lambda = \begin{pmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{21} & \Lambda_{22} \end{pmatrix}, \quad \Lambda_{11} \in \mathcal{K}^{q\times r},\ \Lambda_{12} \in \mathcal{K}^{q\times q},\ \Lambda_{21} \in \mathcal{K}^{r\times r},\ \Lambda_{22} \in \mathcal{K}^{r\times q}.$$

By Lemma 9.1 of [40], if $\det(I_q+\Theta_{11}) \neq 0$, then the plant $P_\delta = (I_q+\Theta_{11}-P\,\Theta_{21})^{-1}\,(-\Theta_{12}+P\,(I_r+\Theta_{22}))$ can be rewritten as $P_\delta = \mathcal{F}_\ell(\Lambda, P)$, where

$$\Lambda = \left(\begin{array}{cc} -(I_q+\Theta_{11})^{-1}\,\Theta_{12} & (I_q+\Theta_{11})^{-1} \\ I_r+\Theta_{22}-\Theta_{21}\,(I_q+\Theta_{11})^{-1}\,\Theta_{12} & \Theta_{21}\,(I_q+\Theta_{11})^{-1} \end{array}\right).$$

Similarly with $P_\delta = (\Xi_{12}+(I_q+\Xi_{11})\,P)\,(I_r+\Xi_{22}+\Xi_{21}\,P)^{-1}$ when $\det(I_r+\Xi_{22}) \neq 0$.

If $I_r+\Theta_{11}$ or $I_r+\Xi_{22}$ is a singular matrix, we need to find out if P_δ can be interpreted as a LFT.

The next corollary of Theorem 5 gives the relation between the projectors Π_C^δ and Π_P^δ − defined by P_δ and a stabilizing controller C of P − and the projectors Π_C and Π_P defined by P and C (see (5)).

Corollary 2. *Using the notations and assumptions of Theorem 5, we then have*

$$\begin{array}{l} \Pi_C^\delta = \left(\begin{array}{cc} (I_q-P_\delta\,C)^{-1} & -(I_q-P_\delta\,C)^{-1}\,P_\delta \\ C\,(I_q-P_\delta\,C)^{-1} & -C\,(I_q-P_\delta\,C)^{-1}\,P_\delta \end{array}\right) = (I_p+\Pi_C\,\Theta)^{-1}\,\Pi_C\,(I_p+\Theta), \\ \Pi_P^\delta = \left(\begin{array}{cc} -P\,(I_r-C\,P_\delta)^{-1}\,C & P\,(I_r-C\,P_\delta)^{-1} \\ -(I_r-C\,P_\delta)^{-1}\,C & (I_r-C\,P_\delta)^{-1} \end{array}\right) = (I_p+\Xi)\,\Pi_P\,(I_p+\Xi\,\Pi_P)^{-1}. \end{array} \tag{36}$$

In particular, $\Pi_C^\delta, \Pi_P^\delta \in \mathcal{A}^{p\times p}$, *which shows (again) that* C *internally stabilizes* P_δ.

Proof. Let $L = (I_q \quad -P) \in \mathcal{K}^{q\times p}$, $M = (P^T \quad I_r^T)^T \in \mathcal{K}^{p\times r}$, and

$$\left\{\begin{array}{l} U = (I_q-P\,C)^{-1} \in \mathcal{K}^{q\times q}, \\ V = C\,(I_q-P\,C)^{-1} \in \mathcal{K}^{r\times q}, \\ F = \left(U^T \quad V^T\right)^T \in \mathcal{K}^{p\times q}, \end{array}\right. \qquad \left\{\begin{array}{l} \widetilde{U} = (I_r-C\,P)^{-1} \in \mathcal{K}^{r\times r}, \\ \widetilde{V} = (I_r-C\,P)^{-1}\,C = V \in \mathcal{K}^{r\times q}, \\ G = \left(-\widetilde{V} \quad \widetilde{U}\right) \in \mathcal{K}^{r\times p}. \end{array}\right.$$

Using the identities $L\,F = U-P\,V = I_q$, $\Delta = L\,\Theta$, $L+\Delta = L+L\,\Theta = L\,(I_p+\Theta)$, $F\,(I_q+\Delta\,F)^{-1} = (I_p+F\,\Delta)^{-1}\,F$, and $F\,L = \Pi_C$, we have

$$\begin{aligned} I_q-P_\delta\,C &= I_q-(I_q+\Delta_1)^{-1}\,(P+\Delta_2)\,V\,U^{-1} \\ &= (I_q+\Delta_1)^{-1}\,((I_q+\Delta_1)\,U-(P+\Delta_2)\,V)\,U^{-1} \\ &= (I_q+\Delta_1)^{-1}\,(U-P\,V+\Delta_1\,U-\Delta_2\,V)\,U^{-1} \\ &= (I_q+\Delta_1)^{-1}\,(I_q+\Delta_1\,U-\Delta_2\,V)\,U^{-1} = (I_q+\Delta_1)^{-1}\,(I_q+\Delta\,F)\,U^{-1} \end{aligned}$$

$$\begin{aligned} \Rightarrow (I_q-P_\delta\,C)^{-1} &= U\,(I_q+\Delta\,F)^{-1}\,(I_q+\Delta_1) \\ \Rightarrow \Pi_C^\delta &= \left(\begin{array}{cc} U\,(I_q+\Delta\,F)^{-1}\,(I_q+\Delta_1) & -U\,(I_q+\Delta\,F)^{-1}\,(P+\Delta_2) \\ V\,(I_q+\Delta\,F)^{-1}\,(I_q+\Delta_1) & -V\,(I_q+\Delta\,F)^{-1}\,(P+\Delta_2) \end{array}\right) \\ &= \left(\begin{array}{c} U \\ V \end{array}\right)(I_q+\Delta\,F)^{-1}\,(I_q+\Delta_1 \quad -P-\Delta_2) \\ &= F\,(I_q+\Delta\,F)^{-1}\,(L+\Delta) = (I_p+F\,\Delta)^{-1}\,F\,L\,(I_p+\Theta) \\ &= (I_p+\Pi_C\,\Theta)^{-1}\,F\,L\,(I_p+\Theta) = (I_p+\Pi_C\,\Theta)^{-1}\,\Pi_C\,(I_p+\Theta). \end{aligned}$$

Similarly, using the identities $G\,M = -\widetilde{V}\,P+\widetilde{U} = I_r$, $\nabla = \Xi\,M$, $M+\nabla = M+\Xi\,M = (I_p+\Xi)\,M$, $(I_r-G\,\nabla)^{-1}\,G = G\,(I_p+\nabla\,G)^{-1}$, and $M\,G = \Pi_P$, we have

$$\begin{aligned} I_r-C\,P_\delta &= I_r-\widetilde{U}^{-1}\,\widetilde{V}\,(P+\nabla_1)\,(I_r+\nabla_2)^{-1} \\ &= \widetilde{U}^{-1}\,(\widetilde{U}\,(I_r+\nabla_2)-\widetilde{V}\,(P+\nabla_1))\,(I_r+\nabla_2)^{-1} \\ &= \widetilde{U}^{-1}\,(\widetilde{U}-\widetilde{V}\,P+\widetilde{U}\,\nabla_2-\widetilde{V}\,\nabla_1)\,(I_r+\nabla_2)^{-1} \\ &= \widetilde{U}^{-1}\,(I_r-\widetilde{V}\,\nabla_1+\widetilde{U}\,\nabla_2)\,(I_r+\nabla_2)^{-1} \\ &= \widetilde{U}^{-1}\,(I_r-G\,\nabla)\,(I_r+\nabla_2)^{-1} \end{aligned}$$

$$\begin{aligned}
&\Rightarrow (I_r - C\,P_\delta)^{-1} = (I_r + \nabla_2)\,(I_r - G\,\nabla)^{-1}\,\widetilde{U} \\
&\Rightarrow \Pi_P^\delta = \begin{pmatrix} -(P+\nabla_1)\,(I_r - G\,\nabla)^{-1}\,\widetilde{V} & (P+\nabla_1)\,(I_r - G\,\nabla)^{-1}\,\widetilde{U} \\ -(I_r+\nabla_2)\,(I_r - G\,\nabla)^{-1}\,\widetilde{V} & (I_r+\nabla_2)\,(I_r - G\,\nabla)^{-1}\,\widetilde{U} \end{pmatrix} \\
&= \begin{pmatrix} P+\nabla_1 \\ I_r+\nabla_2 \end{pmatrix} (I_r - G\,\nabla)^{-1} \begin{pmatrix} -\widetilde{V} & \widetilde{U} \end{pmatrix} \\
&= (M+\nabla)\,(I_r - G\,\nabla)^{-1}\,G = (I_p + \Xi)\;M\,(I_r - G\,\nabla)^{-1}\,G \\
&= (I_p + \Xi)\;M\,G\,(I_p + \nabla\,G)^{-1} = (I_p + \Xi)\;\Pi_P\,(I_p + \Xi\,M\,G)^{-1} \\
&= (I_p + \Xi)\;\Pi_P\,(I_p + \Xi\,\Pi_P)^{-1}.
\end{aligned}$$

Using Θ, $\Xi \in \mathcal{A}^{p\times p}$ and $I_p + \Xi\,\Pi_P \in \mathrm{GL}_p(\mathcal{A})$, $I_p + \Pi_C\,\Theta \in \mathrm{GL}_p(\mathcal{A})$, we finally have

$$\Pi_C^\delta = (I_p + \Pi_C\,\Theta)^{-1}\,\Pi_C\,(I_p + \Theta) \in \mathcal{A}^{p\times p}, \quad \Pi_P^\delta = (I_r + \Xi)\;\Pi_P\,(I_p + \Xi\,\Pi_P)^{-1} \in \mathcal{A}^{p\times p},$$

which proves that C internally stabilizes P_δ. □

Remark 17. The assumptions that $C \in \mathrm{Stab}(P)$ and $\Theta, \Xi \in \mathcal{A}^{p\times p}$ were not used to prove (36). They were only used to show that Π_C^δ, $\Pi_P^\delta \in \mathcal{A}^{p\times p}$. Hence, (36) holds for $\Theta, \Xi \in \mathcal{K}^{p\times p}$ or for a controller C which does not necessarily stabilize P. Note that Corollary 2 gives an elementary proof of Theorem 5.

Example 5. Continuing Example 4 studying the SISO case, we have

$$\begin{aligned}
\Pi_c^\delta &= \frac{1}{1 + u\,\delta_1 - v\,\delta_2} \begin{pmatrix} 1 - v\,\delta_2 & u\,\delta_2 \\ -v\,\delta_1 & 1 + u\,\delta_1 \end{pmatrix} \Pi_c \begin{pmatrix} 1+\theta_{11} & \theta_{12} \\ \theta_{21} & 1+\theta_{22} \end{pmatrix} \\
\Pi_p^\delta &= \begin{pmatrix} 1+\xi_{11} & \xi_{12} \\ \xi_{21} & 1+\xi_{22} \end{pmatrix} \Pi_p \begin{pmatrix} 1 + u\,\partial_2 & -u\,\partial_1 \\ v\,\partial_2 & 1 - v\,\partial_1 \end{pmatrix} \frac{1}{1 - v\,\partial_1 + u\,\partial_2}.
\end{aligned}$$

Example 6. With the notations (31), we have the following results.

1. *Inverse multiplicative output uncertainty*: Considering $\Theta_{12} = 0$, $\Theta_{21} = 0$, and $\Theta_{22} = 0$, Theorem 5 shows that C stabilizes $P_\delta = (I_q + \Theta_{11})^{-1}\,P$ if $\det(I_q + \Theta_{11}) \neq 0$ and the following condition is satisfied

$$I_p + \Pi_C\,\Theta = \begin{pmatrix} I_q + U\,\Theta_{11} & 0 \\ V\,\Theta_{11} & I_r \end{pmatrix} \in \mathrm{GL}_p(\mathcal{A}) \;\Leftrightarrow\; I_q + U\,\Theta_{11} \in \mathrm{GL}_q(\mathcal{A}).$$

Writing $(I_p + \Pi_C\,\Theta)^{-1}$ as a product of two elementary matrices, we have

$$\Pi_C^\delta = \begin{pmatrix} I_q & 0 \\ -V\,\Theta_{11} & I_r \end{pmatrix} \begin{pmatrix} (I_q + U\,\Theta_{11})^{-1} & 0 \\ 0 & I_r \end{pmatrix} \Pi_C \begin{pmatrix} I_q + \Theta_{11} & 0 \\ 0 & I_r \end{pmatrix}.$$

2. *Additive uncertainty*: Considering $\Theta_{11} = 0$, $\Theta_{21} = 0$, and $\Theta_{22} = 0$, Theorem 5 shows that C stabilizes $P_\delta = P - \Theta_{12}$ if the following condition is satisfied

$$I_p + \Pi_C\,\Theta = \begin{pmatrix} I_q & U\,\Theta_{12} \\ 0 & I_r + V\,\Theta_{12} \end{pmatrix} \in \mathrm{GL}_p(\mathcal{A}) \;\Leftrightarrow\; I_r + V\,\Theta_{12} \in \mathrm{GL}_r(\mathcal{A}).$$

Writing $(I_p + \Pi_C\,\Theta)^{-1}$ as a product of two elementary matrices, we have

$$\Pi_C^\delta = \begin{pmatrix} I_q & -U\,\Theta_{12} \\ 0 & I_r \end{pmatrix} \begin{pmatrix} I_q & 0 \\ 0 & (I_r + V\,\Theta_{12})^{-1} \end{pmatrix} \Pi_C \begin{pmatrix} I_q & \Theta_{12} \\ 0 & I_r \end{pmatrix}.$$

3. *Inverse additive uncertainty*: Considering $\Theta_{11} = 0$, $\Theta_{12} = 0$, and $\Theta_{22} = 0$, Theorem 5 shows that C stabilizes $P_\delta = (I_q - P\,\Theta_{21})^{-1}\,P$ if $\det(I_q - P\,\Theta_{21}) \neq 0$ and the following condition is satisfied

$$I_p + \Pi_C\,\Theta = \begin{pmatrix} I_q - U\,P\,\Theta_{21} & 0 \\ -V\,P\,\Theta_{21} & I_r \end{pmatrix} \in \mathrm{GL}_p(\mathcal{A}) \;\Leftrightarrow\; I_q - U\,P\,\Theta_{21} \in \mathrm{GL}_q(\mathcal{A}).$$

Writing $(I_p + \Pi_C\, \Theta)^{-1}$ as a product of two elementary matrices, we have

$$\Pi_C^\delta = \begin{pmatrix} I_q & 0 \\ V\,P\,\Theta_{21} & I_r \end{pmatrix} \begin{pmatrix} (I_q - U\,P\,\Theta_{21})^{-1} & 0 \\ 0 & I_r \end{pmatrix} \Pi_C \begin{pmatrix} I_q & 0 \\ \Theta_{21} & I_r \end{pmatrix}.$$

4. *Input multiplicative uncertainty*: Considering $\Theta_{11} = 0$, $\Theta_{12} = 0$, and $\Theta_{21} = 0$, Theorem 5 shows that C stabilizes $P_\delta = P\,(I_r + \Theta_{22})$ if the following condition is satisfied

$$I_p + \Pi_C\, \Theta = \begin{pmatrix} I_q & -U\,P\,\Theta_{22} \\ 0 & I_r - V\,P\,\Theta_{22} \end{pmatrix} \in \mathrm{GL}_p(\mathcal{A}) \;\Leftrightarrow\; I_r - V\,P\,\Theta_{22} \in \mathrm{GL}_r(\mathcal{A}).$$

Writing $(I_p + \Pi_C\, \Theta)^{-1}$ as a product of two elementary matrices, we have

$$\Pi_C^\delta = \begin{pmatrix} I_q & U\,P\,\Theta_{22} \\ 0 & I_r \end{pmatrix} \begin{pmatrix} I_q & 0 \\ 0 & (I_r - V\,P\,\Theta_{22})^{-1} \end{pmatrix} \Pi_C \begin{pmatrix} I_q & 0 \\ 0 & I_r + \Theta_{22} \end{pmatrix}.$$

5. *Output multiplicative uncertainty*: Considering $\Xi_{12} = 0$, $\Xi_{21} = 0$, and $\Xi_{22} = 0$, Theorem 5 shows that C stabilizes $P_\delta = (I_q + \Xi_{11})\,P$ if the following condition is satisfied

$$I_p + \Xi\,\Pi_P = \begin{pmatrix} I_q - \Xi_{11}\,P\,\widetilde{V} & \Xi_{11}\,P\,\widetilde{U} \\ 0 & I_r \end{pmatrix} \in \mathrm{GL}_p(\mathcal{A}) \;\Leftrightarrow\; I_q - \Xi_{11}\,P\,\widetilde{V} \in \mathrm{GL}_q(\mathcal{A}).$$

Writing $(I_p + \Xi\,\Pi_P)^{-1}$ as a product of two elementary matrices, we have

$$\Pi_P^\delta = \begin{pmatrix} I_q + \Xi_{11} & 0 \\ 0 & I_r \end{pmatrix} \Pi_P \begin{pmatrix} \left(I_q - \Xi_{11}\,P\,\widetilde{V}\right)^{-1} & 0 \\ 0 & I_r \end{pmatrix} \begin{pmatrix} I_q & -\Xi_{11}\,P\,\widetilde{U} \\ 0 & I_r \end{pmatrix}.$$

6. *Additive uncertainty*: Considering $\Xi_{11} = 0$, $\Xi_{21} = 0$, and $\Xi_{22} = 0$, Theorem 5 shows that C stabilizes $P_\delta = P + \Xi_{12}$ if the following condition is satisfied

$$I_p + \Xi\,\Pi_P = \begin{pmatrix} I_q - \Xi_{12}\,\widetilde{V} & \Xi_{12}\,\widetilde{U} \\ 0 & I_r \end{pmatrix} \in \mathrm{GL}_p(\mathcal{A}) \;\Leftrightarrow\; I_q - \Xi_{12}\,\widetilde{V} \in \mathrm{GL}_q(\mathcal{A}).$$

Writing $(I_p + \Xi\,\Pi_P)^{-1}$ as a product of two elementary matrices, we have

$$\Pi_P^\delta = \begin{pmatrix} I_q & \Xi_{12} \\ 0 & I_r \end{pmatrix} \Pi_P \begin{pmatrix} \left(I_q - \Xi_{12}\,\widetilde{V}\right)^{-1} & 0 \\ 0 & I_r \end{pmatrix} \begin{pmatrix} I_q & -\Xi_{12}\,\widetilde{U} \\ 0 & I_r \end{pmatrix}.$$

7. *Inverse additive uncertainty*: Considering $\Xi_{11} = 0$, $\Xi_{12} = 0$, and $\Xi_{22} = 0$, Theorem 5 shows that C stabilizes $P_\delta = P\,(I_r + \Xi_{21}\,P)^{-1}$ if $\det(I_r + \Xi_{21}\,P) \neq 0$ and the following condition is satisfied

$$I_p + \Xi\,\Pi_P = \begin{pmatrix} I_q & 0 \\ -\Xi_{21}\,P\,\widetilde{V} & I_r + \Xi_{21}\,P\,\widetilde{U} \end{pmatrix} \in \mathrm{GL}_p(\mathcal{A}) \;\Leftrightarrow\; I_r + \Xi_{21}\,P\,\widetilde{U} \in \mathrm{GL}_r(\mathcal{A}).$$

Writing $(I_p + \Xi\,\Pi_P)^{-1}$ as a product of two elementary matrices, we have

$$\Pi_P^\delta = \begin{pmatrix} I_q & 0 \\ \Xi_{21} & I_r \end{pmatrix} \Pi_P \begin{pmatrix} I_q & 0 \\ 0 & \left(I_r + \Xi_{21}\,P\,\widetilde{U}\right)^{-1} \end{pmatrix} \begin{pmatrix} I_q & 0 \\ \Xi_{21}\,P\,\widetilde{V} & I_r \end{pmatrix}.$$

8. *Inverse multiplicative input uncertainty*: Considering $\Xi_{11} = 0$, $\Xi_{12} = 0$, and $\Xi_{21} = 0$, Theorem 5 shows that C stabilizes $P_\delta = P\,(I_r + \Xi_{22})^{-1}$ if $\det(I_r + \Xi_{22}) \neq 0$ and the following condition is satisfied

$$I_p + \Xi\,\Pi_P = \begin{pmatrix} I_q & 0 \\ -\Xi_{22}\,\widetilde{V} & I_r + \Xi_{22}\,\widetilde{U} \end{pmatrix} \in \mathrm{GL}_p(\mathcal{A}) \;\Leftrightarrow\; I_r + \Xi_{22}\,\widetilde{U} \in \mathrm{GL}_r(\mathcal{A}).$$

Writing $(I_p + \Xi\,\Pi_P)^{-1}$ as a product of two elementary matrices, we have

$$\Pi_P^\delta = \begin{pmatrix} I_q & 0 \\ 0 & I_r + \Xi_{22} \end{pmatrix} \Pi_P \begin{pmatrix} I_q & 0 \\ 0 & \left(I_r + \Xi_{22}\,\widetilde{U}\right)^{-1} \end{pmatrix} \begin{pmatrix} I_q & 0 \\ \Xi_{22}\,\widetilde{V} & I_r \end{pmatrix}.$$

The homological perturbation lemma can also be applied to a split exact sequence relating the lattices $\mathcal{P}$ and $\mathcal{Q}$ defined in Remark 8 (see Lemma 5 of [28]).

Finally, let us state a few consequences of Theorem 5 in the case where $\mathcal{A}$ be a *Banach algebra*, namely, $\mathcal{A}$ is a $\mathbb{K}$-algebra, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, which is a *Banach space* as a $\mathbb{K}$-vector space for a *submultiplicative norm* $\|\cdot\|_{\mathcal{A}}$ (i.e., $\|a_1\,a_2\|_{\mathcal{A}} \leq \|a_1\|_{\mathcal{A}}\,\|a_2\|_{\mathcal{A}}$ for all $a_1,\, a_2 \in \mathcal{A}$) satisfying $\|1\|_{\mathcal{A}} = 1$ [23].

For example, $H^\infty(\mathbb{C}_+)$ and $\mathcal{W}$, defined in Section 2, are two Banach algebras.

If $\mathcal{A}$ is a Banach algebra, then, for $p \in \mathbb{Z}_{>0}$, $\mathcal{A}^{p\times p}$ is a Banach algebra for the norm defined by $\|M\|_{\mathcal{A}^{p\times p}} = \max_{1\leq i\leq p} \sum_{j=1}^{p} \|M_{ij}\|_{\mathcal{A}}$ for all $M \in \mathcal{A}^{p\times p}$ (or for any equivalent norm).

Let $\mathcal{B}$ be a Banach algebra (e.g., $\mathcal{B} = \mathcal{A}^{p\times p}$). Then, we have the following standard results [23]

- If $\|1 - b\|_{\mathcal{B}} < 1$, then $b^{-1} \in \mathcal{B}$, i.e., $b \in \mathrm{U}(\mathcal{B})$, and $b^{-1} = \sum_{i=0}^{\infty} (1-b)^i$.
- If $b \in \mathrm{U}(\mathcal{B})$, then $\|b^{-1}\|_{\mathcal{B}} \leq \sum_{i=0}^{\infty} \|1 - b\|_{\mathcal{B}}^i = (1 - \|1-b\|_{\mathcal{B}})^{-1}$.

We have the following consequences of Theorem 5.

Corollary 3. *Using the notations of Theorem 5, a sufficient condition for $I_p + \Pi_C\,\Theta \in \mathrm{GL}_p(\mathcal{A})$ and $I_p + \Xi\,\Pi_P \in \mathrm{GL}_p(\mathcal{A})$ are respectively given by*

$$\|\Theta\|_{\mathcal{A}^{p\times p}} < \|\Pi_C\|_{\mathcal{A}^{p\times p}}^{-1}, \quad \|\Xi\|_{\mathcal{A}^{p\times p}} < \|\Pi_P\|_{\mathcal{A}^{p\times p}}^{-1}. \tag{37}$$

Furthermore, we have the bounds for the norms of the idempotents Π_C^δ and Π_P^δ defined by (36)

$$\begin{aligned} \|\Pi_C^\delta\|_{\mathcal{A}^{p\times p}} &\leq \|(I_p + \Pi_C\,\Theta)^{-1}\|_{\mathcal{A}^{p\times p}}\,\|\Pi_C\|_{\mathcal{A}^{p\times p}}\,\|I_p + \Theta\|_{\mathcal{A}^{p\times p}} \\ &\leq (1 - \|\Pi_C\,\Theta\|_{\mathcal{A}^{p\times p}})^{-1}\,\|\Pi_C\|_{\mathcal{A}^{p\times p}}\,\|I_p + \Theta\|_{\mathcal{A}^{p\times p}}, \\ \|\Pi_P^\delta\|_{\mathcal{A}^{p\times p}} &\leq \|I_p + \Xi\|_{\mathcal{A}^{p\times p}}\,\|\Pi_P\|_{\mathcal{A}^{p\times p}}\,\|(I_p + \Xi\,\Pi_P)^{-1}\|_{\mathcal{A}^{p\times p}} \\ &\leq \|I_p + \Xi\|_{\mathcal{A}^{p\times p}}\,\|\Pi_P\|_{\mathcal{A}^{p\times p}}\,(1 - \|\Xi\,\Pi_P\|_{\mathcal{A}^{p\times p}})^{-1}. \end{aligned}$$

Note that the inequalities in Corollary 3 are usually far from being sharp.

Using (27) and (25), we can define the *left/right optimal robust radius* by

$$l_{\text{opt.rad.}} := \inf_{Q\in\Omega} \|\Pi_C(Q)\|_{\mathcal{A}^{p\times p}}, \quad r_{\text{opt.rad.}} := \inf_{Q\in\Omega} \|\Pi_P(Q)\|_{\mathcal{A}^{p\times p}}.$$

They will be studied elsewhere for the different Banach algebras used in stabilization problems.

Example 7. We continue Example 6. C stabilizes all the plants of the form $P_\delta = (I_q + \Theta_{11})^{-1}\,P$ where $\|U\,\Theta_{11}\|_{\mathcal{A}^{q\times q}} < 1$, of the form $P_\delta = P - \Theta_{12}$ where $\|V\,\Theta_{12}\|_{\mathcal{A}^{r\times r}} < 1$, of the form $P_\delta = (I_q - P\,\Theta_{21})^{-1}\,P$ where $\|U\,P\,\Theta_{21}\|_{\mathcal{A}^{q\times q}} < 1$, of the form $P_\delta = P\,(I_r + \Theta_{22})$ where $\|V\,P\,\Theta_{22}\|_{\mathcal{A}^{r\times r}} < 1$, of the form $P_\delta = (I_q + \Xi_{11})\,P$ where $\|\Theta_{11}\,P\,\widetilde{V}\|_{\mathcal{A}^{q\times q}} < 1$, $P_\delta = P + \Xi_{12}$ where $\|\Theta_{12}\,\widetilde{V}\|_{\mathcal{A}^{r\times r}} < 1$, of the form $P_\delta = P\,(I_r + \Xi_{21}\,P)^{-1}$ where $\|\Theta_{21}\,P\,\widetilde{U}\|_{\mathcal{A}^{r\times r}} < 1$, and of the form $P_\delta = P\,(I_r + \Xi_{22})^{-1}$ where $\|\Theta_{22}\,\widetilde{U}\|_{\mathcal{A}^{r\times r}} < 1$.

Finally, we consider $\mathcal{A} = H^\infty(\mathbb{C}_+)$. Recall that the norm of $M \in \mathcal{A}^{r\times s}$ is defined by

$$\|M\|_\infty = \sup_{s\in\mathbb{C}+} \overline{\sigma}(M(s)) = \text{ess.sup}_{\omega\in\mathbb{R}} \overline{\sigma}(M(i\,\omega)),$$

where $\overline{\sigma}(\cdot)$ denotes the *maximal singular value*. It is well-known that $\|M_1\,M_2\|_\infty \leq \|M_1\|_\infty\,\|M_2\|_\infty$ for all $M_1 \in \mathcal{A}^{r\times s}$ and $M_2 \in \mathcal{A}^{s\times t}$. Note that $\mathcal{B} = \mathcal{A}^{p\times p}$ is a Banach algebra for the above norm. Therefore, (37) are sufficient conditions for robust stabilizability, i.e., C stabilizes all the plants of the form (33) where $\|\Theta\|_\infty < \|\Pi_C\|_\infty^{-1}$ and $\|\Xi\|_\infty < \|\Pi_P\|_\infty^{-1}$. We find again an important result in robust control, first obtained in [14, 15, 37] using doubly coprime factorizations for P and C. One can prove that $\|\Pi_C\|_\infty = \|\Pi_P\|_\infty$ [12]. Finally, considering the invertibility conditions for the model uncertainties studied in Example 6 and using (31), we find again the standard results summarized in Table 1. These conditions are also necessary if we consider the set of all possible Θ_{ij}'s satisfying the conditions of Table 1 due to the small gain theorem.

Table 1: Standard robust stability conditions.

Models of uncertainties	**Conditions for** $C \in \mathrm{Stab}(P_\delta)$
$P_\delta = (I_q + \Theta_{11})^{-1}\, P$	$\|\Theta_{11}\|_\infty \, < \, \left\|(I_q - P\, C)^{-1}\right\|_\infty^{-1}$
$P_\delta = P - \Theta_{12}$	$\|\Theta_{12}\|_\infty \, < \, \left\|C\,(I_q - P\, C)^{-1}\right\|_\infty^{-1}$
$P_\delta = (I_q - P\,\Theta_{21})^{-1}\, P$	$\|\Theta_{21}\|_\infty \, < \, \left\|(I_q - P\, C)^{-1}\, P\right\|_\infty^{-1}$
$P_\delta = P\,(I_r + \Theta_{22})$	$\|\Theta_{22}\|_\infty \, < \, \left\|(I_r - C\, P)^{-1}\, C\, P\right\|_\infty^{-1}$
$P_\delta = (I_q + \Xi_{11})\, P$	$\|\Xi_{11}\|_\infty \, < \, \left\|P\, C\,(I_q - P\, C)^{-1}\right\|_\infty^{-1}$
$P_\delta = P + \Xi_{12}$	$\|\Xi_{12}\|_\infty \, < \, \left\|(I_r - C\, P)^{-1}\, C\right\|_\infty^{-1}$
$P_\delta = P\,(I_r + \Xi_{21}\, P)^{-1}$	$\|\Xi_{21}\|_\infty \, < \, \left\|P\,(I_r - C\, P)^{-1}\right\|_\infty^{-1}$
$P_\delta = P\,(I_r + \Xi_{22})^{-1}$	$\|\Xi_{22}\|_\infty \, < \, \left\|(I_r - C\, P)^{-1}\right\|_\infty^{-1}$

Example 8. We continue Example 4. Suppose that $\mathcal{A}$ is a Banach algebra and

$$\| - \theta_{11}\, u - \theta_{12}\, v + \theta_{21}\,(u\, p) + \theta_{22}\,(v\, p)\|_{\mathcal{A}} \, < \, 1,$$

then c stabilizes p_δ for all θ_{11}, θ_{12}, θ_{21}, $\theta_{22} \in \mathcal{A}$ satisfying $1 + \theta_{11} - \theta_{21}\, p \neq 0$. Moreover, if $\mathcal{A} = H^\infty(\mathbb{C}_+)$, then c stabilizes p_δ for all θ_{11}, θ_{12}, θ_{21}, $\theta_{22} \in \mathcal{A}$ satisfying $1 + \theta_{11} - \theta_{21}\, p \neq 0$ and

$$\|\,(\theta_{11} \quad \theta_{12} \quad \theta_{21} \quad \theta_{22})\,\|_\infty \, < \, \left\|(-u \quad -v \quad u\, p \quad v\, p)^T\right\|_\infty^{-1}.$$

7.2 Study of the poles of P_δ

In this section, we shall study the poles of P_δ. First, note that the classical concept of poles makes sense for a plant P having a weakly doubly coprime factorization (extensions are possible).

Note also that the existence of a weakly doubly coprime factorization for a stabilizable plant is equivalent to the existence of a doubly coprime factorization (2) (see Example 1 and Corollary 8 of [28]). Hence, according to Theorem 2, if $\mathcal{L}_\delta = (I_q \quad -P_\delta)\, \mathcal{A}^{p \times 1}$ and $\mathcal{M}_\delta = \mathcal{A}^{1 \times p}\, (P_\delta^T \quad I_r^T)^T$ are the lattices associated with the plant P_δ defined by (33), then we must study the lattices $\mathcal{A} : \mathcal{L}_\delta$ and $\mathcal{A} : \mathcal{M}_\delta$, and connect them to respectively the lattices $\mathcal{A} : \mathcal{L}$ and $\mathcal{A} : \mathcal{M}$ associated with P.

The bi-split short exact sequence (47) shows that

$$\begin{aligned}\mathcal{L} &= (L + \Delta)\, \mathcal{A}^{p \times 1} = (I_q + \Delta_1 \quad -P - \Delta_2)\, \mathcal{A}^{p \times 1} = (I_q + \Delta_1)\,(I_q \quad -P_\delta)\, \mathcal{A}^{p \times 1} \\ &= (I_q + \Delta_1)\, \mathcal{L}_\delta,\end{aligned}$$

where $\det(I_q + \Delta_1) \neq 0$, which gives $\mathcal{L}_\delta \cong \mathcal{L}$. Using 3 of Theorem 2, P_δ has a left coprime factorization if and only if so does P. We also have

$$\begin{aligned}\mathcal{A} : \mathcal{L} &= \{\lambda \in \mathcal{K}^{1 \times q} \mid \lambda\, \mathcal{L} \subseteq \mathcal{A}\} = \{\lambda \in \mathcal{K}^{1 \times q} \mid \lambda\,(I_q + \Delta_1)\, \mathcal{L}_\delta \subseteq \mathcal{A}\}, \\ \mathcal{A} : \mathcal{L}_\delta &= \{\lambda \in \mathcal{K}^{1 \times q} \mid \lambda\, \mathcal{L}_\delta \subseteq \mathcal{A}\} = \{\lambda \in \mathcal{K}^{1 \times q} \mid \lambda\,(I_q + \Delta_1)^{-1} \mathcal{L} \subseteq \mathcal{A}\}\end{aligned}$$

which shows that $\mathcal{A} : \mathcal{L}_\delta = (\mathcal{A} : \mathcal{L})\,(I_q + \Delta_1)$. Similarly, we can prove that we have $\mathcal{M} = \mathcal{M}_\delta\,(I_r + \nabla_2)$, where $\det(I_r + \nabla_2) \neq 0$, which gives $\mathcal{M}_\delta \cong \mathcal{M}$ and $\mathcal{A} : \mathcal{M}_\delta = (I_r + \nabla_2)\,(\mathcal{A} : \mathcal{M})$. Using Theorem 2, P_δ has a right coprime factorization if and only if so has P. Hence, if the internally stabilizable plant P has a (weaky) doubly coprime factorization, then so does P_δ.

Let us give an explicit doubly coprime factorization for P_δ. Using 4 of Theorem 2, we have $\mathcal{A} : \mathcal{L} = \mathcal{A}^{1\times q}\,D$, $\mathcal{L} = D^{-1}\,\mathcal{A}^{q\times 1}$, $\mathcal{A} : \mathcal{M} = \widetilde{D}\,\mathcal{A}^{r\times 1}$, $\mathcal{M} = \mathcal{A}^{1\times r}\,\widetilde{D}^{-1}$, and thus

$$\mathcal{A} : \mathcal{L}_\delta = \mathcal{A}^{1\times q}\,(D\,(I_q + \Delta_1)), \quad \mathcal{A} : \mathcal{M}_\delta = \left((I_r + \nabla_2)\,\widetilde{D}\right)\,\mathcal{A}^{r\times 1},$$

which shows that $P_\delta = D_\delta^{-1}\,N_\delta = \widetilde{N}_\delta\,\widetilde{D}_\delta^{-1}$ is a (weaky) doubly coprime factorization of the stabilizable plant $P_\delta = (I_q + \Delta_1)^{-1}\,(P + \Delta_2) = (P + \nabla_1)\,(I_r + \nabla_2)^{-1}$, where

$$\begin{cases} D_\delta = D\,(I_q + \Delta_1) = D\,(I_q + \Theta_{11}) - N\,\Theta_{21} \in \mathcal{A}^{q\times q},\\ N_\delta = D_\delta\,P_\delta = D\,(P + \Delta_2) = N\,(I_r + \Theta_{22}) - D\,\Theta_{12} \in \mathcal{A}^{q\times r},\\ \widetilde{D}_\delta = (I_r + \nabla_2)\,\widetilde{D} = (I_r + \Xi_{22})\,\widetilde{D} + \Xi_{21}\,\widetilde{N} \in \mathcal{A}^{r\times r},\\ \widetilde{N}_\delta = P_\delta\,\widetilde{D}_\delta = (P + \nabla_1)\,\widetilde{D} = (I_q + \Xi_{11})\,\widetilde{N} + \Xi_{12}\,\widetilde{D} \in \mathcal{A}^{q\times r}, \end{cases} \tag{38}$$

and $\det(D_\delta) = \det(D)\,\det(I_q + \Delta_1) \neq 0$ and $\det(\widetilde{D}_\delta) = \det(I_r + \nabla_2)\,\det(\widetilde{D}) \neq 0$. In particular, we have

$$R_\delta := (D_\delta \quad -N_\delta) = (D \quad -N)\,(I_p + \Theta) = R\,(I_p + \Theta)$$
$$\widetilde{R}_\delta := \begin{pmatrix} \widetilde{N}_\delta \\ \widetilde{D}_\delta \end{pmatrix} = (I_p + \Xi)\begin{pmatrix} \widetilde{N} \\ \widetilde{D} \end{pmatrix} = (I_p + \Xi)\,\widetilde{R}.$$

According to the homological perturbation lemma (see the proof of Theorem 5), we have the identities $(L + \Delta)\,F_\delta = I_q$ and $G_\delta\,(M + \nabla) = I_r$, where the matrices F_δ and G_δ are defined by (46). Considering the first one, we have

$$\begin{aligned} &(I_q + \Delta_1 \quad -P - \Delta_2)\,F_\delta = I_q \;\Leftrightarrow\; (I_q + \Delta_1)\,(I_q \quad -P_\delta)\,F_\delta = I_q\\ \Leftrightarrow\; &(I_q + \Delta_1)\,D_\delta^{-1}\,(D_\delta \quad -N_\delta)\,F_\delta = I_q \;\Leftrightarrow\; (D_\delta \quad -N_\delta)\,F_\delta = \left((I_q + \Delta_1)\,D_\delta^{-1}\right)^{-1}\\ \Leftrightarrow\; &(D_\delta \quad -N_\delta)\,F_\delta\,(I_q + \Delta_1)\,D_\delta^{-1} = I_q. \end{aligned}$$

Therefore, using the identity $D_\delta = D\,(I_q + \Delta_1)$ (see (38)), $U = X\,D$, $V = Y\,D$, $F_\delta\,\mathcal{L} \subseteq \mathcal{A}^{p\times 1}$ (see (47)), where $\mathcal{L} = D^{-1}\,\mathcal{A}^{q\times 1}$, i.e., $F_\delta\,D^{-1} \in \mathcal{A}^{p\times q}$, and F_δ defined in (46), we obtain that

$$\begin{pmatrix} X_\delta \\ Y_\delta \end{pmatrix} := F_\delta\,(I_q + \Delta_1)\,D_\delta^{-1} = F_\delta\,D^{-1} = \begin{pmatrix} X \\ Y \end{pmatrix} D\,\left(I_q + \Delta\,F\right)^{-1}\,D^{-1} \in \mathcal{A}^{p\times q}, \tag{39}$$

satisfies the Bézout identity $D_\delta\,X_\delta - N_\delta\,Y_\delta = I_q$. Finally, we can check this identity again

$$\begin{aligned} (D_\delta \quad -N_\delta)\begin{pmatrix} X_\delta \\ Y_\delta \end{pmatrix} &= D\,((I_q + \Delta_1) \quad -(P + \Delta_2))\begin{pmatrix} U \\ V \end{pmatrix}(I_q + \Delta\,F)^{-1}\,D^{-1}\\ &= D\,(I_q - \Delta\,F)\,\left(I_q + \Delta\,F\right)^{-1}\,D^{-1} = I_q. \end{aligned}$$

Similarly, using $G_\delta\,(M + \nabla) = I_r$ and $\widetilde{D}_\delta = (I_r + \nabla_2)\,\widetilde{D}$, we have $\widetilde{D}_\delta^{-1}\,(I_r + \nabla_2)\,G_\delta\,(\widetilde{N}_\delta^T \quad \widetilde{D}_\delta^T)^T = I_q$, i.e., $\widetilde{D}^{-1}\,G_\delta\,(\widetilde{N}_\delta^T \quad \widetilde{D}_\delta^T)^T = I_q$. Using $G_\delta(\mathcal{A}^{p\times 1}) \subseteq \mathcal{A} : \mathcal{M} = \widetilde{D}\,\mathcal{A}^{r\times 1}$, i.e., $(\widetilde{D}^{-1}\,G_\delta)(\mathcal{A}^{p\times 1}) \subseteq \mathcal{A}^{r\times 1}$ (see (47)), i.e., $\widetilde{D}^{-1}\,G_\delta \in \mathcal{A}^{r\times p}$, and $\widetilde{U} = \widetilde{D}\,\widetilde{X}$, $\widetilde{V} = \widetilde{D}\,\widetilde{Y}$ (see Example 1), and (47), we obtain that

$$\left(-\widetilde{Y}_\delta \quad \widetilde{X}_\delta\right) := \widetilde{D}^{-1}\,G_\delta = \widetilde{D}^{-1}\,(I_p + \nabla\,G)^{-1}\,\widetilde{D}\,\left(-\widetilde{Y} \quad \widetilde{X}\right) \in \mathcal{A}^{r\times p}$$

satisfies the Bézout identity $-\widetilde{Y}_\delta\,\widetilde{N}_\delta + \widetilde{X}_\delta\,\widetilde{D}_\delta = I_r$.

Finally, using $\mathcal{A} : \mathcal{L}_\delta = \mathcal{A}^{1\times q}\,D_\delta$ and $\mathcal{A} : \mathcal{M}_\delta = \widetilde{D}_\delta\,\mathcal{A}^{r\times 1}$, the poles of P_δ are the zeros of $\det(D_\delta)$ or $\det(\widetilde{D}_\delta)$ (and the zeros of P_δ corresponds to the zeros of N_δ or $\widetilde{N}_\delta$). Using the model of uncertainty defined by (33), i.e., a combination of the 7 standard uncertainty models defined in Example 6 (the additive model appearing twice), the poles and the zeros of P can be shifted to those of P_δ.

Example 9. Let $D' \in \mathcal{A}^{q \times q}$ satisfy $\det(D') \neq 0$, $N' \in \mathcal{A}^{q \times r}$, and let us set

$$\Theta = \begin{pmatrix} X \\ Y \end{pmatrix} (D' \quad -N') - I_p \in \mathcal{A}^{p \times p}.$$

Using $D\,X - N\,Y = I_q$, (38) yields $N_\delta = D'$, $N_\delta = N'$, i.e., $P_\delta = {D'}^{-1} N'$. Moreover, using $U = X\,D$ and $V = Y\,D$ (see Example 1), the first condition of (33) is then

$$\begin{aligned} I_p + \Pi_C\,\Theta &= I_q + \begin{pmatrix} X \\ Y \end{pmatrix} D\,(I_q \quad -P) \left(\begin{pmatrix} X \\ Y \end{pmatrix} (D' \quad -N') - I_p \right) \\ &= I_q + \begin{pmatrix} X \\ Y \end{pmatrix} (D' - D \quad -(N' - N)). \end{aligned}$$

Using *Sylvester's determinant theorem*, namely, $\det(I_m + A\,B) = \det(I_n + B\,A)$ for all $A \in \mathcal{A}^{m \times n}$ and $B \in \mathcal{A}^{n \times m}$, we have $I_p + \Pi_C\,\Theta \in \mathrm{GL}_p(\mathcal{A})$ if and only if

$$I_q + (D' - D)\,X - (N' - N)\,Y = D'\,X - N'\,Y \in \mathrm{GL}_r(\mathcal{A}).$$

Hence, C stabilizes all the plants of the form $P_\delta = {D'}^{-1} N'$, where $D' \in \mathcal{A}^{q \times q}$, $\det(D') \neq 0$, and $N' \in \mathcal{A}^{q \times r}$, if and only if $D'\,X - N'\,Y \in \mathrm{GL}_r(\mathcal{A})$. This result is well-known [37]. Moreover, using $I_q + \Delta\,F = D^{-1}\,(D'\,X - N'\,Y)\,D \in \mathrm{GL}_q(\mathcal{A})$ and (39), the plant $P_\delta = {D'}^{-1}\,N$ has the following doubly coprime factorization

$$\begin{pmatrix} X_\delta \\ Y_\delta \end{pmatrix} = \begin{pmatrix} X \\ Y \end{pmatrix} (D'\,X - N'\,Y)^{-1} \in \mathcal{A}^{p \times q}.$$

7.3 Coprime factor uncertainty

Following Example 9, let us now show how Theorem 5 reduces to the well-known robust stability test for coprime factor uncertainty when P has a doubly coprime factorization $P = D^{-1}\,N = \widetilde{N}\,\widetilde{D}^{-1}$.

Using Theorem 2, we have $\mathcal{L} = D^{-1}\,\mathcal{A}^{q \times 1}$, $\mathcal{M} = \mathcal{A}^{1 \times r}\,\widetilde{D}^{-1}$, and thus, $\mathcal{A} : \mathcal{M} = \widetilde{D}\,\mathcal{A}^{r \times 1}$. With the notations of the proof of Theorem 5 and (44), we get

$$\begin{cases} \delta_1 \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{A}^{p \times 1}, \mathcal{L}) \cong \mathcal{L}^{1 \times p} = D^{-1}\,\mathcal{A}^{q \times p}, \\ \delta_2 \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{A} : \mathcal{M}, \mathcal{A}^{p \times 1}) \cong \mathcal{M}^{p \times 1} = \mathcal{A}^{p \times r}\,\widetilde{D}^{-1}, \end{cases}$$

i.e., there exist $\Delta_D \in \mathcal{A}^{q \times q}$, $\Delta_N \in \mathcal{A}^{q \times r}$, $\nabla_{\widetilde{N}} \in \mathcal{A}^{q \times r}$, and $\nabla_{\widetilde{D}} \in \mathcal{A}^{r \times r}$ such that

$$\Delta = (\Delta_1 \quad -\Delta_2) = D^{-1}\,(\Delta_D \quad -\Delta_N), \quad \nabla = \begin{pmatrix} \nabla_1 \\ \nabla_2 \end{pmatrix} = \begin{pmatrix} \nabla_{\widetilde{N}} \\ \nabla_{\widetilde{D}} \end{pmatrix} \widetilde{D}^{-1}.$$

Therefore, we have

$$\begin{aligned} P_\delta &= (I_q + \Delta_1)^{-1}\,(P + \Delta_2) = (I_q + D^{-1}\,\Delta_N)^{-1}\,(D^{-1}\,N + D^{-1}\,\Delta_N) \\ &= (D + \Delta_N)^{-1}\,(N + \Delta_N) \\ &= (P + \nabla_1)\,(I_r + \nabla_2)^{-1} = (\widetilde{N}\,\widetilde{D}^{-1} + \nabla_{\widetilde{N}}\,\widetilde{D}^{-1}) \left(I_r + \nabla_{\widetilde{D}}\,\widetilde{D}^{-1}\right)^{-1} \\ &= (\widetilde{N} + \nabla_{\widetilde{N}})\,(\widetilde{D} + \nabla_{\widetilde{D}})^{-1}. \end{aligned}$$

Moreover, using Example 1, we have $U = X\,D$, $V = Y\,D$, $\widetilde{U} = \widetilde{D}\,\widetilde{X}$, $\widetilde{V} = \widetilde{D}\,\widetilde{Y}$, and $C = V\,U^{-1} = \widetilde{U}^{-1}\,\widetilde{V}$. Then, (35) yields

$$\begin{cases} I_q + \Delta\,F = I_q + \Delta_1\,U - \Delta_2\,V = I_q + D^{-1}\,\Delta_D\,U - D^{-1}\,\Delta_N\,V \\ \qquad\qquad = D^{-1}\,(I_q + \Delta_D\,X - \Delta_N\,Y)\,D \in \mathrm{GL}_q(\mathcal{A}), \\ I_r + G\,\nabla = I_r - \widetilde{V}\,\nabla_1 + \widetilde{U}\,\nabla_2 = I_r + \widetilde{D}\,(-\widetilde{Y}\,\nabla_{\widetilde{N}} + \widetilde{X}\,\nabla_{\widetilde{D}})\,\widetilde{D}^{-1} \\ \qquad\qquad = \widetilde{D}\,(I_r - \widetilde{Y}\,\nabla_{\widetilde{N}} + \widetilde{X}\,\nabla_{\widetilde{D}})\,\widetilde{D}^{-1} \in \mathrm{GL}_r(\mathcal{A}), \end{cases}$$

which is finally equivalent to

$$I_q + \Delta_D\, X - \Delta_N\, Y \in \mathrm{GL}_q(\mathcal{A}), \quad I_r - \widetilde{Y}\, \nabla_{\widetilde{N}} + \widetilde{X}\, \nabla_{\widetilde{D}} \in \mathrm{GL}_r(\mathcal{A}). \tag{40}$$

We find again the robust stability test for coprime factor uncertainty [37]. See also Example 9 with the notations $\Delta_D = D' - D$ and $\Delta_N = N' - N$. Thus, the robust test for coprime factor uncertainty is a direct consequence of Theorem 5.

Finally, if $\mathcal{A}$ is a Banach algebra, then sufficient conditions for (40) are given by

$$\|\Delta_D\, X - \Delta_N\, Y\|_{\mathcal{A}^{q\times q}} < 1, \quad \| - \widetilde{Y}\, \nabla_{\widetilde{N}} + \widetilde{X}\, \nabla_{\widetilde{D}}\|_{\mathcal{A}^{r\times r}} < 1.$$

In particular, if $\mathcal{A} = H^\infty(\mathbb{C}_+)$, then we find again the following standard sufficient conditions for robust stability for coprime factor uncertainty [37]

$$\|\,(\Delta_D \quad -\Delta_N)\,\|_\infty < \left\|\begin{pmatrix} X^T & Y^T\end{pmatrix}^T\right\|_\infty^{-1}, \quad \left\|\begin{pmatrix}\nabla_{\widetilde{N}}^T & \Delta_{\widetilde{D}}^T\end{pmatrix}^T\right\|_\infty < \left\|\begin{pmatrix}-\widetilde{Y} & \widetilde{X}\end{pmatrix}\right\|_\infty^{-1}.$$

Appendix

We first give a proof of Lemma 2.

Proof. Let us suppose that 1 holds. We then have

$$\begin{array}{rl} & (\mathrm{id}_{\mathcal{P}} + h\circ\delta)\circ(\mathrm{id}_{\mathcal{P}} - h\circ(\mathrm{id}_{\mathcal{P}} + \delta\circ h)^{-1}\circ\delta) \\ = & \mathrm{id}_{\mathcal{P}} + h\circ\delta - h\circ(\mathrm{id}_{\mathcal{P}} + \delta\circ h)^{-1}\circ\delta - h\circ\delta\circ h\circ(\mathrm{id}_{\mathcal{P}} + \delta\circ h)^{-1}\circ\delta \\ = & \mathrm{id}_{\mathcal{P}} + h\circ(\mathrm{id}_{\mathcal{P}} - (\mathrm{id}_{\mathcal{P}} + \delta\circ h)^{-1} - \delta\circ h\circ(\mathrm{id}_{\mathcal{P}} + \delta\circ h)^{-1})\circ\delta \\ = & \mathrm{id}_{\mathcal{P}} + h\circ\underbrace{(\mathrm{id}_{\mathcal{P}} - (\mathrm{id}_{\mathcal{P}} + \delta\circ h)\circ(\mathrm{id}_{\mathcal{P}} + \delta\circ h)^{-1})}_{=0}\circ\delta = \mathrm{id}_{\mathcal{P}}, \end{array}$$

$$\begin{array}{rl} & (\mathrm{id}_{\mathcal{P}} - h\circ(\mathrm{id}_{\mathcal{P}} + \delta\circ h)^{-1}\circ\delta)\circ(\mathrm{id}_{\mathcal{P}} + h\circ\delta) \\ = & \mathrm{id}_{\mathcal{P}} + h\circ\delta - h\circ(\mathrm{id}_{\mathcal{P}} + \delta\circ h)^{-1}\circ\delta - h\circ(\mathrm{id}_{\mathcal{P}} - \delta\circ h)^{-1}\circ\delta\circ h\circ\delta \\ = & \mathrm{id}_{\mathcal{P}} + h\circ(\mathrm{id}_{\mathcal{P}} - (\mathrm{id} + \delta\circ h)^{-1} - (\mathrm{id}_{\mathcal{P}} + \delta\circ h)^{-1}\circ\delta\circ h)\circ\delta \\ = & \mathrm{id}_{\mathcal{P}} + h\circ\underbrace{(\mathrm{id}_{\mathcal{P}} - (\mathrm{id}_{\mathcal{P}} + \delta\circ h)^{-1}\circ(\mathrm{id}_{\mathcal{P}} + \delta\circ h))}_{=0}\circ\delta = \mathrm{id}_{\mathcal{P}}. \end{array}$$

Hence, $\mathrm{id}_{\mathcal{P}} + \delta\circ h$ is invertible with inverse $\mathrm{id}_{\mathcal{P}} - h\circ(\mathrm{id}_{\mathcal{P}} + \delta\circ h)^{-1}\circ\delta$.

Point 2 can be proved similarly by permuting δ and h.

Finally, $h\circ(\mathrm{id}_{\mathcal{M}} + \delta\circ h) = h + h\circ\delta\circ h = (\mathrm{id}_{\mathcal{P}} + h\circ\delta)\circ h$, which proves the last identity of Lemma 2. □

We now give a proof of Theorem 4.

Proof. First note that the equalities given in (29) is a consequence of Lemma 2.

Let us prove the following identities

$$(d_2 + \delta_2)\circ H_1 + H_0\circ(d_1 + \delta_1) = \mathrm{id}_{\mathcal{M}_1}, \tag{41}$$

$$H_1\circ(d_2 + \delta_2) = \mathrm{id}_{\mathcal{M}_2}, \tag{42}$$

$$(d_1 + \delta_1)\circ H_0 = \mathrm{id}_{\mathcal{M}_0}. \tag{43}$$

We start with (41). Let $E = \mathrm{id}_{\mathcal{M}_1} - (d_2 + \delta_2)\circ H_1 - H_0\circ(d_1 + \delta_1)$. Using (29), we have

$$\begin{aligned} E &= \mathrm{id}_{\mathcal{M}_1} - (d_2 + \delta_2)\circ h_1\circ(\mathrm{id}_{\mathcal{M}_1} + \delta_2\circ h_1)^{-1} - (\mathrm{id}_{\mathcal{M}_1} + h_0\circ\delta_1)^{-1}\circ h_0\circ(d_1 + \delta_1) \\ &= \mathrm{id}_{\mathcal{M}_1} - (d_2\circ h_1 + \delta_2\circ h_1)\circ(\mathrm{id}_{\mathcal{M}_1} + \delta_2\circ h_1)^{-1} \\ &\quad - (\mathrm{id}_{\mathcal{M}_1} + h_0\circ\delta_1)^{-1}\circ(h_0\circ d_1 + h_0\circ\delta_1) \\ &= (\mathrm{id}_{\mathcal{M}_1} + h_0\circ\delta_1)^{-1} \\ &\quad \circ[(\mathrm{id}_{\mathcal{M}_1} + h_0\circ\delta_1)\circ(\mathrm{id}_{\mathcal{M}_1} + \delta_2\circ h_1) - (\mathrm{id}_{\mathcal{M}_1} + h_0\circ\delta_1)\circ(d_2\circ h_1 + \delta_2\circ h_1) \\ &\quad - (h_0\circ d_1 + h_0\circ\delta_1)\circ(\mathrm{id}_{\mathcal{M}_1} + \delta_2\circ h_1)]\circ(\mathrm{id}_{\mathcal{M}_1} + \delta_2\circ h_1)^{-1} \end{aligned}$$

Let F be the term in the bracket of the previous equation. Using the identity (14) and (28), we obtain

$$\begin{aligned} F = &\,\mathrm{id}_{\mathcal{M}_1} + h_0 \circ \delta_1 + \delta_2 \circ h_1 + h_0 \circ \delta_1 \circ \delta_2 \circ h_1 - d_2 \circ h_1 - \delta_2 \circ h_1 - h_0 \circ \delta_1 \circ d_2 \circ h_1 \\ &- h_0 \circ \delta_1 \circ \delta_2 \circ h_1 - h_0 \circ d_1 - h_0 \circ \delta_1 - h_0 \circ d_1 \circ \delta_2 \circ h_1 - h_0 \circ \delta_1 \circ \delta_2 \circ h_1 \\ = &\,\underbrace{\mathrm{id}_{\mathcal{M}_1} - d_2 \circ h_1 - h_0 \circ d_1}_{=0 \text{ by (14)}} - h_0 \circ (\underbrace{\delta_1 \circ d_2 + d_1 \circ \delta_2 + \delta_1 \circ \delta_2}_{=0 \text{ by (28}})) \circ h_1 = 0. \end{aligned}$$

Thus, we obtain $E = 0$, which proves (41).

For (42), using $h_1 \circ d_2 = \mathrm{id}_{\mathcal{M}_2}$ (see Proposition 1), we have

$$\begin{aligned} \mathrm{id}_{\mathcal{M}_2} - H_1 \circ (d_2 + \delta_2) &= \mathrm{id}_{\mathcal{M}_2} - (\mathrm{id}_{\mathcal{M}_2} + h_1 \circ \delta_2)^{-1} \circ h_1 \circ (d_2 + \delta_2) \\ &= (\mathrm{id}_{\mathcal{M}_2} + h_1 \circ \delta_2)^{-1} \, \underbrace{(\mathrm{id}_{\mathcal{M}_2} + h_1 \circ \delta_2 - h_1 \circ d_2 - h_1 \circ \delta_2)}_{=0} = 0. \end{aligned}$$

Similarly for (43), using $d_1 \circ h_0 = \mathrm{id}_{\mathcal{M}_0}$ (see Proposition 1), we have

$$\begin{aligned} \mathrm{id}_{\mathcal{M}_0} - (d_1 + \delta_1) \circ H_0 &= \mathrm{id}_{\mathcal{M}_0} - (d_1 + \delta_1) \circ h_0 \circ (\mathrm{id}_{\mathcal{M}_0} + \delta_1 \circ h_0)^{-1} \\ &= \underbrace{(\mathrm{id}_{\mathcal{M}_0} + \delta_1 \circ h_0 - d_1 \circ h_0 - \delta_1 \circ h_0)}_{=0} (\mathrm{id}_{\mathcal{M}_0} + \delta_1 \circ h_0)^{-1} = 0. \end{aligned}$$

Using (28), i.e., $\mathrm{im}\,(d_2 + \delta_2) \subseteq \ker\,(d_1 + \delta_1)$, we have the following complex

$$0 \longrightarrow \mathcal{M}_2 \xrightarrow{d_2+\delta_2} \mathcal{M}_1 \xrightarrow{d_1+\delta_1} \mathcal{M}_0 \longrightarrow 0.$$

Let us show that it is exact. The identity (42) (resp., (43)) implies that $d_2 + \delta_2$ (resp., $d_1 + \delta_1$) is injective (resp., surjective). Now, let $x \in \ker\,(d_1 + \delta_1)$. Using (41), we get

$$x = (d_2 + \delta_2)(H_1(x)) + H_0((d_1 + \delta_1)(x)) = (d_2 + \delta_2)(H_1(x)) \in \mathrm{im}\,(d_2 + \delta_2),$$

which shows that $\ker\,(d_1 + \delta_1) \subseteq \mathrm{im}\,(d_2 + \delta_2)$, and thus, $\mathrm{im}\,(d_2 + \delta_2) = \ker\,(d_1 + \delta_1)$. Therefore, the above complex is exact and (30) a split short exact sequence.

Finally, if $h_1 \circ h_0 = 0$, using (29), we then have

$$H_1 \circ H_0 = (\mathrm{id}_{\mathcal{M}_2} + h_1 \circ \delta_2)^{-1} \circ \underbrace{(h_1 \circ h_0)}_{=0} \circ (\mathrm{id}_{\mathcal{M}_0} + \delta_1 \circ h_0)^{-1} = 0,$$

which proves that (30) is a bi-split short exact sequence by 4 of Corollary 1. □

Below, we give a detailed proof of Theorem 5 (although it is a straightforward application of Theorem 4, and thus, could be highly shortened for algebraists).

Proof. Let $L = (I_q \quad -P) \in \mathcal{K}^{q \times p}$, $M = (P^T \quad I_r^T)^T \in \mathcal{K}^{p \times r}$, $\mathcal{L} = L\,\mathcal{A}^{p \times 1}$, $\mathcal{M} = \mathcal{A}^{1 \times p}\,M$, and $\mathcal{A} : \mathcal{M} = \left\{\mu \in \mathcal{A}^{r \times 1} \mid P\,\mu \in \mathcal{A}^{q \times 1}\right\}$ (see Example 3).

Using the fact that $P \in \mathcal{K}^{q \times r}$ is internally stabilizable, Remark 11 shows that the short exact sequence (22) bi-splits, where F and G are defined in (3).

Set $d_1 = L.$ and $d_2 = M.$, i.e., $d_1 \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{A}^{p \times 1}, \mathcal{L})$ is defined by $d_1(\eta) = L\,\eta$ for all $\eta \in \mathcal{A}^{p \times 1}$, and $d_2 \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{A} : \mathcal{M}, \mathcal{A}^{p \times 1})$ by $d_2(\mu) = M\,\mu$ for all $\mu \in \mathcal{A} : \mathcal{M}$.

Set $h_0 = F.$ and $h_1 = G.$, i.e., $h_0 \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{L}, \mathcal{A}^{p \times 1})$ is defined by $h_0(\lambda) = F\,\lambda$ for all $\lambda \in \mathcal{L}$, and $h_1 \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{A}^{p \times 1}, \mathcal{A} : \mathcal{M})$ by $h_1(\eta) = G\,\eta$ for all $\eta \in \mathcal{A}^{p \times 1}$.

Let us now consider a perturbation of (12), i.e., $\delta_1 \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{A}^{p \times 1}, \mathcal{L})$ and $\delta_2 \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{A} : \mathcal{M}, \mathcal{A}^{p \times 1})$ defining the following complex $0 \longrightarrow \mathcal{A} : \mathcal{M} \xrightarrow{d_2+\delta_2} \mathcal{A}^{p \times 1} \xrightarrow{d_1+\delta_1} \mathcal{L} \longrightarrow 0$.

Let us prove that the following isomorphisms

$$\left\{ \begin{array}{l} \mathrm{Hom}_{\mathcal{A}}(\mathcal{A}^{p \times 1}, \mathcal{L}) \cong \mathcal{L}^{1 \times p} = L\,\mathcal{A}^{p \times p}, \\ \mathrm{Hom}_{\mathcal{A}}(\mathcal{A} : \mathcal{M}, \mathcal{A}^{p \times 1}) \cong \mathrm{Hom}_{\mathcal{A}}(\mathcal{A} : \mathcal{M}, \mathcal{A})^{p \times 1} \cong (\mathcal{A} : (\mathcal{A} : \mathcal{M}))^{p \times 1} \cong \mathcal{M}^{p \times 1}. \end{array} \right. \quad (44)$$

The first isomorphism maps $f \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{A}^{p\times 1}, \mathcal{L})$ to $(f(e_1), \ldots, f(e_p)) \in \mathcal{L}^{1\times p}$, where $\{e_i\}_{i=1,\ldots,p}$ is the standard basis of $\mathcal{A}^{p\times 1}$. The second isomorphism maps $g \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{A} : \mathcal{M}, \mathcal{A}^{p\times 1})$ to $(\pi_1 \circ g, \ldots, \pi_p \circ g)^T \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{A} : \mathcal{M}, \mathcal{A})^{p\times 1}$, where $\pi_i : \mathcal{A}^{p\times 1} \longrightarrow \mathcal{A}$ is the canonical projection onto the i^{th} component for $i = 1, \ldots, p$. The third isomorphism is a consequence of the second point of Remark 7. Finally, we prove the last isomorphism by showing that $\mathcal{M} = \mathcal{A} : (\mathcal{A} : \mathcal{M})$. We first prove $\mathcal{A} : \mathcal{M} = G\,\mathcal{A}^{p\times 1}$. If $\mu \in \mathcal{A} : \mathcal{M} = \left\{\mu \in \mathcal{A}^{r\times 1} \mid P\,\mu \in \mathcal{A}^{q\times 1}\right\}$, then set $\vartheta = M\,\mu \in \mathcal{A}^{p\times 1}$ and using $G\,M = I_r$ (see (21)), we have $\mu = G\,\vartheta$, i.e., $\mathcal{A} : \mathcal{M} \subseteq G\,\mathcal{A}^{p\times 1} \subseteq \mathcal{A} : \mathcal{M}$, which proves $\mathcal{A} : \mathcal{M} = G\,\mathcal{A}^{p\times 1}$. Let us deduce from the equality $\mathcal{A} : \mathcal{M} = G\,\mathcal{A}^{p\times 1}$ that $\mathcal{M} = \mathcal{A} : (\mathcal{A} : \mathcal{M})$. We have $\mathcal{A} : (\mathcal{A} : \mathcal{M}) = \mathcal{A} : G\,\mathcal{A}^{p\times 1} = \left\{\lambda \in \mathcal{K}^{1\times r} \mid \lambda\,G \in \mathcal{A}^{1\times p}\right\}$. First, let us consider $m = m_1\,P + m_2 \in \mathcal{M}$, where $m_1 \in \mathcal{A}^{1\times q}$ and $m_2 \in \mathcal{A}^{1\times r}$. Then, $m \in \mathcal{K}^{1\times r}$ and $m\,G = m_1\,(P\,G) + m_2\,G \in \mathcal{A}^{1\times p}$ because $G \in \mathcal{A}^{r\times p}$ and $P\,G \in \mathcal{A}^{q\times p}$ (see (21)), which shows that $\mathcal{M} \subseteq \mathcal{A} : (\mathcal{A} : \mathcal{M})$. Conversely, if $\lambda \in \mathcal{A} : (\mathcal{A} : \mathcal{M})$, then $\lambda = (\lambda\,G)\,M \in \mathcal{M}$ because $G\,M = I_r$ and $\lambda\,G \in \mathcal{A}^{1\times p}$, which shows that $\mathcal{A} : (\mathcal{A} : \mathcal{M}) \subseteq \mathcal{M}$ and proves $\mathcal{A} : (\mathcal{A} : \mathcal{M}) = \mathcal{M} = \mathcal{A}^{1\times p}\,G$. Therefore, we obtain $\mathrm{Hom}_{\mathcal{A}}(\mathcal{A} : \mathcal{M}, \mathcal{A}^{p\times 1}) \cong \mathcal{M}^{p\times 1} = \mathcal{A}^{p\times p}\,M$.

Using $d_1 + \delta_1 \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{A}^{p\times 1}, \mathcal{L}) \cong L\,\mathcal{A}^{p\times p}$ and $d_2 + \delta_2 \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{A} : \mathcal{M}, \mathcal{A}^{p\times 1}) \cong \mathcal{A}^{p\times p}\,M$, there exist two matrices $\Theta, \Xi \in \mathcal{A}^{p\times p}$ such that

$$\forall\,\eta \in \mathcal{A}^{p\times 1}, \quad \delta_1(\eta) = L\,\Theta\,\eta, \quad \forall\,\mu \in \mathcal{A}^{r\times 1}, \quad \delta_2(\mu) = \Xi\,M\,\mu.$$

Let us set $\Delta = L\,\Theta \in \mathcal{K}^{q\times p}$ and $\nabla = \Xi\,M \in \mathcal{K}^{r\times p}$ so that $\delta_1(\eta) = \Delta\,\eta$ for all $\eta \in \mathcal{A}^{p\times 1}$ and $\delta_2(\mu) = \nabla\,\mu$ for all $\mu \in \mathcal{A}^{r\times 1}$. The above complex can be rewritten as follows

$$0 \longrightarrow \mathcal{A} : \mathcal{M} \xrightarrow{(M+\nabla).} \mathcal{A}^{p\times 1} \xrightarrow{(L+\Delta).} \mathcal{L} \longrightarrow 0, \tag{45}$$

i.e., we have $(L + \Delta)\,(M + \nabla)\,\mu = 0$ for all $\mu \in \mathcal{A} : \mathcal{M} = \{\mu \in \mathcal{A}^{r\times 1} \mid P\,\mu \in \mathcal{A}^{q\times 1}\}$. Using Remark 1, in particular, we have $(L + \Delta)\,(M + \nabla)\,d = 0$ for a certain $0 \neq d \in \mathcal{A}$. Thus, using the fact that $\mathcal{A}$ is an integral domain, we get $(L + \Delta)\,(M + \nabla) = 0$, i.e.,

$$\begin{aligned} &(L + \Delta)\,(M + \nabla) = (I_q + \Delta_1 \quad - P - \Delta_2) \begin{pmatrix} P + \nabla_1 \\ I_r + \nabla_2 \end{pmatrix} = 0 \\ \Leftrightarrow \quad &(I_q + \Delta_1)\,(P + \nabla_1) = (P + \Delta_2)\,(I_r + \nabla_2). \end{aligned}$$

Using this last equation with $\det(I_q + \Delta_1) \neq 0$ and $\det(I_r + \nabla_2) \neq 0$ (see (34)), we can define the plant $P_\delta = (I_q + \Delta_1)^{-1}\,(P + \Delta_2) = (P + \nabla_1)\,(I_r + \nabla_2)^{-1}$ and get (33).

Now, using the identities $\Delta = L\,\Theta$, $F\,L = \Pi_C$, $\nabla = \Xi\,M$, and $M\,G = \Pi_P$ (see (18) and (32)), we get

$$\begin{aligned} \forall\,\lambda \in \mathcal{L},\ &(\mathrm{id}_{\mathcal{M}_0} + \delta_1 \circ h_0)(\lambda) = (I_q + \Delta\,F)\,\lambda, \\ \forall\,\eta \in \mathcal{A}^{p\times 1},\ &(\mathrm{id}_{\mathcal{M}_1} + h_0 \circ \delta_1)(\eta) = (I_p + F\,\Delta)\,\eta = (I_p + \Pi_C\,\Theta)\,\eta, \\ \forall\,\eta \in \mathcal{A}^{p\times 1},\ &(\mathrm{id}_{\mathcal{M}_1} + \delta_2 \circ h_1)(\eta) = (I_p + \nabla\,G)\,\eta = (I_p + \Xi\,\Pi_P)\,\eta, \\ \forall\,\mu \in \mathcal{A} : \mathcal{M},\ &(\mathrm{id}_{\mathcal{M}_2} + h_1 \circ \delta_2)(\mu) = (I_r + G\,\nabla)\,\mu. \end{aligned}$$

From the second identity, $\mathrm{id}_{\mathcal{M}_1} + h_0 \circ \delta_1$ is invertible, and so is $\mathrm{id}_{\mathcal{M}_0} + \delta_1 \circ h_0$ by Lemma 2, if and only if the first condition of (34) holds. Finally, using Sylvester's determinant theorem (see Example 9), $\det(I_q + \Delta\,F) = \det(I_p + F\,\Delta)$, which shows that $I_p + F\,\Delta \in \mathrm{GL}_p(\mathcal{A})$ is equivalent to $I_q + \Delta\,F \in \mathrm{GL}_q(\mathcal{A})$.

Similarly, from the above third identity, $\mathrm{id}_{\mathcal{M}_1} + \delta_2 \circ h_1$ is invertible, and so is $\mathrm{id}_{\mathcal{M}_2} + h_1 \circ \delta_2$ by Lemma 2, if and only if the second condition of (34) holds. Finally, Sylvester's determinant theorem yields $\det(I_r + G\,\nabla) = \det(I_p + \nabla\,G)$, which shows that $I_p + \Xi\,\Pi_P \in \mathrm{GL}_p(\mathcal{A})$ if and only if $I_r + G\,\nabla \in \mathrm{GL}_r(\mathcal{A})$.

Hence, if (34) is satisfied, then Theorem 4 shows that (30) is a bi-split short exact sequence, where the contractions H_0 and H_1 defined by (29, are given by

$$\begin{aligned} H_0(\lambda) &= (h_0 \circ (\mathrm{id}_{\mathcal{M}_0} + \delta_1 \circ h_0)^{-1})(\lambda) = F\,(I_q + \Delta\,F)^{-1}\,\lambda \\ &= ((\mathrm{id}_{\mathcal{M}_1} + h_0 \circ \delta_1)^{-1} \circ h_0)(\lambda) = (I_p + F\,\Delta)^{-1}\,F\,\lambda = (I_p + \Pi_C\,\Theta)^{-1}\,F\,\lambda, \\ H_1(\eta) &= (h_1 \circ (\mathrm{id}_{\mathcal{M}_1} + \delta_2 \circ h_1)^{-1})(\eta) = G\,(I_p + \nabla\,G)^{-1}\,\eta = G\,(I_p + \Xi\,\Pi_P)^{-1}\,\eta \\ &= ((\mathrm{id}_{\mathcal{M}_2} + h_1 \circ \delta_2)^{-1} \circ h_1)(\eta) = (I_r + G\,\nabla)^{-1}\,G\,\eta, \end{aligned}$$

for all $\lambda \in \mathcal{L}$ and $\eta \in \mathcal{A}^{p\times 1}$. Let us note

$$\begin{cases} F_\delta := F\,(I_q + \Delta\,F)^{-1} = \begin{pmatrix} U \\ V \end{pmatrix} (I_q + \Delta\,F)^{-1}, \\ G_\delta := (I_p + \nabla\,G)^{-1}\,G = (I_p + \nabla\,G)^{-1} \begin{pmatrix} -\widetilde{V} & \widetilde{U} \end{pmatrix}. \end{cases} \tag{46}$$

Then, (29) yields the following bi-split short exact sequence of $\mathcal{A}$-modules

$$0 \longrightarrow \mathcal{A} : \mathcal{M} \underset{G_\delta.}{\overset{(M+\nabla).}{\rightleftarrows}} \mathcal{A}^{p\times 1} \underset{F_\delta.}{\overset{(L+\Delta).}{\rightleftarrows}} \mathcal{L} \longrightarrow 0. \tag{47}$$

Now, $H_0 \in \hom_{\mathcal{A}}(\mathcal{L}, \mathcal{A}^{p\times 1})$ shows that $H_0(\lambda) = F_\delta\,\lambda_1 - F_\delta\,P\,\lambda_2 \in \mathcal{A}^{p\times 1}$ for all $\lambda = \lambda_1 - P\,\lambda_2 \in \mathcal{L}$, where $\lambda_1 \in \mathcal{A}^{q\times 1}$ and $\lambda_2 \in \mathcal{A}^{r\times 1}$, which yields $F_\delta \in \mathcal{A}^{p\times q}$ and $F_\delta\,P \in \mathcal{A}^{p\times r}$. Thus, $F_\delta\,(I_q \quad -P) \in \mathcal{A}^{p\times p}$, i.e., $F_\delta\,L \in \mathcal{A}^{p\times p}$, which yields, $F_\delta\,L\,(I_p + \Theta) \in \mathcal{A}^{p\times p}$ for all $\Theta \in \mathcal{A}^{p\times p}$, i.e., $F_\delta\,(L + \Delta) \in \mathcal{A}^{p\times p}$.

Furthermore, $(d_1 + \delta_1) \circ H_0 = \mathrm{id}_{\mathcal{L}}$ yields $(L + \Delta)\,F_\delta\,\lambda = \lambda$ for all $\lambda \in \mathcal{L}$ and considering $\lambda \in \mathcal{A}^{p\times 1} \subset \mathcal{L}$, we obtain the identity $(L + \Delta)\,F_\delta = I_q$.

Hence, we have

$$\begin{cases} (I_q + \Delta_1 \quad -P - \Delta_2)\,F_\delta = I_q, \\ F_\delta\,(I_q + \Delta_1 \quad -P - \Delta_2) \in \mathcal{A}^{p\times p}, \end{cases} \Leftrightarrow \begin{cases} (I_q + \Delta_1)\,(I_q \quad -P_\delta)\,F_\delta = I_q, \\ F_\delta\,(I_q + \Delta_1)\,(I_q \quad -P_\delta) \in \mathcal{A}^{p\times p}, \end{cases}$$

$$\begin{cases} (I_q \quad -P_\delta)\,F_\delta = (I_q + \Delta_1)^{-1}, \\ F_\delta\,(I_q + \Delta_1)\,(I_q \quad -P_\delta) \in \mathcal{A}^{p\times p}, \end{cases} \Leftrightarrow \begin{cases} (I_q \quad -P_\delta)\,(F_\delta\,(I_q + \Delta_1)) = I_q, \\ (F_\delta\,(I_q + \Delta_1))\,(I_q \quad -P_\delta) \in \mathcal{A}^{p\times p}. \end{cases}$$

Using Theorem 3, P_δ is then internally stabilizable by the controller defined by

$$C_\delta = (V\,(I_q + \Delta\,F)^{-1}\,(I_q + \Delta_1))\,(U\,(I_q + \Delta\,F)^{-1}\,(I_q + \Delta_1))^{-1} = U\,V^{-1} = C,$$

which finally proves the result. □

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**RESEARCH CENTRE
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2 rue Simone Iff - CS 42112
75589 Paris Cedex 12

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