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This paper is dedicated to Orion, our bright new constellation. With all our love!

§ The revised version of the research report improves the form and the readability of the previous version by adding more mathematical details and more precisely describing the computational aspects, as well as by illustrating the RankFactorization package – dedicated to the rank factorization problem – in the Appendix.

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Abstract: Vibration analysis aims to identify a rotating machinery's potential failures by monitoring its vibration levels, i.e., by measuring the vibrations and comparing them to known failure vibration signals. New demodulation methods have recently been introduced in acoustic and signal processing to diagnose gearboxes. This new approach put forward the mathematical problem of decomposing a given complex matrix M as $\sum_{i=1}^{r} D_i \, u \, v_i$, where $D_1, \ldots, D_r$ are fixed matrices and u (resp., $v_1, \ldots, v_r$) a row vector (resp., column vectors) to be determined. This problem is equivalent to factoring M as $M = (D_1 \, u \; \ldots \; D_r \, u) \left(v_1^T \; \ldots \; v_r^T\right)^T$, where the integer r is larger than or equal to the rank of M. Using methods of algebraic geometry, module theory, homological algebra, and computer algebra, we study this particular rank factorization problem. We characterize the general solutions of the corresponding polynomial system. The results we develop are effective in the sense of computer algebra, i.e., algorithms are obtained that can be implemented in a computer algebra system that effectively handles polynomial systems (Gröbner bases). The symbolic package RankFactorization has thus been developed to effectively study this particular rank factorization problem and the corresponding demodulation problems.

Key-words: Rank factorization problem, polynomial systems, computer algebra, module theory, homological algebra, demodulation, gearbox fault detection/surveillance, vibration analysis

Sur les solutions générales d'un problème de factorisation de rang

Résumé : L'analyse vibratoire a pour but l'identification de potentiels défauts d'une machine tournante par la surveillance de ses niveaux de vibration, c'est-à-dire par la mesure de ses vibrations et la comparaison avec des signaux de défauts connus. Pour le diagnostic d'engrenages, de nouvelles méthodes de démodulation ont récemment été introduites en acoustique et en traitement du signal. Cette nouvelle approche a mis en avant le problème mathématique consistant à écrire une matrice M sous la forme de $\sum_{i=1}^{r} D_i\, u\, v_i$, où $D_1, \ldots, D_r$ sont des matrices fixées et u (resp., $v_1, \ldots, v_r$) un vecteur colonne (resp., des vecteurs lignes) à déterminer. Ce problème est équivalent à factoriser M sous la forme de $M = (D_1\, u \; \ldots \; D_r\, u) \left(v_1^T \; \ldots \; v_r^T\right)^T$, où l'entier r est supérieur ou égal au rang de M. En utilisant des méthodes de géométrie algébrique, de théorie des modules, d'algèbre homologique et de calcul formel, nous étudions ce problème particulier de factorisation de rang. Nous caractérisons les solutions générales du système polynomial associé. Les résultats obtenus sont effectifs au sens du calcul formel, c'est-à-dire nous proposons des algorithmes implantables dans un système de calcul formel permettant l'étude effective des systèmes polynomiaux (bases de Gröbner). La librairie symbolique RankFactorization a ainsi été développée pour l'étude effective du problème de factorisation de rang précédent et des problèmes de démodulation correspondants.

Mots-clés : Problème de factorisation de rang, systèmes polynomiaux, théorie des modules, algèbre homologique, démodulation, détection et surveillance des défauts d'engrenages, analyse vibratoire

1 Introduction

The mathematical problem studied in this paper has recently been introduced in the acoustic, vibration analysis, and signal processing literature. Let us briefly explain the context in which it was introduced.

Within the *frequency domain* [24], the *toothed gearbox vibration* [22] can be interpreted as a *modulation process* of a high-frequency periodic carrier with a low-frequency periodic modulation [3, 4, 14, 15]. For *gearbox fault surveillance*, one has to separate these two time-domain signals and compare them to known failure vibration signals. The deviation of the signals can arise from a change in the tooth's shape due to a damaging effect. Detecting the appearance of these defects is an important issue in vibration analysis and for different industrial applications[1]. To study this problem, *demodulation methods* [24] were used and generalized in [14, 15]. Let us briefly state the main ideas of the approach developed in [14].

In practice, the toothed gearbox vibration is measured and some of the Fourier coefficients of this periodic real-valued time signal are stored into a so-called *centrohermitian matrix* $M \in \mathbb{C}^{m \times n}$ [13, 21], namely, a complex matrix of size $m \times n$ which satisfies the identity $\overline{M} = J_m\, M\, J_n$, where $\overline{M}$ stands for the complex conjugate of M and J_m (resp., J_n) denotes the anti-diagonal matrix of size $m \times m$ (resp., $n \times n$). More precisely, if s is the T-periodic real-valued signal of the toothed gearbox vibration, then s can be expressed by its *Fourier series* $s(t) = \sum_{j \in \mathbb{Z}} c_j(s)\, e^{\frac{2\,\pi\, i\, j\, t}{T}}$, where the *Fourier coefficients* of s, defined by $c_j(s) = \frac{1}{T} \int_0^T s(t)\, e^{\frac{-2\,\pi\, i\, j\, t}{T}} dt$ for $j \in \mathbb{Z}$, then satisfy the identity $\overline{c_j(s)} = c_{-j}(\overline{s}) = c_{-j}(s)$ for all $j \in \mathbb{Z}$. The vectors $C_l = (c_{-l}(s)\ \ldots\ c_0(s)\ \ldots\ c_l(s))^T \in \mathbb{C}^{(2\,l+1) \times 1}$ for $l \geq 0$ and the following matrix

$$M = \begin{pmatrix} c_{q\,(2\,p+1)+p} & \cdots & c_p & \cdots & c_{-q\,(2\,p+1)+p} \\ \vdots & & \vdots & & \vdots \\ c_{q\,(2\,p+1)} & \cdots & c_0 & \cdots & c_{-q\,(2\,p+1)} \\ \vdots & & \vdots & & \vdots \\ c_{q\,(2\,p+1)-p} & \cdots & c_{-p} & \cdots & c_{-q\,(2\,p+1)-p} \end{pmatrix} \in \mathbb{C}^{(2p+1) \times (2q+1)}, \quad p,\, q \geq 0,$$

respectively satisfy the identities $\overline{C_l} = J_{2\,l+1}\, C_l\, J_1$ and $\overline{M} = J_{2\,p+1}\, M\, J_{2\,q+1}$, which shows that C_l and M are both centrohermitian matrices. Fixing $r+1$ centrohermitian matrices $D_1, \ldots, D_r \in \mathbb{C}^{(2\,p+1) \times (2\,p+1)}$ (which depend on the demodulation problem under study) and $M \in \mathbb{C}^{(2\,p+1) \times (2\,q+1)}$ (coming from the measurement of the signal s), the *demodulation problem* introduced in [14, 15] aims at determining − if they exist − a centrohermitian column vector $u \in \mathbb{C}^{(2\,p+1) \times 1}$ and r centrohermitian row vectors $v_1, \ldots, v_r \in \mathbb{C}^{1 \times (2\,q+1)}$ satisfying the following identity:

$$M = \sum_{i=1}^{r} D_i\, u\, v_i. \tag{1}$$

The computation of u and $v_1, \ldots, v_r$ yields the reconstruction of the different signals, and thus of their separation. The mathematical formulation eq. (1) encompasses standard demodulation problems [24]: the *amplitude demodulation problem* corresponds to $r = 1$ and $D_1 = I_{2\,p+1}$, and *the amplitude and phase demodulation problem* corresponds to $r = 2$, $D_1 = I_{2\,p+1}$, and $D_2 = 2\,\pi\, i\, f_c\, \mathrm{diag}(-p, \ldots, 0, \ldots, p)$, where $f_c > 0$, $p \in \mathbb{Z}_{>0}$, and $\mathrm{diag}(c_1, \ldots, c_m)$ denotes the diagonal matrix with $c_1, \ldots, c_m$ on the principal diagonal [14, 15]. Note that eq. (1) bores resemblance to the *Singular Value Decomposition* (SVD) of M, namely, $M = \sum_{i=1}^{r} \sigma_i\, u_i\, v_i^\star$, for which we have $u_i = \sigma_i\, D_i\, u$ for $i = 1, \ldots, r$, where r is the rank of M. Finally, eq. (1) corresponds to the SVD of M in the case of the amplitude demodulation problem [14, 15].

The demodulation problem eq. (1) can be generalized as follows. Let $\mathbb{K}$ denote a field, $\mathbb{K}^{m \times n}$ the $\mathbb{K}$-vector space formed by all the $m \times n$ matrices with entries in $\mathbb{K}$, $r \in \mathbb{Z}_{>0}$, $D_1, \ldots, D_r \in \mathbb{K}^{m \times m}$, and $M \in \mathbb{K}^{m \times n}$. Find − if they exist − a column vector $u \in \mathbb{K}^{m \times 1}$ and r row vectors $v_1, \ldots, v_r \in \mathbb{K}^{1 \times n}$ satisfying eq. (1). This last problem will be called the *rank factorization problem* because if a solution of eq. (1) exists, the matrix M can then be factorized as follows

$$M = (D_1\, u\ \ldots\ D_r\, u) \begin{pmatrix} v_1 \\ \vdots \\ v_r \end{pmatrix}, \tag{2}$$

[1]This work was motivated by a problem investigated by the Safran company.

where the rank of M must be less than or equal to r.

For fixed matrices $D_1, \ldots, D_r$ and M, eq. (2) defines a system of $m\,n$ quadratic equations in $m + r\,n$ unknowns − the entries of the vectors u and v_i for $i = 1, \ldots, r$. Hence, for $\mathbb{K} = \mathbb{Q}$, $\mathbb{R}, \mathbb{C}$, different algebraic geometry methods can be used to study the solutions of the rank factorization problem eq. (2) over the algebraic closure $\overline{\mathbb{K}}$ of $\mathbb{K}$. For more details on these methods, see, e.g., [10, 12, 20, 26].

The main goal of this paper is to characterize the solutions to the rank factorization problem eq. (2). Note that the study of the rank factorization problem was initiated in [2, 17, 18]. The particular set of solutions (u, v) of eq. (2), where the matrix $v = (v_1^T \; \ldots \; v_r^T)^T$ is assumed to have full row rank, were characterized in [17, 18] using linear algebra, and further studied in [2] using module theory. In this paper, we shall characterize the set of all the solutions of the rank factorization problem eq. (2). To do that, we shall exploit the *bilinear structure* of eq. (2) in u and $v = \left(v_1^T \; \ldots \; v_r^T\right)^T$ to explicitly characterize the general solutions of eq. (2) over $\overline{\mathbb{K}}$ in terms of *quasi-affine varieties* and explicit parametrizations. Our approach uses standard algebraic geometry, module theory, and homological algebra [10, 12, 25]. For $\mathbb{K} = \mathbb{Q}$, the general solutions over $\overline{\mathbb{K}}$ can be explicitly characterized using standard computer algebra systems that handle both elimination theory for polynomial systems (e.g., *Gröbner* or *Janet basis methods*) and basic homological methods such as, e.g., `Singular` [12], the `GAP` library `CapAndHomalg` [1], or the `Maple` package OreModules [6]. The different results presented in this paper are implemented in the RankFactorization package [9] built upon the OreModules package.

Characterizing the solutions of a polynomial system is known to be a difficult issue in algebraic geometry and computer algebra especially when its affine algebraic set of complex solutions is not simply formed by a finite number of complex points. As we shall explain in this paper, the special structure of the class of polynomial systems defined by eq. (2) (i.e., its bilinear structure) allows us to characterize its solution space, particularly when $\mathbb{Q} \subseteq \mathbb{K}$, using methods of module theory, homological algebra, and computer algebra. Thus, the rank factorization problem provides an interesting class of polynomial systems for which their solution spaces can be explicitly characterized. Hence, we hope that this paper will draw the (effective) algebraic geometry community's attention to the rank factorization problem and that further investigations will be done on this problem (e.g., its intrinsic geometric characterization).

Finally, note that the demodulation problem eq. (1) corresponds to the rank factorization problem eq. (2) for $\mathbb{K} = \mathbb{C}$, where the solution (u, v) are sought to be centrohermitian vectors. Hence, solving the rank factorization problem eq. (2) for $\mathbb{K} = \mathbb{C}$ does not solve the demodulation problem eq. (1). But a result of [21] shows that the set of centrohermitian matrices is bijectively mapped onto the set of real matrices by an explicit and simple *unitary* transformation ρ. Hence, the demodulation problem eq. (1) is equivalent to solving the rank factorization problem eq. (2) for $\mathbb{K} = \mathbb{R}$. More precisely, the demodulation problem eq. (1), arising in vibration analysis and studied in [14, 15], can be transformed into a rank factorization problem eq. (2) over $\mathbb{K} = \mathbb{R}$ for the transformed real matrices $\rho(M), \rho(D_1), \ldots, \rho(D_r)$. The real solutions u_ρ and $\{v_{i\rho}\}_{i=1,\ldots,r}$, of the latter problem can then be transformed back to obtain the centrohermitian solutions $u = \rho^{-1}(u_\rho)$ and $\{v_i = \rho^{-1}(v_{i,\rho})\}_{i=1,\ldots,r}$ of eq. (1). For more details, see [18, 19], where structured matrices (e.g., *coninvolutory* and *involutory* matrices) play an important role. To completely solve the demodulation problem eq. (1), we are finally led to understand how the solutions of eq. (2), obtained in this paper over $\overline{\mathbb{K}}$, can then be used to characterize the real solutions of eq. (2). This last mathematical problem is out of the scope of the present paper and it will be studied in a forthcoming publication dedicated to the application of the results of this paper to the demodulation problem eq. (1).

Plan

This introductory section introduces the demodulation problem, the rank factorization problem, and the notations. In section 2, we state again the results obtained in [16, 17, 18, 19] which characterize all the solutions (u, v) of the rank factorization problem eq. (2) for which $v = (v_1^T \; \ldots \; v_r^T)^T$ has full row rank. These results use linear algebra and module theory. In section 3, using algebraic geometry, module theory, homological algebra, and computer algebra, we show how the results obtained in section 2 can be extended to characterize all the solutions of the rank factorization problem over $\overline{\mathbb{K}}$, where $\mathbb{K} = \mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$. Explicit examples, computed with the RankFactorization package [9], illustrate the main results of the paper. In section 5, we end the paper by stating problems that will be studied in the future.

Notation

In what follows, $\mathbb{K}$ will denote a field of characteristic 0, $\mathcal{R}$ a commutative unital ring, and $\mathcal{R}^{m\times n}$ the $\mathcal{R}$-module formed by all the $m \times n$ matrices with entries in $\mathcal{R}$. If $M \in \mathcal{R}^{m\times n}$, then we can consider the $\mathcal{R}$-homomorphisms $M. : \mathcal{R}^{n\times 1} \longrightarrow \mathcal{R}^{m\times 1}$ and $.M : \mathcal{R}^{1\times m} \longrightarrow \mathcal{R}^{1\times n}$ respectively defined by $(M.)(\eta) = M\,\eta$ for all $\eta \in \mathcal{R}^{n\times 1}$ and $(.M)(\lambda) = \lambda\, M$ for all $\lambda \in \mathcal{R}^{1\times m}$. Their kernels, images, and cokernels $\mathcal{R}$-modules are respectively denoted by $\ker_{\mathcal{R}}(M.)$, $\mathrm{im}_{\mathcal{R}}(M.)$, $\mathrm{coker}_{\mathcal{R}}(M.)$, $\ker_{\mathcal{R}}(.M)$, $\mathrm{im}_{\mathcal{R}}(.M)$, and $\mathrm{coker}_{\mathcal{R}}(.M)$ [25]. A matrix M is said to have *full column rank* (resp., *full row rank*) if $\ker_{\mathcal{R}}(M.) = 0$ (resp., $\ker_{\mathcal{R}}(.M) = 0$). If $M \in \mathbb{K}^{m\times n}$, then the *rank* of M, i.e., $\dim_{\mathbb{K}}(\mathrm{im}_{\mathbb{K}}(M.))$, is denoted by $\mathrm{rank}_{\mathbb{K}}(M)$. Let I_n denote the identity matrix, i.e., the $n \times n$ matrix with 1 on the first diagonal and 0 elsewhere, and J_n the exchange $n \times n$ matrix, i.e., the $n \times n$ matrix with 1 on the second diagonal and 0 elsewhere. Moreover, $\mathrm{diag}(d_1, \ldots, d_r)$ denotes the matrix with $d_1, \ldots, d_r$ on the first diagonal. If $M \in \mathbb{C}^{m\times n}$, then $\overline{M}$ (resp., $M^\star$) stands for the *conjugate matrix* (resp., the *adjoint matrix*, i.e., $\overline{M}^T \in \mathbb{C}^{n\times m}$). If $M \in \mathbb{K}^{m\times n}$, then we shall denote by l the rank of M, i.e., $l = \mathrm{rank}_{\mathbb{K}}(M) = \dim_{\mathbb{K}}(\mathrm{im}_{\mathbb{K}}(M.))$. If M is a matrix whose entries are functions of the vector variable $x = (x_1\ \ldots\ x_n)^T$, then $M(\psi)$ denotes the evaluation of M at $\psi = (\psi_1\ \ldots\ \psi_n)^T \in \mathbb{K}^{n\times 1}$. Finally, if $\mathcal{I}$ is an ideal of the ring $\mathcal{R}$ generated by the elements $g_1, \ldots, g_t \in \mathcal{R}$, then we shall note $\mathcal{I} = \langle g_1, \ldots, g_t\rangle_{\mathcal{R}}$ or $\mathcal{I} = \sum_{i=1}^t \mathcal{R}\, g_i$.

2 Characterization of a particular set of solutions

2.1 A few preliminary remarks

In this section, we state of few remarks on the rank factorization problem eq. (1).

As stated in section 1, the rank factorization problem eq. (1) corresponds to a system of $m\,n$ quadratic equations in $m+r\,n$ unknowns (namely, the entries of the vectors $u \in \mathbb{K}^{m\times 1}$ and $v_i \in \mathbb{K}^{1\times n}$ for $i = 1, \ldots, r$). Thus, this problem belongs to the realm of (effective) algebraic geometry (see, e.g., [10, 12, 20] and the references therein).

For $n = 1$, using $r \geq 1$, we note that $m + r > m$, which shows that eq. (1) defines a system with more unknowns than algebraic equations.

For $n \geq 2$, the sign of $m\,n - (m + r\,n) = m\,(n-1) - n\,r$ is the sign of the function $\Psi(m, n, r) = m - \left(1 + \frac{1}{n-1}\right) r$, which satisfies $m - 2\,r \leq \Psi(m, n, r) \leq m - r$. Hence, if $m \geq 2\,r$ (resp., $m \leq r$), then eq. (1) defines a polynomial system with more (resp., less) equations than unknowns. Thus, it is expected that the corresponding polynomial system becomes *over-determined* for large m and n. But nothing can be said about the *dimension* of the corresponding polynomial system without having computed the *Hilbert series* of the corresponding polynomial ideal, and thus, without having first computed *a Gröbner/Janet basis* (or an equivalent normal form) for this polynomial system (see, e.g., [10, 12]). Using a computer algebra system, this can be done for fixed matrices $D_1, \ldots, D_r$, and M with rather small m and n. However, this approach does not seem to be useful for the study of the general problem.

In what follows, we shall suppose that $D_1, \ldots, D_r$ are not 0 and we use the following notations:

$$A(u) = (D_1\, u\ \ldots\ D_r\, u) \in \mathbb{K}^{m\times r}, \quad v = \begin{pmatrix} v_1 \\ \vdots \\ v_r \end{pmatrix} \in \mathbb{K}^{r\times n}. \tag{3}$$

The rank factorization problem eq. (1) can then be rewritten as follows:

$$A(u)\, v = M. \tag{4}$$

The bilinear structure in u and v is emphasized in eq. (4). Under the form eq. (4), eq. (1) corresponds to a factorization problem for the matrix $M \in \mathbb{K}^{m\times n}$.

It is important to note that the existence of $u \in \mathbb{K}^{m\times 1}$ and $v_i \in \mathbb{K}^{1\times n}$ for $i = 1, \ldots, r$ satisfying eq. (4) is equivalent to the existence of $u \in \mathbb{K}^{m\times 1}$ satisfying the following inclusion of $\mathbb{K}$-vector spaces:

$$\mathrm{im}_{\mathbb{K}}(M.) \subseteq \mathrm{im}_{\mathbb{K}}(A(u).). \tag{5}$$

Indeed, eq. (5) implies that each column $M_{\bullet i}$'s of M belongs to $\mathrm{im}_{\mathbb{K}}(A(u).)$, i.e., the existence of $v_{\bullet i} \in \mathbb{K}^{r \times 1}$ such that $A(u)\, v_{\bullet i} = M_{\bullet i}$ for $i = 1, \ldots, n$, which yields $A(u)\, v = M$ with $v = (v_{\bullet 1} \; \ldots \; v_{\bullet n}) \in \mathbb{K}^{r \times n}$. Conversely, eq. (4) yields eq. (5).

Note that eq. (5) shows that a necessary condition on M for the solvability of eq. (1) is:

$$l = \mathrm{rank}_{\mathbb{K}}(M) \leq \mathrm{rank}_{\mathbb{K}}(A(u)) \leq \min\{m, r\}. \tag{6}$$

Therefore, if l is not less than or equal to $\min\{m, r\}$, no solution of eq. (4) exists.

The choice of the terminology for Problem eq. (4), namely, the *rank factorization problem*, comes from the factorization condition eq. (4) and the rank condition eq. (6).

Finally, the approach that will be developed in the rest of the paper is based on the characterization of the set of all the vectors u's such that the inclusion eq. (5) holds. In particular, u must be chosen so that not all the $l \times l$ minors of the matrix $A(u)$ vanish.

2.2 Review on the characterization of particular set of solutions

We briefly state again results obtained in [2, 16, 17] which characterize a particular class of solutions of eq. (4). These results use linear algebra and module theory. In section 3, using also homological algebra and computer algebra, this approach will be generalized to characterize all the solutions of eq. (4). Since the approach will be extended in section 3, we now state again the main arguments.

The next lemma gives two necessary conditions for a solution $(u,\, v)$ of eq. (4) to be such that the matrix v has full row rank (i.e., the rows of v are $\mathbb{K}$-linearly independent).

Lemma 1. *If there exists a solution $(u,\, v)$ of eq. (4) such that v has full row rank, then:*

1. $D_i\, u \in \mathrm{im}_{\mathbb{K}}(M.)$ *for* $i = 1, \ldots, r$.
2. $\mathrm{rank}_{\mathbb{K}}(A(u)) = \mathrm{rank}_{\mathbb{K}}(M) = l$, *i.e.,* $\dim_{\mathbb{K}}(\mathrm{span}_{\mathbb{K}}\{D_i\, u\}_{i=1,\ldots,r}) = l$.

Proof. Recall that the matrix v has full row rank if and only if it admits a right inverse $t \in \mathbb{K}^{n \times r}$, i.e., $v\, t = I_r$. Hence, if a solution $(u,\, v)$ of eq. (4) exists with a full row rank matrix v, then eq. (4) yields $A(u) = M\, t$, which, using eq. (5), shows that $\mathrm{im}_{\mathbb{K}}(A(u).) = \mathrm{im}_{\mathbb{K}}(M.)$. Hence, $u \in \mathbb{K}^{m \times 1}$ must satisfy Conditions 1 and 2. □

In this section, we shall characterize the solutions of eq. (4) satisfying Conditions 1 and 2 of Lemma 1. Since these two conditions are usually not sufficient for v to have full row rank, among the solutions $(u,\, v)$ of eq. (4) satisfying Conditions 1 and 2 are the solutions of eq. (4) with full rank matrices v. Note also that these two conditions depend only on u, i.e., no conditions on v appear, which simplifies the search for such solutions as explained below.

Let us first study Condition 1 of Lemma 1.

Lemma 2. *If $l < m$, let $p = m - l$ and $L \in \mathbb{K}^{p \times m}$ be a full row matrix whose rows define a basis of* $\ker_{\mathbb{K}}(.M)$*, i.e., are such that* $\ker_{\mathbb{K}}(.M) = \mathrm{im}_{\mathbb{K}}(.L)$*. If $m = l$, then we set $L = (0 \; \ldots \; 0) \in \mathbb{K}^{1 \times m}$.*

Then, we have:

1. $\ker_{\mathbb{K}}(L.) = \mathrm{im}_{\mathbb{K}}(M.)$.
2. *Condition 1 of Lemma 1 is equivalent to the fact that $u \in \mathbb{K}^{m \times 1}$ satisfies the $\mathbb{K}$-linear system $N\, u = 0$, where:*
$$N = \begin{pmatrix} L\, D_1 \\ \vdots \\ L\, D_r \end{pmatrix} \in \mathbb{K}^{p\, r \times m}. \tag{7}$$
3. *If $Z \in \mathbb{K}^{m \times d}$ is a full column matrix whose columns define a basis of* $\ker_{\mathbb{K}}(N.)$*, where* $d = \dim_{\mathbb{K}}(\ker_{\mathbb{K}}(N.))$ *(if $l = m$, then $d = m$ and we can take $Z = I_m$), then Condition 1 of Lemma 1 is equivalent to $u = Z\, \psi$ for a certain $\psi \in \mathbb{K}^{d \times 1}$.*

Proof. Let us first suppose that $\mathrm{im}_{\mathbb{K}}(M.) \neq \mathbb{K}^{m\times 1}$. Set $p = m - l > 0$. Let $L \in \mathbb{K}^{p\times m}$ be a full row rank matrix whose rows define a basis of $\ker_{\mathbb{K}}(.M)$, i.e., $\ker_{\mathbb{K}}(.M) = \mathrm{im}_{\mathbb{K}}(.L)$. Then, we get $L\,M = 0$, which shows that $\mathrm{im}_{\mathbb{K}}(M.) \subseteq \ker_{\mathbb{K}}(L.)$. Now, $\dim_{\mathbb{K}}(\ker_{\mathbb{K}}(L.)) = m - p = l$ yields $\ker_{\mathbb{K}}(L.) = \mathrm{im}_{\mathbb{K}}(M.)$. Hence, Condition 1 of Lemma 1 is equivalent to $D_i\,u \in \ker_{\mathbb{K}}(L.)$ for $i = 1, \ldots, r$, i.e., u satisfies the system of linear equations $(L\,D_i)\,u = 0$ for $i = 1, \ldots, r$, i.e., $u \in \ker_{\mathbb{K}}(N.)$, where N is given by eq. (7). Now, if $\mathrm{im}_{\mathbb{K}}(M.) = \mathbb{K}^{m\times 1} = \ker_{\mathbb{K}}(L.)$, i.e., $p = 0$, then $D_i\,u \in \mathbb{K}^{m\times 1} = \mathrm{im}_{\mathbb{K}}(M.)$ for $i = 1, \ldots, r$, i.e., Condition 1 of Lemma 1 is equivalent to $u \in \mathbb{K}^{m\times 1}$, which is coherent with the fact that $N = 0$ and $\ker_{\mathbb{K}}(N.) = \mathbb{K}^{m\times 1}$. Finally, the third point is a direct consequence of the second point and the definition of Z. □

According to 3 of Lemma 2, Condition 1 is equivalent to $u = Z\,\psi$ for a certain $\psi \in \mathbb{K}^{d\times 1}$. Condition 2 is then equivalent to $\psi \in \mathcal{P}$, where the set $\mathcal{P}$ is defined by:

$$\mathcal{P} = \left\{\psi \in \mathbb{K}^{d\times 1} \mid \mathrm{rank}_{\mathbb{K}}(A(Z\,\psi)) = l\right\}.$$

We have just shown that Conditions 1 and 2 of Lemma 1 are equivalent to $u = Z\,\psi$, where $\psi \in \mathcal{P}$.

Let us briefly treat the case of $M = 0$. We then have $l = 0$, $p = m$, $L = I_m$, $N = (D_1^T \ \ldots \ D_r^T)^T$, $A(Z\,\psi) = (D_1\,Z\,\psi \ \ldots \ D_r\,Z\,\psi) = (0 \ \ldots \ 0)$ for all $\psi \in \mathbb{K}^{d\times 1}$, where $\ker_{\mathbb{K}}(N.) = \mathrm{im}_{\mathbb{K}}(Z.)$ for a certain matrix $Z \in \mathbb{K}^{m\times d}$, i.e., $\mathcal{P} = \mathbb{K}^{d\times 1}$. Therefore, all the solutions $(u,\ v)$ of eq. (4) are of the form $(Z\,\psi,\ v)$ for all $\psi \in \mathbb{K}^{d\times 1}$ and for all $v \in \mathbb{K}^{r\times n}$.

In the rest of the section, we shall suppose that $M \neq 0$.

Remark 1. If $d = 0$, i.e., $\ker_{\mathbb{K}}(N.) = 0$, then $Z = 0$, and thus, $A(Z\,\psi) = A(0) = 0$. Since $M \neq 0$, then $\mathrm{im}_{\mathbb{K}}(A(u).) = \{0\} \subsetneq \mathrm{im}_{\mathbb{K}}(M.)$ and eq. (4) has no solution satisfying Conditions 1 and 2 of Lemma 1, and thus, no solution $(u,\ v)$ with a full row rank matrix v exists.

Remark 2. Note that $\mathcal{P}$ is a *linear cone*, namely, $\lambda\,\psi \in \mathcal{P}$ for all $\lambda \in \mathbb{K}^{\times} = \mathbb{K} \setminus \{0\}$ and $\psi \in \mathcal{P}$.

The next lemma gives an equivalent characterization of the linear cone $\mathcal{P}$.

Lemma 3. *Let $X \in \mathbb{K}^{m\times l}$ be a full column rank whose columns define a basis of* $\mathrm{im}_{\mathbb{K}}(M.)$*, i.e.,* $\mathrm{im}_{\mathbb{K}}(M.) = \mathrm{im}_{\mathbb{K}}(X.)$*, where $l \geq 1$. Then, we have the following results:*

1. *There exists a unique matrix $Y \in \mathbb{K}^{l\times n}$ such that $M = X\,Y$.*
2. *For $i = 1, \ldots, r$, there exists a unique matrix $W_i \in \mathbb{K}^{l\times d}$ such that $D_i\,Z = X\,W_i$, where Z is defined in Lemma 3.*
3. *Let $B(\psi) = (W_1\,\psi \ \ldots \ W_r\,\psi) \in \mathbb{K}^{l\times r}$ for all $\psi \in \mathbb{K}^{d\times 1}$. Then, the linear cone $\mathcal{P}$ satisfies:*

$$\mathcal{P} = \left\{\psi \in \mathbb{K}^{d\times 1} \mid \mathrm{rank}_{\mathbb{K}}(B(\psi)) = l\right\}. \tag{8}$$

Finally, the linear cone $\mathcal{P}$ does not depend on the choice of the bases of the $\mathbb{K}$-vector spaces $\ker_{\mathbb{K}}(.M)$, $\ker_{\mathbb{K}}(N.)$*, where N is defined by eq.* (7)*, and* $\mathrm{im}_{\mathbb{K}}(M.)$.

Proof. 1. Let $X \in \mathbb{K}^{m\times l}$ be a full column rank whose columns define a basis of $\mathrm{im}_{\mathbb{K}}(M.)$, i.e., $\mathrm{im}_{\mathbb{K}}(M.) = \mathrm{im}_{\mathbb{K}}(X.)$. Let $M_{\bullet i}$ denote the i^{th} column of M for $i = 1, \ldots, n$. Then, there exist unique vector $Y_{\bullet i} \in \mathbb{K}^{l\times 1}$ such that $M_{\bullet i} = X\,Y_{\bullet i}$ for $i = 1, \ldots, n$, which yields $M = X\,Y$, where $Y = (Y_{\bullet 1} \ \ldots \ Y_{\bullet n}) \in \mathbb{K}^{l\times n}$.

2. We have $D_i\,Z\,\psi \in \ker_{\mathbb{K}}(L.) = \mathrm{im}_{\mathbb{K}}(M.) = \mathrm{im}_{\mathbb{K}}(X.)$ for all $\psi \in \mathbb{K}^{d\times 1}$, which shows that there exists a unique matrix $W_i \in \mathbb{K}^{l\times d}$ such that $D_i\,Z = X\,W_i$ for $i = 1, \ldots, r$. We then get:

$$\forall\ \psi \in \mathbb{K}^{d\times 1}, \quad A(Z\,\psi) = (D_1\,Z\,\psi \ \ldots \ D_r\,Z\,\psi) = X\,(W_1\,\psi \ \ldots \ W_r\,\psi). \tag{9}$$

3. The first part of the result is a direct consequence of the identity $A(Z\,\psi) = X\,B(\psi)$ for all $\psi \in \mathbb{K}^{d\times 1}$, where X has full column rank. Finally, if the rows $L' \in \mathbb{K}^{(m-l)\times m}$ define another basis of $\ker_{\mathbb{K}}(.M)$, then there exists an invertible $T \in \mathbb{K}^{(m-l)\times(m-l)}$ such that $L' = T\,L$. If N' is the matrix defined by eq. (7) where L is replaced by L', then $N' = T\,N$, and thus, $\ker_{\mathbb{K}}(N'.) = \ker_{\mathbb{K}}(N.)$. If $Z' \in \mathbb{K}^{m\times d}$ (resp., $X' \in \mathbb{K}^{m\times l}$) is another basis of $\ker_{\mathbb{K}}(N.)$ (resp., $\mathrm{im}_{\mathbb{K}}(M.)$), then there exists an invertible matrix $U \in \mathbb{K}^{d\times d}$ (resp., $V \in \mathbb{K}^{l\times l}$) such that $Z = Z'\,U$ (resp., $X = X'\,V$). Then, $D_i\,Z = X\,W_i$ yields $D_i\,Z' = X'\,(V\,W_i\,U^{-1})$, i.e., $W_i' = V\,W_i\,U^{-1}$ satisfies $D_i\,Z' = W_i'\,X'$ for $i = 1, \ldots, r$. Finally, using the invertibility of U and V, the matrix $B'(\psi) = (W_1'\,\psi \ \ldots \ W_r'\,\psi) = V\,B(U^{-1}\,\psi)$ has the same rank as $B(\psi)$ for all $\psi \in \mathbb{K}^{d\times 1}$, which shows that the linear cone $\mathcal{P}$ is independent of the choice of the different intermediate bases. □

Remark 3. If a $l \times l$ minor of $B(\psi)$ is not 0, then $\mathcal{P}$ is not empty. Since the columns $W_i\,\psi$ of $B(\psi)$ are linear forms in $\psi = (\psi_1 \ \ldots \ \psi_d)^T$, the $l \times l$ minors of $B(\psi)$ are either 0 or homogeneous polynomials in ψ of total degree l. In particular, we find again that $\mathcal{P}$ is a linear cone.

Remark 4. If $\operatorname{im}_{\mathbb{K}}(M.) = \mathbb{K}^{m\times 1}$, i.e., $l = m$, then Condition 1 of Lemma 1 is always satisfied for all $u \in \mathbb{K}^{m\times 1}$ since $D_i\, u \in \mathbb{K}^{m\times 1}$ for all $u \in \mathbb{K}^{m\times 1}$ and $i = 1, \ldots, r$. Equivalently, using Lemma 2, $L = (0 \ \ldots \ 0) \in \mathbb{K}^{1\times m}$, $N = 0$ (see eq. (7)), and thus, we can take $Z = I_m$, and $X = I_m$. Therefore, we have $W_i = D_i$, which yields $B(\psi) = (D_1\,\psi \ \ldots \ D_r\,\psi)$ and $u = \psi \in \mathcal{P} = \{\psi \in \mathbb{K}^{m\times 1} \mid \operatorname{rank}_{\mathbb{K}}(A(\psi)) = m\}$.

Let us state the characterization of the solutions of eq. (4) satisfying Conditions 1 and 2 of Lemma 1.

Theorem 1 ([16]). *Let $D_i \in \mathbb{K}^{m\times m}$ for $i = 1, \ldots, r$ and $M \in \mathbb{K}^{m\times n}$ be such that:*

$$1 \leq l = \operatorname{rank}_{\mathbb{K}}(M) \leq \min\{m, r\}.$$

With the notations of Lemma 2 and Lemma 3, if the linear cone $\mathcal{P}$, defined by eq. (8), is not empty, then

$$\forall\, \psi \in \mathcal{P}, \quad \forall\, Y' \in \mathbb{K}^{(r-l)\times n}, \quad \begin{cases} u = Z\,\psi, \\ v = (E_\psi \quad C_\psi) \begin{pmatrix} Y \\ Y' \end{pmatrix}, \end{cases} \tag{10}$$

are the solutions of eq. (4) satisfying Conditions 1 and 2 of Lemma 1, where:

- *The matrix $E_\psi \in \mathbb{K}^{r\times l}$ is a right inverse of $B(\psi)$, i.e., $B(\psi)\, E_\psi = I_l$,*
- *The columns of the matrix $C_\psi \in \mathbb{K}^{r\times(r-l)}$ define a basis of $\ker_{\mathbb{K}}(B(\psi).)$, i.e., $C_\psi \in \mathbb{K}^{r\times(r-l)}$ has full column matrix and satisfies $\ker_{\mathbb{K}}(B(\psi).) = \operatorname{im}_{\mathbb{K}}(C_\psi.)$.*

In particular, if $l = r$, then $C_\psi = 0$ and the solution eq. (10) is unique.

Moreover, the matrix v defined by eq. (10) has full row rank if and only if $Y' \in \mathbb{K}^{(r-l)\times n}$ is chosen so that $\left(Y^T \quad Y'^T\right)^T \in \mathbb{K}^{r\times n}$ has full row rank.

Finally, the results do not depend on the choice of bases for the different intermediate $\mathbb{K}$-vector spaces.

Proof. Using eq. (6), i.e., $0 < l \leq r$, $\mathcal{P}$ characterizes the ψ's which are so that $B(\psi)$ admits a right inverse $E_\psi \in \mathbb{K}^{r\times l}$, i.e., $B(\psi)\, E_\psi = I_l$. Using that X has full column rank, we get:

$$\forall\, \psi \in \mathbb{K}^{d\times 1}, \quad A(Z\,\psi)\, v = M \iff X\, B(\psi)\, v = X\, Y \iff B(\psi)\, v = Y.$$

Hence, if $\psi \in \mathcal{P}$, then $v_\star = E_\psi\, Y \in \mathbb{K}^{r\times n}$ is a particular solution of the linear inhomogeneous system $B(\psi)\, v = Y$. Let $C_\psi \in \mathbb{K}^{r\times(r-l)}$ be a full column matrix whose columns define a basis of $\ker_{\mathbb{K}}(B(\psi).)$, i.e., $\ker_{\mathbb{K}}(B(\psi).) = \operatorname{im}_{\mathbb{K}}(C_\psi.)$. Then, all the solutions of eq. (1) satisfying Conditions 1 and 2 of Lemma 1 are of the form eq. (10). Note that $(E_\psi \quad C_\psi) \in \mathbb{K}^{r\times r}$ is invertible. Thus, v has full row rank if and only if so has the matrix $\left(Y^T \quad Y'^T\right)^T$. Finally, let us prove that eq. (10) does not depend on the choice of the different intermediate bases. According to 3 of Lemma 3, $\mathcal{P}$ does not depend on them. Using the notations of the proof of 3 of Lemma 3, setting $\psi' = U\,\psi$, we have $u = Z\,\psi = Z'\,\psi'$ and, using the identity $B'(\psi') = V\, B(\psi)$ (see the proof of 3 of Lemma 3), $B(\psi)\, E_\psi = I_l$ yields $B'(\psi')\,(E_\psi\, V^{-1}) = I_l$, i.e., $E'_{\psi'} = E_\psi\, V^{-1}$ is a right inverse of $B'(\psi')$. Using $M = X'\, Y'$, where $X' = X\, V^{-1}$ and $Y' = V\, Y$ (see the proof of 3 of Lemma 3), we get $E'_\psi\, Y' = E_\psi\, Y$. Finally, we clearly have $\ker_{\mathbb{K}}(B'(\psi').) = \ker_{\mathbb{K}}(B(\psi).)$. □

Remark 5. For the demodulation problem eq. (1) studied in vibration analysis, as explained in Section 1, the matrices M and D_i's are then centrohermitian and the solutions u and v_i for $i = 1, \ldots, r$ are also sought to be centrohermitian. We refer the readers to [18, 19] for the extensions of Theorem 1 to the demodulation problem. See also [17]. Note that structured matrices (e.g., *coninvolutory* and *involutory* matrices) then play an important role in the algebraic structure of the corresponding solutions.

Let us now study $\mathcal{P}$ in more detail. We first introduce a few more notations.

Let $\mathcal{R} = \mathbb{K}[x_1, \ldots, x_d]$ be the commutative polynomial ring in $x_1, \ldots, x_d$ with coefficients in $\mathbb{K}$, $x = (x_1 \ \ldots \ x_d)^T$, and $B = (W_1\, x \ \ldots \ W_r\, x) \in \mathcal{R}^{l\times r}$. According to eq. (6), we have $1 \leq l \leq r$, i.e.,

B is a wide matrix. If $\mathcal{I}$ is the ideal of $\mathcal{R}$ generated by all the $l \times l$ minors of B, then $\mathcal{I}$ is either reduced to $\langle 0 \rangle$ or it can be generated by homogeneous polynomials $g_1, \ldots, g_t$ of total degree l. Finally, if $\mathcal{V}_{\mathbb{K}}(\mathcal{I}) = \left\{\psi \in \mathbb{K}^{d \times 1} \mid \forall\, P \in \mathcal{I} : P(\psi) = 0\right\}$ is the *affine algebraic set* associated with $\mathcal{I}$ [10, 12], then the linear cone $\mathcal{P}$, defined by eq. (8), satisfies $\mathcal{P} = \mathbb{K}^{d \times 1} \setminus \mathcal{V}_{\mathbb{K}}(\mathcal{I})$.

Let us consider the finitely presented $\mathcal{R}$-module $\mathcal{B} = \operatorname{coker}_{\mathcal{R}}(B.) = \mathcal{R}^{l \times 1} / \left(B\, \mathcal{R}^{r \times 1}\right)$. Note that the 0^{th}-*Fitting ideal* $\operatorname{Fitt}_0(\mathcal{B})$ of $\mathcal{B}$ is the ideal of $\mathcal{R}$ generated by all the $l \times l$ minors of B [10, 23], i.e., $\mathcal{I} = \operatorname{Fitt}_0(\mathcal{B})$. See the forthcoming Definition 1 for the general definition of *Fitting ideals*. Moreover, if $\operatorname{ann}_{\mathcal{R}}(\mathcal{B}) = \{a \in \mathcal{R} \mid \forall\, b \in \mathcal{B} : a\, b = 0\}$ is the *annihilator* of $\mathcal{B}$, then we have

$$\operatorname{ann}_{\mathcal{R}}(\mathcal{B})^l \subseteq \operatorname{Fitt}_0(\mathcal{B}) \subseteq \operatorname{ann}_{\mathcal{R}}(\mathcal{B}) \implies \sqrt{\operatorname{ann}_{\mathcal{R}}(\mathcal{B})} = \sqrt{\operatorname{Fitt}_0(\mathcal{B})}, \tag{11}$$

where $\sqrt{\mathcal{I}} = \left\{a \in \mathcal{R} \mid \exists\, k \in \mathbb{Z} : a^k \in I\right\}$ denotes the *radical* of $\mathcal{I}$. If $\mathbb{K}$ is an *algebraically closed field* (e.g., $\mathbb{K} = \overline{\mathbb{Q}}$ is the algebraic closure of $\mathbb{Q}$ or $\mathbb{K} = \mathbb{C}$), then $\mathcal{V}_{\mathbb{K}}(\mathcal{I}) = \mathcal{V}_{\mathbb{K}}(\operatorname{ann}_{\mathcal{R}}(\mathcal{B}))$. For more details, see [10, 23].

Corollary 1 ([2])**.** *Let $W_1, \ldots, W_r \in \mathbb{K}^{l \times d}$ be the matrices defined in Theorem 1, $\mathcal{R} = \mathbb{K}[x_1, \ldots, x_d]$, $x = (x_1 \ \ldots \ x_d)^T$, $B = (W_1\, x \ \ldots \ W_r\, x) \in \mathcal{R}^{l \times r}$, $\mathcal{B} = \operatorname{coker}_{\mathcal{R}}(B.)$, $\mathcal{I} = \operatorname{Fitt}_0(\mathcal{B})$, and $\operatorname{ann}_{\mathcal{R}}(\mathcal{B})$ the annihilator of the $\mathcal{R}$-module $\mathcal{B}$. Then, the linear cone $\mathcal{P}$, defined in Theorem 1, is the complementary of the algebraic set $\mathcal{V}_{\mathbb{K}}(\mathcal{I})$ in the affine space $\mathbb{K}^{d \times 1}$, and thus, $\mathcal{P}$ is a quasi-affine variety. Finally, if $\mathbb{K}$ is an algebraically closed field, then we have $\mathcal{V}_{\mathbb{K}}(\mathcal{I}) = \mathcal{V}_{\mathbb{K}}(\operatorname{ann}_{\mathcal{R}}(\mathcal{B}))$.*

We summarize the above results in Algorithm 1 and illustrate this algorithm with an explicit example.

Algorithm 1 PreRankFactorizationProblem

procedure PreRankFactorizationProblem($D_1, \ldots, D_r \in \mathbb{K}^{m \times m}, 0 \neq M \in \mathbb{K}^{m \times n}$)
- Compute a basis of $\ker_{\mathbb{K}}(.M)$ to get a full row rank matrix $L \in \mathbb{K}^{(m-l) \times m}$ satisfying $\ker_{\mathbb{K}}(.M) = \operatorname{im}_{\mathbb{K}}(.L)$, where $l = \operatorname{rank}_{\mathbb{K}}(M)$
- Set the matrix $N \in \mathbb{K}^{p\, r \times m}$ defined by eq. (7) and compute a basis of $\ker_{\mathbb{K}}(N.)$, i.e., a full column rank matrix $Z \in \mathbb{K}^{m \times d}$ such that $\ker_{\mathbb{K}}(N.) = \operatorname{im}_{\mathbb{K}}(Z.)$, where $d = \dim_{\mathbb{K}}(\ker_{\mathbb{K}}(N.))$
- Compute a basis of $\operatorname{im}_{\mathbb{K}}(M.)$ to get a full column rank matrix $X \in \mathbb{K}^{m \times l}$ satisfying $\operatorname{im}_{\mathbb{K}}(M.) = \operatorname{im}_{\mathbb{K}}(X.)$
- Factorize M as $M = X\,Y$, where $Y \in \mathbb{K}^{l \times n}$
- For $i = 1, \ldots, r$, compute the unique matrix $W_i \in \mathbb{K}^{l \times d}$ satisfying $D_i\, Z = X\, W_i$
- Define $\mathcal{R} = \mathbb{K}[x_1, \ldots, x_d]$, $x = (x_1 \ \ldots \ x_d)^T$, $B = (W_1\, x \ \ldots \ W_r\, x) \in \mathcal{R}^{l \times r}$, and $\mathcal{I} = \operatorname{Fitt}_0(\mathcal{B})$ the ideal of $\mathcal{R}$ generated by all the $l \times l$ minors of B
- **return** Z, Y, $\mathcal{I}$, $\mathcal{R}$, and B which are such that eq. (10) defines solutions of eq. (4) satisfying Conditions 1 and 2 of Lemma 1 for all $\psi \notin \mathcal{V}_{\mathbb{K}}(\mathcal{I})$ and for all $Y' \in \mathbb{K}^{(r-l) \times n}$ (where $E_\psi \in \mathbb{K}^{r \times l}$ is a right inverse of $B(\psi) = (W_1\, \psi \ \ldots \ W_r\, \psi)$ and the columns of $C_\psi \in \mathbb{K}^{r \times (r-l)}$ define a basis of $\ker_{\mathbb{K}}(B(\psi).)$)

end procedure

Example 1. Let us consider the following matrices:

$$D_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad D_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad D_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},$$

$$D_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad M = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}.$$

Then, we have $m = n = r = 4$. We can easily check that $l = \mathrm{rank}_{\mathbb{K}}(M) = 1$ and:

$$L = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad X = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad Y = (1 \quad 0 \quad 0 \quad 1),$$
$$W_1 = -1, \quad W_2 = 0, \quad W_3 = 1, \quad W_4 = 0.$$

Hence, we have $d = 1$, $\psi = \psi_1 \in \mathbb{K}$, $\mathcal{R} = \mathbb{K}[x_1]$, and $B = (-x_1 \quad 0 \quad x_1 \quad 0) \in \mathcal{R}^{1\times 4}$. Considering the $\mathcal{R}$-module $\mathcal{B} = \mathcal{R}/(B\,\mathcal{R}^{4\times 1}) = \mathcal{R}/\mathcal{I}$, where $\mathcal{I} = \mathrm{Fitt}_0(\mathcal{B}) = \mathrm{ann}_{\mathcal{R}}(\mathcal{B}) = \langle x_1 \rangle_{\mathcal{R}}$ denotes the principal ideal of $\mathcal{R}$ generated by x_1, $\mathcal{V}_{\mathbb{K}}(\mathcal{I}) = \{0\}$, and $\mathcal{P} = \mathbb{K} \setminus \{0\}$. If we set $W = (-1 \quad 0 \quad 1 \quad 0)$, then $B = x_1\, W$, $F = 1/2\,(-1 \quad 0 \quad 1 \quad 0)^T$ is a right inverse of W, and thus, for $\psi \in \mathcal{P}$, $E_\psi = \psi_1^{-1}\, F$ is a right inverse of the matrix $B(\psi)$. Computing a basis of $\ker_{\mathbb{K}}(W.)$, we get $\ker_{\mathbb{K}}(W.) = \mathrm{im}_{\mathbb{K}}(C.)$, where C is defined by

$$C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathbb{K}^{4\times 3}, \tag{12}$$

and thus, we have $\ker_{\mathbb{K}}(B(\psi).) = \mathrm{im}_{\mathbb{K}}(C.)$ for all $\psi \in \mathcal{P}$. Then, eq. (10) defines solutions of eq. (4), i.e.:

$$\forall\, \psi \in \mathbb{K} \setminus \{0\}, \quad \forall\, Y' \in \mathbb{K}^{3\times 4}, \quad \begin{cases} u = Z\,\psi = (-\psi \quad 0 \quad 0 \quad \psi)^T, \\ v = \dfrac{1}{2\,\psi} \begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} Y'. \end{cases} \tag{13}$$

Finally, all the solutions $(u,\, v)$ of eq. (4) with full row rank matrices v can be written as eq. (13) for all $Y' \in \mathbb{K}^{3\times 4}$ satisfying the condition $\det((Y^T \quad Y'^T)^T) \neq 0$.

Let us state a few comments on Theorem 1 and Corollary 1. Let us suppose that $\mathcal{I} \neq \langle 0 \rangle$ and $g_1, \ldots, g_t$ are homogeneous polynomials that generate the ideal $\mathcal{I}$, i.e., $\mathcal{I} = \langle g_1, \ldots, g_t \rangle_{\mathcal{R}}$. If $D(g_i) = \{\psi \in \mathbb{K}^{d\times 1} \mid g_i(\psi) \neq 0\}$ denotes the *distinguished open Zariski set* defined by g_i [10, 20], then we have $\mathcal{P} = \mathbb{K}^{d\times 1} \setminus \mathcal{V}_{\mathbb{K}}(\langle g_1, \ldots, g_t \rangle) = \bigcup_{i=1}^t D(g_i)$. For instance, if $l = r$, then $B \in \mathcal{R}^{l\times l}$, and thus, $g_1 = \det(B)$, $t = 1$, and $\mathcal{P} = D(g_1)$. For every $\psi \in D(g_i)$, eq. (10) characterizes a set of solutions of eq. (4) satisfying Conditions 1 and 2 of Lemma 1, where E_ψ is a right inverse of the matrix $B(\psi) \in \mathbb{K}^{l\times r}$ and $C_\psi \in \mathbb{K}^{r\times(r-l)}$ is a matrix whose columns define a basis of $\ker_{\mathbb{K}}(B(\psi).)$. Note that this set of solutions is defined only for the points ψ in $D(g_i)$, and thus, it is only a local solution. Moreover, this set of solutions varies with $\psi \in D(g_i)$ because the matrices E_ψ and C_ψ change with ψ in $D(g_i)$. Using effective module theory (see, e.g., [1, 6, 12]), in [2], it is shown how to "glue" these local solutions together to get a global solution on the whole open set $D(g_i)$ of $\mathbb{K}^d$, namely, to get a regular closed-form solution on $D(g_i)$. To do that, we use the existence of a right inverse $E_{g_i} \in \mathcal{R}_{g_i}^{l\times r}$ of B, where $\mathcal{R}_{g_i} = \{a/g_i^k \mid a \in \mathcal{R},\, k \in \mathbb{Z}_{\geq 0}\}$ denotes the *localization* of the integral domain $\mathcal{R}$ at the *multiplicatively closed set* $\{g_i^k\}_{k\in\mathbb{Z}_{\geq 0}}$ [10], and the existence of a matrix $C_{g_i} \in \mathcal{R}_{g_i}^{r\times s_i}$, where $s_i \geq r - l$, whose columns generate the $\mathcal{R}_{g_i}$-module $\ker_{\mathcal{R}_{g_i}}(B.)$, i.e., satisfying $\ker_{\mathcal{R}_{g_i}}(B.) = \mathrm{im}_{\mathcal{R}_{g_i}}(C_{g_i}.)$. We then have $\ker_{\mathbb{K}}(B(\psi).) = \mathrm{im}_{\mathbb{K}}(C_{g_i}(\psi).)$ for all $\psi \in D(g_i)$.

Remark 6. The matrices E_{g_i} and C_{g_i} can be computed using methods of computer algebra [2, 5]. For more details, see Remark 9 in the next section. For instance, they can be computed using the OreModules package [6], the `CapAndHomalg` library [1], and the `Singular` system [12].

In Algorithm 2, we sum up the computation of the set of all the solutions of eq. (4) satisfying Conditions 1 and 2 of Lemma 1.

Remark 7. The fact that $s_i \geq r - l$, where $r - l = \dim_{\mathbb{K}}(\ker_{\mathbb{K}}(B(\psi).))$ for $\psi \in D(g_i)$ comes from the fact that the identity $\ker_{\mathbb{K}}(B(\psi).) = \mathrm{im}_{\mathbb{K}}(C_{g_i}(\psi).)$ holds for all $\psi \in D(g_i)$ and not only for a particular $\psi \in D(g_i)$. More generators than $r - l$ is usually needed for generating the $\mathcal{R}_{g_i}$-module $\ker_{\mathcal{R}_{g_i}}(B.)$. We have $s_i = r - l$ when $\ker_{\mathcal{R}_{g_i}}(B.)$ is a free $\mathcal{R}_{g_i}$-module.

Algorithm 2 RankFactorizationProblemWithConditions

procedure RANKFACTORIZATIONPROBLEMWITHCONDS($D_1, \ldots, D_r \in \mathbb{K}^{m\times m}, M \in \mathbb{K}^{m\times n}$)
  PRERANKFACTORIZATIONPROBLEM($D_1, \ldots, D_r, M$) $= \{Z, Y, \mathcal{I}, \mathcal{R}, B\}$
  Let $\{g_i\}_{i=1,\ldots,t}$ be a set of generators of $\mathcal{I}$, i.e., $\mathcal{I} = \langle g_1, \ldots, g_t\rangle_{\mathcal{R}}$
  For $i = 1, \ldots, t$, compute a right inverse $E_{g_i} \in \mathcal{R}_{g_i}^{l\times r}$ of B
  For $i = 1, \ldots, t$, compute a matrix $C_{g_i} \in \mathcal{R}_{g_i}^{r\times s_i}$ satisfying $\ker_{\mathcal{R}_{g_i}}(B.) = \mathrm{im}_{\mathcal{R}_{g_i}}(C_{g_i}.)$
  return Z, $\mathcal{I} = \langle g_1, \ldots, g_t\rangle$, $\{(E_{g_i},\ C_{g_i})\}_{i=1,\ldots,t}$, which are such that

$$\forall\, \psi \in D(g_i), \quad \forall\, Y' \in \mathbb{K}^{s_i\times m}, \quad \begin{cases} u = Z\,\psi, \\ v = (E_{g_i}(\psi) \quad C_{g_i}(\psi)) \begin{pmatrix} Y \\ Y' \end{pmatrix}, \end{cases} \tag{14}$$

  is a regular solution of eq. (4) on $D(g_i)$ satisfying Conditions 1 and 2 of Lemma 1, where $E_{g_i}(\psi)$ (resp., $C_{g_i}(\psi)$) denotes the evaluation of the matrix E_{g_i} (resp., C_{g_i}) at the point $x = \psi$
end procedure

In [2], it is shown that the $\mathcal{R}_{g_i}$-module $\ker_{\mathcal{R}_{g_i}}(B.)$ is *stably free* of rank $r-l$, i.e., *locally free* [10, 25]. Hence, the study of these regular closed-form solutions on each $D(g_i)$ advocates for the study of the following well-known difficult problems in module theory:

1. Recognizing whether or not a finitely generated stably free $\mathcal{R}_{g_i}$-module is *free*.
2. Effective computation of bases of finitely generated free $\mathcal{R}_{g_i}$-modules (i.e., study possible effective extensions of the well-known *Quillen-Suslin theorem*) [10, 11, 25].
3. Effective computation of a minimal set of generators for $\mathcal{I} = \mathrm{Fitt}_0(\mathcal{B})$ [10].

Points 1 and 2 are related to the possibility of considering $s_i = r - l$ in eq. (14). For instance, the first two points can be effectively solved in the following particular cases:

- $g_i \in \mathbb{K} \setminus \{0\}$, i.e., $\mathcal{R}_{g_i} = \mathcal{R}$, by an effective version of the Quillen-Suslin theorem [10, 11].
- $g_i = x_i$ by an extension of the Quillen-Suslin theorem to *generalized Laurent polynomial ring*. See [11] and the references therein.
- $r = l + 1$ since a stably free module of rank 1 over a commutative ring is free [10, 11].
- $d = 1$ because $\mathcal{R} = \mathbb{K}[x_1]$ is a *principal ideal domain* and stably free $\mathcal{R}$-modules (e.g., $\ker_{\mathcal{R}}(B.)$) are free [25]. Bases of free $\mathcal{R}$-modules can then be computed using *Smith normal forms* [11].
- $d = 2$ because $\ker_{\mathcal{R}}(B.)$ is then a *projective* $\mathcal{R} = \mathbb{K}[x_1, x_2]$-module [10, 25], and thus, a free $\mathcal{R}$-module by the Quillen-Suslin theorem [10, 11, 25].

Point 3 is related to finding a *minimal cover* of $\mathcal{P} = \bigcup_{i=1}^{t} D(g_i)$ by distinguished open sets $D(g_i)$, i.e., to find a representation of the global solution space eq. (10) using a minimal number of regular closed-form solutions on distinguished open sets $D(g_i)$. We have the following facts:

- If $\mu(\mathcal{I})$ denotes the number of elements of a minimal set of generators of $\mathcal{I}$, then we know that $\mu(\mathcal{I}) = \mu\left(\mathcal{I}/\mathcal{I}^2\right)$, where $\mathcal{I}/\mathcal{I}^2$ is the so-called *conormal* $\mathcal{R}/\mathcal{I}$-module [2].
- $d = 1$ since $\mathcal{R} = \mathbb{K}[x_1]$ is principal and $\mathcal{I}$ can then be generated by an element of $\mathcal{R}$. For instance, in Example 1, eq. (12) is the unique regular solution on $\mathcal{P} = D(x_1)$.

For more details, we refer the interested reader to [2].

3 General solutions

The goal of this section is to extend the approach of Section 2 to characterize the general solutions of the rank factorization problem eq. (4). In other words, we shall not assume Conditions 1 and 2 of Lemma 1 anymore. We shall use standard results of module theory and homological algebra [10, 12, 25].

3.1 Case of a full row rank matrix M

In this section, we focus on the case of a matrix M with full row rank.

Lemma 4. *Let $M \in \mathbb{K}^{m\times n}$ be a full row rank matrix, i.e., $\ker_{\mathbb{K}}(.M) = 0$, $D_1, \ldots, D_r \in \mathbb{K}^{m\times m}$, and for $u \in \mathbb{K}^{m\times 1}$, $A(u) = (D_1\, u \ \ldots \ D_r\, u) \in \mathbb{K}^{m\times r}$.*

1. *The rank factorization problem eq.* (4) *has a solution if and only if the following linear cone*
$$\mathcal{U} = \left\{u \in \mathbb{K}^{m\times 1} \mid \operatorname{rank}_{\mathbb{K}}(A(u)) = m\right\} \tag{15}$$
of $\mathbb{K}^{m\times 1}$ is not empty.

2. *If $\mathcal{U} \neq \emptyset$, then for every $u \in \mathcal{U}$, all the solutions of the inhomogeneous linear system $A(u)\, v = M$ in $v \in \mathbb{K}^{r\times n}$ are then defined by $V_u = \left\{F_u\, M + C_u\, Y' \mid Y' \in \mathbb{K}^{(m-r)\times n}\right\}$, where $F_u \in \mathbb{K}^{r\times m}$ is a right inverse of $A(u)$ and the columns of the matrix $C_u \in \mathbb{K}^{r\times(m-r)}$ define a basis of $\ker_{\mathbb{K}}(A(u).)$, i.e., $\ker_{\mathbb{K}}(A(u).) = \operatorname{im}_{\mathbb{K}}(C_u.)$.*

Proof. 1. M having full row rank, i.e., $\operatorname{im}_{\mathbb{K}}(M.) = \mathbb{K}^{m\times 1}$, we have $m \leq n$ and there exists $N \in \mathbb{K}^{n\times m}$ such that $M\, N = I_m$. Now, eq. (4) yields $A(u)\,(v\, N) = I_m$, which shows that $A(u)$ has full row rank and $m \leq r$, i.e., $u \in \mathcal{U}$. Conversely, if there exists $u \in \mathbb{K}^{m\times 1}$ such that $A(u) \in \mathbb{K}^{m\times r}$ has full row rank, then there exists $F_u \in \mathbb{K}^{r\times m}$ such that $A(u)\, F_u = I_m$, which yields $A(u)\,(F_u\, M) = M$ and eq. (4) has a solution. Thus, eq. (4) is solvable if and only if $\mathcal{U} \neq \emptyset$. Finally, we have $\operatorname{rank}_{\mathbb{K}}(A(\lambda\, u)) = \operatorname{rank}_{\mathbb{K}}(A(u))$ for all $\lambda \in \mathbb{K} \setminus \{0\}$, i.e., $\mathcal{U}$ is a linear cone.

2. If $\mathcal{U} \neq \emptyset$, then the set of all the solutions of the inhomogeneous linear system $A(u)\, v = M$ in $v \in \mathbb{K}^{m\times 1}$ is clearly defined by V_u, where E_u is a right inverse of $A(u)$ and the columns of the matrix C_u defines a basis of $\ker_{\mathbb{K}}(A(u).)$. □

Remark 8. If M has full row rank, following the approach of Section 2.2 and using Remark 4, we have $u = \psi \in \mathbb{K}^{m\times 1}$, $B(\psi) = A(u)$, and $\mathcal{U} = \mathcal{P}$.

Proposition 1. *Let $M \in \mathbb{K}^{m\times n}$ be a full row rank matrix and $D_1, \ldots, D_r \in \mathbb{K}^{m\times m}$. Set $\mathcal{R} = \mathbb{K}[x_1, \ldots, x_m]$, $x = (x_1 \ \ldots \ x_m)^T$, $A = (D_1\, x \ \ldots \ D_r\, x) \in \mathcal{R}^{m\times r}$, $\mathcal{A} = \operatorname{coker}_{\mathcal{R}}(A.) = \mathcal{R}^{m\times 1}/(A\, \mathcal{R}^{r\times 1})$, and $\mathcal{I} = \operatorname{Fitt}_0(\mathcal{A})$ the ideal of $\mathcal{R}$ generated by all the $m \times m$ minors of the matrix A. Then, we have:*

1. *$\mathcal{U}$ is a quasi-affine variety of $\mathbb{K}^{m\times 1}$ defined by $\mathcal{U} = \mathbb{K}^{m\times 1} \setminus \mathcal{V}_{\mathbb{K}}(\mathcal{I})$.*

2. *If $\mathbb{K}$ is an algebraically closed field, then the rank factorization problem eq.* (4) *has a solution if and only $\mathcal{I} \neq \langle 0\rangle$.*

3. *Let us suppose that $\mathcal{I} \neq \langle 0\rangle$ and let $h_1, \ldots, h_s \in \mathcal{R} \setminus \{0\}$ be such that $\mathcal{I} = \langle h_1, \ldots, h_s\rangle$. Then, $\mathcal{U} = \bigcup_{i=1}^s D(h_i)$, i.e., $\{D(h_i)\}_{i=1,\ldots,s}$ is a cover of $\mathcal{U}$ by distinguished open subsets of $\mathbb{K}^{m\times 1}$.*

4. *For $i = 1, \ldots, s$, if $\mathcal{R}_{h_i} = \{a/h_i^k \mid a \in \mathcal{R},\, k \in \mathbb{Z}_{\geq 0}\}$ denotes the localization of the polynomial ring $\mathcal{R}$ at the multiplicatively closed set $\{h_i^k\}_{k\in\mathbb{Z}}$, then there exists a right inverse $F_{h_i} \in \mathcal{R}_{h_i}^{r\times m}$ of A, i.e., $A\, F_{h_i} = I_m$.*

5. *For $i = 1, \ldots, s$, there exists a matrix $C_{h_i} \in \mathcal{R}_{h_i}^{r\times t_i}$ whose columns generate the $\mathcal{R}_{h_i}$-module $\ker_{\mathcal{R}_{h_i}}(A.)$, i.e., which is such that $\ker_{\mathcal{R}_{h_i}}(A.) = \operatorname{im}_{\mathcal{R}_{h_i}}(C_{h_i}.)$.*

Proof. 1 is a direct consequence of eq. (15).

2. Using Point 1, eq. (4) has a solution if and only $\mathcal{V}_{\mathbb{K}}(\mathcal{I}) \neq \mathbb{K}^{m\times 1}$, i.e., under the hypothesis that $\mathbb{K}$ is an algebraically closed field, if and only if $\mathcal{I} \neq \langle 0\rangle$.

3. Let $h_1, \ldots, h_s \in \mathcal{R} \setminus \{0\}$ be such that $\mathcal{I} = \langle h_1, \ldots, h_s\rangle$. Then, we have $\mathcal{U} = \bigcup_{i=1}^s D(h_i)$, i.e., $\{D(h_i)\}_{i=1,\ldots,s}$ is a cover of $\mathcal{U}$ by distinguished open subsets of $\mathbb{K}^{m\times 1}$.

4. If $u \in D(h_i) = \left\{\psi \in \mathbb{K}^{m\times 1} \mid h_i(\psi) \neq 0\right\}$, then we have $\operatorname{rank}_{\mathbb{K}}(A(u)) = m$, which shows that there exists a right inverse $F_{h_i} \in \mathbb{K}^{r\times m}$ of $A(u)$ on $D(h_i)$. More generally, to prove the existence of a right inverse of $A(u)$ which is globally well-defined in $D(h_i)$ (and not only at the point $u \in D(h_i)$), we first note that $\operatorname{ann}_{\mathcal{R}}(\mathcal{A})^m \subseteq \operatorname{Fitt}_0(\mathcal{A}) \subseteq \operatorname{ann}_{\mathcal{R}}(\mathcal{A})$ (see, e.g., [10, 20]). Thus, we have $h_i\, \mathcal{A} = 0$, and thus, $S_{h_i}^{-1} \mathcal{A} = 0$,

where $S_{h_i}^{-1}\mathcal{A} = \{m/h_i^k \mid m \in \mathcal{A}, \, k \in \mathbb{Z}\}$ is the localization of $\mathcal{A}$ at S_{h_i} (see, e.g., [10, 20, 25]), i.e., we have $A\,\mathcal{R}_{h_i}^{r\times 1} = \mathcal{R}_{h_i}^{m\times 1}$, which shows the existence of a right inverse $F_{h_i} \in \mathcal{R}_{h_i}^{r\times m}$ of A, i.e., $A\,F_{h_i} = I_m$.

5. Since $\mathcal{R}_{h_i}$ is a *noetherian ring* (see, e.g., [10, 20, 25]), $\ker_{\mathcal{R}_{h_i}}(A.)$ is a finitely generated $\mathcal{R}_{h_i}$-module, and thus, there exists $C_{h_i} \in \mathcal{R}_{h_i}^{r\times t_i}$ which satisfies $\ker_{\mathcal{R}_{h_i}}(A.) = \mathrm{im}_{\mathcal{R}_{h_i}}(C_{h_i}.)$. □

Recall that if P is a matrix whose entries are functions of the vector variable $x = (x_1 \ \dots \ x_m)^T$, then $P(u)$ denotes the evaluation of P at the point $u = (u_1 \ \dots \ u_m) \in \mathbb{K}^{m\times 1}$.

Theorem 2. *Let $M \in \mathbb{K}^{m\times n}$ be a full row rank matrix and $D_1, \dots, D_r \in \mathbb{K}^{m\times m}$. Set $\mathcal{R} = \mathbb{K}[x_1, \dots, x_m]$, $x = (x_1 \ \dots \ x_m)^T$, $A = (D_1\, x \ \dots \ D_r\, x) \in \mathcal{R}^{m\times r}$, $\mathcal{A} = \mathrm{coker}_{\mathcal{R}}(A.) = \mathcal{R}^{m\times 1}/(A\,\mathcal{R}^{r\times 1})$, and $\mathcal{I} = \mathrm{Fitt}_0(\mathcal{A})$ the ideal of $\mathcal{R}$ defined by all the $m \times m$ minors of the matrix A.*

Let us suppose that $\mathcal{I} \neq \langle 0\rangle$ and let $h_1, \dots, h_s \in \mathcal{R} \setminus \{0\}$ be such that $\mathcal{I} = \langle h_1, \dots, h_s\rangle$.

With the notations of 3 and 4 of Proposition 1, all the solutions of the rank factorization problem eq. (4) *with $u \in D(h_i) = \{\psi \in \mathbb{K}^{m\times 1} \mid h_i(\psi) \neq 0\}$ are then defined by:*

$$\begin{cases} u \in D(h_i), \\ v\,(u, Y') = F_{h_i}(u)\, M + C_{h_i}(u)\, Y', \ \forall\, Y' \in \mathbb{K}^{t\times n}, \end{cases} \qquad i = 1, \dots, s. \tag{16}$$

Finally, for $u \in \mathcal{U} = \bigcup_{i=1}^s D(h_i)$, there exists $i \in \{1, \dots, s\}$ such that $u \in D(h_i)$ and eq. (16) *defines the set of all the solutions of eq.* (4) *in $D(h_i)$.*

Proof. Using Proposition 1, we have $\mathcal{U} = \bigcup_{i=1}^s D(h_i)$ and there exist $F_{h_i} \in \mathcal{R}_{h_i}^{r\times m}$ and $C_{h_i} \in \mathcal{R}_{h_i}^{r\times t}$ such that $A\,F_{h_i} = I_m$ and $\ker_{\mathcal{R}_{h_i}}(A.) = \mathrm{im}_{\mathcal{R}_{h_i}}(C_{h_i}.)$ for $i = 1, \dots, s$. Then, we get

$$\forall\, u \in D(h_i), \quad \forall\, Y' \in \mathbb{K}^{t\times n}, \quad A(u)\,(F_{h_i}(u)\, M + C_{h_i}(u)\, Y') = (A\,(F_{h_i}\, M + C_{h_i}\, Y'))(u) = M,$$

which shows that eq. (16) are solutions of eq. (4) with $u \in D(h_i)$.

Let $(u, \, v)$ be a solution of eq. (4) with $u \in D(h_i)$. Then, $v - F_{h_i}(u)\, M$ satisfies $A(u)\,(v - F_{h_i}(u)\, M) = 0$, i.e., $v - F_{h_i}(u)\, M \in \ker_{\mathbb{K}}(A(u).)$. Evaluating the identity $A\, C_{h_i} = 0$ at u, we get $A(u)\, C_{h_i}(u) = 0$, i.e., $\mathrm{im}_{\mathbb{K}}(C_{h_i}(u).) \subseteq \ker_{\mathbb{K}}(A(u).)$. Let us prove the reverse inclusion. Applying the *exact covariant functor* $\mathcal{R}_{h_i} \otimes_{\mathcal{R}} \cdot$ (since $\mathcal{R}_{h_i}$ is a *flat* $\mathcal{R}$-module; see, e.g., [10, 25]) to the following *exact sequence* of $\mathcal{R}$-modules defining a *finite presentation* of $\mathcal{A}$

$$\mathcal{R}^{r\times 1} \xrightarrow{A.} \mathcal{R}^{m\times 1} \xrightarrow{\sigma} \mathcal{A} \longrightarrow 0,$$

we obtain the following *split exact sequence* of $\mathcal{R}_{h_i}$-modules (see, e.g., [10, 25]):

$$\mathcal{R}_{h_i}^{t_i\times 1} \xrightarrow{C_{h_i}.} \mathcal{R}_{h_i}^{r\times 1} \xrightarrow{A.} \mathcal{R}_{h_i}^{m\times 1} \xrightarrow{\mathrm{id}\otimes\sigma} S_{h_i}^{-1}\mathcal{A} = 0. \tag{17}$$

Setting $\Pi = I_r - F_{h_i}\, A \in \mathcal{R}_{h_i}^{r\times r}$ and using the identity $A\, F_{h_i} = I_m$, we obtain $A\,\Pi = 0$, i.e., we have $\mathrm{im}_{\mathcal{R}_{h_i}}(\Pi.) \subseteq \ker_{\mathcal{R}_{h_i}}(A.) = \mathrm{im}_{\mathcal{R}_{h_i}}(C_{h_i}.)$, which proves the existence of $G_{h_i} \in \mathcal{R}_{h_i}^{t_i\times r}$ such that $\Pi = C_{h_i}\, G_{h_i}$, i.e., $C_{h_i}\, G_{h_i} + F_{h_i}\, A = I_r$. Evaluating this identity at u, we then obtain $C_{h_i}(u)\, G_{h_i}(u) + F_{h_i}(u)\, A(u) = I_r$. Now, if $\xi \in \ker_{\mathbb{K}}(A(u).)$, this last identity implies the identity $\xi = C_{h_i}(u)\,(G_{h_i}(u)\,\xi)$, which shows $\ker_{\mathbb{K}}(A(u).) \subseteq \mathrm{im}_{\mathbb{K}}(C_{h_i}(u).)$ and proves that $\ker_{\mathbb{K}}(A(u).) = \mathrm{im}_{\mathbb{K}}(C_{h_i}(u).)$. We then have $v - F_{h_i}(u)\, M \in \mathrm{im}_{\mathbb{K}}(C_{h_i}(u).)$, and thus, there exists $Y' \in \mathbb{K}^{t_i\times n}$ such that $v - F_{h_i}(u)\, M = C_{h_i}(u)\, Y'$, which proves that $(u, \, v)$ is of the form of eq. (16) with $u \in D(h_i)$, i.e., all the solutions of eq. (4) with $u \in D(h_i)$ are defined by eq. (16).

Finally, let $(u, \, v)$ be a solution of eq. (4). By 1 of Lemma 4, $u \in \mathcal{U} \neq \emptyset$, i.e., by 3 of Proposition 1, $u \in \mathcal{U} = \bigcup_{i=1}^s D(h_i)$. Then, there exists $i \in \{1, \dots, s\}$ such that $u \in D(h_i)$ and the last result holds. □

Remark 9. A matrix $F_{h_i} \in \mathcal{R}_{h_i}^{r\times m}$ satisfying $A\, F_{h_i} = I_m$ can be computed as follows: $h_i\, \mathcal{A} = 0$ (see the proof of 4 of Proposition 1), where $\mathcal{A} = \mathcal{R}^{m\times 1}/(A\,\mathcal{R}^{r\times 1})$, yields the identity $h_i\, I_m = A\, G_{h_i}$ for a certain $G_{h_i} \in \mathcal{R}^{r\times m}$, and thus, $F_{h_i} = h_i^{-1}\, G_{h_i} \in \mathcal{R}_{h_i}^{r\times m}$. A matrix G_{h_i} can be obtained using a factorization problem (see, e.g., [6]), i.e., by solving a standard membership problem in $\mathcal{R}^{m\times 1}$, where

$\mathcal{R} = \mathbb{K}[x_1, \ldots, x_m]$. This matrix can be computed by, for instance, the `LocalLeftInverse` command of the OreModules package [6] or the `PreInverse` command of the `CapAndHomalg` library [1].

To compute a matrix C_{h_i} satisfying $\ker_{\mathcal{R}_{h_i}}(A.) = \mathrm{im}_{\mathcal{R}_{h_i}}(C_{h_i}.)$, we can first compute $\ker_{\mathcal{R}}(A.)$ to get $C \in \mathcal{R}^{r\times t}$ such that $\ker_{\mathcal{R}}(A.) = \mathrm{im}_{\mathcal{R}}(C.)$ (see, e.g., [5]). Then, the application of the exact functor $\mathcal{R}_{h_i} \otimes_{\mathcal{R}} \cdot$ to the exact sequence of $\mathcal{R}$-modules $\mathcal{R}^{t\times 1} \xrightarrow{C.} \mathcal{R}^{r\times 1} \xrightarrow{A.} \mathcal{R}^{m\times 1}$ yields the exact sequence of $\mathcal{R}_{h_i}$-modules $\mathcal{R}_{h_i}^{t\times 1} \xrightarrow{C.} \mathcal{R}_{h_i}^{r\times 1} \xrightarrow{A.} \mathcal{R}_{h_i}^{m\times 1}$. Thus, we have $\ker_{\mathcal{R}_{h_i}}(A.) = \mathrm{im}_{\mathcal{R}_{h_i}}(C.)$, which shows that we can take $C_{h_i} = C$. Note that $t_i = t$. Finally, to reduce t_i, we can follow the next approach.

Considering $\mathcal{R}[y] = \mathbb{K}[x_1, \ldots, x_m, y]$ and denoting by $\langle y\, h_i - 1\rangle$ the ideal of $\mathcal{R}[y]$ generated by $y\, h_i - 1$, we then have $\mathcal{R}_{h_i} = \mathcal{R}[y]/\langle y\, h_i - 1\rangle$. The following exact sequence of $\mathcal{R}[y]$-modules

$$0 \longrightarrow \mathcal{R}[y] \xrightarrow{.(y\, h_i - 1)} \mathcal{R}[y] \xrightarrow{\vartheta} \mathcal{R}_{h_i} \longrightarrow 0$$

combined with eq. (17) seen as an exact sequence of $\mathcal{R}[y]$-modules, yields the *commutative exact diagram* of $\mathcal{R}[y]$-modules:

$$\begin{array}{ccccc}
0 & & 0 & & \\
\uparrow & & \uparrow & & \\
\mathcal{R}_{h_i}^{r\times 1} & \xrightarrow{A.} & \mathcal{R}_{h_i}^{m\times 1} & \xrightarrow{\mathrm{id}\otimes\sigma} & S_{h_i}^{-1}\mathcal{A} = 0. \\
{\scriptstyle \mathrm{id}_r \otimes \vartheta}\uparrow & & {\scriptstyle \mathrm{id}_m \otimes \vartheta}\uparrow & & \\
\mathcal{R}[y]^{r\times 1} & \xrightarrow{A.} & \mathcal{R}[y]^{m\times 1} & & \\
{\scriptstyle (y\, h_i - 1)\, I_r.}\uparrow & & {\scriptstyle (y\, h_i - 1)\, I_m.}\uparrow & & \\
\mathcal{R}[y]^{r\times 1} & \xrightarrow{A.} & \mathcal{R}[y]^{m\times 1} & &
\end{array}$$

Computing $\ker_{\mathcal{R}[y]}((A \quad (y\, h_i - 1)\, I_m).)$, we obtain $U \in \mathcal{R}[y]^{r\times t}$ and $V \in \mathcal{R}[y]^{m\times t}$ such that:

$$\ker_{\mathcal{R}[y]}((A \quad (y\, h_i - 1)\, I_m).) = \mathrm{im}_{\mathcal{R}[y]}\left(\begin{pmatrix} U \\ V \end{pmatrix}\right).$$

Now, 1 of Proposition 3.1 of [7] yields $\ker_{\mathcal{R}_{h_i}}(A.) = \mathrm{im}_{\mathcal{R}[y]}(U.)/\mathrm{im}_{\mathcal{R}[y]}((y\, h_i - 1)\, I_r.) = \mathrm{im}_{\mathcal{R}_{h_i}}(C_{h_i}.)$, where C_{h_i} is the matrix obtained by setting $y\, h_i$ to 1 (which corresponds to substituting y by h_i^{-1}) in U. A matrix U can be computed by using, e.g., the `SyzygyModule` command of OreModules or the `WeakKernelEmbedding` command of `CapAndHomalg`.

The different computations are implemented in the RankFactorization package [9].

Example 2. Let us consider the following matrices:

$$M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad D_1 = \begin{pmatrix} -1 & -2 \\ 1 & 2 \end{pmatrix}, \quad D_2 = \begin{pmatrix} -3 & -4 \\ 3 & 4 \end{pmatrix}.$$

Then, we have $m = n = r = 2$, $x = (x_1 \quad x_2)^T$, and:

$$A = \begin{pmatrix} -x_1 - 2\, x_2 & -3\, x_1 - 4\, x_2 \\ x_1 + 2\, x_2 & 3\, x_1 + 4\, x_2 \end{pmatrix} \implies \det(A) = 0 \implies \mathcal{I} = \langle 0 \rangle \implies \mathcal{U} = \emptyset.$$

Hence, the corresponding rank factorization problem eq. (4) has no solutions.

Example 3. Let us consider the following matrices:

$$M = \begin{pmatrix} 15 & 14 & 13 \\ 24 & 20 & 16 \end{pmatrix}, \quad D_1 = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \quad D_2 = \begin{pmatrix} 1 & 2 \\ -1 & 2 \end{pmatrix}, \quad D_3 = \begin{pmatrix} 1 & 3 \\ 4 & 3 \end{pmatrix}.$$

Then, we have $l = \mathrm{rank}_{\mathbb{K}}(M) = 2 = m < r = 3 = n$. In particular, M has full row rank. Let us consider $\mathcal{R} = \mathbb{K}[x_1, x_2]$ and the $\mathcal{R}$-module $\mathcal{A} = \mathcal{R}^{2\times 1}/(A\, \mathcal{R}^{3\times 1})$ finitely presented by the following matrix:

$$A = (D_1\, x \quad D_2\, x \quad D_3\, x) = \begin{pmatrix} x_1 - x_2 & x_1 + 2\, x_2 & x_1 + 3\, x_2 \\ x_1 + x_2 & -x_1 + 2\, x_2 & 4\, x_1 + 3\, x_2 \end{pmatrix} \in \mathcal{R}^{2\times 3}.$$

We can now check that $\mathcal{I} = \mathrm{Fitt}_0(\mathcal{A}) = \langle x_1^2,\, x_1\, x_2,\, x_2^2 \rangle$. Hence, we have:

$$\mathcal{U} = \mathbb{K}^{2\times 1} \setminus \mathcal{V}_{\mathbb{K}}(\mathcal{I}) = D\left(x_1^2\right) \cup D(x_1\, x_2) \cup D\left(x_2^2\right) = \mathbb{K}^{2\times 1} \setminus \{(0 \quad 0)^T\}.$$

Moreover, we have:

$$F_{x_1^2} = \begin{pmatrix} \dfrac{66}{97\,x_1} & \dfrac{6}{97\,x_1} \\ \dfrac{38}{97\,x_1} + \dfrac{33\,x_2}{97\,x_1^2} & -\dfrac{23}{97\,x_1} + \dfrac{3\,x_2}{97\,x_1^2} \\ -\dfrac{7}{97\,x_1} - \dfrac{22\,x_2}{97\,x_1^2} & \dfrac{17}{97\,x_1} - \dfrac{2\,x_2}{97\,x_1^2} \end{pmatrix} \in \mathcal{R}_{x_1^2}^{3\times 2}, \quad R\, F_{x_1^2} = I_2,$$

$$F_{x_1\,x_2} = \begin{pmatrix} -\dfrac{55}{194\,x_2} & -\dfrac{5}{194\,x_2} \\ \dfrac{33}{194\,x_2} + \dfrac{21}{194\,x_1} & \dfrac{3}{194\,x_2} - \dfrac{51}{194\,x_1} \\ \dfrac{11}{97\,x_2} - \dfrac{7}{97\,x_1} & \dfrac{1}{97\,x_2} + \dfrac{17}{97\,x_1} \end{pmatrix} \in \mathcal{R}_{x_1\,x_2}^{3\times 2}, \quad R\, F_{x_1\,x_2} = I_2,$$

$$F_{x_2^2} = \begin{pmatrix} -\dfrac{35\,x_1}{388\,x_2^2} - \dfrac{1}{2\,x_2} & \dfrac{85\,x_1}{388\,x_2^2} + \dfrac{1}{2\,x_2} \\ \dfrac{21\,x_1}{388\,x_2^2} + \dfrac{31}{388\,x_2} & -\dfrac{51\,x_1}{388\,x_2^2} + \dfrac{91}{388\,x_2} \\ \dfrac{7\,x_1}{194\,x_2^2} + \dfrac{11}{97\,x_2} & -\dfrac{17\,x_1}{194\,x_2^2} + \dfrac{1}{97\,x_2} \end{pmatrix} \in \mathcal{R}_{x_2^2}^{3\times 2}, \quad R\, F_{x_2^2} = I_2.$$

If we note

$$C = \begin{pmatrix} 5\,x_1^2 + 12\,x_1\,x_2 \\ -3\,x_1^2 + 5\,x_1\,x_2 + 6\,x_2^2 \\ -2\,x_1^2 - 4\,x_2^2 \end{pmatrix} \in \mathcal{R}^{3\times 1},$$

then we have $\ker_{\mathcal{R}_h}(A.) = \mathrm{im}_{\mathcal{R}_h}(C.)$ for $h = x_1^2,\, x_1\,x_2$, and x_2^2, which shows that all the solutions of eq. (4) are of the form of eq. (16):

$$\begin{cases} \forall\, u \in D\left(x_1^2\right), & \forall\, Y'_{x_1^2} \in \mathbb{K}^{1\times 3}, \quad v\left(u, Y'_{x_1^2}\right) = F_{x_1^2}(u)\, M + C(u)\, Y'_{x_1^2}, \\ \forall\, u \in D(x_1\, x_2), & \forall\, Y'_{x_1\,x_2} \in \mathbb{K}^{1\times 3}, \quad v\left(u, Y'_{x_1\,x_2}\right) = F_{x_1\,x_2}(u)\, M + C(u)\, Y'_{x_1\,x_2}, \\ \forall\, u \in D\left(x_2^2\right), & \forall\, Y'_{x_2^2} \in \mathbb{K}^{1\times 3}, \quad v\left(u, Y'_{x_2^2}\right) = F_{x_2^2}(u)\, M + C(u)\, Y'_{x_2^2}. \end{cases} \tag{18}$$

For $h = x_1^2,\, x_1\,x_2$, or x_2^2, the determinant of $U_h = (F_h \quad C)$ is 1, i.e., U_h is invertible. Hence, the matrix v defined by eq. (18) has full row rank if and only if so has $\left(M^T \quad Y_h'^T\right)^T \in \mathbb{K}^{3\times 3}$, where $Y'_h = (Y'_{h,1} \quad Y'_{h,2} \quad Y'_{h,3}) \in \mathbb{K}^{1\times 3}$, i.e., if and only we have $Y'_{h,1} - 2\, Y'_{h,2} + Y'_{h,3} \neq 0$.

3.2 General case

Let us now consider the case where the matrix M is not full row rank, i.e., $\mathrm{im}_{\mathbb{K}}(M.) \neq \mathbb{K}^{m\times 1}$. Set again $l = \mathrm{rank}_{\mathbb{K}}(M)$ and $p = m - l > 0$. Moreover, let $L \in \mathbb{K}^{p\times m}$ be a full column rank matrix whose rows define a basis of the $\mathbb{K}$-vector space $\ker_{\mathbb{K}}(.M)$, i.e., which satisfies $\ker_{\mathbb{K}}(.M) = \mathrm{im}_{\mathbb{K}}(.L)$.

Remark 10. Note that the rows of the matrix L define a generating set of *compatibility conditions* of the inhomogeneous linear system $M\,\eta = \zeta$, where ζ is a fixed vector of $\mathbb{K}^{m\times 1}$ and η is sought in $\mathbb{K}^{m\times 1}$. Indeed, a necessary (and sufficient) condition on ζ for the solvability of $M\,\eta = \zeta$ is defined by $L\,\zeta = 0$.

3.2.1 Necessary conditions on u

Let us suppose that a solution $(u,\, v)$ of the rank factorization problem eq. (4) exists. If we set $Q(u) = L\,A(u) \in \mathbb{K}^{p\times r}$, then combining $A(u)\,v = M$ with $L\,M = 0$, we obtain $Q(u)\,v = L\,M = 0$, which shows that all the columns of v belong to $\ker_{\mathbb{K}}(Q(u).)$. Thus, $u \in \mathbb{K}^{m\times 1}$ must necessarily be such that:

$$\ker_{\mathbb{K}}(Q(u).) \neq 0.$$

In linear algebra, the rank-nullity theorem yields the *index* of $Q(u).$ is defined by

$$\dim_{\mathbb{K}}(\ker_{\mathbb{K}}(Q(u).)) - \dim_{\mathbb{K}}(\operatorname{coker}_{\mathbb{K}}(Q(u).)) = r - p,$$

which yields $\dim_{\mathbb{K}}(\ker_{\mathbb{K}}(Q(u).)) \geq r - p$. Thus, we have:

1. If $r > p$, i.e., if the matrix $Q(u)$ is wide, then $\ker_{\mathbb{K}}(Q(u).) \neq 0$ for all $u \in \mathbb{K}^{m\times 1}$.

2. If $r \leq p$, i.e., if the matrix $Q(u)$ is tall or square, then $\ker_{\mathbb{K}}(Q(u).)$ can be reduced to 0 for almost all $u \in \mathbb{K}^{m\times 1}$. The u's for which $\ker_{\mathbb{K}}(Q(u).) \neq 0$ are the common zeros in $\mathbb{K}^{m\times 1}$ of all the $r \times r$ minors of the matrix $Q(u) \in \mathbb{K}^{p\times r}$.

Let us state again the definition of the *Fitting ideals* (see, e.g., [10, 23]).

Definition 1. Let $\mathcal{R} = \mathbb{K}[x_1,\ldots,x_m]$, $N \in \mathcal{R}^{s\times r}$, and $\mathcal{N} = \mathcal{R}^{1\times r}/(\mathcal{R}^{1\times s}\,N)$ be the $\mathcal{R}$-module finitely presented by the matrix N. *The Fitting ideals* $\operatorname{Fitt}_i(\mathcal{N})$'s of $\mathcal{N}$ are defined by:

- $\operatorname{Fitt}_i(\mathcal{N})$ is the ideal of $\mathcal{R}$ generated by all the $(r-i)\times(r-i)$ minors of the matrix N for $1 \leq r-i \leq s$,
- $\operatorname{Fitt}_i(\mathcal{N}) = \langle 0\rangle$ for $s < r - i$,
- $\operatorname{Fitt}_i(\mathcal{N}) = \mathcal{R}$ for $r - i \leq 0$.

We can now state a necessary condition on u for the existence of a solution of eq. (4).

Lemma 5. *Let* $\mathcal{R} = \mathbb{K}[x_1,\ldots,x_m]$, $x = (x_1\ \ldots\ x_m)^T$, $A = (D_1\,x\ \ldots\ D_r\,x) \in \mathcal{R}^{m\times r}$, $Q = L\,A \in \mathcal{R}^{p\times r}$. *Moreover, let* $\mathcal{Q} = \operatorname{coker}_{\mathcal{R}}(.Q) = \mathcal{R}^{1\times r}/(\mathcal{R}^{1\times p}\,Q)$ *be the* $\mathcal{R}$*-module finitely presented by the matrix* Q*. Equivalently, the* $\mathcal{R}$*-module* $\mathcal{Q}$ *is defined by the following finite presentation:*

$$\mathcal{R}^{1\times p} \xrightarrow{.Q} \mathcal{R}^{1\times r} \xrightarrow{\pi} \mathcal{Q} \longrightarrow 0. \tag{19}$$

A necessary condition for the solvability of eq. (4) *is* $u \in \mathcal{V}_{\mathbb{K}}(\operatorname{Fitt}_0(\mathcal{Q}))$*, where* $\operatorname{Fitt}_0(\mathcal{Q})$ *is either* $\langle 0\rangle$ *if* $r > m - l$ *or the ideal of* $\mathcal{R}$ *generated by all the* $r \times r$ *minors of* $Q \in \mathcal{R}^{(m-l)\times r}$ *if* $r \leq m - l$*. Finally,* $\operatorname{Fitt}_0(\mathcal{Q})$ *is either generated by homogeneous polynomials of total degree* r *or by 0.*

Proof. By definition, for $r - p \leq i \leq r - 1$, $\operatorname{Fitt}_i(\mathcal{Q})$ is the ideal of $\mathcal{R}$ generated by all the $(r-i)\times(r-i)$ minors of the matrix $Q = (L\,D_1\,x\ \ldots\ L\,D_r\,x)$. Thus, $\operatorname{Fitt}_0(\mathcal{Q})$ is either $\langle 0\rangle$ if $r > p$, or the ideal generated by all the $r \times r$ minors of $Q \in \mathcal{R}^{p\times r}$ if $r \leq p$. Hence, if $r > p$, then $\operatorname{Fitt}_0(\mathcal{Q}) = \langle 0\rangle$ (which then yields $\mathcal{V}_{\mathbb{K}}(\operatorname{Fitt}_0(\mathcal{Q})) = \mathbb{K}^{m\times 1}$) or, if $r \leq p$, $\operatorname{Fitt}_0(\mathcal{Q})$ is generated by homogeneous polynomials of degree r or 0 if all the $r \times r$ minors of $Q(u)$ are reduced to 0. Finally, the fact that $\ker_{\mathbb{K}}(Q(u).) \neq 0$ is equivalent to $u \in \mathcal{V}_{\mathbb{K}}(\operatorname{Fitt}_0(\mathcal{Q}))$ since the rank of the matrix $Q(u)$ is then strictly less than r. □

More generally, for $i = 0,\ldots,r-1$, if $i < r-m+l$, then $\operatorname{Fitt}_i(\mathcal{Q}) = \langle 0\rangle$, or if $i \geq r-m+l$, $\operatorname{Fitt}_i(\mathcal{Q}) = \langle 0\rangle$ is generated by homogeneous polynomials of degree $r - i$ or by 0 if all the $(r-i)\times(r-i)$ minors of $Q(u)$ are reduced to 0. Therefore, $1 \notin \operatorname{Fitt}_i(\mathcal{Q})$ for $i = 0,\ldots,r-1$, which shows that $\operatorname{Fitt}_i(\mathcal{Q}) \neq \mathcal{R}$ for $i = 0,\ldots,r-1$. Finally, we have:

$$\mathcal{V}_{\mathbb{K}}(\operatorname{Fitt}_i(\mathcal{Q})) = \{u \in \mathbb{K}^{m\times 1} \mid \operatorname{rank}_{\mathbb{K}}(Q(u)) < r - i\}, \quad i = 0,\ldots,r-1.$$

Remark 11. We can easily check again that the following chain of Fitting ideals holds

$$\langle 0\rangle \subseteq \mathrm{Fitt}_0(\mathcal{Q}) \subseteq \mathrm{Fitt}_1(\mathcal{Q}) \subseteq \ldots \subseteq \mathrm{Fitt}_{r-1}(\mathcal{Q}) \subseteq \mathrm{Fitt}_r(\mathcal{Q}) = \mathcal{R}, \tag{20}$$

(see, e.g., [10, 12, 20]), which yields the following chain of affine algebraic subsets of $\mathbb{K}^{m\times 1}$:

$$\{0\} \subseteq \mathcal{V}_{\mathbb{K}}(\mathrm{Fitt}_{r-1}(\mathcal{Q})) \subseteq \mathcal{V}_{\mathbb{K}}(\mathrm{Fitt}_{r-2}(\mathcal{Q})) \subseteq \ldots \subseteq \mathcal{V}_{\mathbb{K}}(\mathrm{Fitt}_0(\mathcal{Q})) \subseteq \mathcal{V}_{\mathbb{K}}(\langle 0\rangle) = \mathbb{K}^{m\times 1}. \tag{21}$$

Note that $\mathrm{Fitt}_{r-1}(\mathcal{Q})$ is the ideal generated by all the entries of $Q = (L\,D_1\,x \;\ldots\; L\,D_r\,x)$, and thus, $\mathcal{V}_{\mathbb{K}}(\mathrm{Fitt}_{r-1}(\mathcal{Q})) = \{u \in \mathbb{K}^{m\times 1} \mid L\,D_i\,u = 0,\ i = 1,\ldots,r\}$, i.e., $\mathcal{V}_{\mathbb{K}}(\mathrm{Fitt}_{r-1}(\mathcal{Q})) = \ker_{\mathbb{K}}(N.)$, where N is defined by eq. (7). Therefore, the approach of Section 2 corresponds to the case of $u \in \mathcal{V}_{\mathbb{K}}(\mathrm{Fitt}_{r-1}(\mathcal{Q}))$.

Example 4. Let us consider the following matrices:

$$D_1 = \begin{pmatrix} 0 & 0 & 0 & 2 \\ 3 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}, \; D_2 = \begin{pmatrix} 5 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 5 & 2 & 0 \\ 0 & 3 & 2 & 0 \end{pmatrix}, \; M = \begin{pmatrix} 30 & 0 & 0 \\ 0 & 0 & 0 \\ 12 & 0 & 0 \\ 12 & 0 & 0 \end{pmatrix}.$$

We then have $m = 4$, $n = 3$, $l = 1 \leq r = 2$, and $p = m - l = 3 > r$. Moreover, we get:

$$L = \begin{pmatrix} 2 & 0 & 0 & -5 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}.$$

Now, set $\mathcal{R} = \mathbb{Q}[x_1, x_2, x_3, x_4]$, $x = (x_1 \;\ldots\; x_4)^T$,

$$A = (D_1\,x \quad D_2\,x) = \begin{pmatrix} 2\,x_4 & 5\,x_1 + 3\,x_2 \\ 3\,x_1 + x_4 & 0 \\ 0 & 5\,x_2 + 2\,x_3 \\ 2\,x_4 & 3\,x_2 + 2\,x_3 \end{pmatrix} \in R^{4\times 2},$$

$$Q = L\,A = \begin{pmatrix} -6\,x_4 & 10\,x_1 - 9\,x_2 - 10\,x_3 \\ 3\,x_1 + x_4 & 0 \\ -2\,x_4 & 2\,x_2 \end{pmatrix} \in R^{3\times 2},$$

and let $\mathcal{Q} = \mathcal{R}^{1\times 2}/(\mathcal{R}^{1\times 3}\,Q)$ be the $\mathcal{R}$-module finitely presented by Q. Then, we have:

$$\begin{aligned} &\mathrm{Fitt}_0(\mathcal{Q}) = \langle (10\,x_1 - 9\,x_2 - 10\,x_3)\,(3\,x_1 + x_4), (2\,x_1 - 3\,x_2 - 2\,x_3)\,x_4, (3\,x_1 + x_4)\,x_2\rangle, \\ &\mathrm{Fitt}_1(\mathcal{Q}) = \langle x_1,\, x_2,\, x_3,\, x_4\rangle. \end{aligned}$$

A necessary condition for the existence of a solution of Problem eq. (4) is then $u \in \mathcal{V}_{\mathbb{K}}(\mathrm{Fitt}_0(\mathcal{Q}))$. We can check again that $\mathcal{V}_{\mathbb{K}}(\mathrm{Fitt}_0(\mathcal{Q}))$ is defined by $u = (u_1 \quad u_2 \quad u_1 - 3\,u_2/2 \quad -3\,u_1)^T$, $u = (u_1 \quad 0 \quad u_1 \quad u_4)^T$, and $u = (0 \quad u_2 \quad u_3 \quad 0)^T$ for all u_1, u_2, $u_3 \in \mathbb{K}$.

Finally, we have $\mathcal{V}_{\mathbb{K}}(\mathrm{Fitt}_1(\mathcal{Q})) = \{(0\;0\;0\;0)^T\} = \ker_{\mathbb{K}}(N.)$, where $N = ((L\,D_1)^T \;(L\,D_2)^T)$, which shows that the approach developed in Section 2 cannot be used to solve Problem eq. (4).

3.2.2 Study of the kernel of the matrix Q

As explained in Section 3.2.1, we must have $u \in \mathcal{V}_{\mathbb{K}}(\mathrm{Fitt}_0(\mathcal{Q}))$ and $v \in \ker_{\mathbb{K}}(Q(u).)$. Hence, we now study the kernel of Q. We first set a few notations.

Definition 2. We note $\mathcal{J} = \mathrm{Fitt}_0(\mathcal{Q}) = \langle m_i\rangle_{i=1,\ldots,\alpha}$, where, if $r \leq p = m - l$, $\{m_i\}_{i=1,\ldots,\alpha}$ is the set of all the $r\times r$ minors of Q and $\alpha = p!/(r!\,(p-r)!)$, or $\alpha = 1$ and $m_i = 0$, i.e., $\mathcal{J} = \langle 0\rangle$, else. If $\{e_i\}_{i=1,\ldots,\gamma}$ is another set of generators of $\mathcal{J}$, where $e_i \in \mathcal{R}$ for $i = 1,\ldots,\gamma$, then we shall write $\mathcal{J} = \langle e_1,\ldots,e_\gamma\rangle_{\mathcal{R}}$.

To algebraically emulate the fact that u belongs to $\mathcal{V}_{\mathbb{K}}(\mathcal{J})$, we can work in the non-trivial factor noetherian ring $\mathcal{S} = \mathcal{R}/\mathcal{J}$ of $\mathcal{R}$. Let $\chi : \mathcal{R} \longrightarrow \mathcal{S}$ be the canonical epimorphism of $\mathbb{K}$-algebras which maps $r \in \mathcal{R}$ onto its residue class $\chi(r) \in \mathcal{S}$, simply denoted by $\overline{r}$. Note that the ring $\mathcal{S}$ inherits a $\mathcal{R}$-module structure defined by $r'\,\overline{r} := \overline{r'\,r} = \overline{r'}\,\overline{r}$ for all $r,\, r' \in \mathcal{R}$.

For $k \in \mathbb{Z}_{>0}$, we can define the following $\mathcal{R}$-homomorphism:

$$\begin{array}{rcl} \mathcal{R}^{k\times 1} & \longrightarrow & \mathcal{S}^{k\times 1} \\ \eta = (\eta_1 \ \ldots \ \eta_k)^T & \longmapsto & \overline{\eta} = (\overline{\eta_1} \ \ldots \ \overline{\eta_k})^T . \end{array} \tag{22}$$

More generally, if $C \in \mathcal{R}^{a\times b}$, then we shall use the following notations:

$$\overline{C} = (\overline{C}_{ij})_{1\leq i\leq a, 1\leq j\leq b} = (\chi(C_{ij}))_{1\leq i\leq a, 1\leq j\leq b} = \chi(C) \in \mathcal{S}^{a\times b}.$$

Let $\mathcal{T}(\mathcal{Q}) = \mathrm{coker}_{\mathcal{R}}(Q.) = \mathcal{R}^{p\times 1}/(Q\,\mathcal{R}^{r\times 1})$ be the so-called *Auslander transpose* of $\mathcal{Q} = \mathrm{coker}_{\mathcal{R}}(.Q)$ [10] (i.e., the $\mathcal{R}$-module finitely presented by the transpose Q^T of Q). Applying the *right exact covariant functor* $\mathcal{S} \otimes_{\mathcal{R}} \cdot$ [10, 25] to the exact sequence of $\mathcal{R}$-modules defining the following presentation of $\mathcal{T}(\mathcal{Q})$

$$0 \longleftarrow \mathcal{T}(\mathcal{Q}) \xleftarrow{\kappa} \mathcal{R}^{p\times 1} \xleftarrow{Q.} \mathcal{R}^{r\times 1},$$

and using $\mathcal{S} \otimes_{\mathcal{R}} \mathcal{R}^{t\times 1} \cong \mathcal{S}^{t\times 1}$, we then obtain the following exact sequence of $\mathcal{S}$-modules

$$0 \longleftarrow \mathcal{S} \otimes_{\mathcal{R}} \mathcal{T}(\mathcal{Q}) \xleftarrow{\kappa_{\mathcal{S}}} \mathcal{S}^{p\times 1} \xleftarrow{Q.} \mathcal{S}^{r\times 1}, \tag{23}$$

where, using both the $\mathcal{R}$-module and the ring structures of $\mathcal{S}$, $Q. \in \hom_{\mathcal{S}}(\mathcal{S}^{r\times 1}, \mathcal{S}^{p\times 1})$ is defined by $Q\,\overline{\eta} := \overline{Q\,\eta} = \overline{Q}\,\overline{\eta}$ for all $\eta \in \mathcal{R}^{r\times 1}$. Hence, we have $Q. = \overline{Q}. \in \hom_{\mathcal{S}}(\mathcal{S}^{r\times 1}, \mathcal{S}^{p\times 1})$. See, e.g., [10, 25]. Since $\mathcal{S}$ is a noetherian ring, $\ker_{\mathcal{S}}(\overline{Q}.)$ is a finitely generated $\mathcal{S}$-module [10, 25]. Thus, if $\ker_{\mathcal{S}}(\overline{Q}.) \neq 0$, then there exists $\overline{K} \in \mathcal{S}^{r\times q}$ is such that $\ker_{\mathcal{S}}(\overline{Q}.) = \mathrm{im}_{\mathcal{S}}(\overline{K}.)$.

To prove that $\ker_{\mathcal{S}}(\overline{Q}.) \neq 0$ (even when $\ker_{\mathcal{R}}(Q.) = 0$), we shall use *McCoy's theorem.*

Theorem 3 (Theorem 6, p. 63, [23])**.** *Let $Q \in \mathcal{R}^{p\times r}$ and $\mathcal{F}$ a non-zero $\mathcal{R}$-module. A necessary and sufficient for the existence of $0 \neq \eta \in \mathcal{F}^r$ satisfying $Q\,\eta = 0$ is that there exists a non-zero element ζ of $\mathcal{F}$ that is annihilated by the determinantal ideal $\mathfrak{U}_r(Q)$ − defined by all the $r \times r$ minors of Q if $r \leq p$, or $\langle 0\rangle$ if $r > p$ − i.e., $P\,\zeta = 0$ for all $P \in \mathfrak{U}_r(Q)$.*

Corollary 2. *Let $M \in \mathbb{K}^{m\times n}$ be such that $\mathrm{im}_{\mathbb{K}}(M.) \neq \mathbb{K}^{m\times 1}$ and $L \in \mathbb{K}^{p\times m}$ a full row rank matrix satisfying $\ker_{\mathbb{K}}(.M) = \mathrm{im}_{\mathbb{K}}(.L)$, where $p = m - \mathrm{rank}_{\mathbb{K}}(M)$. Moreover, let $D_i \in \mathbb{K}^{m\times m}$ for $i = 1, \ldots, r$, $\mathcal{R} = \mathbb{K}[x_1, \ldots, x_m]$, $x = (x_1 \ \ldots \ x_m)^T$, $A = (D_1\,x \ \ldots \ D_r\,x) \in \mathcal{R}^{m\times r}$, $Q = L\,A \in \mathcal{R}^{p\times r}$, $\mathcal{Q} = \mathrm{coker}_{\mathcal{R}}(.Q)$, $\mathcal{J} = \mathrm{Fitt}_0(\mathcal{Q})$, and $\mathcal{S} = \mathcal{R}/\mathcal{J}$. Then, we have $\ker_{\mathcal{S}}(\overline{Q}.) \neq 0$, and thus, there exists a non-zero matrix $K \in \mathcal{R}^{r\times q}$ such that $\ker_{\mathcal{S}}(\overline{Q}.) = \mathrm{im}_{\mathcal{S}}(\overline{K}.)$.*

Proof. By Remark 11, $\mathcal{J} = \mathrm{Fitt}_0(\mathcal{Q})$ is a proper ideal of $\mathcal{R}$ so that $\mathcal{S} = \mathcal{R}/\mathcal{J} \neq 0$. McCoy's theorem, i.e., Theorem 3, shows that $\ker_{\mathcal{S}}(Q.) = \ker_{\mathcal{S}}(\overline{Q}.) \neq 0$ if and only if there exists $0 \neq s \in \mathcal{S}$ such that $P\,s = 0$ for all $P \in \mathfrak{U}_r(Q)$.

If $r > p$, then $\mathcal{J} = \mathfrak{U}_r(Q) = \langle 0\rangle$, and thus, $\mathcal{S} = \mathcal{R}$, $\overline{Q} = Q$, and $0 \neq 1 \in \mathcal{R}$ satisfies $0 \times 1 = 0$, which shows that $\ker_{\mathcal{R}}(Q.) \neq 0$.

If $r \leq p$, then $\mathcal{J} = \langle m_i\rangle_{i=1,\ldots,\alpha}$, where $\{m_i\}_{i=1,\ldots,\alpha}$ denotes the set of all the $r \times r$ minors of Q and $\alpha = p!/(r!\,(p-r)!)$. The result holds because $\overline{0} \neq \overline{1} \in \mathcal{S} = \mathcal{R}/\mathcal{J}$ satisfies $m_i \times \overline{1} = \overline{m_i} = 0$ for $i = 1, \ldots, \alpha$.

Finally, using the fact that $\mathcal{S} = \mathcal{R}/\mathcal{J}$ is a noetherian ring [10, 25] and $\ker_{\mathcal{S}}(\overline{Q}.) \neq 0$, there exists a matrix $K \in \mathcal{R}^{r\times q}$ satisfying $\ker_{\mathcal{S}}(\overline{Q}.) = \mathrm{im}_{\mathcal{S}}(\overline{K}.)$. □

We now explain how a matrix $\overline{K}$ can be effectively computed using Gröbner basis methods [10, 12].

Using Definition 2, if $r > p$, then we have $\mathcal{J} = \langle 0\rangle$ and $\mathcal{S} = \mathcal{R}$. A matrix K can be computed by standard elimination theory (e.g., Gröbner basis methods) for the polynomial ring $\mathcal{R} = \mathbb{K}[x_1, \ldots, x_m]$. Note the fact that $\ker_{\mathcal{R}}(Q.) \neq 0$ can also be proved by considering the *Euler-Poincaré characteristic* [10, 25] of the exact sequence of $\mathcal{R}$-modules $0 \longleftarrow \mathcal{T}(\mathcal{Q}) \xleftarrow{\kappa} \mathcal{R}^{p\times 1} \xleftarrow{Q.} \mathcal{R}^{r\times 1} \longleftarrow \ker_{\mathcal{R}}(Q.) \longleftarrow 0$, i.e., $\mathrm{rank}_{\mathcal{R}}(\ker_{\mathcal{R}}(Q.)) - r + p - \mathrm{rank}_{\mathcal{R}}(\mathcal{T}(\mathcal{Q})) = 0$, where $\mathrm{rank}_{\mathcal{R}}(\mathcal{M})$ stands for the *rank* of a finitely generated $\mathcal{R}$-module $\mathcal{M}$ defined as the dimension of the finite-dimensional $\mathbb{K}(x_1, \ldots, x_m)$-vector obtained by extending the coefficients of $\mathcal{M}$ from $\mathcal{R}$ to the *field of fractions* of $\mathcal{R}$, i.e., the field of rational functions $\mathbb{K}(x_1, \ldots, x_m)$ in $x_1, \ldots, x_m$ with coefficients in $\mathbb{K}$ [10, 25]. In other words, we have $\mathrm{rank}_{\mathcal{R}}(\mathcal{M}) = \dim_{\mathbb{K}(x_1,\ldots,x_m)} \mathbb{K}(x_1, \ldots, x_m) \otimes_{\mathcal{R}} \mathcal{M}$. Hence, we have $\mathrm{rank}_{\mathcal{R}}(\ker_{\mathcal{R}}(Q.)) \geq r - p > 0$, which shows that $\ker_{\mathcal{R}}(Q.) \neq 0$ and proves the existence of $K \in \mathcal{R}^{r\times q}$, where $q \geq r-p > 0$, satisfying $\ker_{\mathcal{R}}(Q.) = \mathrm{im}_{\mathcal{R}}(K.)$.

Now, if $r \leq p$, using Definition 2, then let $\mathcal{J} = \mathrm{Fitt}_0(\mathcal{Q}) = \langle e_1, \ldots, e_\gamma\rangle$. If $e = (e_1, \ldots, e_\gamma)^T \in \mathcal{R}^{\gamma\times 1}$, then $\mathcal{J} = \mathrm{im}_{\mathcal{R}}(.e)$ and we have the following exact sequence of $\mathcal{R}$-modules defining a finite presentation of $\mathcal{S}$ as a $\mathcal{R}$-module:

$$\mathcal{R}^{1\times\gamma} \xrightarrow{.e} \mathcal{R} \xrightarrow{\chi} \mathcal{S} \longrightarrow 0. \tag{24}$$

Combining eq. (23) and eq. (24), we obtain the following commutative exact diagram of $\mathcal{R}$-modules

$$\begin{array}{ccccccc}
 & & & & 0 & & 0 \\
 & & & & \uparrow & & \uparrow \\
0 & \longleftarrow & \mathcal{S} \otimes_{\mathcal{R}} \mathcal{T}(\mathcal{Q}) & \xleftarrow{\mathrm{id}_{\mathcal{S}} \otimes \kappa} & \mathcal{S}^{p\times 1} & \xleftarrow{\overline{Q}.} & \mathcal{S}^{r\times 1} \\
 & & & & \uparrow {\scriptstyle \mathrm{id}_p \otimes \chi} & & \uparrow {\scriptstyle \mathrm{id}_r \otimes \chi} \\
 & & & & \mathcal{R}^{p\times 1} & \xleftarrow{Q.} & \mathcal{R}^{r\times 1} \\
 & & & & \uparrow {\scriptstyle I_p \otimes .e} & & \uparrow {\scriptstyle I_r \otimes .e} \\
 & & & & \mathcal{R}^{p\times\gamma} & \xleftarrow{Q.} & \mathcal{R}^{r\times\gamma},
\end{array}$$

where $(\mathrm{id}_p \otimes \chi)(\mu_1 \ \ldots \ \mu_p)^T = (\chi(\mu_1) \ \ldots \ \chi(\mu_p))^T$ for all $(\mu_1 \ \ldots \ \mu_p)^T \in \mathcal{R}^{p\times 1}$ and $I_p \otimes .e$ is defined by:

$$\forall\, \Lambda = \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_p \end{pmatrix} \in \mathcal{R}^{p\times\gamma}, \quad (I_p \otimes .e)(\Lambda) = \begin{pmatrix} \lambda_1\, e \\ \vdots \\ \lambda_p\, e \end{pmatrix} = \Lambda\, e \in \mathcal{R}^{p\times 1}. \tag{25}$$

Similarly for $\mathrm{id}_r \otimes \sigma$ and $I_r \otimes .e$. Let us now characterize $\mathrm{im}_{\mathcal{R}}(I_p \otimes .e)$. Let $\mathrm{col}(\Lambda) = (\lambda_1 \ \ldots \ \lambda_p)^T \in \mathcal{R}^{p\,\gamma\times 1}$ be the *vectorization* of the matrix $\Lambda \in \mathcal{R}^{p\times\gamma}$ obtained by stacking the columns of Λ into a single column vector, and $e^T \otimes I_p \in \mathcal{R}^{p\times p\,\gamma}$ the *Kronecker product* of e^T by I_p, i.e., the diagonal matrix whose p diagonal blocks are the row vector e^T (see, e.g., [25]). Now, the "vec trick", i.e., the standard identity $\mathrm{col}(\Lambda\, e) = (e^T \otimes I_p)\,\mathrm{col}(\Lambda)$ for all $\Lambda \in \mathcal{R}^{p\times\gamma}$ yields:

$$(I_p \otimes .e)(\Lambda) = \Lambda\, e = \mathrm{col}(\Lambda\, e) = (e^T \otimes I_p)\,\mathrm{col}(\Lambda). \tag{26}$$

Note that eq. (26) shows that $\mathrm{im}_{\mathcal{R}}(I_p \otimes .e)$ is exactly the image of the following $\mathcal{R}$-homomorphism

$$\begin{array}{rccc}
(e^T \otimes I_l).: \mathcal{R}^{p\,\gamma\times 1} & \longrightarrow & \mathcal{R}^{p\times 1} \\
\gamma & \longmapsto & (e^T \otimes I_l)\,\gamma,
\end{array}$$

i.e., $\mathrm{im}_{\mathcal{R}}(I_p \otimes .e) = \mathrm{im}_{\mathcal{R}}((e^T \otimes I_p).)$. Now, using 1 of Proposition 3.1 of [7], we have

$$\ker_{\mathcal{S}}(\overline{Q}.) = \mathrm{im}_{\mathcal{R}}(K.)/\mathrm{im}_{\mathcal{R}}((e^T \otimes I_r).) = \mathrm{im}_{\mathcal{S}}(\overline{K}.), \tag{27}$$

where the matrix $K \in \mathcal{R}^{r\times q}$ is defined by:

$$\ker_{\mathcal{R}}((Q \quad e^T \otimes I_p).) = \mathrm{im}_{\mathcal{R}}\left(\begin{pmatrix} K \\ K' \end{pmatrix}.\right). \tag{28}$$

The matrices $K \in \mathcal{R}^{r\times q}$ and $K' \in \mathcal{R}^{r\times p\,\gamma}$ can be computed using, for instance, the `SyzygyModule` command of the OreModules package, the `Ker` (resp., `WeakKernelEmbedding`) command of `Singular` [12] (resp., `CapAndHomalg` [1]). The computation of $\ker_{\mathcal{S}}(\overline{Q}.)$ and of a matrix $\overline{K}$ are available in the RankFactorization package [9] dedicated to the rank factorization problem and its applications.

Remark 12. Considering $\mathcal{S}$ as a $\mathcal{R}$-module and applying the *left exact contravariant functor* $\hom_{\mathcal{R}}(\cdot\,, \mathcal{S})$ to the finite presentation of $\mathcal{Q}$ defined by eq. (19), we obtain the exact sequence of $\mathcal{R}$-modules

$$\mathcal{S}^{p\times 1} \xleftarrow{\overline{Q}.} \mathcal{S}^{r\times 1} \longleftarrow \hom_{\mathcal{R}}(\mathcal{Q}, \mathcal{S}) \longleftarrow 0,$$

i.e., $\ker_{\mathcal{S}}(\overline{Q}.) \cong \hom_{\mathcal{R}}(\mathcal{Q}, \mathcal{S})$, where the $\mathcal{R}$-module $\hom_{\mathcal{R}}(\mathcal{Q}, \mathcal{S})$ inherits a $\mathcal{S}$-module structure defined by $(f\,s)(q) = f(q)\,s$ for all $s \in \mathcal{S}$, $q \in \mathcal{Q}$, and $f \in \hom_{\mathcal{R}}(\mathcal{Q}, \mathcal{S})$ [10, 25]. Note that the finitely presented $\mathcal{R}$-module $\hom_{\mathcal{R}}(\mathcal{Q}, \mathcal{S})$ can be effectively characterized (see, e.g., [12, 7]) and $\overline{K}$ can be computed using, e.g., OreMorphisms [8], `Singular` [12], or `CapAndHomalg` [1].

Example 5. We continue Example 4. We note $\mathcal{J} = \mathrm{Fitt}_0(\mathcal{Q}) = \langle e_1, e_2, e_3\rangle_{\mathcal{R}}$, where

$$e_1 = (10\,x_1 - 9\,x_2 - 10\,x_3)\,(3\,x_1 + x_4),\; e_2 = (2\,x_1 - 3\,x_2 - 2\,x_3)\,x_4,\; e_3 = (3\,x_1 + x_4)\,x_2,$$

$e = (e_1 \quad e_2 \quad e_3)^T$, and $\mathcal{S} = \mathcal{R}/\mathcal{J}$. Computing the left-hand side of eq. (28) with $p = 3$, we obtain:

$$\ker_{\mathcal{S}}(\overline{Q}.) = \mathrm{im}_{\mathcal{S}}(\overline{K}.), \quad K = \begin{pmatrix} x_2 & 0 & 2\,(x_1 - x_3) \\ x_4 & 3\,x_1 + x_4 & 3\,x_4 \end{pmatrix} \in \mathcal{R}^{2\times 3}.$$

The non-zero entries of K are homogeneous polynomials of total degree 1 in $x_1, \ldots, x_4$.

By construction of $\overline{K}$, we have $\ker_{\mathcal{S}}(\overline{Q}.) = \mathrm{im}_{\mathcal{S}}(\overline{K}.)$, and thus, $\overline{Q}\,\overline{K} = 0$, i.e., $Q\,K \in \mathcal{J}^{p\times q}$, which yields $Q(u)\,K(u) = 0$ for all $u \in \mathcal{V}_{\mathbb{K}}(\mathcal{J})$, i.e.: $\mathrm{im}_{\mathbb{K}}(K(u).) \subseteq \ker_{\mathbb{K}}(Q(u).)$ for all $u \in \mathcal{V}_{\mathbb{K}}(\mathcal{J})$.

Remark 13. If $0 \neq \overline{\eta} \in \ker_{\mathcal{S}}(\overline{Q}.)$, where $\eta \in \mathcal{R}^{r\times 1}$, then $Q\,\eta \in \mathcal{J}^{p\times 1}$ and $\eta \notin \mathcal{J}^{r\times 1}$, i.e., $Q(u)\,\eta(u) = 0$, i.e., $\eta(u) \in \ker_{\mathbb{K}}(Q(u).)$ for all $u \in \mathcal{V}_{\mathbb{K}}(\mathcal{J})$, where η is not identically zero as a polynomial map from $\mathcal{V}_{\mathbb{K}}(\mathcal{J})$ to $\mathbb{K}^{r\times 1}$. Using $\ker_{\mathcal{S}}(\overline{Q}.) \subseteq \mathrm{im}_{\mathcal{S}}(\overline{K}.)$, there exists $\overline{\xi} \in \mathcal{S}^{q\times 1}$ such that $\overline{\eta} = \overline{K}\,\overline{\xi}$, i.e., $\eta - K\,\xi \in \mathcal{J}^{r\times 1}$. Thus, we have $\eta(u) = K(u)\,\xi(u) \in \mathrm{im}_{\mathbb{K}}(K(u).)$ for all $u \in \mathcal{V}_{\mathbb{K}}(\mathcal{J})$. But it is important to note that $\zeta \in \ker_{\mathbb{K}}(Q(u).)$, at a fixed $u \in \mathcal{V}_{\mathbb{K}}(\mathcal{J})$, is not necessarily of the form $\zeta = \eta(u)$ for a certain $\overline{\eta} \in \ker_{\mathcal{S}}(\overline{Q}.)$, i.e., for a certain polynomial vector $\eta \in \mathcal{R}^{r\times 1} : \mathcal{V}_{\mathbb{K}}(\mathcal{J}) \longrightarrow \mathbb{K}^{r\times 1}$. Indeed, $\zeta \in \mathbb{K}^{r\times 1}$ can result from a drop of the rank of the matrix Q at a particular $u \in \mathcal{V}_{\mathbb{K}}(\mathcal{J})$. For an explicit example, see Example 6 below.

A complete description of $\ker_{\mathbb{K}}(Q(u).)$ will be given in the next section (see Theorem 4).

3.2.3 General solutions for the case $M = 0$

The following result will play an important role in what follows.

Theorem 4. *Let $\mathcal{Q} = \mathrm{coker}_{\mathcal{R}}(.Q)$ be the $\mathcal{R}$-module finitely presented by the matrix $Q \in \mathcal{R}^{p\times r}$ defined in Corollary 2, $\mathcal{J}_k = \mathrm{Fitt}_k(\mathcal{Q})$ for $k = 0, \ldots, r-1$, $\mathcal{J}_r = \mathcal{R}$, $\mathcal{J} = \mathcal{J}_0$, $\mathcal{S}_k = \mathcal{R}/\mathcal{J}_k$ for $k = 0, \ldots, r-1$, $\mathcal{S} = \mathcal{S}_0 = \mathcal{R}/\mathcal{J}$, and $\chi_k : \mathcal{R} \longrightarrow \mathcal{S}_k$ the canonical ring epimorphisms for $k = 0, \ldots, r-1$. Then, for $k = 0, \ldots, r-1$, there exists a matrix $K_k \in \mathcal{R}_k^{r\times q_k}$ such that:*

$$\ker_{\mathcal{S}_k}(\chi_k(Q).) = \mathrm{im}_{\mathcal{S}_k}(\chi_k(K_k).).$$

In particular, we have $\mathrm{im}_{\mathbb{K}}(K_k(u).) \subseteq \ker_{\mathbb{K}}(Q(u).)$ for all $u \in \mathcal{V}_{\mathbb{K}}(J_k)$ and $k = 0, \ldots, r-1$.

Finally, for $k = 0, \ldots, r-1$, we have:

$$\forall\, u \in \mathcal{V}_{\mathbb{K}}(\mathcal{J}_k) \setminus \mathcal{V}_{\mathbb{K}}(\mathcal{J}_{k+1}), \quad \ker_{\mathbb{K}}(Q(u).) = \mathrm{im}_{\mathbb{K}}(K_k(u).). \tag{29}$$

Proof. Let $\chi_k : \mathcal{R} \longrightarrow \mathcal{S}_k = \mathcal{R}/\mathcal{J}_k$ be the epimorphism of $\mathbb{K}$-algebras for $k = 0, \ldots, r-1$ and note $\chi_k(Q) = (\chi_k(Q_{ij}))_{1\leq r,\, 1\leq j\leq p} \in \mathcal{S}_k^{p\times r}$ and similarly for any matrix with entries in $\mathcal{R}$. Note that we have $\chi_0 = \chi$, where χ is defined by eq. (22), i.e., $\chi_0(r) = \chi(r) = \overline{r}$ for all $r \in \mathcal{R}$. Note that $\mathcal{S}_k$ has a $\mathcal{R}$-module structure defined by $r'\,\chi_k(r) := \chi_k(r'\,r) = \chi_k(r')\,\chi_k(r)$ for all $r, r' \in \mathcal{R}$.

Using both the $\mathcal{R}$-module structure and the ring structure of $\mathcal{S}_k$, we first have:

$$\ker_{\mathcal{S}_k}(Q.) = \{\chi_k(\eta) \mid \eta \in \mathcal{R}^{r\times 1} : Q\,\chi_k(\eta) = \chi_k(Q\,\eta) = \chi_k(Q)\,\chi_k(\eta) = 0\} = \ker_{\mathcal{S}_k}(\chi_k(Q).).$$

Using the fact that $\mathcal{J}_k \neq \mathcal{R}$ for $k = 0, \ldots, r-1$, $\mathcal{S}_k$ is a non-trivial ring. Then, Theorem 3 shows that $\ker_{\mathcal{S}_k}(\chi_k(Q).) \neq 0$ since, by construction of $\mathcal{S}_k = \mathcal{R}/\mathcal{J}_k$, all the $(r-k) \times (r-k)$ minors of $\chi_k(Q)$ vanish in $\mathcal{S}_k$, and thus, so do all the $r \times r$ minors of $\chi_k(Q)$ (see eq. (20)). Using the fact that $\mathcal{S}_k$ is a noetherian ring, there exists a non-zero matrix $K_k \in \mathcal{R}_k^{r \times q_k}$ such that $\ker_{\mathcal{S}_k}(\chi_k(Q).) = \operatorname{im}_{\mathcal{S}_k}(\chi_k(K_k).)$. Thus, we have the following exact sequence of $\mathcal{S}_k$-modules:

$$\mathcal{S}_k^{q_k \times 1} \xrightarrow{\chi_k(K_k).} \mathcal{S}_k^{r \times 1} \xrightarrow{\chi_k(Q).} \mathcal{S}_k^{p \times 1}. \tag{30}$$

Note that $\mathcal{J}_{r-1}$ is the ideal generated by all the entries of Q, which yields $\chi_{r-1}(Q) = 0$, i.e., $\ker_{\mathcal{S}_{r-1}}(\chi_{r-1}(Q).) = \mathcal{S}_{r-1}^{r \times 1}$, i.e., $K_{r-1} = I_r$. This case corresponds to Section 2.2.

Now, we have $\chi_k(Q)\,\chi_k(K_k) = \chi_k(Q\,K_k) = 0$, and thus, we have $Q\,K_k \in \mathcal{J}_k^{p \times q_k}$ and $K_k \notin \mathcal{J}^{r \times q_k}$, which yields $Q(u)\,K_k(u) = 0$ for all $u \in \mathcal{V}(\mathcal{J}_k)$, where $K_k : \mathcal{V}(\mathcal{J}_k) \longrightarrow \mathbb{K}^{r \times q_k}$ is not identically zero, and then proves that $\operatorname{im}_{\mathbb{K}}(K_k(u).) \subseteq \ker_{\mathbb{K}}(Q(u).)$ for all $u \in \mathcal{V}(\mathcal{J}_k)$.

Finally, let us prove that $\ker_{\mathbb{K}}(Q(u).) = \operatorname{im}_{\mathbb{K}}(K_k(u).)$ for all $u \in \mathcal{V}_{\mathbb{K}}(\mathcal{J}_k) \setminus \mathcal{V}_{\mathbb{K}}(\mathcal{J}_{k+1})$[2]. Let us consider $u \in \mathcal{V}_{\mathbb{K}}(\mathcal{J}_k) \setminus \mathcal{V}_{\mathbb{K}}(\mathcal{J}_{k+1})$ so that $\operatorname{rank}_{\mathbb{K}}(Q(u)) = r-k+1$. Hence, there exists a $(r-k-1) \times (r-k-1)$ minor of Q, denoted by $\mathfrak{m}$, such that $\mathfrak{m}(u) \neq 0$. Note that $\mathfrak{m} \in \mathcal{J}_{k+1}$ and $f = \chi_k(\mathfrak{m})$ is not a *nilpotent element* of $\mathcal{S}_k$, i.e., $\mathfrak{m} \notin \sqrt{\mathcal{J}_k}$ (since $\mathfrak{m} \in \sqrt{\mathcal{J}_k}$ yields $\mathfrak{m}(u) = 0$). We can consider the non-trivial ring $\mathcal{S}_{k,f} = \{s/f^k \mid s \in \mathcal{S},\ k \in \mathbb{Z}_{\geq 0}\}$ defined as the localization of $\mathcal{S}_k$ at the multiplicatively closed set $\{f^k\}_{k \in \mathbb{Z}_{\geq 0}}$ and the canonical ring homomorphism $j_f : \mathcal{S}_k \longrightarrow \mathcal{S}_{k,f}$ defined by $j_f(s) = s/1$ for all $s \in \mathcal{S}_k$, whose kernel is given by $\ker j_f = \{s \in \mathcal{S}_k \mid \exists\, c \in \mathbb{Z}_{\geq 0} :\ f^c\, s = 0\}$ [10, 25]. Note $K_{k,f} = j_f(\chi_k(K_k))$ and $Q_f = j_f(\chi_k(Q))$. Using the fact that $\mathcal{S}_{k,f}$ is a flat $\mathcal{S}_k$-module [10, 25], applying the covariant exact functor $\mathcal{S}_{k,f} \otimes_{\mathcal{S}_k} \cdot$ to eq. (30), we obtain the following exact sequence of $\mathcal{S}_{k,f}$-modules:

$$\mathcal{S}_{k,f}^{q_k \times 1} \xrightarrow{K_{k,f}.} \mathcal{S}_{k,f}^{r \times 1} \xrightarrow{Q_f.} \mathcal{S}_{k,f}^{p \times 1} \longrightarrow \operatorname{coker}_{\mathcal{S}_{k,f}}(Q_f.) \longrightarrow 0. \tag{31}$$

Using the fact that f is invertible in $\mathcal{S}_{k,f}$, the matrix Q_f is then equivalent to the following matrix:

$$\begin{pmatrix} I_{r-k-1} & 0 \\ 0 & 0 \end{pmatrix} \in \mathcal{S}_{k,f}^{p \times r}.$$

Thus, we have $\operatorname{coker}_{\mathcal{S}_{k,f}}(Q_f.) \cong \mathcal{S}_{k,f}^{(p-r+k+1) \times 1}$, i.e., $\operatorname{coker}_{\mathcal{S}_{k,f}}(Q_f.)$ is a free $\mathcal{S}_{k,f}$-module, which yields that eq. (31) *splits* [10, 25]. Therefore, there exist $U \in \mathcal{R}^{r \times p}$, $V \in \mathcal{R}^{q_k \times r}$, $\alpha, \beta \in \mathbb{Z}_{\geq 0}$ such that $U_f = f^{-\alpha}\,\chi_k(U) \in \mathcal{S}_{k,f}^{r \times p}$ and $V_f = f^{-\beta}\,\chi_k(V) \in \mathcal{S}_{k,f}^{q_k \times r}$ satisfy $K_{k,f}\,V_f + U_f\,Q_f = I_r$. Let $\delta = \operatorname{lcm}(\alpha, \beta)$. Using the characterization of $\ker j_f$, the last identity is equivalent to the existence of $\gamma \in \mathbb{Z}_{\geq 0}$ satisfying

$$f^{\gamma}\,(f^{\delta-\beta}\,\chi_k(K_k\,V) + f^{\delta-\alpha}\,\chi_k(U\,Q) - f^{\delta}\,I_r) = 0,$$

i.e., to $\mathfrak{m}^{\gamma}\,(\mathfrak{m}^{\delta-\beta}\,K_k\,V + \mathfrak{m}^{\delta-\alpha}\,U\,Q - \mathfrak{m}^{\delta}\,I_r) \in \mathcal{J}_k^{r \times r}$. Now, using $u \in \mathcal{V}_{\mathbb{K}}(\mathcal{J}_k)$ and $\mathfrak{m}(u) \neq 0$, we then have $\mathfrak{m}(u)^{\gamma}\,(\mathfrak{m}(u)^{\delta-\beta}\,K_k(u)\,V(u) + \mathfrak{m}(u)^{\delta-\alpha}\,U(u)\,Q(u) - \mathfrak{m}(u)^{\delta}\,I_r) = 0$, and thus:

$$K_k(u)\,(\mathfrak{m}(u)^{-\beta}\,V(u)) + (\mathfrak{m}(u)^{-\alpha}\,U(u))\,Q(u) = I_r.$$

Finally, if $v \in \ker_{\mathbb{K}}(Q(u).)$, then the last identity yields $v = K_k(u)\,(\mathfrak{m}(u)^{-\beta}\,V(u))\,v \in \operatorname{im}_{\mathbb{K}}(K_k(u).)$ and shows that $\ker_{\mathbb{K}}(Q(u).) = \operatorname{im}_{\mathbb{K}}(K_k(u).)$, which concludes the proof. □

Let us illustrate eq. (29) with an explicit example.

Example 6. Let $\mathcal{Q} = \operatorname{coker}_{\mathcal{R}}(.Q)$ be the $\mathcal{R} = \mathbb{K}[x_1, x_2]$-module finitely presented by the following matrix:

$$Q = \begin{pmatrix} x_1 & x_2 & 0 \\ x_2 & x_1 & 0 \end{pmatrix} \in \mathcal{R}^{2 \times 3}.$$

Then, we have:

$$\mathcal{J}_0 = \operatorname{Fitt}_0(\mathcal{Q}) = \langle 0 \rangle,\ \mathcal{J}_1 = \operatorname{Fitt}_1(\mathcal{Q}) = \langle (x_1 - x_2)\,(x_1 + x_2) \rangle,\ \mathcal{J}_2 = \operatorname{Fitt}_2(\mathcal{Q}) = \langle x_1, x_2 \rangle,$$
$$\mathcal{V}_{\mathbb{K}}(\mathcal{J}_0) = \mathbb{K}^{2 \times 1},\ \mathcal{V}_{\mathbb{K}}(\mathcal{J}_1) = \{(1 \quad 1)^T\,u_1 \mid u_1 \in \mathbb{K}\} \cup \{(1 \quad -1)^T\,u_1 \mid u_1 \in \mathbb{K}\},\ \mathcal{V}_{\mathbb{K}}(\mathcal{J}_2) = \{0\}.$$

[2]The proof of this point was communicated to us by Prof. David Eisenbud (University of California Berkeley). We are grateful to him for authorizing its reproduction here.

We can check that $\operatorname{rank}_{\mathbb{K}}(Q(u)) = 2$ if $u \in \mathcal{V}_{\mathbb{K}}(\mathcal{J}_0) \setminus \mathcal{V}_{\mathbb{K}}(\mathcal{J}_1)$, $\operatorname{rank}_{\mathbb{K}}(Q(u)) = 1$ if $u \in \mathcal{V}_{\mathbb{K}}(\mathcal{J}_1) \setminus \mathcal{V}_{\mathbb{K}}(\mathcal{J}_2)$, and $\operatorname{rank}_{\mathbb{K}}(Q(u)) = 0$ if $u \in \mathcal{V}_{\mathbb{K}}(\mathcal{J}_2)$.

Let us now characterize $\ker_{\mathbb{K}}(Q(u).)$ for $u \in \mathbb{K}^{2\times 1}$. If $u \in \mathbb{K}^{2\times 1}$, then we have:

$$\ker_{\mathbb{K}}(Q(u).) = \left\{\xi = (\xi_1 \quad \xi_2 \quad \xi_3)^T \in \mathbb{K}^{3\times 1} \mid u_1\,\xi_1 + u_2\,\xi_2 = 0,\, u_2\,\xi_1 + u_1\,\xi_2 = 0\right\}.$$

If we set $K = (0 \quad 0 \quad 1)^T$, $K' = (1 \quad -1 \quad 0)^T$, and $K'' = (1 \quad 1 \quad 0)^T$, then we can check that:

$$\ker_{\mathbb{K}}(Q(u).) = \begin{cases} \operatorname{im}_{\mathbb{K}}(K.) = K\,\mathbb{K}, & u \in \mathbb{K}^{2\times 1} \setminus \mathcal{V}_{\mathbb{K}}(\mathcal{J}_1), \\ \operatorname{im}_{\mathbb{K}}((K \quad K').) = K\,\mathbb{K} + K'\,\mathbb{K}, & u_2 = u_1 \neq 0, \\ \operatorname{im}_{\mathbb{K}}((K \quad K'').) = K\,\mathbb{K} + K''\,\mathbb{K}, & u_2 = -u_1 \neq 0, \\ \operatorname{im}_{\mathbb{K}}(I_3.), & u_1 = u_2 = 0. \end{cases}$$

Let now us first consider $\mathcal{S}_0 = \mathcal{R}$. We can ckeck again that $\ker_{\mathcal{R}}(Q.) = \operatorname{im}_{\mathcal{R}}(K.)$.

Let us now consider the ring $\mathcal{S}_1 = \mathcal{R}/\mathcal{J}_1$ and $\chi_1 : \mathcal{R} \longrightarrow \mathcal{S}_1$ the canonical ring epimorphism. Then, we have $\ker_{\mathcal{S}_1}(\chi_1(Q).) = \operatorname{im}_{\mathcal{S}_1}(\chi_1(K_1).)$, where:

$$K_1 = \begin{pmatrix} 0 & -x_2 & x_1 \\ 0 & x_1 & -x_2 \\ 1 & 0 & 0 \end{pmatrix} \in \mathcal{R}^{3\times 3}.$$

Let us finally consider the ring $\mathcal{S}_2 = \mathcal{R}/\mathcal{J}_2$ and $\chi_2 : \mathcal{R} \longrightarrow \mathcal{S}_2$ the canonical ring epimorphism. Then, we have $\chi_2(Q) = 0$ and $\ker_{\mathcal{S}_2}(\chi_2(Q)) = \mathcal{S}_2^{3\times 3}$, i.e., $K_2 = I_3$ satisfies $\ker_{\mathcal{S}_2}(\chi_2(Q)) = \operatorname{im}_{\mathcal{S}_2}(\chi_2(K_2).)$.

Finally, let us check again that:

$$\ker_{\mathbb{K}}(Q(u).) = \begin{cases} \operatorname{im}_{\mathbb{K}}(K(u).), & u \in \mathcal{V}_{\mathbb{K}}(\mathcal{J}_0) \setminus \mathcal{V}_{\mathbb{K}}(\mathcal{J}_1), \\ \operatorname{im}_{\mathbb{K}}(K_1(u).), & u \in \mathcal{V}_{\mathbb{K}}(\mathcal{J}_1) \setminus \mathcal{V}_{\mathbb{K}}(\mathcal{J}_2), \\ \operatorname{im}_{\mathbb{K}}(K_2(u).), & u \in \mathcal{V}_{\mathbb{K}}(\mathcal{J}_2). \end{cases}$$

It is clear for $i = 0$ and $i = 2$. Thus, if we consider $i = 1$, then we have

$$\forall\, u \in \mathcal{V}_{\mathbb{K}}(\mathcal{J}_1) \setminus \mathcal{V}_{\mathbb{K}}(\mathcal{J}_2), \quad \operatorname{im}_{\mathbb{K}}(K_1(u).) = \begin{cases} K\,\mathbb{K} + K'\,\mathbb{K}, & u_2 = u_1 \neq 0, \\ K\,\mathbb{K} + K''\,\mathbb{K}, & u_2 = -u_1 \neq 0, \end{cases}$$

which finally illustrates Theorem 4 on a simple example.

Let us now study the connections between the matrices K_k's for $k = 0, \ldots, r-1$, where $K_0 = K$.

Using $\mathcal{J}_r = \mathcal{R}$, for $k = 0, \ldots, r-1$, we have the following exact sequence of ring homomorphisms:

$$0 \longrightarrow \mathcal{J}_{k+1}/\mathcal{J}_k \longrightarrow \mathcal{S}_k \xrightarrow{\delta_k} \mathcal{S}_{k+1} \longrightarrow 0.$$

Thus, $\mathcal{S}_{k+1}$ inherits a $\mathcal{S}_k$-module structure and we have $\chi_{k+1} = \delta_k \circ \chi_k$, where $\chi_k : \mathcal{R} \longrightarrow \mathcal{S}_k$ is the canonical ring epimorphism. Applying the covariant functor $\mathcal{S}_{k+1} \otimes_{\mathcal{S}_k} \cdot$ to the exact sequence $\mathcal{S}_k^{p\times 1} \xleftarrow{\chi_k(Q).} \mathcal{S}_k^{r\times 1} \xleftarrow{\chi_k(K_k).} \mathcal{S}_k^{q_k\times 1}$ of $\mathcal{S}_k$-modules, we obtain the complex of $\mathcal{S}_{k+1}$-modules

$$\mathcal{S}_{k+1}^{p\times 1} \xleftarrow{\chi_{k+1}(Q).} \mathcal{S}_{k+1}^{r\times 1} \xleftarrow{\chi_{k+1}(K_k).} \mathcal{S}_{k+1}^{q_k\times 1},$$

which yields

$$\operatorname{im}_{\mathcal{S}_{k+1}}(\chi_{k+1}(K_k).) \subseteq \ker_{\mathcal{S}_{k+1}}(\chi_{k+1}(Q).) = \operatorname{im}_{\mathcal{S}_{k+1}}(\chi_{k+1}(K_{k+1}).),$$

and proves the existence of a matrix $L_{k,k+1} \in \mathcal{R}^{q_{k+1}\times q_k}$ such that:

$$\chi_{k+1}(K_k) = \chi_{k+1}(K_{k+1})\,\chi_{k+1}(L_{k,k+1}), \quad k = 0, \ldots, r-2. \tag{32}$$

The next result gives the solutions of the rank factorization problem eq. (4) when $M = 0$.

Corollary 3. *If $M = 0$, then all the solutions of the rank factorization problem eq. (4) are defined by*

$$\forall\, k = 0, \ldots, r-1,\ \forall\, u \in \mathcal{V}_{\mathbb{K}}(\mathcal{J}_k) \setminus \mathcal{V}_{\mathbb{K}}(\mathcal{J}_{k+1}),\ \forall\, Y' \in \mathbb{K}^{q_k \times n},\ (u,\, v_k(u, Y') = K_k(u)\, Y')\,,$$

where the matrix K_k is defined in Theorem 4. Moreover, we have

$$\forall\, u \in \mathcal{V}_{\mathbb{K}}(\mathcal{J}_{k+1}),\ \forall\, Y' \in \mathbb{K}^{q_k \times n},\ v_k(u, Y') = K_{k+1}(u)\ (L_{k,k+1}(u)\, Y') = v_{k+1}(u, L_{k,k+1}(u)\, Y'),$$

and thus, all the solutions of the rank factorization problem eq. (4) are defined, up to redundancy, by:

$$\forall\, k = 0, \ldots, r-1,\ \forall\, u \in \mathcal{V}_{\mathbb{K}}(\mathcal{J}_k),\ \forall\, Y' \in \mathbb{K}^{q_k \times n},\ (u,\, v_k(u, Y') = K_k(u)\, Y')\,.$$

Proof. If $M = 0$, then we have $L = I_m$, $Q = A$, and $A(u)\, v = 0$ shows that the each column of $v \in \mathbb{K}^{r \times n}$ must belong to $\ker_{\mathbb{K}}(A(u).)$. Using Theorem 4, for $k = 0, \ldots, r-1$, $(u,\, v(u) = K_k(u)\, Y')$ for all $u \in \mathcal{V}_{\mathbb{K}}(\mathcal{J}_k)$ and for all $Y' \in \mathbb{K}^{q_k \times n}$ are all the solutions of eq. (4).

Using eq. (32), we have $K_k - K_{k+1}\, L_{k,k+1} \in \mathcal{J}_{k+1}^{r \times q_k}$, and thus, $K_k(u) = K_{k+1}(u)\, L_{k,k+1}(u)$ for all $u \in \mathcal{V}_{\mathbb{K}}(\mathcal{J}_{k+1})$, which finally shows $v_k(u, Y') = v_{k+1}(u, L_{k,k+1}(u)\, Y')$ for all $u \in \mathcal{V}_{\mathbb{K}}(\mathcal{J}_{k+1})$. ☐

The computation of all the solutions of the rank factorization problem for $M = 0$, characterized in Corollary 3, are implemented in the RankFactorization package [9].

Example 7. Let us consider the following matrices:

$$D_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad D_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad D_3 = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \quad M = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then, $\mathcal{R} = \mathbb{K}[x_1, x_2]$, $L = I_2$, and $\mathcal{Q} = \operatorname{coker}_{\mathcal{R}}(.Q)$ is the $\mathcal{R}$-module finitely presented by:

$$Q = A = \begin{pmatrix} x_1 & x_2 & 2\,x_1 + x_2 \\ x_2 & x_1 & x_1 + 2\,x_2 \end{pmatrix} \in \mathcal{R}^{2 \times 2}.$$

We have $\mathcal{J}_0 = \operatorname{Fitt}_0(\mathcal{Q}) = \langle 0 \rangle$, $\mathcal{J}_1 = \operatorname{Fitt}_1(\mathcal{Q}) = \langle (x_1 - x_2)\,(x_1 + x_2) \rangle$, $\mathcal{J}_2 = \operatorname{Fitt}_2(\mathcal{Q}) = \langle x_1, x_2 \rangle$, $\mathcal{V}_{\mathbb{K}}(\mathcal{J}_0) = \mathbb{K}^{2 \times 1}$, $\mathcal{V}_{\mathbb{K}}(\mathcal{J}_1) = \{(u_1 \quad u_1)^T, (u_1 \quad -u_1)^T \mid u_1 \in \mathbb{K}\}$, and $\mathcal{V}_{\mathbb{K}}(\mathcal{J}_2) = \{(0 \quad 0)^T\}$.

If we note $\mathcal{S}_k = \mathcal{R}/\mathcal{J}_k$, for $k = 0, 1, 2$, then we have $\ker_{\mathcal{S}_k}(Q.) = \operatorname{im}_{\mathcal{S}_k}(K_k.)$, where:

$$K_0 = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}, \quad K_1 = \begin{pmatrix} 2 & 0 & 0 \\ 1 & -3\,x_2 & 2\,x_1 + x_2 \\ -1 & 2\,x_1 - x_2 & -x_2 \end{pmatrix}, \quad K_2 = I_3.$$

Moreover, we have $K_0 = K_1\, L_{0,1}$ and $K_1 = K_2\, L_{1,2}$, where $L_{0,1} = (1 \quad 0 \quad 0)^T$ and $L_{1,2} = K_1$. All the solutions to the rank factorization problem are then defined by:

$$\left\{ \begin{array}{lll} \forall\, u \in \mathcal{V}_{\mathbb{K}}(\mathcal{J}_0) \setminus \mathcal{V}_{\mathbb{K}}(\mathcal{J}_1), & \forall\, Y' \in \mathbb{K}^{1 \times 3}, & v_0(u, Y') = K_0\, Y', \\ \forall\, u \in \mathcal{V}_{\mathbb{K}}(\mathcal{J}_1) \setminus \mathcal{V}_{\mathbb{K}}(\mathcal{J}_2), & \forall\, Y' \in \mathbb{K}^{3 \times 3}, & v_1(u, Y') = K_1(u)\, Y', \\ \forall\, u \in \mathcal{V}_{\mathbb{K}}(\mathcal{J}_2), & \forall\, Y' \in \mathbb{K}^{3 \times 3}, & v_2(u, Y') = Y'. \end{array} \right. \tag{33}$$

We can check again that $(u \in \mathcal{V}_{\mathbb{K}}(\mathcal{J}_1), v_0(u, Y') = K_0\, Y')$ and $(u \in \mathcal{V}_{\mathbb{K}}(\mathcal{J}_2), v_1(u, Y') = K_1\, Y')$ are also solutions, which are respectively contained in the second and third set of eq. (33). Therefore, up to redundancy, all the solutions are also defined by:

$$\left\{ \begin{array}{lll} \forall\, u \in \mathcal{V}_{\mathbb{K}}(\mathcal{J}_0), & \forall\, Y' \in \mathbb{K}^{1 \times 3}, & v_0(u, Y') = K_0\, Y', \\ \forall\, u \in \mathcal{V}_{\mathbb{K}}(\mathcal{J}_1, & \forall\, Y' \in \mathbb{K}^{3 \times 3}, & v_1(u, Y') = K_1(u)\, Y', \\ \forall\, u \in \mathcal{V}_{\mathbb{K}}(\mathcal{J}_2), & \forall\, Y' \in \mathbb{K}^{3 \times 3}, & v_2(u, Y') = Y'. \end{array} \right.$$

Finally, note that, e.g., $u = (1 \quad 1)^T$ and

$$v = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -3 & 3 \\ 0 & 1 & -1 \end{pmatrix}$$

define a solution of eq. (4) which is not of the form $v_0(u, Y') = K_0\, Y'$ for $Y' \in \mathbb{K}^{1 \times 3}$. It shows that we also have to consider the solutions of eq. (4) over $\mathcal{S}_1$ and $\mathcal{S}_2$ to get the complete set of solutions.

3.2.4 Construction of the matrix B as a pullback

Corollary 3 solving the case $M = 0$, in the rest of the paper, we shall assume that $M \neq 0$.

Let $\mathcal{J} = \mathcal{J}_0$, $\mathcal{S} = \mathcal{R}/\mathcal{J}$, and $K \in \mathcal{R}^{r \times q}$ be such that $\ker_{\mathcal{S}}(\overline{Q}.) = \operatorname{im}_{\mathcal{S}}(\overline{K}.)$ (see Corollary 2). In this section, we extend the construction of the matrix B used in Section 2.2. In the next Section 3.2.5, and Section 3.2.6, this matrix B will play an important role in the characterization of solutions of the rank factorization problem eq. (4). We have the following proposition.

Proposition 2. *With the notations of Lemma 3 and Corollary 2, if we set* $\overline{A} = \chi(A)$ *and* $\overline{X} = \chi(X)$*, then there exists a unique* $\overline{B} = \chi(B) \in \mathcal{S}^{l \times q}$*, where* $B \in \mathcal{R}^{l \times q}$*, such that:*

$$\overline{A}\,\overline{K} = \overline{X}\,\overline{B}. \tag{34}$$

Moreover, if $V \in \mathbb{K}^{l \times m}$ *is a left inverse of* $X \in \mathbb{K}^{m \times l}$ *and* $\overline{V} = \chi(V)$*, then we have:*

$$\overline{B} = \overline{V}\,\overline{A}\,\overline{K}. \tag{35}$$

In particular, eq. (35) *does not depend on the choice of the left inverse* V *of* X*.*

Proof. As explained in Section 2, the full row matrix $L \in \mathbb{K}^{p \times m}$, where $p = m - l$, is such that $\ker_{\mathbb{K}}(L.) = \operatorname{im}_{\mathbb{K}}(M.)$. We thus have the following exact sequence of $\mathbb{K}$-vector spaces:

$$0 \longleftarrow \mathbb{K}^{p \times 1} \xleftarrow{L.} \mathbb{K}^{m \times 1} \xleftarrow{M.} \mathbb{K}^{n \times 1}.$$

By definition of the matrices $X \in \mathbb{K}^{m \times l}$ and $Y \in \mathbb{K}^{l \times n}$ (see Lemma 3), we have $M = X\,Y$, where the columns of X define a basis of $\operatorname{im}_{\mathbb{K}}(M.)$, i.e., $\operatorname{im}_{\mathbb{K}}(M.) = \operatorname{im}_{\mathbb{K}}(X.)$ and $\ker_{\mathbb{K}}(X.) = 0$. Hence, we have $\ker_{\mathbb{K}}(M.) = \ker_{\mathbb{K}}(Y.)$. Moreover, since $\operatorname{im}_{\mathbb{K}}(X.) \subseteq \operatorname{im}_{\mathbb{K}}(M.)$, there exists $H \in \mathbb{K}^{n \times l}$ such that $X = M\,H$, which yields $X = X\,Y\,H$, and thus, $Y\,H = I_l$ because X has full column rank. Thus, $\operatorname{im}_{\mathbb{K}}(Y.) = \mathbb{K}^{l \times 1}$ and Y has full row rank. Hence, we have the following commutative exact diagram of finite-dimensional $\mathbb{K}$-vector spaces:

$$\begin{array}{ccccccccc}
0 & \longleftarrow & \mathbb{K}^{p \times 1} & \xleftarrow{L.} & \mathbb{K}^{m \times 1} & \xleftarrow{M.} & \mathbb{K}^{n \times 1} & \longleftarrow & \ker_{\mathbb{K}}(M.) \longleftarrow 0 \\
 & & \| & & \| & & \downarrow Y. & & \\
0 & \longleftarrow & \mathbb{K}^{p \times 1} & \xleftarrow{L.} & \mathbb{K}^{m \times 1} & \xleftarrow{X.} & \mathbb{K}^{l \times 1} & \longleftarrow & 0. \\
 & & & & & & \downarrow & & \\
 & & & & & & 0 & &
\end{array}$$

The second horizontal short exact sequence of finite-dimensional $\mathbb{K}$-vector spaces of the above commutative exact diagram splits [10, 25], i.e., there exist two matrices $U \in \mathbb{K}^{m \times p}$ and $V \in \mathbb{K}^{l \times m}$ such that:

$$(U \quad X) \begin{pmatrix} L \\ V \end{pmatrix} = I_m. \tag{36}$$

Since $\mathbb{K}$ is a field and the matrices $(U \quad X)$ and $\left(L^T \quad V^T\right)^T$ are square, eq. (36) is then equivalent to:

$$\begin{pmatrix} L \\ V \end{pmatrix} (U \quad X) = \begin{pmatrix} I_p & 0 \\ 0 & I_l \end{pmatrix} = I_m. \tag{37}$$

Using the fact that $\mathcal{S}$ is a $\mathbb{K}$-vector space, applying the *exact functor* $\mathcal{S} \otimes_{\mathbb{K}} \cdot$ (see, e.g., [10, 25]) to the above diagram, we obtain the following commutative exact diagram of $\mathcal{S}$-modules:

$$\begin{array}{ccccccccc}
0 & \longleftarrow & \mathcal{S}^{p \times 1} & \xleftarrow{\overline{L}.} & \mathcal{S}^{m \times 1} & \xleftarrow{\overline{M}.} & \mathcal{S}^{n \times 1} & & \\
 & & \| & & \| & & \downarrow \overline{Y}. & & \\
0 & \longleftarrow & \mathcal{S}^{p \times 1} & \xleftarrow{\overline{L}.} & \mathcal{S}^{m \times 1} & \xleftarrow{\overline{X}.} & \mathcal{S}^{l \times 1} & \longleftarrow & 0. \\
 & & & & & & \downarrow & & \\
 & & & & & & 0 & &
\end{array}$$

Combining $Q = L\,A$ and $\overline{Q}\,\overline{K} = 0$, we get $\overline{L}\,(\overline{A}\,\overline{K}) = 0$, i.e., $\operatorname{im}_{\mathcal{S}}((\overline{A}\,\overline{K}).) \subseteq \ker_{\mathcal{S}}(\overline{L}.) = \operatorname{im}_{\mathcal{S}}(\overline{X}.)$, which using $\ker_{\mathcal{S}}(\overline{X}.) = 0$ shows that there exists a unique matrix $\overline{B} \in \mathcal{S}^{l\times q}$, where $B \in \mathcal{R}^{l\times q}$, satisfying $\overline{A}\,\overline{K} = \overline{X}\,\overline{B}$, and we have the commutative exact diagram of $\mathcal{S}$-modules:

$$\begin{array}{ccccccccc}
 & & 0 & \longleftarrow & \mathcal{S}^{p\times 1} & \xleftarrow{\overline{L}.} & \mathcal{S}^{m\times 1} & \xleftarrow{\overline{M}.} & \mathcal{S}^{n\times 1} & & \\
 & & & & \| & & \| & & \downarrow \overline{Y}. & & \\
 & & 0 & \longleftarrow & \mathcal{S}^{p\times 1} & \xleftarrow{\overline{L}.} & \mathcal{S}^{m\times 1} & \xleftarrow{\overline{X}.} & \mathcal{S}^{l\times 1} & \longleftarrow & 0. \\
 & & & & \| & & \uparrow \overline{A}. & & \uparrow \overline{B}. & & \\
0 & \longleftarrow & \mathcal{S}\otimes_{\mathcal{R}} \mathcal{T}(\mathcal{Q}) & \xleftarrow{\operatorname{id}_{\mathcal{S}}\otimes\kappa} & \mathcal{S}^{p\times 1} & \xleftarrow{\overline{Q}.} & \mathcal{S}^{r\times 1} & \xleftarrow{\overline{K}.} & \mathcal{S}^{q\times 1} & &
\end{array} \tag{38}$$

Using the identity $V\,X = I_l$ (see eq. (37)), eq. (34) then yields eq. (35).

Finally, the identity eq. (35) does not depend on a particular left inverse V of X. This can be checked again by considering a second left inverse V' of X, i.e., $V'\,X = I_l$. Then, we have $(V' - V)\,X = 0$, which shows that the rows of $V' - V$ belong to $\ker_{\mathbb{K}}(.X) = \operatorname{im}_{\mathbb{K}}(.L)$, and thus, that there exists $L' \in \mathbb{K}^{l\times p}$ such that $V' = V + L'\,L$, which using $Q = L\,A$ and $\overline{Q}\,\overline{K} = 0$, yields $\overline{V'}\,\overline{A}\,\overline{K} = \overline{V}\,\overline{A}\,\overline{K} + \overline{L'}\,(\overline{L}\,\overline{A})\,\overline{K} = \overline{V}\,\overline{A}\,\overline{K}$. ☐

Proposition 2 shows that there exists a unique matrix $\overline{B} \in \mathcal{S}^{l\times q}$ satisfying $\overline{A}\,\overline{K} = \overline{X}\,\overline{B}$ defined by $\overline{B} = \overline{V}\,\overline{A}\,\overline{K}$. But the pre-image $B \in \mathcal{R}^{l\times q}$ of $\overline{B}$ is not uniquely defined. It is defined by any matrix $B \in \mathcal{R}^{l\times q}$ satisfying $B - V\,A\,K \in \mathcal{J}^{l\times q}$. To simplify, we shall thus set $B = V\,A\,K$.

Remark 14. Note that the construction of the commutative exact diagram eq. (38) corresponds to finding an $\mathcal{R}$-algebra $\mathcal{S}$ for which the *pullback* (see, e.g., [10, 25]) of the $\mathcal{S}$-homomorphisms $\overline{A}. : \mathcal{S}^{r\times 1} \longrightarrow \mathcal{S}^{m\times 1}$ and $\overline{X}. : \mathcal{S}^{l\times 1} \longrightarrow \mathcal{S}^{m\times 1}$ is non-trivial, i.e., such that $\ker_{\mathcal{S}}\left((\overline{A} \quad \overline{X}).\right) \neq 0$.

Example 8. We continue Example 5. We can first check that:

$$X = (5 \quad 0 \quad 2 \quad 2)^T, \quad Y = (6 \quad 0 \quad 0), \quad V = \frac{1}{2}\,(0 \quad 0 \quad 0 \quad 1), \quad V\,X = 1.$$

Let $\overline{x_i}$ be the residue class of x_i in $\mathcal{S} = \mathcal{R}/\mathcal{J}$ for $i = 1, \ldots, 4$. Computing $\overline{B} = \overline{V}\,\overline{A}\,\overline{K}$, we get:

$$\overline{B} = \left(\overline{x_4}\,\left(\frac{5}{2}\,\overline{x_2} + \overline{x_3}\right) \quad \overline{x_3}\,(3\,\overline{x_1} + \overline{x_4}) \quad 3\,\overline{x_4}\,\left(\frac{5}{2}\,\overline{x_2} + \overline{x_3}\right)\right) \in \mathcal{S}^{1\times 3}.$$

Finally, we can check again that the identity of matrices $\overline{A}\,\overline{K} = \overline{X}\,\overline{B}$ with entries in $\mathcal{S}$.

Let $\mathcal{B} = \operatorname{coker}_{\mathcal{S}}(\overline{B}.) = \mathcal{S}^{l\times 1}/(\overline{B}\,\mathcal{S}^{q\times 1})$ be the $\mathcal{S}$-module finitely presented by $\overline{B}$, i.e., defined by the following finite presentation:

$$\mathcal{S}^{q\times 1} \xrightarrow{\overline{B}.} \mathcal{S}^{l\times 1} \xrightarrow{\sigma} \mathcal{B} \longrightarrow 0. \tag{39}$$

The next lemma shows that $\mathcal{B}$ depends only on the rank factorization problem eq. (2).

Lemma 6. *With the above notations, the $\mathcal{S}$-module $\mathcal{B} = \operatorname{coker}_{\mathcal{S}}(\overline{B}.)$ does not depend on the choice of the bases defining L and X, and on the choice of a generating set of $\ker_{\mathcal{S}}(\overline{Q}.)$ defining $\overline{K}$. Hence, the Fitting ideals of $\mathcal{B}$ defined by*

$$\operatorname{Fitt}_i(\mathcal{B}) = \operatorname{Fitt}_i\left(\operatorname{coker}_{\mathcal{S}}\left(\overline{B}.\right)\right), \quad i = 0, \ldots, l. \tag{40}$$

depend only on the matrices M and $D_1, \ldots, D_r$.

Proof. Let $L' \in \mathbb{K}^{p\times m}$ be a matrix whose rows define another basis of $\ker_{\mathbb{K}}(.M)$. Then, there exists an invertible matrix $T \in \mathbb{K}^{p\times p}$ such that $L' = T\,L$. Similarly, let $X' \in \mathbb{K}^{m\times l}$ be a matrix whose columns define another basis of $\operatorname{im}_{\mathbb{K}}(M.)$. Then, there exists an invertible matrix $W \in \mathbb{K}^{l\times l}$ such that $X' = X\,W$. Finally, let $Q' = L'\,A = T\,Q$, where $Q = L\,A$, and $\overline{K'} \in \mathcal{S}^{r\times q'}$ be such that $\ker_{\mathcal{S}}(\overline{Q'}.) = \operatorname{im}_{\mathcal{S}}(\overline{K'}.)$. Using the invertibility of T, we clearly have $\ker_{\mathcal{S}}(\overline{Q'}.) = \ker_{\mathcal{S}}(\overline{Q}.)$, and thus, $\operatorname{im}_{\mathcal{S}}(\overline{K'}.) = \operatorname{im}_{\mathcal{S}}(\overline{K}.)$, which shows that there exists $\Lambda \in \mathcal{R}^{q\times q'}$ such that $\overline{K'} = \overline{K}\,\overline{\Lambda}$. Let $\overline{B'} \in \mathcal{S}^{l\times q'}$ be the unique matrix such that $\overline{A}\,\overline{K'} = \overline{X'}\,\overline{B'}$. Using $\overline{A}\,\overline{K} = \overline{X}\,\overline{B}$ and the fact that X has full column rank, we then get:

$$\overline{X}\,\overline{W}\,\overline{B'} = \overline{X'}\,\overline{B'} = \overline{A}\,\overline{K'} = \overline{A}\,\overline{K}\,\overline{\Lambda} = \overline{X}\,\overline{B}\,\overline{\Lambda} \implies \overline{W}\,\overline{B'} = \overline{B}\,\overline{\Lambda}.$$

If $\mathcal{B}' = \operatorname{coker}_{\mathcal{S}}(B'.)$, then we have the following commutative exact diagram of $\mathcal{S}$-modules

$$\begin{array}{ccccccc}
\mathcal{S}^{q'\times 1} & \xrightarrow{\overline{B'}.} & \mathcal{S}^{l\times 1} & \xrightarrow{\sigma'} & \mathcal{B}' & \longrightarrow & 0\\
\downarrow \overline{\Lambda}. & & \downarrow \overline{W}. & & \downarrow \gamma & & \\
\mathcal{S}^{q\times 1} & \xrightarrow{\overline{B}.} & \mathcal{S}^{l\times 1} & \xrightarrow{\sigma} & \mathcal{B} & \longrightarrow & 0,
\end{array}$$

where $\gamma \in \hom_{\mathcal{S}}(\mathcal{B}', \mathcal{B})$ is the $\mathcal{S}$-homomorphism defined by $\gamma(\sigma'(\overline{\mu})) = \sigma(\overline{W}\,\overline{\mu})$ for all $\overline{\mu} \in \mathcal{S}^{l\times 1}$. Since $W \in \mathbb{K}^{l\times l}$ is an invertible matrix, γ is an isomorphism and $\gamma^{-1}(\sigma(\overline{\nu})) = \sigma'(\overline{W}^{-1}\,\overline{\nu})$ for all $\overline{\nu} \in \mathcal{S}^{l\times 1}$, i.e., $\mathcal{B}' \cong \mathcal{B}$, which yields $\operatorname{Fitt}_i(\mathcal{B}') = \operatorname{Fitt}_i(\mathcal{B})$ for $i = 0, \ldots, l$ (see, e.g., [10, 25]), where, using Definition 1, the $\operatorname{Fitt}_i(\mathcal{B})$'s are defined by eq. (40). Hence, these ideals depend only on the matrices M, $D_1, \ldots, D_r$, and not particular choices in the construction of the matrix $\overline{B}$. □

Finally, using eq. (40), let us introduce a few more notations that will be used in the next section.

Definition 3. We note $\mathcal{I} = \operatorname{Fitt}_0(\mathcal{B})$, i.e., if $l < q$, $\mathcal{I} = \langle 0 \rangle_{\mathcal{S}}$ or if $l \geq q$, $\mathcal{I} = \operatorname{Fitt}_0(\mathcal{B}) = \langle n_1, \ldots, n_\rho \rangle_{\mathcal{S}}$, where $\{n_i\}_{i=1,\ldots,\rho}$ denotes the set formed by all the $l \times l$ minors of the matrix $\overline{B}$ and $\rho = l!/(q!\,(l-q)!)$.

If $\{h_1, \ldots, h_\beta\}$ is another set of generators of $\mathcal{I}$, then we shall note $\mathcal{I} = \langle h_1, \ldots, h_\beta \rangle_{\mathcal{S}}$, where according to the notation eq. (22), we can write $h_i = \overline{g_i}$ for $g_i \in \mathcal{R}$ and $i = 1, \ldots, \beta$.

Remark 15. $\mathcal{J}$ is generated by all the $r \times r$ minors of $Q = (L\,D_1\,x\ \ldots\ L\,D_r\,x) \in \mathcal{R}^{p\times r}$, i.e., $\mathcal{J} = \langle 0 \rangle$ if $r > p$ or $\mathcal{J} = \langle e_1, \ldots, e_\gamma \rangle$ if $r \leq p$, where the e_i's are either homogeneous polynomials of total degree r in the x_i's or all 0. The non-zero entries of K can then be chosen to be homogeneous polynomials. Using $V \in K^{l\times m}$, the non-zero entries of the matrix $\overline{B} = \overline{V}\,\overline{A}\,\overline{K}$ are then homogeneous polynomials. Finally, if $l < q$, then $\mathcal{I} = \langle 0 \rangle_{\mathcal{S}}$ or if $l \geq q$, $\mathcal{I} = \langle h_1, \ldots, h_\beta \rangle_{\mathcal{S}}$, where the h_i's are either homogeneous polynomials in the $\overline{x_i}$'s (and thus, the g_i's are homogeneous polynomials in the x_i's) or all 0.

Example 9. We continue Example 8. Let us denote by $\mathcal{B}$ the $\mathcal{S}$-module finitely presented by $\overline{B}$, i.e., $\mathcal{B} = \mathcal{S}/(\overline{B}\,\mathcal{S}^{3\times 1}) = \mathcal{S}/\left\langle \overline{B_1}, \overline{B_2}, \overline{B_3} \right\rangle_{\mathcal{S}}$ where $\overline{B_i}$ stands for the i^{th} entry of $\overline{B}$. Using $\overline{B_3} = 3\,\overline{B_1}$, we then have $\mathcal{I} = \operatorname{Fitt}_0(\mathcal{B}) = \langle \overline{B_1}, \overline{B_2} \rangle_{\mathcal{S}}$.

Remark 16. Using eq. (11), i.e., $\operatorname{ann}_{\mathcal{S}}(\mathcal{B})^l \subseteq \operatorname{Fitt}_0(\mathcal{B}) \subseteq \operatorname{ann}_{\mathcal{S}}(\mathcal{B})$, we then have $h_i\,\mathcal{B} = 0$ for $i = 1, \ldots, \beta$. Note that $\operatorname{ann}_{\mathcal{S}}(\mathcal{B}) = \langle 0 \rangle$ if and only if $\operatorname{Fitt}_0(\mathcal{B}) = \langle 0 \rangle$ since $\operatorname{ann}_{\mathcal{S}}(\mathcal{B}) \subseteq \sqrt{\operatorname{ann}_{\mathcal{S}}(\mathcal{B})} = \sqrt{\operatorname{Fitt}_0(\mathcal{B})}$.

Remark 17. Combining eq. (24) and eq. (39), we get the commutative exact diagram of $\mathcal{R}$-modules

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \ker_{\mathcal{S}}(\overline{B}.) & \longrightarrow & \mathcal{S}^{q\times 1} & \xrightarrow{\overline{B}.} & \mathcal{S}^{l\times 1} & \xrightarrow{\sigma} & \mathcal{B} \longrightarrow 0,\\
 & & & & \uparrow \mathrm{id}_q \otimes \chi & & \uparrow \mathrm{id}_l \otimes \chi & & \\
 & & & & \mathcal{R}^{q\times 1} & \xrightarrow{B.} & \mathcal{R}^{l\times 1} & & \\
 & & & & \uparrow I_q \otimes .e & & \uparrow I_l \otimes .e & & \\
 & & & & \mathcal{R}^{q\times \gamma} & \xrightarrow{B.} & \mathcal{R}^{l\times \gamma} & &
\end{array}$$

where $I_l \otimes .e$ and $I_q \otimes .e$ are defined by eq. (25). Now, using 4 of Proposition 3.1 of [7]), we have $\mathcal{B} \cong \mathcal{R}^{l\times 1}/(B\,\mathcal{R}^{q\times 1} + \mathcal{R}^{l\times\gamma}\,(I_l \otimes .e))$. Using eq. (26), we have $\operatorname{im}_{\mathcal{R}}(I_l \otimes .e) = \operatorname{im}_{\mathcal{R}}((e^T \otimes I_l).)$ (see Section 3.2.2). Thus, as a $\mathcal{R}$-module, $\mathcal{B}$ has the following finite presentation:

$$\mathcal{B} \cong \mathcal{R}^{l\times 1}/\left(B\,\mathcal{R}^{q\times 1} + (e^T \otimes I_l)\,\mathcal{R}^{l\,\gamma\times 1}\right) = \mathcal{R}^{l\times 1}/\left(\begin{pmatrix} B & e^T \otimes I_l \end{pmatrix}\mathcal{R}^{(q+l\,\gamma)\times 1}\right). \tag{41}$$

Doing similarly as in Section 3.2.2 for the computation of a matrix $\overline{K}$ satisfying $\ker_{\mathcal{S}}(\overline{Q}.) = \mathrm{im}_{\mathcal{S}}(\overline{K}.)$ (see eq. (27)), using 1 of Proposition 3.1 of [7], we then have

$$\ker_{\mathcal{S}}(\overline{B}.) = \mathrm{im}_{\mathcal{R}}(C.)/\mathrm{im}_{\mathcal{R}}((e^T \otimes I_q).) = \mathrm{im}_{\mathcal{S}}(\overline{C}.),$$

where $C \in \mathcal{R}^{q\times t}$ is such that:

$$\ker_{\mathcal{R}}((B \quad e^T \otimes I_l).) = \mathrm{im}_{\mathcal{R}}\left(\left(\begin{array}{c} C \\ C' \end{array}\right).\right).$$

The matrices $C \in \mathcal{R}^{q\times t}$ and $C' \in \mathcal{R}^{l\,\gamma\times t}$ can be computed using, e.g., the `SyzygyModule` command of the OreModules package. The matrix $\overline{C}$ can also be computed by the `Ker` (resp., `WeakKernelEmbedding`) command of `Singular` (resp., `CapAndHomalg`). Finallly, the effective characterization eq. (41) of $\mathcal{B}$ and the computation of $\ker_{\mathcal{S}}(\overline{B}.)$ are implemented in the RankFactorization package.

Finally, using the notations of Theorem 4, instead of considering $\mathcal{J} = \mathcal{J}_0$, we can repeat the same construction as above with $\mathcal{J}_k$, $\mathcal{S}_k = \mathcal{R}/\mathcal{J}_k$, and $K_k \in \mathcal{R}^{r\times q}$ satisfying $\ker_{\mathcal{S}}(\chi_k(Q).) = \mathrm{im}_{\mathcal{S}}(\chi_k(K_k).)$ for $k = 0, \ldots, r-1$ to obtain a matrix $B_k \in \mathcal{R}^{l\times q_k}$ satisfying:

$$\chi_k(A)\,\chi_k(K_k) = \chi_k(X)\,\chi_k(B_k) \;\Leftrightarrow\; A\,K_k - X\,B_k \in \mathcal{J}_k^{m\times q}. \tag{42}$$

Combining this last identity with $V\,X = I_l$ (see eq. (36)), we then get $V\,A\,K_k - B_k \in \mathcal{J}_k^{l\times q}$, and thus, $\chi_k(B_k) = \chi_k(V)\,\chi_k(A)\,\chi_k(K_k)$. We can define the finitely presented $\mathcal{S}_k$-module $\mathcal{B}_k = \mathrm{coker}_{\mathcal{S}_k}(\chi_k(B_k).)$, which depends only on the matrices M, $D_1, \ldots, D_r$. Finally, as for $\mathcal{J} = \mathcal{J}_0$, to simply the notations, we shall set $B_k = V\,A\,K_k \in \mathcal{R}^{l\times q_k}$ for $k = 0, \ldots, r-1$.

3.2.5 Characterization of the existence of a right inverse of B

Let us sum up the results obtained so far. Let us suppose that the rank factorization problem eq. (4) has a solution (u, v), where $M \in \mathbb{K}^{m\times n}$ has not full column rank. As explained at the beginning of Section 3.2.1, if the rows of $L \in \mathbb{K}^{p\times m}$ define a basis of $\ker_{\mathbb{K}}(.M)$, then $L\,A(u)\,v = 0$, i.e., $v \in \ker_{\mathbb{K}}(Q(u).)$, where $Q(u) = L\,A(u)$. Thus, $u \in \mathbb{K}^{m\times 1}$ must be so that $\ker_{\mathbb{K}}(Q(u).) \neq 0$, i.e., by Lemma 5, $u \in \mathcal{V}_{\mathbb{K}}(\mathcal{J}_0)$. Let $0 \leq k \leq r-1$ be such that $u \in \mathcal{V}_{\mathbb{K}}(\mathcal{J}_k) \setminus \mathcal{V}_{\mathbb{K}}(\mathcal{J}_{k+1})$. Then, by Theorem 4, there exists $K_k \in \mathcal{R}^{s\times q}$ satisfying $\ker_{\mathbb{K}}(Q(u).) = \mathrm{im}_{\mathbb{K}}(K_k(u).)$. Hence, v must necessary be of the form of $v = K_k(u)\,T_u$ for a certain $T_u \in \mathbb{K}^{q_k\times n}$. We then have $A(u)\,v = A(u)\,K_k(u)\,T_u$ and using the identity eq. (42), i.e., $A\,K_k - X\,B_k \in \mathcal{J}_k^{m\times q}$, where $B_k \in \mathcal{R}^{l\times q_k}$, we get $A(u)\,K_k(u) = X\,B_k(u)$ for all $u \in \mathcal{V}_{\mathbb{K}}(\mathcal{J}_k)$, and thus, $A(u)\,K_k(u)\,T_u = X\,B_k(u)\,T_u = M = X\,Y$, which yields $B_k(u)\,T_u = Y$ because $X \in \mathbb{K}^{m\times l}$ has full column rank. Hence, if (u, v) is a solution of the rank factorization problem eq. (4), then necessarily $u \in \mathcal{V}_{\mathbb{K}}(\mathcal{J}_k)$, $0 \leq k \leq r-1$, and $v = K_k(u)\,T_u$ for a matrix $T_u \in \mathbb{K}^{q_k\times n}$ satisfying $B_k(u)\,T_u = Y$. Conversely, if we consider u and v satisfying these conditions, we then have

$$A(u)\,v = A(u)\,K_k(u)\,T_u = X\,B_k(u)\,T_u = X\,Y = M,$$

i.e., (u, v) is a solution of the rank factorization problem eq. (4). Hence, we are led to study the existence of a matrix $T_k \in \mathcal{R}^{q\times m}$ satisfying the following inhomogeneous linear system

$$\chi_k(B_k)\,\chi_k(T_k) = \chi_k(Y), \tag{43}$$

where $\chi_k : \mathcal{R} \longrightarrow \mathcal{S}_k$ denotes the canonical ring epimorphism, i.e., satisfying $B_k\,T_k - Y \in \mathcal{J}_k^{l\times n}$.

Again, to simplify the exposition, we shall only consider below the case $\mathcal{S} = \mathcal{S}_0$, i.e., $\mathcal{J} = \mathcal{J}_0$. The general case exactly follows the same arguments.

More generally, we might want to solve eq. (43), i.e., $\overline{B}\,\overline{T}_0 = \overline{Y}$, in a different ring than just $\mathcal{S}$. To do that, let $\varphi : \mathcal{S} \longrightarrow \mathcal{T}$ be a ring homomorphism. If we note $B_\varphi = \varphi(\overline{B}) \in \mathcal{T}^{l\times q}$ and $Y_\varphi = \varphi(\overline{Y}) \in \mathcal{T}^{l\times n}$, then eq. (43) yields the problem of finding $T \in \mathcal{T}^{q\times m}$ satisfying:

$$B_\varphi\,T = Y_\varphi. \tag{44}$$

The next lemma shows that the existence of a solution of eq. (44) in $\mathcal{T}$ is equivalent to the existence of a right inverse of B_φ with entries in $\mathcal{T}$, namely, the existence of $E \in \mathcal{T}^{q\times l}$ such that $B_\varphi\,E = I_l$.

Lemma 7. *With the above notations, the following assertions are equivalent:*

1. *There exists $T \in \mathcal{T}^{q\times n}$ satisfying $B_\varphi\, T = Y_\varphi$.*
2. *There exists $E \in \mathcal{T}^{q\times l}$ satisfying $B_\varphi\, E = I_l$.*

Proof. Using the fact that Y has a right inverse $H \in \mathbb{K}^{n\times l}$ (see the proof of Proposition 2), i.e., $Y\,H = I_l$, we then have $Y_\varphi\, H_\varphi = I_l$ where $H_\varphi = \varphi(H)$. Point 1 then yields $B_\varphi\,(T\,H_\varphi) = I_l$, which shows that B_φ has a right inverse with entries in $\mathcal{T}$.

Conversely, if B_φ has a right inverse $E \in \mathcal{T}^{q\times l}$, then $T = E\,Y_\varphi$ satisfies $B_\varphi\,T = Y_\varphi$. □

The next lemma parametrizes all the solutions $T \in \mathcal{T}^{q\times l}$ of $B_\varphi\,T = Y_\varphi$.

Lemma 8. *If $\mathcal{T}$ is a noetherian ring, $C \in \mathcal{T}^{q\times t}$ is such that $\ker_{\mathcal{T}}(B_\varphi.) = \mathrm{im}_{\mathcal{T}}(C.)$, $E \in \mathcal{T}^{q\times l}$ is a right inverse of B_φ, then all the solutions $T \in \mathcal{T}^{q\times l}$ of $B_\varphi\,T = Y_\varphi$ are given by:*

$$\forall\, Y' \in \mathcal{T}^{t\times n}, \quad T(Y') = E\,Y_\varphi + C\,Y'. \tag{45}$$

Proof. $E\,Y_\varphi$ is a particular solution of the inhomogeneous linear system $B_\varphi\,T = Y_\varphi$. If $T' \in \mathcal{T}^{q\times l}$ is another solution, then we get $B_\varphi\,(T' - E\,Y_\varphi) = 0$, which shows that $\mathrm{im}_{\mathcal{T}}((T' - E\,Y_\varphi).) \subseteq \ker_{\mathcal{T}}(B_\varphi.) = \ker_{\mathcal{T}}(C.)$, which yields the existence of $Y' \in \mathcal{T}^{t\times l}$ such that $T' - E\,Y_\varphi = C\,Y'$, i.e., such as $T' = T(Y')$ for a certain $Y' \in \mathcal{T}^{t\times n}$. Finally, we have $B_\varphi\,T(Y') = B_\varphi\,E\,Y_\varphi + B_\varphi\,C\,Y' = Y_\varphi$ for all $Y' \in \mathcal{T}^{t\times n}$, which shows that eq. (45) parametrizes all the solutions of eq. (44) in $\mathcal{T}^{q\times l}$. □

Using Lemma 7 and Lemma 8, we can then reduce the rank factorization problem eq. (4) to the study of the existence of a right inverse of the matrix B_φ. Let us finally study this last problem.

Applying the right exact covariant functor $\mathcal{T} \otimes_{\mathcal{S}} \cdot$ [10, 25] to the exact sequence eq. (39), we obtain the following exact sequence of $\mathcal{T}$-modules:

$$\mathcal{T}^{q\times 1} \xrightarrow{B_\varphi.} \mathcal{T}^{l\times 1} \xrightarrow{\mathrm{id}_{\mathcal{T}} \otimes \sigma} \mathcal{T} \otimes_{\mathcal{S}} \mathcal{B} \longrightarrow 0. \tag{46}$$

Note that we then have $\mathcal{T} \otimes_{\mathcal{S}} \mathcal{B} \cong \mathrm{coker}_{\mathcal{T}}(B_\varphi.)$. Thus, $\mathcal{T} \otimes_{\mathcal{S}} \mathcal{B} \cong 0$, i.e., $B_\varphi\,\mathcal{T}^{q\times 1} = \mathcal{T}^{l\times 1}$, if and only if there exists a matrix $E \in \mathcal{T}^{q\times l}$ satisfying $B_\varphi\,E = I_l$, i.e., if and only if the matrix B_φ has a right inverse with entries in $\mathcal{T}$. Let us now investigate when the $\mathcal{T}$-module $\mathcal{T} \otimes_{\mathcal{S}} \mathcal{B}$ is reduced to 0.

We state again that a *prime ideal* $\mathfrak{p}$ of $\mathcal{T}$ is an ideal of $\mathcal{T}$ which is such that the factor ring $\mathcal{T}/\mathfrak{p}$ is an integral domain (i.e., $\mathcal{T}/\mathfrak{p}$ has no non-zero zero divisors). A $\mathcal{T}$-module $\mathcal{M}$ is said to be *projective of constant rank* r if the $\mathcal{T}_{\mathfrak{p}} = \{t/s \mid t \in \mathcal{T},\, s \notin \mathfrak{p}\}$-module $\mathcal{M}_{\mathfrak{p}} = \{m/s \mid m \in \mathcal{M},\, s \notin \mathfrak{p}\}$ (i.e., the localization of $\mathcal{M}$ at the multiplicatively closed set $S = \mathcal{T} \setminus \mathfrak{p}$) is a free module of rank r, i.e., $\mathcal{M}_{\mathfrak{p}} \cong \mathcal{T}_{\mathfrak{p}}^r$, for all prime ideals $\mathfrak{p}$ of $\mathcal{T}$. For more details, see, e.g., [10, 20].

Proposition 3 (Proposition 20.7, [10])**.** *A finitely presented $\mathcal{T}$-module $\mathcal{M}$ is projective of constant rank r if and only if* $\mathrm{Fitt}_r(\mathcal{M}) = \mathcal{T}$ *and* $\mathrm{Fitt}_{r-1}(\mathcal{M}) = \langle 0 \rangle$.

Note that the zero module is a projective $\mathcal{T}$-module of rank 0. Hence, Proposition 3 shows that $\mathcal{T} \otimes_{\mathcal{S}} \mathcal{B} \cong \mathrm{coker}_{\mathcal{T}}(B_\varphi.) = 0$ if and only if $\mathrm{Fitt}_0(\mathrm{coker}_{\mathcal{T}}(B_\varphi.)) = \mathcal{T}$.

The next proposition is a direct consequence of the right exactness of the covariant functor $\mathcal{T} \otimes_{\mathcal{S}} \cdot$.

Proposition 4 (Corollary 20.5 of [10])**.** *Let $\varphi : \mathcal{S} \longrightarrow \mathcal{T}$ be a ring homomorphism and $\mathcal{L}$ a finitely presented $\mathcal{S}$-module. Then, we have* $\mathrm{Fitt}_i(\mathcal{T} \otimes_{\mathcal{S}} \mathcal{L}) = \mathcal{T} \otimes_{\mathcal{S}} \mathrm{Fitt}_i(\mathcal{L})$ *for all $i \geq 0$, where* $\mathcal{T} \otimes_{\mathcal{S}} \mathrm{Fitt}_i(\mathcal{L})$ *denotes the ideal of $\mathcal{T}$ generated by* $\varphi(\mathrm{Fitt}_i(\mathcal{L}))$*, i.e., by the images of the generators of* $\mathrm{Fitt}_i(\mathcal{L})$ *by φ.*

The next result studies when the matrix B_φ has a right inverse with entries in $\mathcal{T}$.

Corollary 4. *With the above notations, let* $\mathcal{I} = \mathrm{Fitt}_0(\mathcal{B}) = \langle h_1, \ldots, h_\beta \rangle_{\mathcal{S}}$ *be the ideal of $\mathcal{S}$ defined by all the $l \times l$ minors of $\overline{B}$, where $h_i \in \mathcal{S}$ for $i = 1, \ldots, \beta$. Then, the matrix $B_\varphi \in \mathcal{T}^{l\times q}$ has a right inverse with entries in $\mathcal{T}$ if and only if $\langle \varphi(h_1), \ldots, \varphi(h_\beta) \rangle_{\mathcal{T}} = \mathcal{T}$, i.e., if and only if there exist $t_i \in \mathcal{T}$, $i = 1, \ldots, \beta$, satisfying the following Bézout equation:*

$$\sum_{i=1}^{\beta} t_i\,\varphi(h_i) = 1. \tag{47}$$

Finally, if $\mathcal{I} = \langle 0 \rangle_{\mathcal{S}}$, then the matrix B_φ has no right inverses for all non-trivial rings $\mathcal{T}$.

Proof. Using Proposition 4, $\mathcal{T} \otimes_{\mathcal{S}} \mathcal{B} \cong \operatorname{coker}_{\mathcal{T}}(B_{\varphi}.) = 0$ if and only if:

$$\operatorname{Fitt}_0(\operatorname{coker}_{\mathcal{T}}(B_{\varphi}.)) = \mathcal{T} \otimes_{\mathcal{S}} \operatorname{Fitt}_0(\operatorname{coker}_{\mathcal{S}}(\overline{B}.)) = \mathcal{T} \otimes_{\mathcal{S}} \operatorname{Fitt}_0(\mathcal{B}) = \varphi(\operatorname{Fitt}_0(\mathcal{B})) = \mathcal{T}.$$

The last equality is equivalent to the Bézout equation eq. (47). Finally, if $\mathcal{I} = \langle 0 \rangle_{\mathcal{S}}$, then $\varphi(\mathcal{I}) = \langle 0 \rangle_{\mathcal{T}}$ for all ring homomorphisms $\varphi : \mathcal{S} \longrightarrow \mathcal{T}$. Hence, if $\mathcal{T}$ is a non-trivial ring, we cannot have $\mathcal{T} = \langle 0 \rangle_{\mathcal{T}}$. □

In what follows, we shall suppose that $\mathcal{I} \neq \langle 0 \rangle_{\mathcal{S}}$ and all the h_i's are not 0.

Let us first study Corollary 4 in the case of $\mathcal{T} = \mathcal{S}$. According to Section 3.2.2, we have $\mathcal{S} = \mathcal{R}/\mathcal{J}$, $\mathcal{R} = \mathbb{K}[x_1, \ldots, x_m]$, and $\mathcal{J} = \operatorname{Fitt}_0(\mathcal{Q}) = \langle e_1, \ldots, e_\gamma \rangle$. Writing $h_i = \overline{g_i}$ and $t_i = \overline{r_i}$, where $g_i,\, r_i \in \mathcal{R}$ for $i = 1, \ldots, \beta$, eq. (47) is equivalent to the existence of $r'_1, \ldots, r'_\gamma \in \mathcal{R}$ such that $\sum_{i=1}^{\beta} r_i\, g_i + \sum_{j=1}^{\gamma} r'_j\, e_j = 1$, i.e., $\langle g_1, \ldots, g_\beta \rangle + \langle e_1, \ldots, e_\gamma \rangle = \mathcal{R}$. Using Gröbner basis methods, the above Bézout identity can be effectively checked and the r_i's, and thus, the t_i's satisfying eq. (47), computed. Therefore, eq. (47) can be effectively tested for $\mathcal{T} = \mathcal{S}$ using standard effective elimination methods (see, e.g., [10, 12]).

Let us now suppose that eq. (47) holds in $\mathcal{T} = \mathcal{S}$. In this case, note that $\operatorname{Fitt}_0(\mathcal{B}) = \langle 0 \rangle_{\mathcal{S}}$ if and only if $\mathcal{B} = 0$. Then, using eq. (41), $\mathcal{B} = 0$ if and only if $(B \quad e^T \otimes I_l)\, \mathcal{R}^{(q+l\,\gamma)\times 1} = \mathcal{R}^{l \times 1}$, i.e., if and only if $(B \quad e^T \otimes I_l)$ has a right inverse with entries in $\mathcal{R}$. Using Gröbner basis methods, this last problem can be effectively solved (see, e.g., [5]). See the `RightInverse` command of the OreModules package [6] or the `PreInverse` command of the `CapAndHomalg` library [1]. Hence, two matrices $E \in \mathcal{R}^{q \times l}$ and $E' \in \mathcal{R}^{l\,\gamma \times l}$ can be obtained satisfying $B\,E + (e^T \otimes I_l)\,E' = I_l$, which yields $\overline{B}\,\overline{E} = I_l$, where $\overline{E} = \chi(E)$. Hence, we have $B(u)\,E(u) = I_l$ for all $u \in \mathcal{V}_{\mathbb{K}}(\mathcal{J})$. Using Lemma 8, where the matrix $C \in \mathcal{R}^{q \times t}$ is such that $\ker_{\mathcal{S}}(\overline{B}.) = \operatorname{im}_{\mathcal{S}}(\overline{C}.)$ (see also Remark 17), then $\{T(Y') = \overline{E}\,\overline{Y} + \overline{C}\,Y' \mid \forall\, Y' \in \mathcal{T}^{t \times n}\}$ parametrizes all the solutions of $\overline{B}\,T = \overline{Y}$ in $\mathcal{S}^{q \times l}$. Hence, the solutions of the rank factorization problem eq. (4) are then parametrized as follows:

$$\begin{cases} u \in \mathcal{V}_{\mathbb{K}}(\mathcal{J}), \\ v(u, Y') = K(u)\,(E(u)\,Y + C(u)\,Y'(u)),\ \forall\, Y' \in \mathcal{S}^{t \times n}. \end{cases}$$

Note that $\mathbb{K}^{t \times n} \subset \mathcal{S}^{t \times n}$ and $Y'(u) \in \mathbb{K}^{t \times n}$ for all $u \in \mathcal{V}_{\mathbb{K}}(\mathcal{J})$ and for all $Y' \in \mathcal{S}^{t \times n}$, which shows that the above parametrization is also defined as follows:

$$\begin{cases} u \in \mathcal{V}_{\mathbb{K}}(\mathcal{J}), \\ v(u, Y') = K(u)\,(E(u)\,Y + C(u)\,Y'),\ \forall\, Y' \in \mathbb{K}^{t \times n}. \end{cases} \tag{48}$$

Hence, if eq. (47) has a solution in $\mathcal{S}$, then we can obtain the above explicit parametrization of solutions of the rank factorization problem eq. (4). Using Remark 15, we know that the h_i's and the e_j's are either homogeneous polynomials or 0. Therefore, 0 is a common zero of the h_i's and the e_j's, which shows that eq. (47) does not hold in $\mathcal{S}$, and thus, the solutions of Problem eq. (4) cannot simply be parametrized by a single closed-form of the form eq. (48).

Example 10. We continue Example 9. We have $\mathcal{I} = \langle \overline{B}_1, \overline{B}_2 \rangle_{\mathcal{S}}$, i.e., $\overline{g_1} = \overline{B}_1$ and $\overline{g_2} = \overline{B}_2$, where $g_1 = 5\,x_2\,x_4/2 + x_3\,x_4$ and $g_2 = 3\,x_1\,x_3 + x_3\,x_4$. Considering e_1, e_2, and e_3 defined in Example 5, we can check again that $(0\ 0\ 0\ 0)^T$ is a common zero of the polynomials g_1, g_2, e_1, e_2, and e_3, which shows that $1 \notin \langle g_1,\, g_2,\, e_1,\, e_2,\, e_3 \rangle$ and proves that $\overline{B}$ has no right inverse with entries in $\mathcal{S}$ by Corollary 4.

3.2.6 Solutions of the rank factorization problem

Since eq. (47) has no solutions in $\mathcal{S}$, in this section, we seek the solutions of eq. (47) in a different ring $\mathcal{T}$.

We suppose that at least one of the generators of $\mathcal{I} = \langle h_1, \ldots, h_\beta \rangle_{\mathcal{S}}$, say h_i, is not a nilpotent element of $\mathcal{S}$, i.e., $h_i^k \neq 0$ for all $k \in \mathbb{Z}_{>0}$. We can consider the non-trivial ring $\mathcal{S}_{h_i} = \{s/h_i^k \mid s \in \mathcal{S},\ k \in \mathbb{Z}_{\geq 0}\}$ defined as the localization of $\mathcal{S}$ at the multiplicatively closed set $\{h_i^k\}_{k \in \mathbb{Z}_{\geq 0}}$ and the canonical ring homomorphism $j_{h_i} : \mathcal{S} \longrightarrow \mathcal{S}_{h_i}$ defined by $j_{h_i}(s) = s/1$ for all $s \in \mathcal{S}$, whose kernel $\ker j_{h_i} = \{s \in \mathcal{S} \mid \exists\, k \in \mathbb{Z}_{\geq 0} :\ h_i^k\, s = 0\}$ (see, e.g., [10, 25]). Then, we have $j_{h_i}(h_j) = h_j/1$ for $j = 1, \ldots, \beta$ and $j_{h_i}(h_i) = h_i/1 \neq 0/1$ because h_i is not a nilpotent element of $\mathcal{S}$. If we set $t_i = (h_i/1)^{-1} \in \mathcal{S}_{h_i}$ and $t_j = 0/1$ for $j \neq i$, then $(t_1, \ldots, t_\beta)$ is a solution of eq. (47) in $\mathcal{S}_{h_i}$, which shows that $j_{h_i}(\overline{B}) = B_{h_i}$ has a right inverse with entries in $\mathcal{S}_{h_i}$.

Let $E \in \mathcal{S}_{h_i}^{q\times l}$ be a right inverse of B_{h_i}. Writing $E_{h_i} = E'_{h_i}/h_i^a$, where $E'_{h_i} \in \mathcal{S}^{q\times l}$ and $a \in \mathbb{Z}_{\geq 0}$, then $B_{h_i}\, E_{h_i} = I_l$ is equivalent to $B_{h_i}\, E'_{h_i} - h_i^a\, I_l = 0$ in $\mathcal{S}_{h_i}^{l\times l}$, and thus, to $h_i^b\, (\overline{B}\, E'_{h_i} - h_i^a\, I_l) = 0$ in $\mathcal{S}^{l\times l}$ for a certain $b \in \mathbb{Z}_{\geq 0}$. Writing $h_i = \overline{g_i}$ and $E'_{h_i} = \chi(E''_{h_i})$, where $E''_{h_i} \in \mathcal{R}^{q\times l}$, i.e., $E'_{h_i} = \overline{E''_{h_i}}$, then the last identity is equivalent to:

$$g_i^b\, (B\, E''_{h_i} - g_i^a\, I_l) \in \mathcal{J}^{l\times l}. \tag{49}$$

Using eq. (49), we then obtain:

$$\forall\, u \in \mathcal{V}_{\mathbb{K}}(\mathcal{J}), \quad g_i(u)^b\, (B(u)\, E''_{h_i}(u) - g_i(u)^a\, I_l) = 0.$$

Hence, if $g_i(u) \neq 0$, then $E''_{h_i}(u)/g_i(u)^a$ is a right inverse of the matrix $B(u)$.

If $\mathbb{K}$ is an algebraically closed field of characteristic 0, using Hilbert's Nullstellensatz, the condition $\mathcal{V}_{\mathbb{K}}(\mathcal{J}) \subseteq \mathcal{V}_{\mathbb{K}}(\langle g_i\rangle)$, i.e., $g_i(u) = 0$ for all $u \in \mathcal{V}_{\mathbb{K}}(\mathcal{J})$, is equivalent to $\langle g_i\rangle \subseteq \sqrt{\langle g_i\rangle} \subseteq \sqrt{\mathcal{J}}$, and thus, to $g_i^k \in \mathcal{J}$ for a certain $k \in \mathbb{Z}_{\geq 0}$, i.e., to $h_i^k = 0$, i.e., h_i is a nilpotent element of $\mathcal{S}$. We state again that $\mathbb{K}$ is supposed to be of characteristic 0. Hence, if $\mathbb{K}$ is an algebraically closed field and h_i is not an idempotent element of $\mathcal{S}$, then there exists $u \in \mathcal{V}_{\mathbb{K}}(\mathcal{J})$ such that $g_i(u) \neq 0$. Thus, a right inverse of E of B_{h_i} exist "locally", i.e., for all $u \in \mathcal{V}_{\mathbb{K}}(\mathcal{J})$ satisfying $g_i(u) \neq 0$ but not "globally", i.e., not for all $u \in \mathcal{V}_{\mathbb{K}}(\mathcal{J})$.

Proposition 5. *With the above notations, let* $\mathrm{Fitt}_0(\mathcal{B}) = \langle h_1, \ldots, h_\beta\rangle_{\mathcal{S}}$. *If* h_i *is not a nilpotent element of* $\mathcal{S}$, *then* B_{h_i} *has a right inverse* $E_{h_i} \in \mathcal{S}_{h_i}^{q\times l}$, *i.e.,* $B_{h_i}\, E_{h_i} = I_l$, *where* $\mathcal{S}_{h_i} = \{s/h_i^k \mid s \in \mathcal{S},\ k \in \mathbb{Z}_{\geq 0}\}$ *denotes the localization of* $\mathcal{S}$ *at the multiplicatively closed set* $\{h_i^k\}_{k\in\mathbb{Z}_{\geq 0}}$.

Moreover, there exist two matrices $C_{h_i} \in \mathcal{S}_{h_i}^{q\times t}$ *and* $F_{h_i} \in \mathcal{S}_{h_i}^{t\times q}$ *satisfying:*

$$\ker_{\mathcal{S}_{h_i}}(B_{h_i}.) = \mathrm{im}_{\mathcal{S}_{h_i}}(C_{h_i}.), \quad C_{h_i}\, F_{h_i} + E_{h_i}\, B_{h_i} = I_q. \tag{50}$$

Finally, all the right inverses of the matrix B_{h_i} *with entries in* $\mathcal{S}_{h_i}$ *are of the form:*

$$\forall\, Q \in \mathcal{S}_{h_i}^{t\times l}, \quad E_{h_i}(Q) = E_{h_i} + C_{h_i}\, Q. \tag{51}$$

Proof. Let $\mathcal{S}_{h_i} = \{s/h_i^k \mid s \in \mathcal{S},\ k \in \mathbb{Z}_{\geq 0}\}$ and $\mathcal{S}_{h_i}^{-1}\mathcal{B} = \{b/h_i^k \mid b \in \mathcal{B},\ k \in \mathbb{Z}_{\geq 0}\}$ the $\mathcal{S}_{h_i}$-module defined by the localization of $\mathcal{B}$ at $\{h_i^k\}_{k\in\mathbb{Z}_{\geq 0}}$. Using Remark 16, we have $h_i\, \mathcal{B} = 0$, and thus, $\mathcal{S}_{h_i}^{-1}\mathcal{B} = 0$, i.e., $B_{h_i}\, \mathcal{S}_{h_i}^{q\times 1} = \mathcal{S}_{h_i}^{l\times 1}$, which proves the existence of $E_{h_i} \in \mathcal{S}_{h_i}^{q\times l}$ satisfying $B_{h_i}\, E_{h_i} = I_l$.

Now, using the fact that $\mathcal{S}_{h_i}$ is a noetherian ring (see, e.g., [10, 25]), $\ker_{\mathcal{S}_{h_i}}(B_{h_i}.)$ is a finitely generated $\mathcal{S}_{h_i}$-module, and thus, there exist $t \in \mathbb{Z}_{\geq 0}$ and $C_{h_i} \in \mathcal{S}_{h_i}^{q\times t}$ such that $\ker_{\mathcal{S}_{h_i}}(B_{h_i}.) = \mathrm{im}_{\mathcal{S}_{h_i}}(C_{h_i}.)$.

Setting $\Pi = I_q - E_{h_i}\, B_{h_i} \in \mathcal{S}_{h_i}^{q\times q}$, we have $B_{h_i}\, \Pi = 0$, i.e., $\mathrm{im}_{\mathcal{S}_{h_i}}(\Pi.) \subseteq \ker_{\mathcal{S}_{h_i}}(B_{h_i}.) = \mathrm{im}_{\mathcal{S}_{h_i}}(C_{h_i}.)$, which shows that there exists $F_{h_i} \in \mathcal{S}_{h_i}^{t\times q}$ satisfying $\Pi = C_{h_i}\, F_{h_i}$, i.e., $C_{h_i}\, F_{h_i} + E_{h_i}\, B_{h_i} = I_q$.

Now, let $E'_{h_i} \in \mathcal{S}_{h_i}^{q\times l}$ be a second right inverse of $\overline{B}$, i.e., $B_{h_i}\, E'_{h_i} = I_l$. Then, $B_{h_i}\, (E'_{h_i} - E_{h_i}) = 0$, i.e., $\mathrm{im}_{\mathcal{S}_{h_i}}((E'_{h_i} - E_{h_i}).) \subseteq \ker_{\mathcal{S}_{h_i}}(B_{h_i}.) = \mathrm{im}_{\mathcal{S}_{h_i}}(C_{h_i}.)$, which shows that there exists $Q \in \mathcal{S}_{h_i}^{t\times l}$ such that $E'_{h_i} - E_{h_i} = C_{h_i}\, Q$, i.e., using eq. (51), such that $E'_{h_i} = E_{h_i}(Q)$.

Finally, we have $B_{h_i}\, E_{h_i}(Q) = B_{h_i}\, E_{h_i} + (B_{h_i}\, C_{h_i})\, Q = I_l$ for all $Q \in \mathcal{S}_{h_i}^{t\times l}$, which finally proves that eq. (51) parametrizes all the right inverses of B_{h_i} with entries in the ring $\mathcal{S}_{h_i}$. □

Remark 18. Using the fact that $\mathcal{S}_{h_i}$ is a flat $\mathcal{S}$-module (see, e.g., [10, 25]), applying the exact functor $\mathcal{S}_{h_i} \otimes_{\mathcal{S}} \cdot$ to eq. (39), we obtain the following split exact sequence of $\mathcal{S}_{h_i}$-modules (see, e.g., [10, 25])

$$\mathcal{S}_{h_i}^{t\times 1} \underset{F_{h_i}.}{\overset{C_{h_i}.}{\rightleftarrows}} \mathcal{S}_{h_i}^{q\times 1} \underset{E_{h_i}.}{\overset{B_{h_i}.}{\rightleftarrows}} \mathcal{S}_{h_i}^{l\times 1} \longrightarrow \mathcal{S}_{h_i}^{-1}\mathcal{B} = 0,$$

where $C_{h_i} \in \mathcal{S}_{h_i}^{q\times t}$ and $F_{h_i} \in \mathcal{S}_{h_i}^{t\times q}$ satisfy eq. (50), which gives another proof of Proposition 5.

Using Lemma 8 with $\mathcal{T} = \mathcal{S}_{h_i}$, all the solutions $T \in \mathcal{S}_{h_i}^{q\times n}$ of $B_{h_i}\, T = \overline{Y}$ are defined by:

$$\forall\, Y' \in \mathcal{S}_{h_i}^{t\times n}, \quad T_{h_i}(Y') = E_{h_i}\, \overline{Y} + C_{h_i}\, Y'.$$

As explained in the first paragraph of Section 3.2.5, all the solutions $v \in \mathcal{S}_{h_i}^{r\times n}$ of $A\,v = M$ are then:

$$\forall\, Y' \in \mathcal{S}_{h_i}^{t\times n}, \quad v_{h_i}(Y') = K\,T_{h_i}(Y') = K\,(E_{h_i}\,Y + C_{h_i}\,Y').$$

We can combine the different abobe exact sequences in the following commutative exact diagram

$$\begin{array}{ccccccccc}
 & & 0 & \longleftarrow & \mathcal{S}_{h_i}^{p\times 1} & \xleftarrow{\overline{L}.} & \mathcal{S}_{h_i}^{m\times 1} & \xleftarrow{\overline{M}.} & \mathcal{S}_{h_i}^{n\times 1} & & \\
 & & & & \Vert & & \Vert & & \downarrow \overline{Y}. & & \\
 & & 0 & \longleftarrow & \mathcal{S}_{h_i}^{p\times 1} & \xleftarrow{\overline{L}.} & \mathcal{S}_{h_i}^{m\times 1} & \xleftarrow{\overline{X}.} & \mathcal{S}_{h_i}^{l\times 1} & \longleftarrow & 0 \\
 & & & & \Vert & & \uparrow \overline{A}. & & B_{h_i}.\uparrow\downarrow E_{h_i}. & & \\
0 & \longleftarrow & \mathcal{S}_{h_i}\otimes_{\mathcal{R}} \mathcal{T}(\mathcal{Q}) & \xleftarrow{\mathrm{id}_{\mathcal{S}_{h_i}}\otimes\kappa} & \mathcal{S}_{h_i}^{p\times 1} & \xleftarrow{\overline{Q}.} & \mathcal{S}_{h_i}^{r\times 1} & \xleftarrow{\overline{K}.} & \mathcal{S}_{h_i}^{q\times 1} & & \\
 & & & & & & & & C_{h_i}.\uparrow\downarrow F_{h_i}. & & \\
 & & & & & & & & \mathcal{S}_{h_i}^{t\times 1} & &
\end{array}
\qquad \mathcal{S}_{h_i}^{n\times 1} \xrightarrow{Y'.} \mathcal{S}_{h_i}^{t\times 1} \tag{52}$$

up to the fact that the composition of the homomorphisms $\overline{Y}.$ and $E_{h_i}.$ does not form a complex.

First note that the u's corresponding to v_{h_i} are the elements of $\mathcal{V}_{\mathbb{K}}(\mathcal{J})$ which satisfy $g_i(u) \neq 0$, i.e., $u \in \mathcal{V}_{\mathbb{K}}(\mathcal{J}) \setminus \mathcal{V}_{\mathbb{K}}(\langle g_i\rangle)$. Let us set $\mathcal{U}_{g_i} = \mathcal{V}_{\mathbb{K}}(\mathcal{J}) \setminus \mathcal{V}_{\mathbb{K}}(\langle g_i\rangle)$. If $h_i = \overline{g_i'}$, where $g_i' \in \mathcal{R}$, then $g_i' - g_i \in \mathcal{J}$, then $g_i'(u) = g_i(u)$ for all $u \in \mathcal{V}_{\mathbb{K}}(\mathcal{J})$, and thus, $\mathcal{V}_{\mathbb{K}}(\mathcal{J}) \setminus \mathcal{V}_{\mathbb{K}}(\langle g_i'\rangle) = \mathcal{V}_{\mathbb{K}}(\mathcal{J}) \setminus \mathcal{V}_{\mathbb{K}}(\langle g_i\rangle)$, which shows that $\mathcal{U}_{g_i}$ does not depend on the choice of a representative $g_i \in \mathcal{R}$ of h_i. Finally, note that we also have:

$$\forall\, u \in \mathcal{U}_{g_i}, \quad \ker_{\mathbb{K}}(B_{h_i}(u).) = \mathrm{im}_{\mathbb{K}}(C_{h_i}(u).). \tag{53}$$

Indeed, using $B_{h_i}\,C_{h_i} = 0$ (see eq. (50)), we have $B_{h_i}(u)\,C_{h_i}(u) = 0$ for all $u \in \mathcal{U}_{g_i}$, and thus, $\mathrm{im}_{\mathbb{K}}(C_{h_i}(u).) \subseteq \ker_{\mathbb{K}}(B_{h_i}(u).)$ for all $u \in \mathcal{U}_{g_i}$. Now, if $u \in \mathcal{U}_{g_i}$ and $w \in \ker_{\mathbb{K}}(B_{h_i}(u).)$, then, using the second identity of eq. (50), we get $w = C_{h_i}(u)(F_{h_i}(u)\,w) \in \mathrm{im}_{\mathbb{K}}(C_{h_i}(u).)$, which proves eq. (53).

Thus, the solutions of eq. (4) are locally defined by

$$\begin{cases} u \in \mathcal{U}_{g_i} = \mathcal{V}_{\mathbb{K}}(\mathcal{J}) \setminus \mathcal{V}_{\mathbb{K}}(\langle g_i\rangle), \\ v_{h_i}(u, Y') = K(u)\,(E_{h_i}(u)\,Y(u) + C_{h_i}(u)\,Y'(u)), \ \forall\, Y' \in \mathcal{S}_{h_i}^{t\times n}, \end{cases}$$

for all the $\overline{g_i}$'s which are not nilpotent elements of $\mathcal{S}$ and $\mathcal{I} = \langle h_1, \ldots, h_\beta\rangle_{\mathcal{S}}$. As explained in Section 3.2.5, we can replace the condition $Y' \in \mathcal{S}_{h_i}^{t\times n}$ in the above parametrization by $Y' \in \mathbb{K}^{t\times n}$. Hence, we have:

$$\begin{cases} u \in \mathcal{U}_{g_i} = \mathcal{V}_{\mathbb{K}}(\mathcal{J}) \setminus \mathcal{V}_{\mathbb{K}}(\langle g_i\rangle), \\ v_{h_i}(u, Y') = K(u)\,(E_{h_i}(u)\,Y(u) + C_{h_i}(u)\,Y'(u)), \ \forall\, Y' \in \mathbb{K}^{t\times n}. \end{cases} \tag{54}$$

Let us note $\mathcal{I}_{\mathcal{R}} = \langle g_1, \ldots, g_\beta\rangle_{\mathcal{R}}$ the ideal of $\mathcal{R}$ generated by the g_i's, where $h_i = \overline{g_i}$. Then, we have:

$$\mathcal{U} = \mathcal{V}_{\mathbb{K}}(\mathcal{J}) \setminus \mathcal{V}_{\mathbb{K}}(\mathcal{I}_{\mathcal{R}}) = \mathcal{V}_{\mathbb{K}}(\mathcal{J}) \setminus \bigcap_{i=1}^{\beta} \mathcal{V}_{\mathbb{K}}(\langle g_i\rangle) = \bigcup_{i=1}^{\beta} (\mathcal{V}_{\mathbb{K}}(\mathcal{J}) \setminus \mathcal{V}_{\mathbb{K}}(\langle g_i\rangle)) = \bigcup_{i=1}^{\beta} \mathcal{U}_{g_i}.$$

If h_i is an nilpotent element of $\mathcal{S}$, then $h_i^k = 0$ for a certain integer k, which yields $g_i^k \in \mathcal{J}$, i.e., $g_i \in \sqrt{\mathcal{J}}$. Thus, we have $\langle g_i\rangle \subseteq \sqrt{\mathcal{J}}$, and thus, $\mathcal{V}_{\mathbb{K}}(\mathcal{J}) \subseteq \mathcal{V}_{\mathbb{K}}(\langle g_i\rangle)$, i.e., $\mathcal{V}_{\mathbb{K}}(\mathcal{J}) \setminus \mathcal{V}_{\mathbb{K}}(\langle g_i\rangle) = \emptyset$, i.e., eq. (54) becomes the empty set (which is consistent with the fact that $\mathcal{S}_{h_i}$ is then the trivial ring 0). Hence, if we set $I = \left\{ i \in [\![1, \ldots, \beta]\!] \mid g_i \notin \sqrt{\mathcal{J}} \right\}$, then we have:

$$\mathcal{U} = \bigcup_{i\in I} \mathcal{V}_{\mathbb{K}}(\mathcal{J}) \setminus \mathcal{V}_{\mathbb{K}}(\langle g_i\rangle) = \bigcup_{i\in I} \mathcal{U}_{g_i}.$$

If $\mathbb{K}$ is an algebraically closed field, then $\mathcal{V}_{\mathbb{K}}(\langle g_i\rangle) = \mathcal{V}_{\mathbb{K}}(\mathcal{J})$ yields $\langle g_i\rangle \subseteq \sqrt{\langle g_i\rangle} = \sqrt{\mathcal{J}}$, i.e., $g_i \in \sqrt{\mathcal{J}}$, which shows that the rank factorization problem eq. (4) has no solutions if and only if h_i is nilpotent.

If all the h_i's are nilpotent elements, then the ideal $\mathcal{I} = \langle h_1, \ldots, h_\beta \rangle_{\mathcal{S}}$ is *nilpotent*, namely, there exists $N \in \mathbb{Z}_{>0}$ such that $\mathcal{I}^N = 0$ (if $h_i^{\nu_i} = 0$ for $i = 1, \ldots, \beta$, then we can take $N = \nu_1 + \ldots + \nu_\beta$). Conversely, if $\mathcal{I}$ is a nilpotent ideal of $\mathcal{S}$, then the h_i's are idempotent elements of $\mathcal{S}$. Hence, if $\mathcal{I}$ is a nilpotent ideal of $\mathcal{S}$, then $\mathcal{U} = \emptyset$ and the rank factorization problem eq. (4) has no solutions.

More generally, if $\mathcal{I}$ is nilpotent and $\varphi : \mathcal{S} \longrightarrow \mathcal{T}$ is a ring homomorphism, then the ideal $\varphi(\mathcal{I}) = \langle \varphi(h_1), \ldots, \varphi(h_\beta) \rangle_{\mathcal{T}}$ of $\mathcal{T}$ is nilpotent. According to Corollary 4, the rank factorization problem eq. (4) has a solution in $\mathcal{T}$ if and only if $\varphi(\mathcal{I}) = \mathcal{T}$. This last condition implies that there exists $N \in \mathbb{Z}_{>0}$ such that $\mathcal{T}^N = 0$. In particular, we have $1^N = 0$ (equivalently, take the N^{th} power of eq. (47)), which yields $1 = 0$ in $\mathcal{T}$, i.e., $\mathcal{T}$ is the trivial ring and the rank factorization problem eq. (4) has no solutions in $\mathcal{T}$.

Remark 19. If $\mathbb{K}$ is an algebraically closed field, then the smallest affine algebraic set containing the set $\mathcal{V}_{\mathbb{K}}(\mathcal{J}) \setminus \mathcal{V}_{\mathbb{K}}(\langle g_i \rangle)$, namely, its *Zariski closure* $\overline{\mathcal{V}_{\mathbb{K}}(\mathcal{J}) \setminus \mathcal{V}_{\mathbb{K}}(\langle g_i \rangle)}$, is defined by

$$\overline{\mathcal{V}_{\mathbb{K}}(\mathcal{J}) \setminus \mathcal{V}_{\mathbb{K}}(\langle g_i \rangle)} = \mathcal{V}_{\mathbb{K}}\left(\mathcal{J} : \langle g_i \rangle^{\infty}\right), \tag{55}$$

where $\mathcal{J} : \langle g_i \rangle^{\infty}$ denotes the *saturation* of $\mathcal{J}$ with respect to $\langle g_i \rangle$ defined by:

$$\mathcal{J} : \langle g_i \rangle^{\infty} = \left\{ a \in \mathcal{R} \mid \exists\, k \in \mathbb{Z}_{\geq 0} :\, a\, g_i^k \in \mathcal{J} \right\}.$$

For more details, see, e.g., [10, 20]. Now, using that h_i is not a nilpotent of $\mathcal{S}$, i.e., $g_i^k \notin \mathcal{J}$ for all $k \in \mathbb{Z}_{\geq 0}$, then $1 \notin \mathcal{J} : \langle g_i \rangle^{\infty}$, i.e., $\mathcal{J} : \langle g_i \rangle^{\infty}$ is not equal to $\mathcal{R}$, which yields $\overline{\mathcal{V}_{\mathbb{K}}(\mathcal{J}) \setminus \mathcal{V}_{\mathbb{K}}(\langle g_i \rangle)} \neq \emptyset$.

Finally, if $\mathcal{R}[t]$ is the polynomial ring $\mathbb{K}[x_1, \ldots, x_m, t]$ and $\mathcal{R}[t]\, \mathcal{J} = \langle g_1, \ldots, g_\gamma \rangle_{\mathcal{R}[t]}$ is the ideal of $\mathcal{R}[t]$ generated by $\mathcal{J}$, then $\mathcal{J} : \langle g_i \rangle^{\infty}$ can be characterized as follows:

$$\mathcal{J} : \langle g_i \rangle^{\infty} = \langle g_1, \ldots, g_\gamma, t\, g_i - 1 \rangle_{\mathcal{R}[t]} \cap \mathcal{R}. \tag{56}$$

See, e.g., [10, 12]. Finally, eq. (56) can be computed by a Gröbner basis computation.

We can now sum up the main result of the paper.

Theorem 5. *Let $D_i \in \mathbb{K}^{m \times m}$ for $i = 1, \ldots, r$ and $M \in \mathbb{K}^{m \times n}$ be such that:*

$$1 \leq l = \operatorname{rank}_{\mathbb{K}}(M) \leq \min\{m, r\}.$$

Let $X \in \mathbb{K}^{m \times l}$ be a full column rank matrix such that $\operatorname{im}_{\mathbb{K}}(M.) = \operatorname{im}_{\mathbb{K}}(X.)$, i.e., the columns of X define a basis of $\operatorname{im}_{\mathbb{K}}(M.)$. Let $V \in \mathbb{K}^{l \times m}$ be any left inverse of X. Moreover, let $Y \in \mathbb{K}^{l \times n}$ be such that $M = X\, Y$. Hence, we have $\ker_{\mathbb{K}}(M.) = \ker_{\mathbb{K}}(Y.)$ and Y has full row rank.

Let $L \in \mathbb{K}^{(m-l) \times m}$ be a full row matrix whose rows define a basis of $\ker_{\mathbb{K}}(.M)$ (with the convention that $L = 0$ if $l = m$), $\mathcal{R} = \mathbb{K}[x_1, \ldots, x_m]$, $x = (x_1\ \ldots\ x_m)^T$, $A = (D_1\, x\ \ldots\ D_r\, x) \in \mathcal{R}^{m \times r}$, $Q = L\, A \in R^{p \times r}$, $\mathcal{Q} = \mathcal{R}^{1 \times r} / (\mathcal{R}^{1 \times p}\, Q)$, $\mathcal{J} = \operatorname{Fitt}_0(\mathcal{Q})$ the 0^{th} Fitting ideal of the $\mathcal{R}$-module $\mathcal{Q}$ ($\mathcal{J} = \langle 0 \rangle$ if $r > p$ or if $l = m$), $\mathcal{S} = \mathcal{R}/\mathcal{J}$, $\chi : \mathcal{R} \longrightarrow \mathcal{S}$ the canonical ring epimorphism, and $\overline{H} = \chi(H)$ for all $H \in \mathcal{R}^{a \times b}$. Then, we have $\ker_{\mathcal{S}}(\overline{Q}.) \neq 0$.

Let $K \in \mathcal{R}^{r \times q}$ be such that $\ker_{\mathcal{S}}(\overline{Q}.) = \operatorname{im}_{\mathcal{S}}(\overline{K}.)$, $B = V\, A\, K \in \mathcal{R}^{l \times q}$, $\mathcal{B} = \mathcal{S}^{l \times 1} / (\overline{B}\, \mathcal{S}^{q \times 1})$ be the $\mathcal{S}$-module finitely presented by $\overline{B}$, and $\mathcal{I} = \operatorname{Fitt}_0(\mathcal{B}) = \langle h_1, \ldots, h_\beta \rangle_{\mathcal{S}}$.

If $\mathcal{I}$ is a nilpotent ideal of $\mathcal{S}$, then the rank factorization problem eq. (4) has no solution.

If $\mathbb{K}$ is an algebraically closed field, then rank factorization problem eq. (4) has no solution if and only if $\mathcal{I}$ is a nilpotent ideal of $\mathcal{S}$.

Finally, let $h_i = \overline{g_i}$, where $g_i \in \mathcal{R}$, $\mathcal{I}_{\mathcal{R}} = \langle g_1, \ldots, g_\beta \rangle_{\mathcal{R}}$ be the ideal of $\mathcal{R}$ generated by the g_i's, $I = \left\{ i \in [\![1, \ldots, \beta]\!] \mid g_i \notin \sqrt{\mathcal{J}} \right\}$, and $\mathcal{U}_{g_i} = \mathcal{V}_{\mathbb{K}}(\mathcal{J}) \setminus \mathcal{V}_{\mathbb{K}}(\langle g_i \rangle)$ for $i \in I$. If $I \neq \emptyset$, then solutions (u, v) of the rank factorization problem eq. (4) are defined by

$$\begin{array}{c} u \in \mathcal{U} = \bigcup_{i \in I} \mathcal{U}_{g_i}, \\ \forall\, i \in I, \quad \left\{ \begin{array}{l} u \in \mathcal{U}_{g_i}, \\ v_{h_i}(u, Y') = K(u)\, (E_{h_i}(u)\, Y + C_{h_i}(u)\, Y'),\ \forall\, Y' \in \mathbb{K}^{t_i \times n}, \end{array} \right. \end{array} \tag{57}$$

where $E_{h_i} \in \mathcal{S}_{h_i}^{q \times l}$ is a right inverse of $B_{h_i} = j_{h_i}(\overline{B})$, i.e., $B_{h_i}\, E_{h_i} = I_l$, $C_{h_i} \in \mathcal{S}_{h_i}^{q \times t_i}$ is such that $\ker_{\mathcal{S}_{h_i}}(B_{h_i}.) = \operatorname{im}_{\mathcal{S}_{h_i}}(C_{h_i}.)$, and $j_{h_i} : \mathcal{S} \longrightarrow \mathcal{S}_{h_i}$ is the canonical ring homomorphism from $\mathcal{S}$ to its localization $\mathcal{S}_{h_i} = \{s/h_i^k \mid s \in S,\, k \in \mathbb{Z}_{\geq 0}\}$ at the multiplicatively set $\{h_i^k \mid k \in \mathbb{Z}_{\geq 0}\}$.

Remark 20. The results of Section 3.1 obtained for a full column rank matrix M can be seen as a particular case of Theorem 5. Indeed, if M has full row rank, then $L = 0$ and $X = I_m$, which yields $Q = L\,A = 0$ and $\ker_{\mathcal{R}}(Q.) = \mathcal{R}^{r\times 1}$, i.e., $K = I_r$, and thus, $B = A$, $\mathcal{B} = \mathcal{A}$, $\mathcal{Q} = \operatorname{coker}_{\mathcal{R}}(.Q) = \mathcal{R}^{1\times r}$, $\mathcal{J} = \operatorname{Fitt}_0(\mathcal{Q}) \subseteq \operatorname{ann}_{\mathcal{R}}(\mathcal{Q}) = \langle 0\rangle$, i.e., $\mathcal{J} = \langle 0\rangle$, $\mathcal{S} = \mathcal{R}$, $\mathcal{I} = \operatorname{Fitt}_0(\mathcal{B}) = \operatorname{Fitt}_0(\mathcal{A})$, and:

$$\mathcal{U} = \mathcal{V}_{\mathbb{K}}(\mathcal{J}) \setminus \mathcal{V}_{\mathbb{K}}(\mathcal{I}) = \mathbb{K}^{m\times 1} \setminus \mathcal{V}_{\mathbb{K}}(\mathcal{I}), \quad \mathcal{I} = \mathcal{I}_{\mathcal{R}} = \operatorname{Fitt}_0(\mathcal{A}) = \langle h_1, \ldots, h_\beta\rangle, \quad h_i \in \mathcal{R},\ i = 1, \ldots, \beta.$$

Thus, Problem eq. (4) has a solution if and only if $\mathcal{V}_{\mathbb{K}}(\mathcal{I}) \neq \mathbb{K}^{m\times 1}$. Finally, in the case of an infinite algebraic closed field $\mathbb{K}$, $\mathcal{V}_{\mathbb{K}}(\mathcal{I}) \neq \mathbb{K}^{m\times 1}$ is equivalent to $\mathcal{I} \neq \langle 0\rangle$.

Let us now explain how to effectively check whether or not h_i is a nilpotent element of $\mathcal{S}$ and how the matrices C_{h_i}, E_{h_i}, and F_{h_i} can be computed.

Let us start with recognizing nilpotent elements of $\mathcal{S}$. As stated again at the beginning of the section, the kernel of the canonical ring homomorphism $j_{h_i} : \mathcal{S} \longrightarrow \mathcal{S}_{h_i}$, defined by $j_{h_i}(s) = s/1$ for all $s \in \mathcal{S}$, is defined by $\ker j_{h_i} = \{s \in \mathcal{S} \mid \exists\, k \in \mathbb{Z}_{\geq 0} :\ h_i^k\, s = 0\}$ (see, e.g., [10, 25]). Hence, $\mathcal{S}_{h_i}$ is the trivial ring if and only if $1 \in \ker j_{h_i}$, i.e., if and only if h_i is nilpotent. To test whether or not $\mathcal{S}_{h_i}$ is the trivial ring, we shall need the following standard lemma.

Lemma 9. *Let $\mathcal{R}_{g_i} = \mathbb{K}[x_1, \ldots, x_m, g_i^{-1}] = \{a/g_i^k \mid a \in \mathcal{R},\ k \in \mathbb{Z}_{\geq 0}\}$ be the localization of $\mathcal{R}$ at the multiplicatively closed set $\{g_i^k\}_{k\in\mathbb{Z}_{\geq 0}}$, $\mathcal{J} = \langle e_1, \ldots, e_\gamma\rangle$ an ideal of $\mathcal{R}$, $\mathcal{S} = \mathcal{R}/\mathcal{J}$, $h_i = \overline{g_i}$, $\mathcal{S}_{h_i}$ the localization of $\mathcal{S}$ at the multiplicatively closed set $\{h_i^k\}_{k\in\mathbb{Z}_{\geq 0}}$, and $\mathcal{J}_{g_i}$ the ideal of $\mathcal{R}_{g_i}$ generated by $\mathcal{J}$, i.e., $\mathcal{J}_{g_i} = \langle e_1, \ldots, e_\gamma\rangle_{\mathcal{R}_{g_i}}$. Then, the map $\rho : \mathcal{S}_{h_i} \longrightarrow \mathcal{R}_{g_i}/\mathcal{J}_{g_i}$ defined by*

$$\forall\, s = r + \mathcal{J} \in \mathcal{S}, \quad r \in \mathcal{R}, \quad \rho\left(\frac{s}{h_i^k}\right) = \rho\left(\frac{r + \mathcal{J}}{g_i^k + \mathcal{J}}\right) = \frac{r}{g_i^k} + \mathcal{J}_{g_i}$$

is a ring isomorphism, i.e., $\mathcal{S}_{h_i} \cong \mathcal{R}_{g_i}/\mathcal{J}_{g_i}$.

Proof. See, e.g., Rule 4.16 on page 83 of [20]. □

Using Lemma 9, let us now characterize $\mathcal{R}_{g_i}/\mathcal{J}_{g_i}$. We have $\mathcal{R}_{g_i} = \mathcal{R}[y]/\langle y\,g_i - 1\rangle$ and the following exact sequence of $\mathcal{R}[y]$-modules:

$$0 \longrightarrow \mathcal{R}[y] \xrightarrow{(y\,g_i - 1).} \mathcal{R}[y] \xrightarrow{\vartheta} \mathcal{R}_{g_i} \longrightarrow 0.$$

Let $e = (e_1\ \ldots\ e_\gamma)^T \in \mathcal{R}^{\gamma\times 1}$ and $j_{g_i} : \mathcal{R} \longrightarrow \mathcal{R}_{g_i}$ be the canonical ring homomorphism defined by $j_{g_i}(a) = a/1$ for all $a \in \mathcal{R}$. Using the fact that $\mathcal{R}$ is an integral domain, we get $\ker j_{g_i} = 0$, and identifying e_i with $e_i/1$ in $\mathcal{R}_{g_i}$, we have the following finite presentation of the $\mathcal{R}_{g_i}$-module $\mathcal{R}_{g_i}/\mathcal{J}_{g_i}$:

$$\mathcal{R}_{g_i}^{1\times\gamma} \xrightarrow{.e} \mathcal{R}_{g_i} \xrightarrow{\varpi} \mathcal{R}_{g_i}/\mathcal{J}_{g_i} \longrightarrow 0.$$

Combining the last exact sequences, we obtain the following commutative exact diagram of $\mathcal{R}[y]$-modules:

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \uparrow & & \\
 & & & & & & \mathcal{R}_{g_i}/\mathcal{J}_{g_i} & & \\
 & & & & & & \uparrow{\scriptstyle \varpi} & & \\
0 & \longrightarrow & \mathcal{R}[y] & \xrightarrow{(y\,g_i-1).} & \mathcal{R}[y] & \xrightarrow{\vartheta} & \mathcal{R}_{g_i} & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle .e} & & \uparrow{\scriptstyle .e} & & \uparrow{\scriptstyle .e} & & \\
0 & \longrightarrow & \mathcal{R}[y]^{1\times\gamma} & \xrightarrow{(y\,g_i-1)\,I_\gamma.} & \mathcal{R}[y]^{1\times\gamma} & \xrightarrow{\vartheta} & \mathcal{R}_{g_i}^{1\times\gamma} & \longrightarrow & 0.
\end{array}$$

Using Lemma 9 and 4 of Proposition 3.1 of [7]), we then have:

$$\mathcal{S}_{h_i} \cong \mathcal{R}_{g_i}/\mathcal{J}_{g_i} \cong \mathcal{R}[y]/\left((y\,g_i - 1)\,\mathcal{R}[y] + \mathcal{R}[y]^{1\times\gamma}\,e\right) = \mathcal{R}[y]/\langle y\,g_i - 1, e_1, \ldots, e_\gamma\rangle_{\mathcal{R}[y]}. \tag{58}$$

Therefore, h_i is a nilpotent element of $\mathcal{S}$ if and only if $\langle y\,g_i - 1, e_1, \ldots, e_\gamma\rangle_{\mathcal{R}[y]} = \mathcal{R}[y]$, i.e., if and only if $1 \in \langle y\,g_i - 1, e_1, \ldots, e_\gamma\rangle_{\mathcal{R}[y]}$, a result that can be checked by a Gröbner basis computation.

Example 11. If $\mathcal{R} = \mathbb{K}[x]$, $e = x^2$, and $g_i = x$, then $\mathcal{J} = \langle x^2 \rangle$, $\mathcal{S} = \mathbb{K}[x]/\langle x^2 \rangle$, $\mathcal{R}_x = \mathbb{K}[x, y]/\langle y\, x - 1 \rangle$, $\mathcal{J}_x = \mathcal{R}_x\, x^2$, and $\mathcal{S}_{\overline{x}} \cong \mathbb{K}[x, y]/\langle y\, x - 1,\, x^2 \rangle$. We can check that $1 \in \langle y\, x - 1,\, x^2 \rangle$ because we have $-(x\, y + 1)\,(y\, x - 1) + y^2\, x^2 = 1$. Therefore, $\mathcal{S}_{\overline{x}} = 0$ and $\overline{x}$ is a nilpotent element of $\mathcal{S}$.

Using eq. (41) and writing $h_i = \overline{g_i}$, where $g_i \in \mathcal{R}$, $h_i\, \mathcal{S} = 0$ is then equivalent to the existence of two matrices $D \in \mathcal{R}^{q \times l}$ and $D' \in \mathcal{R}^{l\, \gamma \times l}$ satisfying:

$$g_i\, I_l = \begin{pmatrix} B & e^T \otimes I_l \end{pmatrix} \begin{pmatrix} D \\ D' \end{pmatrix} = B\, D + (e^T \otimes I_l)\, D'.$$

Using Gröbner basis methods, the matrices D and D' can be computed (see, e.g., [5]) using, e.g., the `Factorize` command of the OreModules package. This factorization yields $h_i\, I_l = \overline{B}\, \overline{D}$ as an identity of matrices with entries in $\mathcal{S}$, which finally shows that we can take $E_{h_i} = h_i^{-1}\, \overline{D}$.

Using eq. (58) and doing similarly as explained in Section 3.2.2 (see eq. (27)), we can compute $C_{h_i} \in \mathcal{R}^{q \times t}$ satisfying $\ker_{\mathcal{S}_{h_i}}(B_{h_i}.) = \mathrm{im}_{\mathcal{S}_{h_i}}(C_{h_i}.)$ using, e.g., OreModules. The matrices C_{h_i}, E_{h_i}, and F_{h_i} can also be computed by `CapAndHomalg` [1]. Note that `CapAndHomalg` directly works in the factor ring $\mathcal{S} = \mathcal{R}/\mathcal{J}$ or the localization $\mathcal{S}_{h_i}$ of $\mathcal{S}$.

Finally, using eq. (56), $\overline{\mathcal{V}_{\mathbb{K}}(\mathcal{J}) \setminus \mathcal{V}_{\mathbb{K}}(\langle g_i \rangle)} = \mathcal{V}_{\mathbb{K}}\,(\mathcal{J} : \langle g_i \rangle^{\infty})$ can also be computed.

Example 12. Let us continue Example 4, Example 5, Example 8, and Example 9. Using $\mathcal{I} = \mathrm{Fitt}_0(\mathcal{B}) = \langle \overline{B_1},\, \overline{B_2} \rangle_{\mathcal{S}}$, where $\overline{B_1}$ and $\overline{B_2}$ are the first two entries of the matrix $\overline{B}$ defined in Example 8, we then have to consider the following two cases:

- If $h_1 = \overline{B_1}$, where $B_1 = g_1 = x_4\,(5\, x_2/2 + x_3) \in \mathcal{R}$, then h_1 is not an nilpotent element of $\mathcal{S}$. Considering the ring $\mathcal{S}_{h_1} = \{s/h_1^k \mid s \in \mathcal{S},\, k \in \mathbb{Z}_{\geq 0}\}$, the matrix $E_{h_1} = (h_1^{-1} \quad 0 \quad 0)^T \in \mathcal{S}_{h_1}^{3 \times 1}$ then satisfies $B\, E_{h_1} = 1$. Using eq. (57), we obtain that

$$\left\{ \begin{array}{l} u \in \mathcal{U}_{g_1} = \mathcal{V}_{\mathbb{K}}(\mathcal{J}) \setminus \mathcal{V}_{\mathbb{K}}(\langle g_1 \rangle), \\ v_{h_1}(u) = K(u)\, E_{h_1}(u)\, Y = \begin{pmatrix} \dfrac{6\, u_2}{g_1(u)} & 0 & 0 \\ \dfrac{6\, u_4}{g_1(u)} & 0 & 0 \end{pmatrix}, \end{array} \right.$$

  is a solution of Problem eq. (4). More generally, using $\ker_{\mathcal{S}_{h_1}}(\overline{B}.) = \mathrm{im}_{\mathcal{S}_{h_1}}(C_{h_1}.)$, where

$$C_{h_1} = \begin{pmatrix} 3 & 0 \\ 0 & -6\, x_4 \\ -1 & 9\, x_2 + 6\, x_3 + 2\, x_4 \end{pmatrix} \in \mathcal{S}_{h_1}^{3 \times 2},$$

  and x_i simply stands for $j_{h_1}(\overline{x_i}) = \overline{x_i}/1$ for $i = 1, \ldots, 4$, then the solutions $(u,\, v)$ of Problem eq. (4) in $\mathcal{U}_{g_1}$ are defined by:

$$\forall\, Y' \in \mathbb{K}^{2 \times 3}, \quad \left\{ \begin{array}{l} u \in \mathcal{U}_{g_1}, \\ v_{h_1}(u, Y') = K(u)\, (E_{h_1}(u)\, Y + C_{h_1}(u)\, Y'). \end{array} \right.$$

  Finally, we have $\mathcal{J} : \langle g_1 \rangle^{\infty} = \langle 2\, x_1 - 3\, x_2 - 2\, x_3,\; x_2\,(9\, x_2 + 6\, x_3 + 2\, x_4) \rangle$, which yields:

$$\begin{array}{c} \overline{\mathcal{V}_{\mathbb{K}}(\mathcal{J}) \setminus \mathcal{V}_{\mathbb{K}}(\langle g_1 \rangle)} = \mathcal{V}_{\mathbb{K}}\,(\mathcal{J} : \langle g_1 \rangle^{\infty}) = \\ \{(u_1 \quad 0 \quad u_1 \quad u_4)^T \mid u_1,\, u_4 \in \mathbb{K}\} \cup \{(-u_4/3 \quad -2\, u_3/3 - 2\, u_4/9 \quad u_3 \quad u_4)^T \mid u_3,\, u_4 \in \mathbb{K}\}. \end{array}$$

- If $h_2 = \overline{B_2}$, where $B_2 = g_2 = (3\, x_1 + x_4)\, x_3 \in \mathcal{R}$, then h_2 is not an nilpotent element of $\mathcal{S}$. Considering the ring $\mathcal{S}_{h_2} = \{s/h_2^k \mid s \in \mathcal{S},\, k \in \mathbb{Z}_{\geq 0}\}$, the matrix $E_{h_2} = (0 \quad h_2^{-1} \quad 0)^T \in \mathcal{S}_{h_2}^{3 \times 1}$ then satisfies $B\, E_{h_2} = 1$. Using eq. (57), we obtain that

$$\left\{ \begin{array}{l} u \in \mathcal{U}_{g_2} = \mathcal{V}_{\mathbb{K}}(\mathcal{J}) \setminus \mathcal{V}_{\mathbb{K}}(\langle g_2 \rangle), \\ v_{h_2}(u) = K(u)\, E_{h_2}(u)\, Y = \begin{pmatrix} 0 & 0 & 0 \\ \dfrac{6\,(3\, u_3 + u_4)}{h_2(u)} & 0 & 0 \end{pmatrix}, \end{array} \right.$$

is a solution of Problem eq. (4). More generally, using $\ker_{\mathcal{S}_{h_2}}(\overline{B}.) = \operatorname{im}_{\mathcal{S}_{h_2}}(C_{h_2}.)$, where

$$C_{h_2} = \begin{pmatrix} 3 & 0 \\ 0 & -3\,x_4 \\ -1 & 3\,x_3 + x_4 \end{pmatrix} \in \mathcal{S}_{h_2}^{3\times 2},$$

then the solutions $(u,\, v)$ of Problem eq. (4) in $\mathcal{U}_{g_2}$ are defined by:

$$\begin{cases} u \in \mathcal{U}_{g_2}, \\ v_{h_2}(u, Y') = K(u)\,(E_{h_2}(u)\,Y + C_{h_2}(u)\,Y'),\ \forall\, Y' \in \mathbb{K}^{2\times 3}. \end{cases}$$

Finally, we have $\overline{\mathcal{V}_{\mathbb{K}}(\mathcal{J}) \setminus \mathcal{V}_{\mathbb{K}}(\langle g_2\rangle)} = \overline{\mathcal{V}_{\mathbb{K}}(\mathcal{J}) \setminus \mathcal{V}_{\mathbb{K}}(\langle g_1\rangle)}$.

In Algorithm 3, we present a pseudocode summarizing Theorem 5. It can easily be implemented in any standard computer algebra system. See the RankFactorization package [9] dedicated to the rank factorization problem and its applications (built upon the OreModules package [6]). Note that an index k – fixed to 0 – is given in this input of RankFactorizationProblem command. It corresponds to the computation of $\operatorname{Fitt}_0(\mathcal{Q})$. In Section 4, we shall show that all the solutions of the rank factorization eq. (4) can be obtained by considering the different $\operatorname{Fitt}_k(\mathcal{Q})$ for $k = 0, \ldots, r-1$ (see also Remark 11).

Algorithm 3 RankFactorizationProblem

procedure RankFactorizationProblem($D_1, \ldots, D_r \in \mathbb{K}^{m\times m}, 0 \neq M \in \mathbb{K}^{m\times n}$, $k = 0$)
  Define $\mathcal{R} = \mathbb{K}[x_1, \ldots, x_m]$, $x = (x_1\ \ldots\ x_m)^T$, $A = (D_1\, x\ \ldots\ D_r\, x) \in \mathcal{R}^{m\times r}$
  Compute a basis of $\ker_{\mathbb{K}}(.M)$ and stack its row vectors into a full row rank matrix $L \in \mathbb{K}^{p\times m}$ satisfying $\ker_{\mathbb{K}}(.M) = \operatorname{im}_{\mathbb{K}}(.L)$, where $p = m - l$ and $l = \operatorname{rank}_{\mathbb{K}}(M)$
  Define $Q = L\,A \in \mathcal{R}^{p\times r}$ and the finitely presented $\mathcal{R}$-module $\mathcal{Q} = \operatorname{coker}_{\mathcal{R}}(.Q)$
  Compute the ideal $\mathcal{J} = \operatorname{Fitt}_k(\mathcal{Q}) = \operatorname{Fitt}_0(\mathcal{Q})$ and define the ring $\mathcal{S} = \mathcal{R}/\mathcal{J}$
  Compute $K \in \mathcal{R}^{r\times q}$ such that $\ker_{\mathcal{S}}(Q.) = \operatorname{im}_{\mathcal{S}}(\overline{K}.)$, where $\overline{K} = \chi(K) \in \mathcal{S}^{r\times q}$ and $\chi : \mathcal{R} \longrightarrow \mathcal{S}$ is the canonical ring epimorphism
  Compute a basis of $\operatorname{im}_{\mathbb{K}}(M.)$ and stack its column vectors into a full column rank matrix $X \in \mathbb{K}^{m\times l}$ satisfying $\operatorname{im}_{\mathbb{K}}(M.) = \operatorname{im}_{\mathbb{K}}(X.)$
  Compute a full row rank matrix $Y \in \mathbb{K}^{l\times n}$ such that $M = X\,Y$
  Compute a left inverse $V \in \mathbb{K}^{l\times m}$ of X
  Define $B = V\,A\,K \in \mathcal{R}^{l\times q}$ and the finitely presented $\mathcal{S}$-module $\mathcal{B} = \operatorname{coker}_{\mathcal{S}}(\overline{B}.)$, where $\overline{B} = \chi(B)$
  Compute the ideal $\mathcal{I} = \operatorname{Fitt}_0(\mathcal{B}) = \langle h_1, \ldots, h_\beta\rangle_{\mathcal{S}}$ of $\mathcal{S}$
  if $\mathcal{I} = \langle 0\rangle_{\mathcal{S}}$ **then return** "No solutions exist"
  else
    Set $I = \emptyset$
    for $i \leftarrow 1, \ldots, \beta$ **do**
      if h_i is nilpotent element of $\mathcal{S}$ **then** $i := i + 1$
      else
        Define the localization $\mathcal{S}_{h_i}$ of $\mathcal{S}$ at the multiplicatively closed set $\{h_i^k\}_{k\in\mathbb{Z}_{\geq 0}}$
        Compute a right inverse $E_{h_i} \in \mathcal{S}_{h_i}^{q\times l}$ of $B_{h_i} = j_{h_i}(\overline{B})$, where $j_{h_i} : \mathcal{S} \longrightarrow \mathcal{S}_{h_i}$ is the canonical ring homomorphism
        Compute $C_{h_i} \in \mathcal{S}_{h_i}^{q\times t_i}$ such that $\ker_{\mathcal{S}_{h_i}}(B_{h_i}.) = \operatorname{im}_{\mathcal{S}_{h_i}}(C_{h_i}.)$
        Let $g_i \in \mathcal{R}$ define the residue class of $h_i \in \mathcal{S}$, i.e., $h_i = \overline{g_i}$, and $I := I \cup \{g_i\}$
      end if
    end for
    if $I = \emptyset$ **then return** "No solutions exist"
    else return $\mathcal{J}$, K, Y, $\{h_i\}_{i\in I}$, $\{E_{h_i}\}_{i\in I}$, and $\{C_{h_i}\}_{i\in I}$ defining the solutions eq. (57)
    end if
  end if
end procedure

4 Getting all the solutions of the rank factorization problem

4.1 The solutions of the rank factorization problem

In this section, we shall use the notations of Theorem 4 and of the last paragraph of Section 3.2.4.

As explained in Section 3.2.1, the solutions $(u,\, v)$ of the rank factorization problem necessarily satisfy the condition $u \in \mathcal{V}_{\mathbb{K}}(\mathcal{J})$, where $\mathcal{J} = \mathcal{J}_0 = \mathrm{Fitt}_0(\mathcal{Q})$. Using module theory over the ring $\mathcal{S} = \mathcal{R}/\mathcal{J}$, in Section 3.2, solutions of the rank factorization problem were characterized. We also explained that a similar approach could be developed for the rings $\mathcal{S}_k = \mathcal{R}/\mathcal{J}_k$ for $k = 0, \ldots, r-1$. More precisely, Theorem 5 also holds if $\mathcal{J} = \mathrm{Fitt}_0(\mathcal{Q})$ is replaced by $\mathcal{J}_k = \mathrm{Fitt}_k(\mathcal{Q})$ for $0 \leq k \leq r-1$. Hence, we can characterize solutions of the rank factorization problem which satisfy $u \in \mathcal{V}_{\mathbb{K}}(\mathcal{J}_k)$ by considering the $\mathcal{S}_k = \mathcal{R}/\mathcal{J}_k$-module $\mathcal{B}_k = \mathrm{coker}_{\mathcal{S}_k}(.B_k)$, where $B_k = V\, A\, K_k$. It is important to note that not all those solutions are the restrictions of the solutions obtained by considering the ring $\mathcal{S} = \mathcal{R}/\mathcal{J}$ to $\mathcal{V}_{\mathbb{K}}(\mathcal{J}_k)$. Indeed, if $(u,\, v)$ is a solution of the rank factorization problem and $\mathrm{rank}_{\mathbb{K}}(Q(u).) = r-k-1$, then $u \in \mathcal{V}(\mathcal{J}_k) \setminus \mathcal{V}(\mathcal{J}_{k+1})$ and using Theorem 4 (see eq. (29)), $v \in \ker_{\mathbb{K}}(Q(u).) = \mathrm{im}_{\mathbb{K}}(K_k(u).)$, i.e., v is of the form $v = K_k(u)\, T_u$, for a certain $T_u \in \mathbb{K}^{q_k \times n}$ satisfying $B_k(u)\, T_u = Y$. But v does not necessarily belong to $\mathrm{im}_{\mathbb{K}}(K(u).)$ since we usually only have $\mathrm{im}_{\mathbb{K}}(K(u).) \subseteq \ker_{\mathbb{K}}(Q(u).)$. See Theorem 4 and eq. (29). Hence, by considering the solutions over $\mathcal{S}$, we are only sure to parametrize all the solutions $(u,\, v)$ which are such that $\mathrm{rank}_{\mathbb{K}}(Q(u)) = r-1$. Similarly, considering the solutions over $\mathcal{S}_k$, we are only sure to parametrizing all the solutions $(u,\, v)$ which are such that $\mathrm{rank}_{\mathbb{K}}(Q(u)) = r-k-1$. Finally, the case $k = r-1$ corresponds to Section 2.2. Indeed, as explained at the end of Remark 11, $\mathcal{V}_{\mathbb{K}}(\mathcal{J}_{r-1}) = \ker_{\mathbb{K}}(N.)$, where $\mathcal{J}_{r-1} = \mathrm{Fitt}_{r-1}(\mathcal{Q})$ and N is defined by eq. (7). In Section 2.2, all the solutions of the rank factorization problem with the condition $u \in \mathcal{V}_{\mathbb{K}}(\mathcal{J}_{r-1})$, were found using linear algebra methods. This approach correspond to the approach developed in Section 3 for the ring $\mathcal{S}_{r-1} = \mathcal{R}/\mathcal{J}_{r-1}$. Indeed, we first note that $\chi_{r-1}(Q) = 0$, which yields $K_{r-1} = I_r$, and then, using eq. (42), we have $\chi_{r-1}(A) = \chi_{r-1}(X)\, \chi_{r-1}(B_{r-1})$ and $\chi_{r-1}(B_{r-1}) = \chi_{r-1}(V)\, \chi_{r-1}(A)$, i.e., $B_{r-1} - V\, A \in \mathcal{J}_{r-1}^{l \times q}$. Using the fact that $\mathcal{V}_{\mathbb{K}}(\mathcal{J}_{r-1}) = \ker_{\mathbb{K}}(N.) = \mathrm{im}_{\mathbb{K}}(Z.)$, where $Z \in \mathbb{K}^{m \times d}$ is defined in Lemma 2, then $u = Z\, \psi$ for all $\psi \in \mathbb{K}^{d \times 1}$, and thus, using eq. (9) and $V\, X = I_l$, we have $B_{r-1}(Z\, \psi) = V\, A(Z\, \psi) = V\, X\, B(\psi) = B(\psi) = (W_1\, \psi \;\ldots\; W_r\, \psi)$ for all $\psi \in \mathbb{K}^{d \times 1}$.

Theorem 6. *The complete set of solutions of the rank factorization problem eq.* (4) *is the union of the solutions given in Theorem 5, where $\mathcal{J}$ is replaced by $\mathcal{J}_k$ for $k = 0, \ldots, r-1$.*

Proof. To get the complete set of solutions for the rank factorization problem, we have to collect all the solutions defined in Theorem 5 for the different ideals $\mathcal{J}_k$ for $k = 0, \ldots, r-1$. Indeed, all these sets of solutions are, by construction, solutions of the rank factorization problem eq. (4). Conversely, if $(u,\, v)$ is a solution to the rank factorization problem, then we have $u \in \mathcal{V}_{\mathbb{K}}(\mathcal{J}_0)$. Let $k \in [\![0, \ldots, r-1]\!]$ be such that $\mathrm{rank}_{\mathbb{K}}(Q(u)) = r-k-1$. Then, we have $u \in \mathcal{V}_{\mathbb{K}}(\mathcal{J}_k) \setminus \mathcal{V}_{\mathbb{K}}(\mathcal{J}_{k+1})$. Let $K_k \in \mathcal{R}^{r \times q_k}$ be defined as in Theorem 4. By Theorem 4, each column of v belongs to $\ker_{\mathbb{K}}(Q(u).) = \mathrm{im}_{\mathbb{K}}(K_k(u).)$. Therefore, there exists $T_u \in \mathbb{K}^{q_k \times n}$ such that $v = K_k(u)\, T_u$, where T_u satisfies $B_k(u)\, T_u = Y$. Since Y has a right inverse $H \in \mathbb{K}^{n \times l}$ (see the proof of Proposition 2), i.e., $Y\, H = I_l$, then we have $B_k(u)\, (T_u\, H) = I_l$, which shows that $\mathrm{rank}_{\mathbb{K}}(B_k(u)) = l$. Let $B_k = V\, A\, K_k \in \mathcal{R}^{l \times q_k}$ and $\mathcal{B}_k = \mathrm{coker}_{\mathcal{S}_k}(\chi_k(B_k).)$ be the $\mathcal{S}_k$-module finitely presented by $\chi_k(B_k)$, where $\chi_k : \mathcal{R} \longrightarrow \mathcal{S}_k$ is the canonical ring epimorphism. Moreover, let $\mathcal{I}_k = \mathrm{Fitt}_0(\mathcal{B}_k) = \langle h_{k,1}, \ldots, h_{k,\beta_k} \rangle_{\mathcal{S}_k}$, $g_{k,j} \in \mathcal{R}$ be such that $h_{k,j} = \chi_k(g_{k,j})$ for $j = 1, \ldots, \beta_k$, and $\mathcal{I}_{k,\mathcal{R}} = \langle g_{k,1}, \ldots, g_{k,\beta_k} \rangle_{\mathcal{R}}$. Set $I_k = \left\{ i \in [\![1, \ldots, \beta_k]\!] \mid g_{k,i} \notin \sqrt{\mathcal{J}_k} \right\}$. By Theorem 5, the solutions of the rank factorization problem defined over $\mathcal{S}_k$ are of the form

$$\begin{array}{l} u \in \mathcal{V}_{\mathbb{K}}(\mathcal{J}_k) \setminus \mathcal{V}_{\mathbb{K}}(\mathcal{I}_{k,\mathcal{R}}) = \bigcup_{i \in I_k} \left(\mathcal{V}_{\mathbb{K}}(\mathcal{J}_k) \setminus \mathcal{V}_{\mathbb{K}}(\langle g_{k,i} \rangle) \right), \\ \forall\, i \in I_k, \quad \left\{ \begin{array}{l} u \in \mathcal{V}_{\mathbb{K}}(\mathcal{J}_k) \setminus \mathcal{V}_{\mathbb{K}}(\langle g_{k,i} \rangle), \\ v_{h_{k,i}}(u, Y') = K_k(u)\, (E_{h_{k,i}}(u)\, Y + C_{h_{k,i}}(u)\, Y'),\ \forall\, Y' \in \mathbb{K}^{t_{k,i} \times n}, \end{array} \right. \end{array} \tag{59}$$

where for $i \in I_k$, $\ker_{\mathbb{K}}(B_k(u).) = \mathrm{im}_{\mathbb{K}}(C_{h_{k,i}}(u).)$, $C_{h_{k,i}}(u) \in \mathbb{K}^{q \times t_{k,i}}$, and $B_k(u)\, E_{h_{k,i}}(u) = I_l$ for all $u \in \mathcal{V}_{\mathbb{K}}(\mathcal{J}_k) \setminus \mathcal{V}_{\mathbb{K}}(\langle g_{k,i} \rangle)$. Finally, coming back to the above solution $(u,\, v)$ of the rank factorization problem satisfying $u \in \mathcal{V}_{\mathbb{K}}(\mathcal{J}_k)$, note that $u \notin \mathcal{V}_{\mathbb{K}}(\mathcal{I}_{k,\mathcal{R}})$. Indeed, if so, then $\mathrm{rank}_{\mathbb{K}}(B_k(u)) < l$, which contradicts the fact that $(u,\, v)$ is a solution of the rank factorization problem. Thus, there exists $i \in I_k$ such that $g_{k,i}(u) \neq 0$, i.e., $u \in \mathcal{V}_{\mathbb{K}}(\mathcal{J}_k) \setminus \mathcal{V}_{\mathbb{K}}(\langle g_{k,i} \rangle)$, and thus, $v = v_{h_{k,i}}(u, Y')$ for a certain matrix $Y' \in \mathbb{K}^{t_{k,i} \times n}$, which finally proves the result. □

Algorithm 4 AllSolutions

procedure $\textsc{AllSolutions}(D_1, \ldots, D_r \in \mathbb{K}^{m\times m}, 0 \neq M \in \mathbb{K}^{m\times n})$
  For $k = 0, \ldots, r-1$, $\mathrm{Sol}_k = \textsc{RankFactorizationProblem}(D_1, \ldots, D_r, M, k)$
  return $\cup_{k=0}^{r-1} \mathrm{Sol}_k$
end procedure

If $r = 1$ (see [14, 15]), then we only have to consider $\mathcal{J} = \mathcal{J}_0 = \mathcal{J}_{r-1}$ and the approach of Section 2.2 is enough to find all the solutions of the rank factorization problem eq. (4).

Example 13. We continue Example 1. Let $x = (x_1 \ \ldots \ x_4)^T$, $\mathcal{R} = \mathbb{Q}[x_1, x_2, x_3, x_4]$,

$$A = (D_1\, x \quad D_2\, x \quad D_3\, x \quad D_4\, x) = \begin{pmatrix} x_1 & 0 & x_4 & 0 \\ 0 & x_2 & 0 & x_3 \\ 0 & -x_3 & 0 & -x_2 \\ -x_4 & 0 & -x_1 & 0 \end{pmatrix} \in \mathcal{R}^{4\times 4},$$

$\mathcal{Q} = \mathrm{coker}_{\mathcal{S}}(.Q) = \mathcal{R}^{1\times 4}/(\mathcal{R}^{1\times 3}\, Q)$ be the $\mathcal{R}$ finitely presented by the following matrix

$$Q = L\, A = \begin{pmatrix} 0 & -x_3 & 0 & -x_2 \\ 0 & x_2 & 0 & x_3 \\ x_1 + x_4 & 0 & x_1 + x_4 & 0 \end{pmatrix} \in \mathcal{R}^{3\times 4},$$

and the Fitting ideals of $\mathcal{Q}$ defined by:

$$\begin{cases} \mathcal{J} = \mathcal{J}_0 = \mathrm{Fitt}_0(\mathcal{Q}) = \langle 0 \rangle, \\ \mathcal{J}_1 = \mathrm{Fitt}_1(\mathcal{Q}) = \langle (x_1 + x_4)\,(x_2 - x_3)\,(x_2 + x_3) \rangle, \\ \mathcal{J}_2 = \mathrm{Fitt}_2(\mathcal{Q}) = \langle (x_2 - x_3)\,(x_2 + x_3),\, x_3\,(x_1 + x_4),\, x_2\,(x_1 + x_4) \rangle, \\ \mathcal{J}_3 = \mathrm{Fitt}_3(\mathcal{Q}) = \langle x_2,\, x_3,\, x_1 + x_4 \rangle. \end{cases}$$

Hence, $\mathrm{rank}_{\mathbb{K}}(Q(u))$ for $u \in \mathcal{V}_{\mathbb{K}}(\mathcal{J}_k)$ is less than or equal to $3 - k$ for $k = 0, \ldots, 3$. In particular, we have $Q(u) = 0$ for $u \in \mathcal{V}_{\mathbb{K}}(\mathcal{J}_3) = \{(1\; 0\; 0\; -1)^T\, u_1 \mid u_1 \in \mathbb{K}\}$. See Example 1.

Let us consider the rings $\mathcal{S}_k = \mathcal{R}/\mathcal{J}_k$ for $k = 0, \ldots, 3$, with the notation $\mathcal{S}_0 = \mathcal{S} = \mathcal{R}$, and $\chi_k : \mathcal{S} \longrightarrow \mathcal{S}_k$ the canonical ring epimorphisms for $k = 1, 2, 3$. Denote the residue class $\overline{x_i}$ (resp., $\chi_k(x_i)$) of x_i in $\mathcal{S}$ (resp., $\mathcal{S}_k$) simply by x_i. Let K_k be a matrix satisfying $\ker_{\mathcal{S}_k}(\chi_k(Q).) = \mathrm{im}_{\mathcal{S}_k}(\chi_k(K_k).)$ for $k = 0, \ldots, 3$:

$$K = K_0 = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} \in \mathcal{S}^{4\times 1}, \quad K_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -x_3 & x_2 & x_1 + x_4 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & x_3 & x_2 & 0 \\ 0 & x_2 & -x_3 & 0 & 0 & 0 & x_1 + x_4 \end{pmatrix} \in \mathcal{S}_2^{4\times 7},$$

$$K_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -x_3\,(x_1 + x_4) & x_2\,(x_1 + x_4) & 0 \\ -1 & 0 & 0 & (x_2 - x_3)\,(x_2 + x_3) \\ 0 & x_2\,(x_1 + x_4) & -x_3\,(x_1 + x_4) & 0 \end{pmatrix} \in \mathcal{S}_1^{4\times 4}, \quad K_3 = I_4.$$

Using the left inverse $V = (0 \quad 0 \quad 0 \quad 1)$ of X, then we have:

$$\begin{cases} B = V\, A\, K_0 = (x_1 - x_4) \in \mathcal{R}, \\ B_1 = V\, A\, K_1 = (x_1 - x_4 \quad 0 \quad 0 \quad -x_1\,(x_2 - x_3)\,(x_2 + x_3)) \in \mathcal{R}^{1\times 4}, \\ B_2 = V\, A\, K_2 = (x_1 - x_4 \quad 0 \quad 0 \quad 0 \quad -x_1\, x_3 \quad -x_1\, x_2 \quad 0) \in \mathcal{R}^{1\times 7}, \\ B_3 = V\, A\, K_3 = (-x_4 \quad 0 \quad -x_1 \quad 0) \in \mathcal{R}^{1\times 4}. \end{cases}$$

Introducing the $\mathcal{S}_k$-modules $\mathcal{B}_k = \mathrm{coker}_{\mathcal{S}_k}(\chi_k(B_k).)$ for $k = 1, 2, 3$, we then get:

$$\begin{cases} \mathcal{I} = \mathrm{Fitt}_0(\mathcal{B}) = \langle x_1 - x_4 \rangle_{\mathcal{S}}, \\ \mathcal{I}_1 = \mathrm{Fitt}_0(\mathcal{B}_1) = \langle x_1 - x_4,\, -x_1\,(x_2 - x_3)\,(x_2 + x_3) \rangle_{\mathcal{S}_1} = \langle x_1 - x_4 \rangle_{\mathcal{S}_1}, \\ \mathcal{I}_2 = \mathrm{Fitt}_0(\mathcal{B}_2) = \langle x_1 - x_4,\, -x_1\, x_3,\, -x_1\, x_2 \rangle_{\mathcal{S}_2} = \langle x_1 - x_4 \rangle_{\mathcal{S}_2}, \\ \mathcal{I}_3 = \mathrm{Fitt}_0(\mathcal{B}_3) = \langle x_1,\, x_4 \rangle_{\mathcal{S}_3} = \langle x_4 \rangle_{\mathcal{S}_3}. \end{cases}$$

We then have $\mathcal{S}_k \otimes_{\mathcal{S}} \mathcal{I} = \langle x_1 - x_4 \rangle_{\mathcal{S}_k} = \mathcal{I}_k$ for $k = 1, 2, 3$.

First consider $\mathcal{I}$. Let $h_0 = g_0 = x_1 - x_4$ and consider the localization $\mathcal{S}_{h_0} \cong \mathcal{R}[y]/\langle y\, h_0 - 1 \rangle$ of $\mathcal{S} = \mathcal{R}$ (see eq. (58)). Then, B has the inverse $E_{h_0} = y$. Moreover, we have $C_{h_0} = 0$ because $\ker_{\mathcal{S}_{h_0}}(B.) = 0$. Hence, using $Y = (1 \quad 0 \quad 0 \quad 1)$, the corresponding solutions of Problem eq. (4) are defined by:

$$\left\{ \begin{array}{l} u = (u_1 \ \dots \ u_4)^T \in \mathcal{U}_0 = \mathbb{K}^{4 \times 1} \setminus \mathcal{V}_{\mathbb{K}}(\langle x_1 - x_4 \rangle) = \left\{ (u_1 \ \dots \ u_4)^T \in \mathbb{K}^{4 \times 1} \mid u_1 \neq u_4 \right\} \\ v_{h_0}(u) = K_0(u)\, E_{h_0}(u)\, Y = \dfrac{1}{u_1 - u_4} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \end{array} \right.$$

Now, consider the ideal $\mathcal{I}_1$ of $\mathcal{S}_1 = \mathcal{R}/\langle (x_1 + x_4)\,(x_2 - x_3)\,(x_2 + x_3) \rangle$, $g_1 = x_1 - x_4$, $h_1 = \chi_1(g_1)$, and the localization $\mathcal{S}_{1h_1} \cong \mathcal{R}[y]/\langle y\, g_1 - 1, (x_1 + x_4)\,(x_2 - x_3)\,(x_2 + x_3) \rangle$ of $\mathcal{S}_1$ (see eq. (58)). We can check that h_1 is not a nilpotent element of $\mathcal{S}_1$, i.e., $\mathcal{S}_{1h_1}$ is not the trivial ring. Then, B_1 has a right inverse $E_{h_1} = y\,(1 \quad 0 \quad 0 \quad 0)^T \in {\mathcal{S}_{1h_1}}^{4 \times 1}$ and the following matrix

$$C_{h_1} = \begin{pmatrix} x_2^2 - x_3^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 0 & 0 \end{pmatrix} \in {\mathcal{S}_{1h_1}}^{4 \times 3}$$

satisfies $\ker_{\mathcal{S}_{1h_1}}(B_1.) = \operatorname{im}_{\mathcal{S}_{1h_1}}(C_{h_1}.)$. Let us define the following quasi-affine variety:

$$\mathcal{U}_1 = \mathcal{V}_{\mathbb{K}}(\langle (x_1 + x_4)\,(x_2 - x_3)\,(x_2 + x_3) \rangle) \setminus \mathcal{V}_{\mathbb{K}}(\langle (x_1 + x_4)\,(x_2 - x_3)\,(x_2 + x_3),\, x_1 - x_4 \rangle)$$
$$=$$
$$\left\{ \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ -u_1 \end{pmatrix}, \begin{pmatrix} u_1 \\ u_2 \\ u_2 \\ u_4 \end{pmatrix}, \begin{pmatrix} u_1 \\ u_2 \\ -u_2 \\ u_4 \end{pmatrix} \mid u_i \in \mathbb{K} \right\} \setminus \left\{ \begin{pmatrix} 0 \\ u_2 \\ u_3 \\ 0 \end{pmatrix}, \begin{pmatrix} u_1 \\ u_2 \\ u_2 \\ u_1 \end{pmatrix}, \begin{pmatrix} u_1 \\ u_2 \\ -u_2 \\ u_1 \end{pmatrix} \mid u_i \in \mathbb{K} \right\}.$$

Then, the corresponding solutions of Problem eq. (4) are defined by:

$$\left\{ \begin{array}{l} u = (u_1 \ \dots \ u_4)^T \in \mathcal{U}_1, \\ v_{h_1}(u, Y_1') = K_1(u)\ (E_{h_1}(u)\, Y + C_{h_1}(u)\, Y_1') \\ \quad = \dfrac{1}{u_1 - u_4} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} u_2^2 - u_3^2 & 0 & 0 \\ 0 & -u_3\,(u_1 + u_4) & u_2\,(u_1 + u_4) \\ u_2^2 - u_3^2 & 0 & 0 \\ 0 & u_2\,(u_1 + u_4) & -u_3\,(u_1 + u_4) \end{pmatrix} Y_1', \\ \hfill \forall\, Y_1' \in \mathbb{K}^{4 \times 4}. \end{array} \right.$$

Consider $\mathcal{S}_2 \cong \mathcal{R}/\langle (x_2 - x_3)\,(x_2 + x_3),\, x_3\,(x_1 + x_4),\, x_2\,(x_1 + x_4) \rangle$, $g_2 = x_1 - x_4$, $h_2 = \chi_2(g_2)$, and the localization $\mathcal{S}_{2h_2} \cong \mathcal{R}[y]/\langle y\, g_2 - 1, (x_2 - x_3)\,(x_2 + x_3),\, x_3\,(x_1 + x_4),\, x_2\,(x_1 + x_4) \rangle$ of $\mathcal{S}_2$ (see eq. (58)). We can check that h_2 is not a nilpotent element of $\mathcal{S}_2$, i.e., $\mathcal{S}_{2h_2}$ is not the trivial ring. Then, B_2 has a right inverse $E_{h_2} = y\,(1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0)^T \in {\mathcal{S}_{2h_2}}^{7 \times 1}$ and the following matrix

$$C_{h_2} = \begin{pmatrix} x_3 & x_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \in {\mathcal{S}_{2h_2}}^{7 \times 6}$$

satisfies $\ker_{\mathcal{S}_{2h_2}}(B_2.) = \mathrm{im}_{\mathcal{S}_{2h_2}}(C_{h_2}.)$. Let us define the following quasi-affine variety:

$$\mathcal{U}_2 = \mathcal{V}_{\mathbb{K}}(\langle x_2^2 - x_3^2,\, x_3\,(x_1+x_4),\, x_2\,(x_1+x_4)\rangle) \setminus \mathcal{V}_{\mathbb{K}}(\langle x_2^2 - x_3^2,\, x_3\,(x_1+x_4),\, x_2\,(x_1+x_4),\, x_1 - x_4\rangle) =$$
$$\left\{ \begin{pmatrix} u_1 \\ 0 \\ 0 \\ u_4 \end{pmatrix}, \begin{pmatrix} u_1 \\ u_2 \\ u_2 \\ -u_1 \end{pmatrix}, \begin{pmatrix} u_1 \\ u_2 \\ -u_2 \\ -u_1 \end{pmatrix} \mid u_i \in \mathbb{K} \right\} \setminus \left\{ \begin{pmatrix} u_1 \\ 0 \\ 0 \\ u_1 \end{pmatrix}, \begin{pmatrix} 0 \\ u_2 \\ u_2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ u_2 \\ -u_2 \\ 0 \end{pmatrix} \mid u_i \in \mathbb{K} \right\}.$$

Then, the corresponding solutions of Problem eq. (4) are:

$$\begin{cases} u = (u_1 \ \dots \ u_4)^T \in \mathcal{U}_2, \\ v_{h_2}(u, Y_2') = K_2(u)\ (E_{h_2}(u)\, Y + C_{h_2}(u)\, Y_2') \\ \quad = \dfrac{1}{u_1 - u_4} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} u_3 & u_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -u_3 & u_2 & u_1+u_4 & 0 \\ u_3 & u_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & u_2 & -u_3 & 0 & u_1+u_4 \end{pmatrix} Y_2', \\ \hfill \forall\, Y_2' \in \mathbb{K}^{6\times 4}. \end{cases}$$

The solutions obtained in Example 1 correspond to $\mathcal{S}_3 = \mathcal{R}/\langle x_2,\, x_3,\, x_1+x_4\rangle$, $g_3 = x_4$, $h_3 = \chi_3(g_3)$, $\mathcal{S}_{3h_3} \cong \mathcal{R}[y]/\langle y\, g_3 - 1, x_2,\, x_3,\, x_1+x_4\rangle$ (see eq. (58)), $E_{h_3} = y/2\,(-1 \quad 0 \quad 1 \quad 0)^T \in \mathcal{S}_{3h_3}^{4\times 1}$, $\ker_{\mathcal{S}_{3h_3}}(B_3.) = \mathrm{im}_{\mathcal{S}_{3x_4}}(C_{h_3}.)$, where C_{h_3} is defined by eq. (12). We find again eq. (13) because we have:

$$\mathcal{U}_3 = \mathcal{V}_{\mathbb{K}}(\langle x_2,\, x_3,\, x_1+x_4\rangle) \setminus \mathcal{V}_{\mathbb{K}}(\langle x_2,\, x_3,\, x_1+x_4,\, x_4\rangle) = \left\{ \begin{pmatrix} -u_4 \\ 0 \\ 0 \\ u_4 \end{pmatrix} \mid u_4 \in \mathbb{K} \setminus \{0\} \right\}.$$

4.2 A few final remarks on the solution space

In this section, we state a few remarks on the solution space of the rank factorization problem.

We first study connections between the modules $\mathcal{B}_k$'s.

Applying the right exact covariant functor $\mathcal{S}_{k+1} \otimes_{\mathcal{S}_k} \cdot$ to the exact sequence of $\mathcal{S}_k$-modules

$$\mathcal{S}_k^{q_k\times 1} \xrightarrow{\chi_k(B_k).} \mathcal{S}_k^{l\times 1} \xrightarrow{\sigma_k} \mathcal{B}_k \longrightarrow 0,$$

we obtain the following exact sequence of $\mathcal{S}_k$-modules

$$\mathcal{S}_{k+1}^{q_k\times 1} \xrightarrow{\chi_{k+1}(B_k).} \mathcal{S}_{k+1}^{l\times 1} \xrightarrow{\mathrm{id}_{\mathcal{S}_{k+1}} \otimes \sigma_k} \mathcal{S}_{k+1} \otimes_{\mathcal{S}_k} \mathcal{B}_k \longrightarrow 0,$$

i.e., $\mathcal{S}_{k+1} \otimes_{\mathcal{S}_k} \mathcal{B}_k \cong \mathrm{coker}_{\mathcal{S}_k}(\chi_{k+1}(B_k).)$. Using Proposition 4, we then have:

$$\mathcal{S}_{k+1} \otimes_{\mathcal{S}} \mathrm{Fitt}_j(\mathcal{B}_k) = \mathrm{Fitt}_j(\mathcal{S}_{k+1} \otimes_{\mathcal{S}_k} \mathcal{B}_k) = \mathrm{Fitt}_j(\mathrm{coker}_{\mathcal{S}_{k+1}}(\chi_{k+1}(B_k).)), \quad j = 0, \dots, l-1.$$

Using eq. (32), we obtain the following commutative exact diagram of $\mathcal{S}_{k+1}$-modules

$$\begin{array}{ccccccccc} & & \mathcal{S}_{k+1}^{q_k\times 1} & \xrightarrow{\chi_{k+1}(B_k).} & \mathcal{S}_{k+1}^{l\times 1} & \xrightarrow{\mathrm{id}_{\mathcal{S}_{k+1}} \otimes \sigma_k} & \mathcal{S}_{k+1} \otimes_{\mathcal{S}_k} \mathcal{B}_k & \longrightarrow & 0 \\ & & \downarrow {\scriptstyle \chi_k(L_{k,k+1}).} & & \| & & \downarrow {\scriptstyle \vartheta_{k+1}} & & \\ \mathcal{S}_{k+1}^{t_{k+1}\times 1} & \xrightarrow{\chi_{k+1}(C_{k+1}).} & \mathcal{S}_{k+1}^{q_{k+1}\times 1} & \xrightarrow{\chi_{k+1}(B_{k+1}).} & \mathcal{S}_{k+1}^{l\times 1} & \xrightarrow{\sigma_{k+1}} & \mathcal{B}_{k+1} & \longrightarrow & 0, \end{array}$$

where $C_{k+1} \in \mathcal{R}^{t_{k+1}\times q_{k+1}}$ is such that $\ker_{\mathcal{S}_{k+1}}(\chi_{k+1}(B_{k+1}).) = \mathrm{im}_{\mathcal{S}_{k+1}}(\chi_{k+1}(C_{k+1}).)$.

Using 4 of Proposition 3.1 of [7], we have

$$\operatorname{coker} \vartheta_{k+1} = \mathcal{S}_{k+1}^{l\times 1} / \left((\chi_{k+1}(B_{k+1}) \quad I_l)\, \mathcal{S}_{k+1}^{(q_{k+1}+l)\times 1} \right) = 0,$$

i.e., $\vartheta_{k+1} : \mathcal{S}_{k+1} \otimes_{\mathcal{S}_k} \mathcal{B}_k \longrightarrow \mathcal{B}_{k+1}$ is an epimorphism for $k = 0, \ldots, r-2$. Thus, we have the following short exact sequences of $\mathcal{S}_{k+1}$-modules:

$$0 \longrightarrow \ker \vartheta_{k+1} \longrightarrow \mathcal{S}_{k+1} \otimes_{\mathcal{S}_k} \mathcal{B}_k \xrightarrow{\vartheta_{k+1}} \mathcal{B}_{k+1} \longrightarrow 0, \quad k = 0, \ldots, r-2. \tag{60}$$

Using 1 of Proposition 3.1 of [7], we obtain the following presentation matrix for ker ϑ_{k+1}:

$$\begin{aligned} \ker\, \vartheta_{k+1} &= \operatorname{im}_{\mathcal{S}_{k+1}}(\chi_{k+1}(B_{k+1}).)/\operatorname{im}_{\mathcal{S}_{k+1}}(\chi_{k+1}(B_k).) \\ &\cong \operatorname{coker}_{\mathcal{S}_{k+1}}((\chi_{k+1}(L_{k,k+1}) \quad \chi_{k+1}(C_{k+1})).). \end{aligned}$$

If $C_{k+1} = 0$, then note that ker $\vartheta_{k+1} \cong \operatorname{coker}_{\mathcal{S}_{k+1}}(\chi_{k+1}(L_{k,k+1}).)$, a result which is a direct consequence of the application of the snake lemma to the above commutative exact diagram.

Proposition 6 ([23]). *If $g : \mathcal{M} \longrightarrow \mathcal{M}''$ is an epimorphism of modules, then we have:*

$$\operatorname{Fitt}_0(\mathcal{M}) \subseteq \operatorname{Fitt}_0(\mathcal{M}'').$$

Note $\mathcal{I}_k = \operatorname{Fitt}_0(\mathcal{B}_k)$ for $k = 0, \ldots, r-1$. Applying Proposition 6 to ϑ_{k+1}, we obtain

$$\mathcal{S}_{k+1} \otimes_{\mathcal{S}_k} \mathcal{I}_k = \mathcal{S}_{k+1} \otimes_{\mathcal{S}_k} \operatorname{Fitt}_0(\mathcal{B}_k) = \operatorname{Fitt}_0(\operatorname{coker}_{\mathcal{S}_{k+1}}(\chi_{k+1}(B_k).)) \subseteq \mathcal{I}_{k+1} = \operatorname{Fitt}_0(\mathcal{B}_{k+1}), \tag{61}$$

for $k = 0, \ldots, r-2$. More precisely, if $\mathcal{I}_k = \langle h_{k,1}, \ldots, h_{k,\beta_k} \rangle_{\mathcal{S}_k}$, where $g_{k,j} \in \mathcal{R}$ is such that $h_{k,j} = \chi_k(g_{k,j})$ for $j = 1, \ldots, \beta_k$ and $k = 0, \ldots, r-1$, then eq. (61) yields:

$$\langle \chi_{k+1}(g_{k,1}), \ldots, \chi_{k+1}(g_{k,\beta_k}) \rangle_{\mathcal{S}_{k+1}} \subseteq \langle \chi_{k+1}(g_{k+1,1}), \ldots, \chi_{k+1}(g_{k+1,\beta_{k+1}}) \rangle_{\mathcal{S}_{k+1}}.$$

Let $\Theta_{k+1} = \{ j \in [\![1, \ldots, \beta_{k+1}]\!] \mid \chi_{k+1}(g_{k+1,j}) \notin \mathcal{S}_{k+1} \otimes_{\mathcal{S}_k} \mathcal{I}_k \}$, i.e., Θ_{k+1} is the set of the indices of the generators $\chi_{k+1}(g_{k+1,j})$ of $\mathcal{I}_{k+1}$ which are not zero in $\mathcal{I}_{k+1}/\,(\mathcal{S}_{k+1} \otimes_{\mathcal{S}_k} \mathcal{I}_k)$. Therefore, we have:

$$\mathcal{I}_{k+1} = \langle \chi_{k+1}(g_{k,1}), \ldots, \chi_{k+1}(g_{k,\beta_k}) \rangle_{\mathcal{S}_{k+1}} + \sum_{j \in \Theta_{k+1}} \mathcal{S}_{k+1}\, \chi_{k+1}(g_{k+1,j}).$$

In other words, $\chi_{k+1}(g_{k,1}), \ldots, \chi_{k+1}(g_{k,\beta_k})$ and the $\chi_{k+1}(g_{k+1,j})$'s, where $j \in \Theta_{k+1}$, is a set of generators of $\mathcal{I}_{k+1}$. If we set $\beta'_{k+1} = \beta_k + \operatorname{card}(\Theta_{k+1})$, $g'_{k+1,j} = g_{k,j}$ for $j = 1, \ldots, \beta_k$, and $g'_{k+1,\beta_k+j} = g_{k+1,j}$ for $j \in \Theta_{k+1}$, then we have:

$$\mathcal{I}_{k+1} = \langle \chi_{k+1}(g'_{k+1,1}), \ldots, \chi_{k+1}(g'_{k+1,\beta'_{k+1}}) \rangle_{\mathcal{S}_{k+1}}. \tag{62}$$

In the rest of the text, while considering a set of generators of $\mathcal{I}_{k+1}$, we shall assume eq. (62) for $k = 0, \ldots, r-2$, i.e., $\mathcal{I}_{k+1} = \langle \chi_{k+1}(g_{k+1,1}), \ldots, \chi_{k+1}(g_{k+1,\beta_{k+1}}) \rangle_{\mathcal{S}_{k+1}}$, where $g_{k+1,j} = g_{k,j}$ for $j = 1, \ldots, \beta_k$.

Remark 21. If $0 \longrightarrow \mathcal{M}' \xrightarrow{f} \mathcal{M} \xrightarrow{g} \mathcal{M}'' \longrightarrow 0$ is a short exact sequence of modules, then we have $\operatorname{Fitt}_i(\mathcal{M}')\operatorname{Fitt}_j(\mathcal{M}'') \subseteq \operatorname{Fitt}_{i+j}(\mathcal{M})$ for all $i, j \in \mathbb{Z}_{\geq 0}$. Thus, we get $\operatorname{Fitt}_0(\mathcal{M}')\operatorname{Fitt}_0(\mathcal{M}'') \subseteq \operatorname{Fitt}_0(\mathcal{M})$. See, e.g., [23]. Hence, if, for $k = 0, \ldots, r-2$, we note $\mathcal{L}_k = \operatorname{Fitt}_0(\operatorname{coker}_{\mathcal{S}_k}((\chi_k(L_{k-1,k}) \quad \chi_k(C_k)).))$, then $\mathcal{L}_{k+1}\,\mathcal{I}_{k+1} \subseteq \mathcal{S}_{k+1} \otimes_{\mathcal{S}_k} \mathcal{I}_k$ for $k = 0, \ldots, r-2$.

Corollary 5. *With the notations of the proof of Theorem 6 and, without loss of generality, assuming $\mathcal{I}_{k+1} = \langle \chi_{k+1}(g_{k+1,1}), \ldots, \chi_{k+1}(g_{k+1,\beta_{k+1}}) \rangle_{\mathcal{S}_{k+1}}$, where $g_{k+1,j} = g_{k,j}$ for $j = 1, \ldots, \beta_k$, then the restriction of a solution (u, v) of the rank factorization problem eq. (4) defined by eq. (59) from $\mathcal{V}_{\mathbb{K}}(\mathcal{J}_k)$ to $\mathcal{V}_{\mathbb{K}}(\mathcal{J}_{k+1})$ satisfy the following property:*

$$\forall\, u \in \mathcal{V}_{\mathbb{K}}(\mathcal{J}_{k+1}) \setminus \mathcal{V}_{\mathbb{K}}(\langle g_{k,i} \rangle),\ \forall\, Y' \in \mathbb{K}^{t_{k,i} \times n},\ \exists\, Y'' \in \mathbb{K}^{t_{k+1,i} \times n} :\ v_{h_{k,i}}(u, Y') = v_{h_{k+1,i}}(u, Y'').$$

In other words, the restrictions of the solutions of the rank factorization problem eq. (4) from $\mathcal{V}_{\mathbb{K}}(\mathcal{J}_k)$ to $\mathcal{V}_{\mathbb{K}}(\mathcal{J}_{k+1})$ are among the solutions defined in $\mathcal{V}_{\mathbb{K}}(\mathcal{J}_{k+1})$.

Proof. Let $B_k = V\,A\,K_k \in \mathcal{R}^{l\times q_k}$, $\mathcal{B}_k = \operatorname{coker}_{\mathcal{S}_k}(\chi_k(B_k).)$, and $\chi_k : \mathcal{R} \longrightarrow \mathcal{S}_k$ be the canonical ring epimorphism. Let $\mathcal{I}_k = \operatorname{Fitt}_0(\mathcal{B}_k) = \langle h_{k,1},\ldots,h_{k,\beta_k}\rangle_{\mathcal{S}_k}$, $g_{k,j} \in \mathcal{R}$ be such that $h_{k,j} = \chi_k(g_{k,j})$ for $j = 1,\ldots,\beta_k$. By notation (see the comment after eq. (62)), we have:

$$\mathcal{I}_{k+1} = \langle \chi_{k+1}(g_{k+1,1}),\ldots,\chi_{k+1}(g_{k+1,\beta_{k+1}})\rangle_{\mathcal{S}_{k+1}}, \quad g_{k+1,j} = g_{k,j}, \quad = 1,\ldots,\beta_k.$$

By Theorem 5, the solutions of the rank factorization problem defined over $\mathcal{S}_k$ are eq. (59).

Combining eq. (32), i.e., $\chi_{k+1}(K_k) = \chi_{k+1}(K_{k+1})\,\chi_{k+1}(L_{k,k+1})$, with $B_k = V\,A\,K_k$, we have

$$\begin{aligned}\chi_{k+1}(B_k) &= \chi_{k+1}(V)\,\chi_{k+1}(A)\,\chi_{k+1}(K_k) = \chi_{k+1}(V)\,\chi_{k+1}(A)\,\chi_{k+1}(K_{k+1})\,\chi_k(L_{k,k+1})\\ &= \chi_{k+1}(B_{k+1})\,\chi_k(L_{k,k+1}), \quad k = 0,\ldots,r-2,\end{aligned}$$

i.e., $B_k - B_{k+1}\,L_{k,k+1} \in \mathcal{J}_{k+1}^{l\times q_k}$, and thus, $B_k(u) = B_{k+1}(u)\,L_{k,k+1}(u)$ for all $u \in \mathcal{V}_{\mathbb{K}}(\mathcal{J}_{k+1})$.

We state again that $\chi_{k+1}(K_k) = \chi_{k+1}(K_{k+1})\,\chi_{k+1}(L_{k,k+1})$ and:

$$\begin{aligned}&\forall\, u \in \mathcal{V}_{\mathbb{K}}(\mathcal{J}_k) \setminus \mathcal{V}_{\mathbb{K}}(\langle g_{k,i}\rangle), && B_k(u)\,E_{h_{k,i}}(u) = I_l,\\ &\forall\, u \in \mathcal{V}_{\mathbb{K}}(\mathcal{J}_{k+1}) \setminus \mathcal{V}_{\mathbb{K}}(\langle g_{k+1,i}\rangle), && B_{k+1}(u)\,E_{h_{k+1,i}}(u) = I_l,\\ &\forall\, u \in \mathcal{V}_{\mathbb{K}}(\mathcal{J}_k) \setminus \mathcal{V}_{\mathbb{K}}(\langle g_{k,i}\rangle), && B_k(u)\,C_{h_{k,i}}(u) = 0.\end{aligned}$$

Using $g_{k+1,i} = g_{k,i}$ for $i = 1,\ldots,\beta_k$ and $\mathcal{V}_{\mathbb{K}}(\mathcal{J}_{k+1}) \subseteq \mathcal{V}_{\mathbb{K}}(\mathcal{J}_k)$, we then have:

$$\forall\, u \in \mathcal{V}_{\mathbb{K}}(\mathcal{J}_{k+1}) \setminus \mathcal{V}_{\mathbb{K}}(\langle g_{k,i}\rangle), \quad \left\{\begin{array}{l} B_{k+1}(u)\,(L_{k,k+1}(u)\,E_{h_{k,i}}(u)) = I_l,\\ B_{k+1}(u)\,E_{h_{k+1,i}}(u) = I_l,\\ B_{k+1}(u)\,(L_{k,k+1}(u)\,C_{h_{k,i}}(u)) = 0,\\ K_k(u) = K_{k+1}(u)\,L_{k,k+1}(u).\end{array}\right.$$

From the first two identities, we get $B_{k+1}(u)\,(E_{h_{k+1,i}}(u) - L_{k,k+1}(u)\,E_{h_{k,i}}(u))) = 0$. Using eq. (53), i.e., $\ker_{\mathbb{K}}(B_{k+1}(u).) = \operatorname{im}_{\mathbb{K}}(C_{h_{k+1,i}}(u).)$ for all $u \in \mathcal{V}_{\mathbb{K}}(\mathcal{J}_{k+1}) \setminus \mathcal{V}_{\mathbb{K}}(\langle g_{k,i}\rangle)$, there exists $O \in \mathbb{K}^{t_k\times q_{k+1}}$ – which depends on k, i, $h_{i,k}$, and u – such that such that $E_{h_{k+1,i}}(u) = L_{k,k+1}(u)\,E_{h_{k,i}} + C_{h_{k+1,i}}(u)\,O$. Similarly, there exists $O' \in \mathbb{K}^{t_{k+1}\times t_k}$ – which depends on k, i, $h_{i,k}$, and u – such that the identity $L_{k,k+1}(u)\,C_{h_{k,i}}(u) = C_{h_{k+1,i}}(u)\,O'$. We then obtain

$$\begin{aligned}&\forall\, u \in \mathcal{V}_{\mathbb{K}}(\mathcal{J}_{k+1}) \setminus \mathcal{V}_{\mathbb{K}}(\langle g_{k,i}\rangle),\ \forall\, Y' \in \mathbb{K}^{t_{k,i}\times n},\\ v_{h_{k,i}}(u,Y') &= K_k(u)\,(E_{h_{k,i}}(u)\,Y + C_{h_{k,i}}(u)\,Y')\\ &= K_{k+1}(u)\,(L_{k,k+1}(u)\,E_{h_{k,i}}(u)\,Y + L_{k,k+1}(u)\,C_{h_{k,i}}(u)\,Y')\\ &= K_{k+1}(u)\,(E_{h_{k+1,i}}(u)\,Y + C_{h_{k+1,i}}(u)\,(O'\,Y' - O\,Y))\\ &= v_{h_{k+1,i}}(u, O'\,Y' - O\,Y),\end{aligned}$$

which finally proves the result. □

Example 14. Let us illustrate Corollary 5 on Example 13. We can first check the identities $\chi_{k+1}(K_k) = \chi_{k+1}(K_{k+1})\,\chi_k(L_{k,k+1})$ for $k = 0, 1, 2$, where:

$$L_{0,1} = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \quad L_{1,2} = \begin{pmatrix}1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}, \quad L_{2,3} = \begin{pmatrix}1&0&0&0&0&0&0\\0&0&0&0&0&0&0\\-1&0&0&0&0&0&0\\0&0&0&0&0&0&0\end{pmatrix}.$$

We can also check again that all the entries of the matrices $K_k - K_{k+1}\,L_{k,k+1}$ belong $\mathcal{J}_{k+1}$ for $k = 0, 1, 2$. We also have $\chi_k(B_k) = \chi_k(B_{k+1})\,\chi_k(L_{k,k+1})$ for $k = 0, 1, 2$, i.e., all the entries of the matrix $B_k - B_{k+1}\,L_{k,k+1}$ belong to $\mathcal{J}_{k+1}$ for $k = 0, 1, 2$. Moreover, all the entries of $L_{k,k+1}\,C_{h_k}$ belong to $\mathcal{J}_{k+1}$, i.e., $L_{k,k+1}(u)\,C_{h_k}(u) = 0$ for all $u \in \mathcal{V}_{\mathbb{K}}(\mathcal{J}_{k+1})$, i.e., we have $L_{k,k+1}(u)\,C_{h_k}(u) = C_{h_{k+1}}(u)\,O'$, where O' is a zero matrix of the appropriated size. Similarly, we have $E_{h_1} = L_{0,1}\, E_{h_0}$ and $E_{h_2} = L_{1,2}\, E_{h_1}$. Note that $E_{h_3} - L_{2,3}\, E_{h_2} = 3/2\,(-y \quad 0 \quad y \quad 0)^T$ is not in the image of the matrix C_{h_3} defined by eq. (12). It does not contradict Corollary 5 once we have noticed that $g_2 = x_1 - x_4$ and $g_3 = x_4$. As explained in eq. (12) (see eq. (62)), we must assume that the generator of $\mathcal{I}_3$ is $\chi_3(g_2) = -2\, x_4$. In this case, a right inverse of $\chi_3(B_3) = (-x_4 \quad 0 \quad x_4 \quad 0)$ with entries in $\mathcal{S}_{3\chi_3(g_2)}$ is, for instance, $E_{h_3} = (y \quad 0 \quad -y \quad 0)^T$. Hence, we obtain $E_{h_3} = L_{2,3}\, E_{h_2}$. We thus have $E_{h_{k+1}}(u) = L_{k,k+1}(u)\, E_{h_k} + C_{h_{k+1}}(u)\, O$ for all $u \in \mathcal{V}_{\mathbb{K}}(\mathcal{J}_{k+1})$, where O is a zero matrix of the appropriate size. Finally, we can check again that $v_{h_k}(u, Y') = v_{h_{k+1}}(u, 0)$ for all $u \in \mathcal{V}_{\mathbb{K}}(\mathcal{J}_{k+1}) \setminus \mathcal{V}_{\mathbb{K}}(\langle g_k \rangle)$ and for all Y'.

5 Conclusion

In this paper, we have studied the general solutions of the rank factorization problem eq. (4). This last problem is connected to demodulation problems studied in gearbox vibration analysis [14, 15]. Using module theory, homological, and computer algebra methods, we have shown how to characterize the general solutions of this rank factorization problem. The results obtained in [16, 17, 18, 19], which characterize a particular class of solutions, are found again and generalized. The characterization of the general solutions obtained here is effective and can be implemented in modern computer algebra systems. In particular, it was implemented in the RankFactorization package (built upon the OreModules package [6]) dedicated to the rank factorization problem eq. (4) and its applications.

Many important issues on the rank factorization problem eq. (4) still have to be investigated such as, for instance, the algebraic geometric interpretation of its solution space using projective geometry.

A first remark in this direction is that if (u, v) is a solution of eq. (4), then so is $(\lambda\, u, \lambda^{-1}\, v)$ for all $\lambda \in \mathbb{K}^\times = \mathbb{K} \setminus \{0\}$. Hence, $\mathbb{K}^\times$ defines a *group action* on the solution space of eq. (4) and the *orbit* $\mathcal{O}_{(u,\,v)} = \{(\lambda\, u, \lambda^{-1}\, v) \mid \lambda \in \mathbb{K}^\times\}$ could be considered instead of the solution (u, v) only.

A second remark is that eq. (4) can be transformed into a *multi-homogeneous polynomial system* over a *multi-projective space* [26]. Indeed, writing $v = (v_{\bullet 1} \ \ldots \ v_{\bullet n})$ (resp., $M = (M_{\bullet 1} \ \ldots \ M_{\bullet n})$), where $v_{\bullet i} \in \mathbb{K}^{r \times 1}$ (resp., $M_{\bullet i} \in \mathbb{K}^{m \times 1}$) denotes the i^{th} column of v (resp., M), eq. (1) can be rewritten as $A(u)\, v_{i\bullet} = M_{\bullet i}$ for $i = 1, \ldots, n$, and introducing the new variables u_0 and v_{0i} for $i = 1, \ldots, n$, and $M_h = u_0\,(v_{01}\, M_{\bullet 1} \ \ldots \ v_{0n}\, M_{\bullet n})$, the change of variables $u \longleftarrow u/u_0$ and $v_{\bullet i} \longleftarrow v_{\bullet i}/v_{0i}$, $i = 1, \ldots, n$, in $A(u)\, v_{i\bullet} = M_{\bullet i}$ for $i = 1, \ldots, n$ then yields the following multi-homogeneous polynomial system

$$A(u)\, v = M_h \tag{63}$$

of degree $(1, 1)$ with respect to the partition $\{u_0, u_1, \ldots, u_m\} \cup \{v_{0j}, v_{1j}, \ldots, v_{rj}\}$ of the variables. Since $(u_0, u_1, \ldots, u_n)$ (resp., $(v_{0j}, v_{1j}, \ldots, v_{rj})$) is a point of the *projective space* $\mathbb{P}^n(\mathbb{K})$ (resp., $\mathbb{P}^r(\mathbb{K})$), the solutions of eq. (63) can be sought in the multi-projective space [26]:

$$\mathbb{P}^m(\mathbb{K}) \times \underbrace{\mathbb{P}^r(\mathbb{K}) \times \ldots \times \mathbb{P}^r(\mathbb{K})}_{n}.$$

In vibration analysis, the complex matrices D_i's and M are centrohermitian. The solutions of the corresponding rank factorization problem eq. (4) are sought to also be centrohermitian. As explained in Section 1, the corresponding rank factorization problem can be transformed into an equivalent rank factorization problem for real matrices $\rho(D_i)$'s and $\rho(M)$. Symbolic-numeric methods will be used in the future to obtain certified numerical solutions to the rank factorization problem eq. (4) for $\mathbb{K} = \overline{\mathbb{Q}}$ or $\mathbb{R}$, and thus, to the corresponding vibration problem by transforming back the obtained real solutions to centrohermitian solutions of the original problem. For more details, see [18].

In practice, the matrices D_i's are explicitly known because they are fixed by the demodulation problem under study, contrary to M which is only measured. Hence, the matrix M is obtained through noisy measurement. Thus, Problem eq. (1) corresponds to the ideal demodulation problem, i.e., to the case with no perturbations and noise corrupting the measured signal. In practice, i.e., when M is not supposed to be exactly known, one usually prefers to consider the following optimization problem

$$\underset{u \in \mathrm{CH}_{n,1},\, v_i \in \mathrm{CH}_{1,m}}{\arg\min} \left\| \sum_{i=1}^{r} D_i\, u\, v_i - M \right\|_{\mathrm{Frob}}, \tag{64}$$

where $\mathrm{CH}_{n,1}$ (resp., $\mathrm{CH}_{1,m}$) denotes the set of all the centrohermitian column (resp., row) vectors of length n (resp., m) and the *Frobenius norm* of a complex matrix A is defined by $\|A\|_{\mathrm{Frob}} = \sqrt{\mathrm{trace}(A^\star A)}$, where $A^\star$ stands for the *adjoint matrix*. Using the fact that the transformation ρ is unitary and the Frobenius norm is invariant by unitary transformations, we then have:

$$\begin{aligned} & \min_{u \in \mathrm{CH}_{n,1},\, v_i \in \mathrm{CH}_{1,m}} \|\textstyle\sum_{i=1}^r D_i\, u\, v_i - M\|_{\mathrm{Frob}} \\ = \; & \min_{u_\rho \in \mathbb{R}^{n \times 1},\, v_{i\rho} \in \mathbb{R}^{1 \times m}} \|\textstyle\sum_{i=1}^r \rho(D_i)\, u_\rho\, v_{i\rho} - \rho(M)\|_{\mathrm{Frob}}\,. \end{aligned} \tag{65}$$

For more details, see [18]. Hence, the optimization problem eq. (64) is reduced to a real polynomial optimization problem eq. (65). This last problem will be studied in the future. In particular, the results developed in the paper on the solution space of the rank factorization problem eq. (4) should provide important information for investigating the corresponding polynomial optimization problem eq. (65).

Finally, we want to develop a noise sensitivity analysis of the method proposed here, i.e., analyze the continuity of our method to small variations of the matrix M that model measurement errors.

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6 Appendix: The RankFactorization package

The RankFactorization package is dedicated to the rank factorization problem eq. (4) and its applications to demodulation problems and vibration analysis. The main algorithmic aspects of the rank factorization problem eq. (4) developed in this paper are implemented in this package. In particular, the general solutions of the rank factorization problem can be computed following Algorithm 3. More commands concerning the applications to the demodulation problems and vibration analysis will soon be added. The package is written in `Maple` and is built upon the OreModules package [6]. Its binary is freely available at `https://who.rocq.inria.fr/Alban.Quadrat/RankFactorizationProblem.html`.

In the next table, we list the main functions of the RanKFactorization package.

`RankFactorization`(M, L, k)	Compute the outputs of Algorithm 3, where $M \in \mathbb{K}^{m\times n}$, L is a list of matrices $D_1, \ldots, D_r \in \mathbb{K}^{m\times m}$, and $k \in [\![0, \ldots, r-1]\!]$. Using the option "reduced" as the last argument of the function, a reduction of the sizes of the parameters q and t_i in Algorithm 3 is attempted but at the cost of calculation time.
`Solutions(M,L,k)`	Compute the solutions eq. (57) of the rank factorization problem eq. (4), where $M \in \mathbb{K}^{m\times n}$, L is a list of r matrices $D_1, \ldots, D_r \in \mathbb{K}^{m\times m}$, and $k \in [\![0, \ldots, r-1]\!]$. Using the option "reduced" as the last argument of the function, a reduction of the sizes of the parameters q and t_i in Algorithm 3 is attempted but at the cost of calculation time.
`IsSolution`	Check again that the outputs of `Solutions(M,L,k)` define solutions of the corresponding rank factorization problem eq. (4).

Table 1: Main functions of the RankFactorization package

In the next table, we list low-level functions of the RankFactorization package. In this table, we shall note $\mathcal{R} = \mathbb{Q}[x_1, \ldots, x_m]$.

`Factorization(M1,M2,V,`$\mathcal{R}$`)`	Left factorize $M_1 \in \mathcal{S}^{a\times b}$ by $M_2 \in \mathcal{S}^{c\times b}$, i.e., find (when possible) $F \in \mathcal{S}^{a\times c}$ such that $M_1 = F\, M_2$, where $\mathcal{S} = \mathcal{R}/\langle V_1, \ldots, V_s\rangle$ and $V_i \in \mathcal{R}$ is the i^{th} entry of the column matrix V.
`FittingIdeal(M,i,`$\mathcal{R}$`)`	Compute a set of generators for the i^{th} Fitting ideal $\text{Fitt}_i(\mathcal{M})$ of the $\mathcal{R}$-module $\mathcal{M} = \text{coker}_{\mathcal{R}}(.M)$ finitely presented by $M \in \mathcal{R}^{q\times p}$. With the option "reduced", it returns a Gröbner basis for this set for the `tdeg` monomial order.
`IsInvertible(P,V,`$\mathcal{R}$`)`	Check whether or not the residue class of P in the factor ring $\mathcal{R}/\langle V_1, \ldots, V_s\rangle$ is invertible, where $V_i \in \mathcal{R}$ is the i^{th} entry of the column matrix V.
`IsNilpotent(P,V,`$\mathcal{R}$`)`	Check whether or not the residue class of P in the factor ring $\mathcal{R}/\langle V_1, \ldots, V_s\rangle$ is nilpotent, where $V_i \in \mathcal{R}$ is the i^{th} entry of the column matrix V.
`Saturation(P,L,`$\mathcal{R}$`)`	Compute the saturation $\langle L_1, \ldots, L_r\rangle : \langle P\rangle^\infty$ of the ideal $\langle L_1, \ldots, L_r\rangle$ w.r.t. P, where L_i is the i^{th} entry of the list L and $P, L_1, \ldots, L_r \in \mathcal{R}$
`Simplification`$(M, V, \mathcal{R})$	Simplify the entries of $M \in \mathcal{R}^{q\times p}$ by computing their normal forms in the factor ring $\mathcal{R}/\langle V_1, \ldots, V_s\rangle$, where $V_i \in \mathcal{R}$ is the i^{th} entry of the column matrix V.
`Syzygies(M,V,`$\mathcal{R}$`)`	Compute $P \in \mathcal{S}^{r\times q}$ such that $\ker_{\mathcal{S}}(.M) = \text{im}_{\mathcal{S}}(.P)$, where $\mathcal{S} = \mathcal{R}/\langle V_1, \ldots, V_s\rangle$, $V_i \in \mathcal{R}$ is the i^{th} entry of the column matrix V, and $M \in \mathcal{R}^{q\times p}$. If $\mathcal{R} = \mathcal{T}[Y]$ and $J_s = Y\,P - 1$, where $P, V_1, \ldots, V_{s-1} \in \mathcal{T}$, then $\mathcal{S}$ is the localization $\mathcal{A}_P$ of $\mathcal{A} = \mathcal{T}/\langle V_1, \ldots, V_{s-1}\rangle$ at P.
`ReducedSyzygies(M,V,`$\mathcal{R}$`)`	Reduce the output of the `Syzygies` function, i.e., reduce the integer r by removing trivial syzygies among the syzygies (but at the cost of calculation time). This function is used by the `RankFactorization` and `Solutions` functions when the option "reduced" is added.

Table 2: Low-level functions of the RankFactorization package

Finally, Table 3 gives functions that are useful for studying the demodulation problems. More functions will be added in the future.

`AntiDiagonal`(n)	Compute the antidiagonal matrix of the size n
`LeeMatrix`(n)	Compute a Lee matrix of size n. If the option "unitary" is added, then a unitary Lee matrix is returned. If the option "unitary_symbolic" is added, then a symbolic unitary Lee matrix is returned which depends on a parameter q given as the third argument satisfying $q^2 = 2$
`IsCentroHermitian`(M)	Test whether or not a complex matrix M is centrohermitian
`CentroHermitian`(M)	Compute a centrohermitian matrix from M

Table 3: Functions of the RankFactorization package for the demodulation problem

Let us illustrate the functions of the RankFactorization package with explicit examples.

To use the RankFactorization package, the OreModules package has to be called. The Maple `LinearAlgebra` package can also be helpful to handle matrices.

```
> with(LinearAlgebra): with(OreModules): with(RankFactorization):
```

6.1 Low-level functions

Let us first demonstrate low-level functions of the RankFactorization package (see Table 2).

6.1.1 FittingIdeal

Let us first introduce the commutative polynomial ring $\mathcal{R} = \mathbb{Q}[x_1, x_2, x_3 x_4]$ in the OreModules package. To do that, the simplest way is to consider the Weyl algebra $A_4(\mathbb{Q})$ of partial differential operators in $x_i = \partial/\partial t_i$ for $i = 1, \ldots, 4$, and consider ideals and matrices defined by polynomials in the x_i's (which commutes with each other). In other words, we can do as follows:

```
> R := DefineOreAlgebra(seq(diff=[x[i],t[i]],i=1..4),polynom=[seq(t[i],i=1..4)]):
```

Let us now consider the following matrix

```
> M := Matrix([[a,b,c],[d,e,f],[g,h,j],[l,m,n]]);
```

$$M := \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & j \\ l & m & n \end{bmatrix}$$

whose entries are symbols but can also be elements of the ring R. Let $\mathcal{M} = \operatorname{coker}_{\mathcal{R}}(.M) = \mathcal{R}^{1\times 3}/(\mathcal{R}^{1\times 4}\, M)$ be the $\mathcal{R}$-module finitely presented by the matrix M. Let us now compute the different Fitting ideals of $\mathcal{M}$, i.e., $\mathrm{Fitt}_i(\mathcal{M})$ for $i \in \mathbb{Z}_{\geq 0}$. We have:

```
> FittingIdeal(M,0,R);
```

$$[a\,e\,j - a\,f\,h - b\,d\,j + b\,f\,g + c\,d\,h - c\,e\,g,\ a\,e\,n - a\,f\,m - b\,d\,n + b\,f\,l + c\,d\,m - c\,e\,l,\ a\,h\,n - a\,j\,m - b\,g\,n + b\,j\,l + c\,g\,m - c\,h\,l,\ d\,h\,n - d\,j\,m - e\,g\,n + e\,j\,l + f\,g\,m - f\,h\,l]$$

```
> FittingIdeal(M,1,R);
```

$$[a\,e - b\,d,\ a\,h - b\,g,\ a\,m - b\,l,\ a\,f - c\,d,\ a\,j - c\,g,\ a\,n - c\,l,\ b\,f - c\,e,\ b\,j - c\,h,\ b\,n - c\,m,\ d\,h - e\,g,\ d\,m - e\,l,\ d\,j - f\,g,\ d\,n - f\,l,\ e\,j - f\,h,\ e\,n - f\,m,\ g\,m - h\,l,\ g\,n - j\,l,\ h\,n - j\,m]$$

```
> FittingIdeal(M,2,R);
```

$$[a,\ b,\ c,\ d,\ e,\ f,\ g,\ h,\ j,\ l,\ m,\ n]$$

```
> FittingIdeal(M,3,R);
```

$$[1]$$

```
> FittingIdeal(M,4,R);
```

$$[1]$$

Let us now consider a more explicit matrix M with entries in $\mathcal{R}$.

```
> M := Matrix([[0,-x[3],0,-x[2]],[0,x[2],0,x[3]],[x[1]+x[4],0,x[1]+x[4],0]]);
```

$$M := \begin{bmatrix} 0 & -x_3 & 0 & -x_2 \\ 0 & x_2 & 0 & x_3 \\ x_1 + x_4 & 0 & x_1 + x_4 & 0 \end{bmatrix}$$

Let us consider the $\mathcal{R}$-module $\mathcal{M} = \operatorname{coker}_{\mathcal{R}}(.M) = \mathcal{R}^{1\times 4}/(\mathcal{R}^{1\times 3}\, M)$ and let us compute $\operatorname{Fitt}_i(\mathcal{M})$ for $i \in \mathbb{Z}_{\geq 0}$. We have:

```
> FittingIdeal(M,0,R);
```

$$[0]$$

```
> FittingIdeal(M,1,R);
```

$$[0, (x_1 + x_4)\left(x_2^2 - x_3^2\right), -x_2^2\, x_1 + x_3^2\, x_1 - x_2^2\, x_4 + x_3^2\, x_4]$$

Note that the elements of the above list are the different minors of all the 3×3 minors of M. These elements form a family of generators of the ideal $\operatorname{Fitt}_1(\mathcal{M})$. But we can also compute a Gröbner basis for the total degree of this set to obtain a more tractable family of generators of $\operatorname{Fitt}_1(\mathcal{M})$. This can be done by adding the option ``‘‘reduced’’`` in the `FittingIdeal` function as follows:

```
> FittingIdeal(M,1,R,"reduced");
```

$$[x_2^2\, x_1 - x_3^2\, x_1 + x_2^2\, x_4 - x_3^2\, x_4]$$

Similarly, we have:

```
> FittingIdeal(M,2,R);
```

$$[0, x_2\,(x_1 + x_4)\,, x_3\,\,(x_1 + x_4)\,, -x_2\,(x_1 + x_4)\,, -x_3\,\,(x_1 + x_4)\,, x_2^2 - x_3^2]$$

```
> FittingIdeal(M,2,R,"reduced");
```

$$[x_3 x_1 + x_3 x_4, x_2^2 - x_3^2, x_2 x_1 + x_2 x_4]$$

```
> FittingIdeal(M,3,R);
```

$$[0, x_2, x_3, -x_2, -x_3, x_1 + x_4]$$

```
> FittingIdeal(M,3,R,"reduced");
```

$$[x_3, x_2, x_1 + x_4]$$

```
> FittingIdeal(M,4,R);
```

$$[1]$$

6.1.2 `Saturation`

The `Saturation` function computes the saturation $\mathcal{I} : \langle P \rangle^{\infty}$ of the ideal $\mathcal{I}$ defined by the elements of the list given in the second argument of the `Saturation` function by a polynomial P given in the first argument (the last one being the ring $\mathcal{R}$). Let us illustrate this function with simple examples.

```
> Saturation(x[1],[x[1]^2],R);
```

$$[1]$$

Therefore, we have $\langle x_1^2 \rangle : \langle x_1 \rangle^{\infty} = \{r \in \mathcal{R} \mid \; \exists\, k \in \mathbb{Z}_{\geq 0},\; r\, x_1^k \in \langle x_1^2 \rangle\} = \langle 1 \rangle = \mathcal{R}$.

```
> Saturation(x[1]*x[2],[x[1]*x[2]*x[3]],R);
```

$$[x_3]$$

Therefore, we have $\langle x_1\, x_2\, x_3 \rangle : \langle x_1\, x_2 \rangle^{\infty} = \langle x_2 \rangle$.

```
> Saturation(x[1],[x[2]^2,x[1]*x[3]-x[2]^2],R);
```

$$[x_3, x_2^2]$$

Therefore, we have $\langle x_2^2,\, x_1\, x_3 - x_2^2\rangle : \langle x_1\, x_2\rangle^\infty = \langle x_3,\, x_2^2\rangle$.

```
>  Saturation(x[3],[x[1]^5*x[3]^3,x[1]*x[2]*x[3],x[2]*x[3]^4],R);
```

$$[x_2, x_1^5]$$

Therefore, we have $\langle x_1^5\, x_3^3,\, x_1\, x_2\, x_3,\, x_2\, x_3^4\rangle : \langle x_3\rangle^\infty = \langle x_2,\, x_1^5\rangle$.

6.1.3 IsNilpotent & IsInvertible

Let us first consider the IsNilpotent function which checks whether or not the residue class of an element $r \in \mathcal{R}$ − given as the first argument of the function − in the factor ring $\mathcal{S} = \mathcal{R}/\mathcal{I}$ is nilpotent, where $\mathcal{I}$ is the ideal generated by the entries of the column matrix given in the second argument of the function (the last one being the ring $\mathcal{R}$). Equivalently, this function tests whether or not the ring $\mathcal{S}_r$ − defined as the localization of the ring $\mathcal{S}$ at the multiplicatively closed set $\{r^k\}_{k\in\mathbb{Z}}$ − is trivial, i.e., $\mathcal{S}_r = 0$.

Let us check again that the residue class of x_1 in $\mathcal{S} = \mathbb{Q}[x_1, \ldots, x_4]/\langle x_1^2\rangle$ is nilpotent

```
>  IsNilpotent(x[1],Matrix([[x[1]^2]]),R);
```

true

whereas the residue class of $x_1 + 1$ in $\mathcal{S}$ is not:

```
>  IsNilpotent(x[1]+1,Matrix([[x[1]^2]]),R);
```

false

Let us give a few more simple examples by now considering the factor ring $\mathcal{S} = \mathcal{R}/\langle x_1^3,\, x_1^2\, x_2,\, x_1\, x_2^2,\, x_2^3\rangle$:

```
>  IsNilpotent(x[1],Matrix([[x[1]^3],[x[1]^2*x[2]],[x[1]*x[2]^2],[x[2]^3]]),R);
```

true

Thus, the residue class of x_1 in $\mathcal{S}$ is nilpotent.

```
>  IsNilpotent(x[1]*x[2]+x[1],Matrix([[x[1]^3],[x[1]^2*x[2]],[x[1]*x[2]^2],[x[2]^3]]),R);
```

true

Thus, the residue class of $x_1\, x_2 + x_1$ in $\mathcal{S}$ is nilpotent. We can check again that $(x_1^2\, x_2 + x_1)^3$ is a polynomial combination of the generators of $\mathcal{I}$:

```
>  Factorize(Matrix([[(x[1]*x[2]+x[1])^3]]),Matrix([[x[1]^3],[x[1]^2*x[2]],[x[1]*x[2]^2],
>  [x[2]^3]]),R);
```

$$\begin{bmatrix} 1 & 3\, x_1 & 3\, x_1^2 & x_1^3 \end{bmatrix}$$

i.e., we have $(x_1^2\, x_2 + x_1)^3 = x_1^3 + 3\, x_1\, (x_1^2\, x_2) + 3\, x_1^2\, (x_1\, x_2^2) + x_1^3\, (x_2^3)$.

```
>  IsNilpotent(0,Matrix([[x[1]^3],[x[1]^2*x[2]],[x[1]*x[2]^2],[x[2]^3]]),R);
```

true

Let us now consider the IsInvertible function which checks whether or not the residue class of an element $r \in \mathcal{R}$ − given as the first argument of the function − in the factor ring $\mathcal{S} = \mathcal{R}/\mathcal{I}$ is invertible, where $\mathcal{I}$ is the ideal generated by the entries of the column matrix given in the second argument of the function (the last one being the ring $\mathcal{R}$). For instance, let us test whether or not the residue class of x_1 in $\mathcal{S} = \mathbb{Q}[x_1]/\langle x_1^2\rangle$ is invertible:

```
>  IsInvertible(x[1],Matrix([[x[1]^2]]),R);
```

false

Therefore, the residue class of x_1 in $\mathcal{S}$ is not invertible (since it is nilpotent).

Similarly, let us test the invertibility of the residue class of $x_1 + 1$ in $\mathcal{S}$.

```
>  IsInvertible(x[1]+1,Matrix([[x[1]^2]]),R);
```

$$true$$

We obtain that it is invertible. We can check again this result by noticing that $(-x_1+1)\,(x_1+1) = 1-x_1^2$ shows that the inverse of the residue class of x_1+1 in $\mathcal{S}$ is the residue class of $-x_1+1$.

Let us now check whether or not x_1+1 is invertible in the ring $\mathcal{S} = \mathcal{R}/\langle x_1^3,\, x_1^2\,x_2,\, x_1\,x_2^2,\, x_2^3\rangle$:

```
> IsInvertible(x[1]+1,Matrix([[x[1]^3],[x[1]^2*x[2]],[x[1]*x[2]^2],[x[2]^3]]),R);
```

$$true$$

To check this last point, we can note that the identity $(x_1^2 - x_1 + 1)\,(x_1+1) = x_1^3 + 1$ shows that the residue class of $x_1^2 - x_1 + 1$ is the inverse of the residue class of $x_1 + 1$ in $\mathcal{S}$. This last identity can be obtained using the `LeftInverse` function of the OreModules package as follows:

```
> LeftInverse(Matrix([[x[1]+1],[x[1]^3],[x[1]^2*x[2]],[x[1]*x[2]^2],[x[2]^3]]),R);
```

$$\begin{bmatrix} x_1^2 - x_1 + 1 & -1 & 0 & 0 & 0 \end{bmatrix}$$

6.1.4 `Syzygies`, `ReducedSygyzies`, `Factorization` & `Simplification`

The `Syzygies` function computes the left kernel $\ker_{\mathcal{S}}(.M)$ of a matrix M, given as the first input of the function, whose entries belong to the ring $\mathcal{S} = \mathcal{R}/\mathcal{J}$, where $\mathcal{R}$ is a commutative polynomial ring defined in the third argument and $\mathcal{J}$ is the ideal generated by the entries of the column matrix V given in the second argument of the function.

The `ReducedSygyzies` function tries to reduce the number of generators of the left kernel $\ker_{\mathcal{S}}(.M)$ of a matrix $M \in \mathcal{S}^{q\times p}$, where $\mathcal{S} = \mathcal{R}/\mathcal{J}$, $\mathcal{R}$ is a commutative polynomial ring defined in the third argument and $\mathcal{J}$ is the ideal generated by the entries of the column matrix V given in the second argument of the function.

Note that if $\mathcal{R} = \mathcal{T}[Y]$, where $\mathcal{T}$ is a commutative polynomial ring, $V = (V_1 \ \ldots \ V_{r-1} \quad V_r)^T$, where $V_r = Y\,P - 1$ and $P, V_1, \ldots, V_{r-1} \in \mathcal{T}$, then $\mathcal{S}$ corresponds to the localization $\mathcal{A}_P$ of $\mathcal{A} = \mathcal{T}/\langle V_1, \ldots, V_{r-1}\rangle$ at P. Note that the index of $Y\,P - 1$ does not need to be the last one for the functions (it can be any).

The `Simplification` function computes the normal form of all the entries of a matrix M $-$ given in the first argument of the function $-$ in the ring $\mathcal{S} = \mathcal{R}/\mathcal{J} = \mathcal{R}/\langle V_1, \ldots, V_r\rangle$, where $\mathcal{R}$ is a commutative polynomial ring defined in the third argument and $\mathcal{J}$ is the ideal generated by the entries of the column matrix V given in the second argument of the function.

Let us first illustrate these functions with a simple example.

```
> M := Matrix([[x[1]+1,0],[0,x[1]-1]]);
```

$$\begin{bmatrix} x_1+1 & 0 \\ 0 & x_1-1 \end{bmatrix}$$

Let us first compute the left kernel of M when V is the empty list, i.e., when $\mathcal{S} = \mathcal{R}$:

```
> Syzygies(M,[],R);
```

$$INJ(2)$$

Thus, $\ker_{\mathcal{R}}(.M) = 0$, i.e., the rows of M are $\mathcal{R}$-linearly independent or equivalently M has full row rank. Since $\mathcal{J} = \langle 0\rangle$, the same result can be obtained by considering the matrix 0 in the second entry:

```
> Syzygies(M,Matrix([[0]]),R);
```

$$INJ(2)$$

Let us now consider $V = [x_1^2 - 1]$, which is the determinant of M. Then, we have $\mathcal{S} = \mathcal{R}/\langle x_1^2 - 1\rangle$, and $\ker_{\mathcal{S}}(.M)$ is defined by:

```
> K := Syzygies(M,Matrix([[x[1]^2-1]]),R);
```

$$K := \begin{bmatrix} x_1-1 & 0 \\ 0 & x_1+1 \end{bmatrix}$$

Equivalently, we have $\ker_{\mathcal{S}}(.M) = \mathrm{im}_{\mathcal{S}}(.K)$, i.e., the $\mathcal{S}$-module $\ker_{\mathcal{S}}(.M)$ can be generated by the two rows of K. In particular, the residue classes of the entries of the matrix $K\,M$ in $\mathcal{S}$ is 0. But if we use the standard product of the matrices K and M in $\mathcal{R}$, we obtain

```
> P := Mult(K,M,R);
```

$$P := \begin{bmatrix} x_1^2-1 & 0 \\ 0 & x_1^2-1 \end{bmatrix}$$

which is not the zero matrix in $\mathcal{R}$. To compute the product $K\,M$ in $\mathcal{S}$, we have to use the `Simplification` function which computes the residue classes of the entries of the matrix $K\,M$ in $\mathcal{S}$:

```
> Simplification(P,Matrix([[x[1]^2-1]]),R);
```

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Let us now check whether or not the rows of the matrix K are $\mathcal{S}$-linearly independent, i.e., whether or not K has full row rank or equivalently whether or not $\ker_{\mathcal{S}}(.M)$ is a free $\mathcal{S}$-module.

```
> Syzygies(K,Matrix([[x[1]^2-1]]),R);
```

$$\begin{bmatrix} x_1+1 & 0 \\ 0 & x_1-1 \end{bmatrix}$$

We obtain $\ker_{\mathcal{S}}(.K) = \mathrm{im}_{\mathcal{S}}(.M)$, which shows that the $\mathcal{S}$-module $\mathcal{M} = \mathrm{coker}_{\mathcal{S}}(.M) = \mathcal{S}^{1\times 2}/(\mathcal{S}^{1\times 2}\,M)$ has the following cyclic free resolution:

$$\ldots \xrightarrow{.M} \mathcal{S}^{1\times 2} \xrightarrow{.K} \mathcal{S}^{1\times 2} \xrightarrow{.M} \mathcal{S}^{1\times 2} \xrightarrow{.K} \mathcal{S}^{1\times 2} \xrightarrow{.M} \mathcal{S}^{1\times 2} \xrightarrow{\pi} \mathcal{M} \longrightarrow 0.$$

Let us consider the localization $\mathcal{S}_{x_1+1}$ of $\mathcal{S}$ at the multiplicatively closed set $\{(x_1+1)^k\}_{k\in\mathbb{Z}}$ and let us compute the left kernel $\ker_{\mathcal{S}_{x_1+1}}(.K)$ of the matrix K over $\mathcal{S}_{x_1+1}$:

```
> L := Syzygies(K,Matrix([[x[1]^2-1],[x[2]*(x[1]+1)-1]]),R);
```

$$L := \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}$$

We obtain that $\ker_{\mathcal{S}_{x_1+1}}(.K) = \mathrm{im}_{\mathcal{S}_{x_1+1}}(.L)$. We note that the second row of L is twice of the first one, i.e., $\ker_{\mathcal{S}_{x_1+1}}(.K) = \mathcal{S}_{x_1+1}\,(1 \quad 0)$, which shows that $\ker_{\mathcal{S}_{x_1+1}}(.K)$ is a free $\mathcal{S}_{x_1+1}$-module.

The trivial linear dependence of the rows of the output of the `Syzygies` function can be removed using the `ReducedSyzygies` function (see more below):

```
> ReducedSyzygies(K,Matrix([[x[1]^2-1],[x[2]*(x[1]+1)-1]]),R);
```

$$\begin{bmatrix} 2 & 0 \end{bmatrix}$$

Now, using the fact that $\ker_{\mathcal{S}}(.M) = \mathrm{im}_{\mathcal{S}}(.K)$ (see the above free resolution of $\mathcal{M}$) and the fact that $\mathcal{S}_{x_1+1}$ is a flat $\mathcal{S}$-module, we have $\ker_{\mathcal{S}_{x_1+1}}(.M) = \mathrm{im}_{\mathcal{S}_{x_1+1}}(.K) = \mathcal{S}_{x_1+1}\,(1 \quad 0) \cong \mathcal{S}_{x_1+1}$, i.e., $\ker_{\mathcal{S}_{x_1+1}}(.M)$ is a free $\mathcal{S}_{x_1+1}$-module. This result is coherent with the fact that over the ring $\mathcal{S}_{x_1+1}$, M is reduced to

```
> N := Simplification(M, Matrix([[x[1]^2-1],[x[2]*(x[1]+1)-1]]),R);
```

$$N := \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$$

and thus, $\mathcal{S}_{x_1+1} \otimes_{\mathcal{S}} \mathcal{M} = \mathrm{coker}_{\mathcal{S}_{x_1+1}}(.M) = \mathrm{coker}_{\mathcal{S}_{x_1+1}}(.N) \cong \mathcal{S}_{x_1+1}$ is a free $\mathcal{S}_{x_1+1}$-module. For more details, see the proof of Theorem 4. Finally, note that a similar comment holds if we consider the localization $\mathcal{S}_{x_1-1}$ of the ring $\mathcal{S}$ at the multiplicatively closed set $\{(x_1-1)^k\}_{k\in\mathbb{Z}}$.

Let us now consider Example 6, i.e., the following matrix:

```
> Q := Matrix([[x[1],x[2],0],[x[2],x[1],0]]);
```

$$Q := \begin{bmatrix} x_1 & x_2 & 0 \\ x_2 & x_1 & 0 \end{bmatrix}$$

Let us compute the Fitting ideals $\mathrm{Fitt}_i(\mathcal{Q})$'s of the $\mathcal{R}$-module $\mathcal{Q} = \mathrm{coker}_{\mathcal{R}}(.Q) = \mathcal{R}^{1\times 3}/(\mathcal{R}^{1\times 2}\, Q)$:

```
> J0 := FittingIdeal(Q,0,R,"reduced");
```

$$[0]$$

```
> J1 := FittingIdeal(Q,1,R,"reduced");
```

$$[x_1^2 - x_2^2]$$

```
> J2 := FittingIdeal(Q,2,R,"reduced");
```

$$[x_2, x_1]$$

```
> J3 := FittingIdeal(Q,3,R,"reduced");
```

$$[1]$$

Let us now consider the rings $\mathcal{S}_k = \mathcal{R}/\mathrm{Fitt}_k(\mathcal{Q})$ for $k = 0$, 1, 2, and let us compute $\ker_{\mathcal{S}_k}(Q.)$. Since the `Syzygies` function only computes left kernels, we shall compute $\ker_{\mathcal{S}_k}\left(.Q^T\right)$, where Q^T is given by

```
> Q_t := Transpose(Q);
```

$$Q_t := \begin{bmatrix} x_1 & x_2 \\ x_2 & x_1 \\ 0 & 0 \end{bmatrix}$$

and finally transpose the obtained matrix. Hence, we first have

```
> K0 := Transpose(Syzygies(Q_t,Transpose(convert(J0,Matrix)),R));
```

$$K0 := \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

which shows that $\ker_{\mathcal{S}_0}(Q.) = \mathrm{im}_{\mathcal{S}_0}(K_0.)$. We have

```
> K1 := Transpose(Syzygies(Q_t,Transpose(convert(J1,Matrix)),R));
```

$$K1 := \begin{bmatrix} -x_2 & x_1 & 0 \\ x_1 & -x_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

which shows that $\ker_{\mathcal{S}_1}(Q.) = \mathrm{im}_{\mathcal{S}_1}(K_1.)$. In particular, we can check again that all the entries of $Q\,K_1$ are reduced to zero in $\mathcal{S}_1$:

```
> Simplification(Mult(Q,K1,R),Transpose(convert(J1,Matrix)),R);
```

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

We finally have

```
> K2 := Transpose(Syzygies(Q_t,Transpose(convert(J2,Matrix)),R));
```

$$K2 := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

which shows that $\ker_{\mathcal{S}_2}(Q.) = \mathrm{im}_{\mathcal{S}_2}(K_2.) = \mathcal{S}_2^{3\times 1}$.

Let us now consider Example 7, i.e., let us repeat the same computations with the following matrix:

```
> Q := Matrix([[x[1],x[2],2*x[1]+x[2]],[x[2],x[1],x[1]+2*x[2]]]);
```

$$Q := \begin{bmatrix} x_1 & x_2 & 2\,x_1+x_2 \\ x_2 & x_1 & x_1+2\,x_2 \end{bmatrix}$$

We can check that that the Fitting ideals $\mathrm{Fitt}_i(\mathcal{Q})$'s of the $\mathcal{R}$-module $\mathcal{Q} = \mathrm{coker}_{\mathcal{R}}(.Q)$ are the same as the previous example (i.e., Example 6):

```
>  J0 := FittingIdeal(Q,0,R,"reduced");
```

$$[0]$$

```
>  J1 := FittingIdeal(Q,1,R,"reduced");
```

$$[x_1^2 - x_2^2]$$

```
>  J2 := FittingIdeal(Q,2,R,"reduced");
```

$$[x_2, x_1]$$

```
>  J3 := FittingIdeal(Q,3,R,"reduced");
```

$$[1]$$

Now, we have $\ker_{\mathcal{S}_0}(Q.) = \mathrm{im}_{\mathcal{S}_0}(K_0.)$, where K_0 is defined by:

```
>  K0 := Transpose(Syzygies(Transpose(Q),Transpose(convert(J0,Matrix)),R));
```

$$K0 := \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$$

We also have $\ker_{\mathcal{S}_1}(Q.) = \mathrm{im}_{\mathcal{S}_1}(K_1.)$, where K_1 is defined by:

```
>  K1 := Transpose(Syzygies(Transpose(Q),Transpose(convert(J1,Matrix)),R));
```

$$K1 := \begin{bmatrix} 2 & 0 & 0 \\ 1 & -3\,x_2 & 2\,x_1+x_2 \\ -1 & 2\,x_1-x_2 & -x_2 \end{bmatrix}$$

We then have $\ker_{\mathcal{S}_2}(Q.) = \mathrm{im}_{\mathcal{S}_2}(K_2.)$, where K_2 is defined by

```
>  K2 := Transpose(Syzygies(Transpose(Q),Transpose(convert(J2,Matrix)),R));
```

$$K2 := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

which shows that $\ker_{\mathcal{S}_2}(Q.) = \mathrm{im}_{\mathcal{S}_2}(K_2) = \mathcal{S}_2^{3\times 1}$.

Finally, let us illustrate the `ReducedSyzygies` and `Factorization` functions.

Let us first consider the matrix Q defined in Example 4, i.e.:

```
>  Q := Matrix(3, 2, [[-6*x[4], 10*x[1] - 9*x[2] - 10*x[3]], [3*x[1] + x[4], 0],
>  [-2*x[4], 2*x[2]]]);
```

$$Q := \begin{bmatrix} -6\,x_4 & 10\,x_1-9\,x_2-10\,x_3 \\ 3\,x_1+x_4 & 0 \\ -2\,x_4 & 2\,x_2 \end{bmatrix}$$

As in Example 5, we are interesting in computing $\ker_{\mathcal{S}}(Q.)$, i.e., $\ker_{\mathcal{S}}\left(.Q^T\right)$, where Q^T is defined by

```
>  Q_t := Transpose(Q);
```

$$Q_t := \begin{bmatrix} -6\,x_4 & 3\,x_1+x_4 & -2\,x_4 \\ 10\,x_1-9\,x_2-10\,x_3 & 0 & 2\,x_2 \end{bmatrix}$$

and the ring $\mathcal{S}$ is defined by $\mathcal{R}/\langle e_1, e_2, e_3\rangle$, where e_i is the i^{th} entry of the following matrix:

```
> e := Vector[column](4, [2*x[1]*x[4] - 3*x[2]*x[4] - 2*x[3]*x[4], 3*x[1]*x[2]
> + x[2]*x[4], 2*x[1]^2 - 2*x[1]*x[3] + x[2]*x[4], 9*x[2]^2*x[4] + 6*x[2]*x[3]*x[4]
> + 2*x[2]*x[4]^2]);
```

$$e := \left[\begin{array}{c} 2\,x_1\,x_4 - 3\,x_2\,x_4 - 2\,x_3\,x_4 \\ 3\,x_1\,x_2 + x_2\,x_4 \\ 2\,x_1^2 - 2\,x_1\,x_3 + x_2\,x_4 \\ 9\,x_2^2\,x_4 + 6\,x_3\,x_4\,x_2 + 2\,x_2\,x_4^2 \end{array}\right]$$

Using the `Syzygies` function, we obtain

```
> K_t := Syzygies(Q_t,e,R);
```

$$K_t := \left[\begin{array}{cc} x_2 & x_4 \\ 2\,x_1 - 2\,x_3 & 3\,x_4 \\ 0 & 3\,x_1 + x_4 \\ 0 & 9\,x_2\,x_4 + 6\,x_3\,x_4 + 2\,x_4^2 \end{array}\right]$$

i.e., $\ker_{\mathcal{S}}(.Q^T) = \operatorname{im}_{\mathcal{S}}(.K_t)$. Equivalently, we have $\ker_{\mathcal{S}}(Q.) = \operatorname{im}_{\mathcal{S}}\left(K_t^T.\right)$, where K_t^T is defined by:

```
> Transpose(K_t);
```

$$\left[\begin{array}{cccc} x_2 & 2\,x_1 - 2\,x_3 & 0 & 0 \\ x_4 & 3\,x_4 & 3\,x_1 + x_4 & 9\,x_2\,x_4 + 6\,x_3\,x_4 + 2\,x_4^2 \end{array}\right]$$

We note that the above matrix has a column more than the matrix K given in Example 5. It comes from the fact that the rows of K_t have trivial relations as it can be checked by computing $\ker_{\mathcal{S}}(.K_t)$:

```
> K_t2 := Syzygies(K_t,e,R);
```

$$K_{t2} := \left[\begin{array}{cccc} 3\,x_4 & -x_4 & 0 & 0 \\ 18\,x_3 & 9\,x_2 + 2\,x_4 & 0 & -3 \\ 18\,x_1 & 2\,x_4 & 0 & -3 \\ 0 & 6\,x_1 + 2\,x_4 & 0 & -3 \\ 0 & 0 & 2\,x_4 & -1 \\ 0 & 0 & x_2 & 0 \\ 0 & 0 & 2\,x_1 - 2\,x_3 & 0 \\ 0 & 0 & 0 & x_2 \\ 0 & 0 & 0 & x_1 - x_3 \end{array}\right]$$

Thus, we get $\ker_{\mathcal{S}}(.K_t) = \operatorname{im}_{\mathcal{S}}(.K_{t2})$, where some entries of K_{t2} are invertible in $\mathcal{S}$ as it can be checked:

```
> map(IsInvertible,K_t2,e,R);
```

$$\left[\begin{array}{cccc} \mathit{false} & \mathit{false} & \mathit{false} & \mathit{false} \\ \mathit{false} & \mathit{false} & \mathit{false} & \mathit{true} \\ \mathit{false} & \mathit{false} & \mathit{false} & \mathit{true} \\ \mathit{false} & \mathit{false} & \mathit{false} & \mathit{true} \\ \mathit{false} & \mathit{false} & \mathit{false} & \mathit{true} \\ \mathit{false} & \mathit{false} & \mathit{false} & \mathit{false} \\ \mathit{false} & \mathit{false} & \mathit{false} & \mathit{false} \\ \mathit{false} & \mathit{false} & \mathit{false} & \mathit{false} \\ \mathit{false} & \mathit{false} & \mathit{false} & \mathit{false} \end{array}\right]$$

Hence, certain syzygies defined by the rows of the matrix K can be removed. There are many strategies to remove these "trivial syzygies". The `ReducedSyzygies` function implements one method to do that. Applying thee `ReducedSyzygies` function to the matrix Q_t, we obtain a set of generators of $\ker_{\mathcal{S}}(.Q_t)$ with three generators, namely, the rows of the following matrix

```
> K_tbis := ReducedSyzygies(Q_t,e,R);
```

$$K_{tbis} := \begin{bmatrix} x_2 & x_4 \\ 2\,x_1 - 2\,x_3 & 3\,x_4 \\ 0 & 3\,x_1 + x_4 \end{bmatrix}$$

whereas the `Syzygies` function returned four generators, namely, the four rows of the matrix K_t. Therefore, we have $\ker_{\mathcal{S}}(.Q_t) = \mathrm{im}_{\mathcal{S}}(.K_{tbis})$, i.e., $\ker_{\mathcal{S}}(Q.) = \mathrm{im}_{\mathcal{S}}\left(K_{tbis}^T.\right)$, where K_{tbis}^T is defined by:

```
>  Transpose(K_tbis);
```

$$\begin{bmatrix} x_2 & 2\,x_1 - 2\,x_3 & 0 \\ x_4 & 3\,x_4 & 3\,x_1 + x_4 \end{bmatrix}$$

We find again the matrix K given in Example 5. Finally, since $\ker_{\mathcal{S}}(.Q_t) = \mathrm{im}_{\mathcal{S}}(.K_t) = \mathrm{im}_{\mathcal{S}}(.K_{tbis})$, the rows of K_t (resp., K_{tbis}) belong to $\mathrm{im}_{\mathcal{S}}(.K_{tbis})$ (resp., $\mathrm{im}_{\mathcal{S}}(.K_t)$), which means that $K_t = F\,K_{tbis}$ and $K_{tbis} = G\,K_t$ for certain matrices F and G having entries in $\mathcal{S}$. These two matrices can be computed using the `Factorization` function as follows:

```
>  Factorization(K_t,K_tbis,e,R);
```

$$F := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 2\,x_4 \end{bmatrix}$$

```
>  G := Factorization(K_tbis,K_t,e,R);
```

$$G := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Finally, let us consider Example 12, where the following matrix B is considered

```
>  B := Matrix([[x[4]*(5*x[2]/2+x[3]), x[3]*(3*x[1]+x[4]), 3*x[4]*(5*x[2]/2+x[3])]]);
```

$$B := \begin{bmatrix} x_4 \left(\frac{5\,x_2}{2} + x_3\right) & x_3\,(3\,x_1 + x_4) & 3\,x_4 \left(\frac{5\,x_2}{2} + x_3\right) \end{bmatrix}$$

whose transpose matrix is defined by

```
>  B_t := Transpose(B);
```

$$B_t := \begin{bmatrix} x_4 \left(\frac{5\,x_2}{2} + x_3\right) \\ x_3\,(3\,x_1 + x_4) \\ 3\,x_4 \left(\frac{5\,x_2}{2} + x_3\right) \end{bmatrix}$$

Considering the ring $\mathcal{S} = \mathcal{R}/\langle e_1,\, e_2,\, e_3\rangle$, where the e_i's are defined by:

```
>  e1 := (10*x[1]-9*x[2]-10*x[3])*(3*x[1]+x[4]);
>  e2 := (2*x[1]-3*x[2]-2*x[3])*x[4];
>  e3 := (3*x[1]+x[4])*x[2];
```

$$(10\,x_1 - 9\,x_2 - 10\,x_3)\,(3\,x_1 + x_4)$$
$$(2\,x_1 - 3\,x_2 - 2\,x_3)\,x_4$$
$$(3\,x_1 + x_4)\,x_2$$

Let us compute the right kernel of B, i.e., the left kernel of B^T, over two localizations of the ring $\mathcal{S}$ at multiplicative set $\{h_i^k\}_{k\in\mathbb{Z}}$, where h_1 and h_2 are respectively defined by:

```
>  h1 := x[4]*(5*x[2]/2+x[3]);
>  h2 := (3*x[1]+x[4])*x[3];
```

$$x_4 \left(\frac{5\,x_2}{2} + x_3\right)$$
$$x_3\,(3\,x_1 + x_4)$$

To do that, we first introduce the polynomial ring $\mathcal{R}[y]$:

```
> R2 := DefineOreAlgebra(seq(diff=[x[i],t[i]],i=1..4),diff=[_y,s],polynom=
> [seq(t[i],i=1..4),s]):
```

We can now compute $\ker_{\mathcal{S}_{h_1}}(.B^T)$ as follows:

```
> C_1t := Syzygies(B_t,Matrix([[_y*h1-1],[e1],[e2],[e3]]),R2);
```

$$C_{1t} := \begin{bmatrix} 3 & 0 & -1 \\ 0 & -12\,x_4 & 18\,x_2 + 12\,x_3 + 4\,x_4 \\ 0 & x_2 & 0 \\ 0 & -6\,x_4 & 9\,x_2 + 6\,vx_3 + 2\,x_4 \\ 0 & 15\,x_2 & 0 \\ 0 & -30\,\_\,y x_4^2 & 12\,\_\,y x_3\,x_4 + 10\,\_\,y x_4^2 + 18 \\ 0 & 2\,\_\,y x_3\,x_4 - 2 & 0 \end{bmatrix}$$

Thus, we have $\ker_{\mathcal{S}_{h_1}}(.B^T) = \operatorname{im}_{\mathcal{S}_{h_1}}(.C_{1t})$, i.e., the set defined by the seven rows of the matrix C_{1t} generates $\ker_{\mathcal{S}_{h_1}}(.B^T)$. We can try to find a set of generators containing fewer elements by using the `ReducedSyzygies` function as follows:

```
> C_1tbis := ReducedSyzygies(B_t,Matrix([[_y*h1-1],[e1],[e2],[e3]]),R2);
```

$$C_{1tbis} := \begin{bmatrix} 3 & 0 & -1 \\ 0 & -30\,\_\,y x_4^2 & 12\,\_\,y x_3\,x_4 + 10\,\_\,y x_4^2 + 18 \end{bmatrix}$$

Thus, we have $\ker_{\mathcal{S}_{h_1}}(.B^T) = \operatorname{im}_{\mathcal{S}_{h_1}}(.C_{1tbis})$, which shows that $\ker_{\mathcal{S}_{h_1}}(.B^T)$ is generated by the two rows of the matrix C_{1tbis}. Finally, let us check again that $\operatorname{im}_{\mathcal{S}_{h_1}}(.C_{1t}) = \operatorname{im}_{\mathcal{S}_{h_1}}(.C_{1tbis})$ by verifying that the identities $C_{1t} = F\,C_{1tbis}$ and $C_{1tbis} = G\,C_{1t}$ hold for certain matrices F and G with entries in $\mathcal{S}_{h_1}$:

```
> F := Factorization(C_1t,C_1tbis,Matrix([[_y*h1-1],[e1],[e2],[e3]]),R2);
```

$$F := \begin{bmatrix} 1 & 0 \\ 0 & x_2 + \frac{2\,x_3}{5} \\ 0 & \frac{1}{27}\,\_\,y x_1\,x_2\,x_3 + \frac{2}{135}\,\_\,y x_1\,x_3^2 - \frac{1}{27}\,\_\,y x_2\,x_3^2 - \frac{2}{135}\,\_\,y x_3^3 + \frac{1}{54} x_2 \\ 0 & \frac{x_2}{2} + \frac{x_3}{5} \\ 0 & \frac{5}{9}\,\_\,y x_1\,x_2\,x_3 + \frac{2}{9}\,\_\,y x_1\,x_3^2 - \frac{5}{9}\,\_\,y x_2\,x_3^2 - \frac{2}{9}\,\_\,y x_3^3 + \frac{5}{18}\,x_2 \\ 0 & 1 \\ 0 & \frac{5}{18}\,\_\,y x_1\,x_2 + \frac{1}{9}\,\_\,y x_1\,x_3 - \frac{5}{18}\,\_\,y x_2\,x_3 - \frac{1}{9}\,\_\,y x_3^2 \end{bmatrix}$$

```
> Factorization(C_1tbis,C_1t,Matrix([[_y*h1-1],[e1],[e2],[e3]]),R2);
```

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

In Example 12, a different matrix C_{h_1} was given as a set of generators for $\ker_{\mathcal{S}_{h_1}}(.B^T)$, whose transpose matrix is defined by.

```
> C_h1t := SubMatrix(C_1t,[1,4],1..3);
```

$$C_{h1t} := \begin{bmatrix} 3 & 0 & -1 \\ 0 & -6\,x_4 & 9\,x_2 + 6\,x_3 + 2\,x_4 \end{bmatrix}$$

In other words, the matrix C_{h_1} given in Example 12 is defined by:

```
> Transpose(C_h1t);
```

$$\begin{bmatrix} 3 & 0 \\ 0 & -6\,x_4 \\ -1 & 9\,x_2+6\,x_3+2\,x_4 \end{bmatrix}$$

Let us check that $\mathrm{im}_{\mathcal{S}_{h_1}}(.C_{h_1}) = \mathrm{im}_{\mathcal{S}_{h_1}}(.C_{1t})$. To do that, using `Factorization` function, we can check that C_{1t} (resp., C_{h_1}) is a left factor of C_{h_1} (resp., C_{1t}):

```
> Factorization(C_h1t,C_1tbis,Matrix([[_y*h1-1],[e1],[e2],[e3]]),R2);
```

$$\begin{bmatrix} 1 & 0 \\ 0 & \frac{x_2}{2}+\frac{x_3}{5} \end{bmatrix}$$

```
> Factorization(C_1tbis,C_h1t,Matrix([[_y*h1-1],[e1],[e2],[e3]]),R2);
```

$$\begin{bmatrix} 1 & 0 \\ 0 & 5\,\_y\,x_4 \end{bmatrix}$$

Finally, let us compute $\ker_{\mathcal{S}_{h_2}}\left(.B^T\right)$. Using the `Syzygies` function, we obtain that the rows of the matrix

```
> C_2t := Syzygies(B_t,Matrix([[_y*h2-1],[e1],[e2],[e3]]),R2);
```

$$C_{2t} := \begin{bmatrix} 3 & 0 & -1 \\ 0 & -12\,x_4 & 12\,x_3+4\,x_4 \\ 0 & -6\,x_4 & 6\,x_3+2\,x_4 \\ 0 & 6\,\_y\,x_3\,x_4 & -2 \\ 0 & -30\,\_y\,x_3\,x_4 & 10 \\ 0 & 6\,x_4 & -6\,x_3-2\,x_4 \\ 0 & -6\,x_4 & 6\,x_3+2\,x_4 \end{bmatrix}$$

generate $\ker_{\mathcal{S}_{h_2}}\left(.B^T\right)$. Hence, the seven rows of the matrix C_{2t} generate $\ker_{\mathcal{S}_{h_2}}\left(.B^T\right)$. Let us search for a set of generators containing fewer elements by using `ReducedSyzygies` function:

```
> C_2tbis := ReducedSyzygies(B_t,Matrix([[_y*h2-1],[e1],[e2],[e3]]),R2);
```

$$C_{2tbis} := \begin{bmatrix} 3 & 0 & -1 \\ 0 & -6\,x_4 & 6\,x_3+2\,x_4 \end{bmatrix}$$

Thus, we have $\ker_{\mathcal{S}_{h_2}}\left(.B^T\right) = \mathrm{im}_{\mathcal{S}_{h_2}}(.C_{2tbis})$, which shows that $\ker_{\mathcal{S}_{h_2}}(.B^T)$ can be generated by the two rows of the matrix C_{2tbis}. Transposing this last matrix

```
> Transpose(C_2tbis);
```

$$\begin{bmatrix} 3 & 0 \\ 0 & -6\,x_4 \\ -1 & 6\,x_3+2\,x_4 \end{bmatrix}$$

we obtain $\ker_{\mathcal{S}_{h_2}}(B.) = \mathrm{im}_{\mathcal{S}_{h_2}}(C_{2tbis}^T.)$. Using the `Factorization` function, we can check again that C_{2tbis} (resp., C_{2t}) is a left factor of C_{2t} (resp., C_{2tbis}):

```
> Factorization(C_2t,C_2tbis,Matrix([[_y*h2-1],[e1],[e2],[e3]]),R2);
```

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 1 \\ 0 & -\_y\,x_1+\frac{3}{2}\,\_y\,x_2 \\ 0 & 5\,\_y\,x_1-\frac{15}{2}\,\_y\,x_2 \\ 0 & -1 \\ 0 & 1 \end{bmatrix}$$

```
> Factorization(C_2tbis,C_2t,Matrix([[_y*h2-1],[e1],[e2],[e3]]),R2);
```

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Therefore, we have $\mathrm{im}_{\mathcal{S}_{h_2}}(.C_{2tbis}) = \mathrm{im}_{\mathcal{S}_{h_2}}(.C_{2t})$. Finally, in Example 12, the matrix $C_{h_2}^T$, defined by

```
> C_h2t :=  Matrix([[Row(C_2t,1)],[Row(C_2t,3)/2]]);
```

$$C_{h2t} := \begin{bmatrix} 3 & 0 & -1 \\ 0 & -3\,x_4 & 3\,x_3 + x_4 \end{bmatrix}$$

is given. Up to a factor of 2 for the second row, we obtain again C_{2tbis}. Such an esthetical cleaning will be added to the ReducedSyzygies function in the future.

6.2 Main commands for solving the rank factorization problem

6.2.1 Description of the main functions of the RankFactorization package

Let us now illustrate the main functions of the RankFactorization package (see Table 1).

The RankFactorization(M, L, k) function computes the outputs of Algorithm 3, where $M \in \mathbb{K}^{m\times n}$ and the list L of matrices $D_1, \ldots, D_r \in \mathbb{K}^{m\times m}$ define the rank factorization problem eq. (4), and the index $k \in [\![0, \ldots, r-1]\!]$ fixes the "leaf of the solution space" we are considering in the sense that the solutions that are computed are defined over the ring $\mathcal{S}_k = \mathcal{R}/\mathcal{J}_k$, where $\mathcal{J}_k = \mathrm{Fitt}_k(\mathcal{Q})$.

The first output is a list of elements of $\mathcal{R}$ which generates the ideal $\mathcal{J}_k$, the second (resp., third) one is the matrix K (resp., Y), the fourth is a list $\{g_{k,i}\}_{i\in I_k}$, where $g_{k,i} \in \mathcal{R}$ is a preimage of $h_{k,i}$, where $\mathcal{I} = \langle h_{k,1}, \ldots, h_{k,\beta_k}\rangle_{\mathcal{S}_k}$ and $I_k \subseteq [\![1, \ldots, \beta_k]\!]$ is the set of the indices of the non-nilpotents elements $h_{k,i}$, the fifth (resp., the sixth) is a list of right inverses $\{E_{k,h_i}\}_{i\in I_k}$ (resp., of kernels $\{C_{k,h_i}\}_{i\in I_k}$) of the matrix B over the localization of the ring $\mathcal{S}_k$ at the multiplicatively closed set $\{h_{k,i}^k\}_{k\in\mathbb{Z}}$ for $i \in I_k$. The seventh is the polynomial ring $\mathcal{R}[y]$ (which allows one to work with the ring $\mathcal{R}_{g_{k,i}}$, and thus, with the ring $\mathcal{S}_{h_{k,i}}$; see the comment after Lemma 9), and the last one is the matrix $A = (D_1\,x \ \ldots \ D_r\,x) \in \mathcal{R}^{m\times r}$.

If the option "reduced" is used as the last argument of the RankFactorizationfunction, i.e., if RankFactorization$(M, L, k,$"reduced"$)$ is used, then a reduction of the parameters q_k and $t_{k,i}$ − respectively defining the matrices $K_k \in \mathcal{S}_k^{r\times q_k}$ and $C_{h_{k,i}} \in \mathbb{K}^{q\times t_{k,i}}$ (see the proof of Theorem 6 and the general expression eq. (59) of the solutions of the rank factorization problem) − is attempted by reducing trivial syzygies (usually at the cost of computational cost).

The Solutions function builds the explicit solutions eq. (59) of the rank factorization eq. (4) from the data obtained from the RankFactorization function (see Theorem 6). Its entries are the same as the RankFactorization function, namely, the matrix $M \in \mathbb{K}^{m\times n}$, a list of matrices $D_1, \ldots, D_r \in \mathbb{K}^{m\times m}$, and $k \in [\![0, \ldots, r-1]\!]$. The first output is a set of elements of $\mathcal{R}$ defining the ideal $\mathcal{J}_k$, the second one is $\{g_{k,i}\}_{i\in I_k}$, the third one is $A = (D_1\,x \ \ldots \ D_r\,x)$, the fourth is a list formed by the $v_{k,h_{k,i}}$'s for $i \in I_k$ defined by eq. (59), and the last one is the polynomial ring $\mathcal{R}[y]$.

As for RankFactorization, the option "reduced" can be used to reduce the sizes of the outputs of the v-components of the solutions (u, v) of the rank factorization problem eq. (4).

Finally, the Isolution function checks whether or not the outputs of the Solutions function define solutions of the corresponding rank factorization problem eq. (4) by substituting the expressions returned by Solutions into eq. (4) and checking whether or not the normal forms of the obtained expressions exactly reduce to 0 in the corresponding ring $\mathcal{S}_{k,h_i}$.

6.2.2 Computation of the solutions of the rank factorization for Example 1 & Example 13

Let us first enter the different matrices considered in Example 1 for the rank factorization problem eq. (4):

```
> M := Matrix([[1,0$2,1],[0$4],[0$4],[1,0$2,1]]);
```

$$M := \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

```
> D1 := Matrix([[1,0$3],[0$4],[0$4],[0$3,-1]]);
```

$$D1 := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

```
> D2 := Matrix([[0$4],[0,1,0$2],[0$2,-1,0],[0$4]]);
```

$$D2 := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

```
> D3 := Matrix([[0$3,1],[0$4],[0$4],[-1,0$3]]);
```

$$D3 := \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}$$

```
> D4 := Matrix([[0$4],[0$2,1,0],[0,-1,0$2],[0$4]]);
```

$$D4 := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We have $m = n = 4$ and $r = 4$. Moreover, we can easily check that $\operatorname{rank}_{\mathbb{Q}}(M) = 1$ or the standard `Rank` function of the `LinearAlgebra` package can be used.

`RankFactorization`$(M, [D1, D2, D3, D4], 0)$ & `Solutions`$(M, [D1, D2, D3, D4], 0)$ Let us now apply the `RankFactorization` function for the above matrices and $k = 0$. Since the outputs are too long to be shown in a single line, we display the data in separate lines.

```
> RF0 := RankFactorization(M,[D1,D2,D3,D4],0):
> nops(RF0);
```

$$8$$

The first output is a set of generators for the ideal $\mathcal{J}_0 = \operatorname{Fitt}_0(\mathcal{Q})$ of $\mathcal{R}$:

```
> RF0[1];
```

$$[0]$$

Thus, we have $\mathcal{J}_0 = \langle 0 \rangle$, and thus, $\mathcal{S}_0 = \mathcal{R}$. The second output is the matrix K_0 defined by:

```
> RF0[2];
```

$$\begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}$$

The third one is the matrix Y defined by:

```
> RF0[3];
```

$$\begin{bmatrix} 1 & 0 & 0 & 1 \end{bmatrix}$$

The fourth output is a list of the non-nilpotent elements of a set of generators $\{g_{0,i}\}_{i \in I_0}$ of $\mathcal{I}_0 = \operatorname{Fitt}_0(\mathcal{B}_0)$:

```
> RF0[4];
```

$$[x_1 - x_4]$$

We thus get $I_0 = \{1\}$ and $g_{0,1} = x_1 - x_4$. In particular, a unique solution of the rank factorization problem eq. (4) can be found over the localization of $\mathcal{S}_0 = \mathcal{R}$ with respect to the multiplicative set $\{g_{0,1}^k\}_{k\in\mathbb{Z}}$, i.e., $\mathcal{S}_{0,g_{0,1}} = \mathcal{S}_0[y]/\langle y\, g_{0,1} - 1\rangle = \mathcal{S}_0\left[g_{0,1}^{-1}\right]$.

In the RankFactorization package, we use the notation $\_y$ instead of y to protect this variable and to avoid any possible confusion with a variable y which could have been used in the `Maple` worksheet.

The next output is a right inverse of the matrix B_0 with entries in the ring $\mathcal{S}_{0,g_{0,1}}$:

```
> RF0[5];
```

$$\mathrm{table}\left([1 = [\ \_y\ ]]\right)$$

Thus, $\_y$ is the inverse of $B_0 = (x_1 - x_4)$.

The sixth output is a matrix defining a set of generators of $\ker_{\mathcal{S}_{0,g_{0,1}}}(B_0 .)$.

```
> RF0[6];
```

$$\mathrm{table}\left([1 = []]\right)$$

We thus have $\ker_{\mathcal{S}_{0,g_{0,1}}}(B_0 .) = 0$.

The next output is the polynomial ring $\mathcal{R}[y]$. It is internally displayed in OreModules as follows:

```
> RF0[7];
```

$$[\mathit{Ore\_algebra}, [\text{"diff"}, \text{"diff"}, \text{"diff"}, \text{"diff"}, \text{"diff"}], [t_1, t_2, t_3, t_4, \_t], [x_1, x_2, x_3, x_4, \_y], [t_1, t_2, t_3, t_4, \_t], [],$$
$$0, [], [], [t_1, t_2, t_3, t_4, \_t], [], [], [\mathit{diff} = [x_1, t_1], \mathit{diff} = [x_2, t_2], \mathit{diff} = [x_3, t_3], \mathit{diff} = [x_4, t_4], \mathit{diff} = [\_y, \_t]],$$
$$\Big[\_a \to \_a * x_1 - \left(\frac{\partial}{\partial t_1}\_a\right), \_a \to \_a * x_2 - \left(\frac{\partial}{\partial t_2}\_a\right), \_a \to \_a * x_3 - \left(\frac{\partial}{\partial t_3}\_a\right),$$
$$\_a \to \_a * x_4 - \left(\frac{\partial}{\partial t_4}\_a\right), \_a \to \_a * \_y - \left(\frac{\partial}{\partial \_t}\_a\right)\Big]]$$

Finally, the last output is the matrix $A = (D_1\, x\ \ldots\ D_4\, x)$, where $x = (x_1\ \ldots\ x_4)^T$, then defined by:

```
> RF0[8];
```

$$\begin{bmatrix} x_1 & 0 & x_4 & 0 \\ 0 & x_2 & 0 & x_3 \\ 0 & -x_3 & 0 & -x_2 \\ -x_4 & 0 & -x_1 & 0 \end{bmatrix}$$

From the above data of the `RankFactorization`, we can then form the explicit solutions eq. (59). The `Solutions` function first computes `RankFactorization` and then builds the corresponding solutions. Again, the outputs of `Solutions` are too long to be displayed in a single line. Hence, we show the data in separate lines.

```
> Sol0 := Solutions(M,[D1,D2,D3,D4],0):
> nops(Sol0);
```

$$5$$

The first output is a list of generators of the ideal $\mathcal{J}_0$.

```
> Sol0[1];
```

$$[0]$$

As above, we have $\mathcal{J}_0 = \langle 0\rangle$, and thus, $\mathcal{S} = \mathcal{R}$ and $\mathcal{V}(\mathcal{J}_0) = \mathbb{K}^{4\times 1}$.

The second output

```
> Sol0[2];
```

$$[x_1 - x_4]$$

shows that there is one solution defined over $\mathcal{R}_{g_{0,1}} = \mathcal{R}[y]/\langle y\, g_{0,1} - 1\rangle = \mathcal{R}\left[g_{0,1}^{-1}\right]$, where $g_{0,1} = x_1 - x_4$, i.e., in $\mathcal{V}(\mathcal{J}_0) \setminus \mathcal{V}(\langle g_0\rangle) = \mathbb{K}^{4\times 1} \setminus \{(u_1\quad u_2\quad u_3\quad u_1)^T \mid u_1, u_2, u_3 \in \mathbb{K}\} = \left\{(u_1\ \ldots\ u_4)^T \in \mathbb{K}^{4\times 1} \mid u_1 \neq u_4\right\}$.

The next output is the matrix $A = (D_1\, x\ \ldots\ D_4\, x)$, where $x = (x_1\ \ldots\ x_4)^T$:

```
>  Sol0[3];
```

$$\begin{bmatrix} x_1 & 0 & x_4 & 0 \\ 0 & x_2 & 0 & x_3 \\ 0 & -x_3 & 0 & -x_2 \\ -x_4 & 0 & -x_1 & 0 \end{bmatrix}$$

The fourth output gives the v-component of the solution $(u,\, v)$ of the rank factorization problem:

```
>  Sol0[4];
```

$$\mathrm{table}\left(\left[1 = \begin{bmatrix} \_y & 0 & 0 & \_y \\ 0 & 0 & 0 & 0 \\ -\_y & 0 & 0 & -\_y \\ 0 & 0 & 0 & 0 \end{bmatrix}\right]\right)$$

Finally, the last output is the ring $\mathcal{R}[y]$ which is used to check again that the above expressions for u and v define solutions to the rank factorization problem using the `IsSolution` function.

```
>  Sol0[5];
```

$$\mathrm{table}([1 = [\ \mathit{Ore\_algebra}, [\text{“diff”}, \text{“diff”}, \text{“diff”}, \text{“diff”}, \text{“diff”}], [t_1, t_2, t_3, t_4, \_t], [x_1, x_2, x_3, x_4, \_y], [t_1, t_2, t_3, t_4, \_t], [y_{1,1}, y_{1,2}, y_{1,3}, y_{1,4}, y_{2,1}, y_{2,2}, y_{2,3}, y_{2,4}, y_{3,1}, y_{3,2}, y_{3,3}, y_{3,4}], 0, [], [], [t_1, t_2, t_3, t_4, \_t], [], [], [\mathit{diff} = [x_1, t_1], \mathit{diff} = [x_2, t_2], \mathit{diff} = [x_3, t_3], \mathit{diff} = [x_4, t_4], \mathit{diff} = [\_y, \_t]],$$

$$\left[\_a \to \_a * x_1 - \left(\frac{\partial}{\partial t_1}\_a\right), \_a \to \_a * x_2 - \left(\frac{\partial}{\partial t_2}\_a\right), \_a \to \_a * x_3 - \left(\frac{\partial}{\partial t_3}\_a\right), \_a \to \_a * x_4 - \left(\frac{\partial}{\partial t_4}\_a\right), \_a \to \_a * \_y - \left(\frac{\partial}{\partial \_t}\_a\right)\right]\ ]])$$

We find again the solution $(u,\, v)$, where $u \in \left\{(u_1\ \ldots\ u_4)^T \in \mathbb{K}^{4\times 1} \mid u_1 \neq u_4\right\}$ and $v_{h_{0,1}}$ defined by the above matrix, where $\_y = (x_1 - x_4)^{-1}$, obtained in Example 13. Finally, using the `IsSolution` function, we can check again that $(u,\, v)$ defines a solution of the corresponding rank factorization problem eq. (4).

```
>  IsSolution(Sol0);
```

$$\mathrm{table}([1 = [\mathit{true}]])$$

If the option “reduced” is added to the `RankFactorization` or the `Solutions` functions, then we obtain the same solution.

As explained in Theorem 6, `Sol0` is the component of the solution space of the corresponding rank factorization problem corresponding the affine algebraic set $\mathcal{V}(\mathcal{J}_0)$, i.e., the 0^{th} leaf of the solution space. We can also get other solutions by considering $\mathcal{J}_k$ for $k = 1, 2, r-1 = 3$, and their corresponding affine algebraic sets $\mathcal{V}(\mathcal{J}_k)$, i.e., the other k^{th} leaves of the solution space for $k = 1, 2, 3$.

`RankFactorization`$(M, [D1, D2, D3, D4], 1)$ & `Solutions`$(M, [D1, D2, D3, D4], 1)$ Let us briefly display the different outputs of the `RankFactorization` for $k = 1$.

```
>  RF1 := RankFactorization(M,[D1,D2,D3,D4],1):
```

The first output is a set of generators for the ideal $\mathcal{J}_1 = \mathrm{Fitt}_1(\mathcal{Q})$ of $\mathcal{R}$:

```
>  RF1[1];
```

$$[x_2^2\, x_1 - x_1\, x_3^2 + x_2^2\, x_4 - x_3^2\, x_4]$$

Thus, we have $\mathcal{J}_1 = \langle (x_2 - x_3)\,(x_2 + x_3)\,(x_1 + x_4)\rangle$ and $\mathcal{S}_1 = \mathcal{R}/\mathcal{J}_1$.

The second output is the matrix K_1 defined by:

```
>  RF1[2];
```

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -x_3\,x_1 - x_3\,x_4 & x_2\,x_1 + x_2\,x_4 & 0 \\ -1 & 0 & 0 & x_2^2 - x_3^2 \\ 0 & x_2\,x_1 + x_2\,x_4 & -x_3\,x_1 - x_3\,x_4 & 0 \end{bmatrix}$$

The third one is the matrix Y defined by:

```
> RF1[3];
```

$$\begin{bmatrix} 1 & 0 & 0 & 1 \end{bmatrix}$$

The fourth output is a list of the non-nilpotent elements of a set of generators $\{g_{1,i}\}_{i\in I_1}$ of $\mathcal{I}_1 = \mathrm{Fitt}_0(\mathcal{B}_1)$:

```
> RF1[4];
```

$$[x_1 - x_4,\, x_2^2\,x_4 - x_3^2\,x_4]$$

We thus have $I_1 = \{1, 2\}$, $g_{1,1} = x_1 - x_4$, and $g_{1,2} = x_4\,(x_2 - x_3)\,(x_2 + x_3)$. If we denote by $h_{1,i}$ the residue class of $g_{1,i}$ in $\mathcal{S}_1$ for $i = 1, 2$, then two solutions exist respectively over the localization $\mathcal{S}_{1,h_{1,i}} = \mathcal{S}_1[y]/\langle y\,h_{1,i} - 1\rangle = \mathcal{S}_1\left[h_{1,i}^{-1}\right]$ for $i = 1, 2$.

The next output is a right inverse of the matrix B_1 over respectively the ring $\mathcal{S}_{1,g_{1,i}}$ for $i = 1, 2$:

```
> RF1[5];
```

$$\mathrm{table}\left(\left[1 = \begin{bmatrix} \_y \\ 0 \\ 0 \\ 0 \end{bmatrix}, 2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \_y \end{bmatrix}\right]\right)$$

The sixth output is a table containing the right kernel of the matrix B_1 over respectively the ring $\mathcal{S}_{1,h_{1,i}}$, i.e., $\ker_{\mathcal{S}_{1,h_{1,i}}}(B_1.)$ for $i = 1, 2$.

```
> RF1[6];
```

$$\mathrm{table}\left(\left[1 = \begin{bmatrix} x_2^2 - x_3^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & x_1 + x_4 & 2\_y x_4 + 1 & -2\_y x_4 - 1 \end{bmatrix}, 2 = \begin{bmatrix} 1 & x_2^2 - x_3^2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 2\_y x_4 & 2 & 0 & 0 \end{bmatrix}\right]\right)$$

Similarly, the next output is the polynomial ring $\mathcal{R}[y]$.

```
> RF1[7];
```

$$[\mathit{Ore\_algebra}, [\text{"diff"}, \text{"diff"}, \text{"diff"}, \text{"diff"}, \text{"diff"}], [t_1, t_2, t_3, t_4, \_t], [x_1, x_2, x_3, x_4, \_y], [t_1, t_2, t_3, t_4, \_t], [],$$
$$0, [], [], [t_1, t_2, t_3, t_4, \_t], [], [], [\mathit{diff} = [x_1, t_1], \mathit{diff} = [x_2, t_2], \mathit{diff} = [x_3, t_3], \mathit{diff} = [x_4, t_4], \mathit{diff} = [\_y, \_t]],$$
$$\Big[\_a \to \_a * x_1 - \left(\frac{\partial}{\partial t_1}\_a\right), \_a \to \_a * x_2 - \left(\frac{\partial}{\partial t_2}\_a\right), \_a \to \_a * x_3 - \left(\frac{\partial}{\partial t_3}\_a\right),$$
$$\_a \to \_a * x_4 - \left(\frac{\partial}{\partial t_4}\_a\right), \_a \to \_a * \_y - \left(\frac{\partial}{\partial \_t}\_a\right)\Big]]$$

Again, the last output is the matrix $A = (D_1\,x\ \ldots\ D_4\,x)$, where $x = (x_1\ \ldots\ x_4)^T$, defined by:

```
> RF1[8];
```

$$\begin{bmatrix} x_1 & 0 & x_4 & 0 \\ 0 & x_2 & 0 & x_3 \\ 0 & -x_3 & 0 & -x_2 \\ -x_4 & 0 & -x_1 & 0 \end{bmatrix}$$

If the option "reduced" is added to the `RankFactorization`, then the same matrix K_1, returned in `RF1`[2], is obtained. However, the matrices $C_{h_{1,i}}$ defining $\ker_{\mathcal{S}_{1,h_{1,i}}}(B_1.)$ are then shorter:

```
> RF1bis := RankFactorization(M,[D1,D2,D3,D4],1,"reduced"):
> RF1bis[6];
```

$$\text{table}\left(\left[1=\begin{bmatrix} x_2^2-x_3^2 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\\ 2 & 0 & 0\end{bmatrix}, 2=\begin{bmatrix} x_2^2-x_3^2 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\\ 2 & 0 & 0\end{bmatrix}\right]\right)$$

As we shall later, the corresponding expressions for the solutions will thus be shorter with the "reduced" option, i.e., fewer free parameters in the matrices Y' will be needed (even if both expressions define the same set of solutions).

```
> Sol1 := Solutions(M,[D1,D2,D3,D4],1):
```

We first obtain that the ideal $\mathcal{J}_1$ is generated by:

```
> Sol1[1];
```

$$[x_2^2\,x_1 - x_1\,x_3^2 + x_2^2\,x_4 - x_3^2\,x_4]$$

i.e., $\mathcal{J}_1 = \langle (x_2 - x_3)\,(x_2 + x_3)\,(x_1 + x_4)\rangle$. The second output

```
> Sol1[2];
```

$$[x_1 - x_4,\, x_2^2\,x_4 - x_3^2\,x_4]$$

shows that two solutions can be found respectively over the localization $\mathcal{S}_{1,h_{1,1}}$ (resp., $\mathcal{S}_{1,h_{1,2}}$) of the ring $\mathcal{S}_1 = \mathcal{R}/\mathcal{J}_1$ with respect respectively to the multiplicatively closed set $\{h_{1,1}^k\}_{k\in\mathbb{Z}}$ (resp., $\{h_{1,2}^k\}_{k\in\mathbb{Z}}$), where $h_{1,1}$ (resp., $h_{1,2}$) denotes the residue class of $g_{1,1} = x_1 - x_4$ (resp., $g_{1,2} = x_4\,(x_2 - x_3)\,(x_2 + x_3)$) in the ring $\mathcal{S}_{1,h_{1,1}}$ (resp., $\mathcal{S}_{1,h_{1,2}}$). Thus, the u-component of the two solutions $(u,\, v)$ respectively belongs to $\mathcal{V}(\mathcal{J}_1) \setminus \mathcal{V}(\langle g_{1,1}\rangle)$ and $\mathcal{V}(\mathcal{J}_1) \setminus \mathcal{V}(\langle g_{1,2}\rangle)$.

As above, the third output is the matrix $A = (D_1\,x\ \dots\ D_4\,x)$, where $x = (x_1\ \dots\ x_4)^T$:

```
> Sol1[3];
```

$$\begin{bmatrix} x_1 & 0 & x_4 & 0\\ 0 & x_2 & 0 & x_3\\ 0 & -x_3 & 0 & -x_2\\ -x_4 & 0 & -x_1 & 0\end{bmatrix}$$

which is useful for the `IsSolution` function to test if the expressions returned by the `Solutions` function are solutions of the corresponding rank factorization problem. The fourth output is a table with the two v-components of solutions.

```
> nops(Sol1[4]);
```

$$2$$

For a better display, we successively show the rows of these two solutions. Let us start with the first one:

```
> Row(Sol1[4][1],1);
```

$$[\ \_y + \left(x_2^2 - x_3^2\right) y_{1,1} \quad \left(x_2^2 - x_3^2\right) y_{1,2} \quad \left(x_2^2 - x_3^2\right) y_{1,3} \quad \_y + \left(x_2^2 - x_3^2\right) y_{1,4}\ ]$$

```
> Row(Sol1[4][1],2);
```

$$\begin{aligned}&[\,(-x_3\,x_1 - x_3\,x_4)\ y_{2,1} + (x_2\,x_1 + x_2\,x_4)\ y_{3,1} \quad (-x_3\,x_1 - x_3\,x_4)\ y_{2,2} + (x_2\,x_1 + x_2\,x_4)\ y_{3,2}\\ &(-x_3\,x_1 - x_3\,x_4)\ y_{2,3} + (x_2\,x_1 + x_2\,x_4)\ y_{3,3} \quad (-x_3\,x_1 - x_3\,x_4)\ y_{2,4} + (x_2\,x_1 + x_2\,x_4)\ y_{3,4}\,]\end{aligned}$$

```
> Row(Sol1[4][1],3);
```

$$\begin{aligned}&[\,-\_y - \left(x_2^2 - x_3^2\right)\ y_{1,1} + \left(x_2^2 - x_3^2\right)\ (2\,y_{1,1} + (x_1 + x_4)\ y_{4,1} + (2\,\_y\,x_4 + 1)\ y_{5,1} + (-2\,\_y\,x_4 - 1)\ y_{6,1})\\ &- \left(x_2^2 - x_3^2\right)\ y_{1,2} + \left(x_2^2 - x_3^2\right)\ (2\,y_{1,2} + (x_1 + x_4)\ y_{4,2} + (2\,\_y\,x_4 + 1)\ y_{5,2} + (-2\,\_y\,x_4 - 1)\ y_{6,2})\\ &- \left(x_2^2 - x_3^2\right)\ y_{1,3} + \left(x_2^2 - x_3^2\right)\ (2\,y_{1,3} + (x_1 + x_4)\ y_{4,3} + (2\,\_y\,x_4 + 1)\ y_{5,3} + (-2\,\_y\,x_4 - 1)\ y_{6,3})\\ &-\_y - \left(x_2^2 - x_3^2\right)\ y_{1,4} + \left(x_2^2 - x_3^2\right)\ (2\,y_{1,4} + (x_1 + x_4)\ y_{4,4} + (2\,\_y\,x_4 + 1)\ y_{5,4} + (-2\,\_y\,x_4 - 1)\ y_{6,4})\,]\end{aligned}$$

```
> Row(Sol1[4][1],4);
```

$$\begin{aligned}&[\,(x_2\,x_1 + x_2\,x_4)\ y_{2,1} + (-x_3\,x_1 - x_3\,x_4)\ y_{3,1} \quad (x_2\,x_1 + x_2\,x_4)\ y_{2,2} + (-x_3\,x_1 - x_3\,x_4)\ y_{3,2}\\ &(x_2\,x_1 + x_2\,x_4)\ y_{2,3} + (-x_3\,x_1 - x_3\,x_4)\ y_{3,3} \quad (x_2\,x_1 + x_2\,x_4)\ y_{2,4} + (-x_3\,x_1 - x_3\,x_4)\ y_{3,4}\,]\end{aligned}$$

where the $y_{i,j}$'s are arbitrary elements of $\mathbb{K}$, and then the second solution:

```
> Row(Sol1[4][2],1);
```

$$\left[\ y_{1,1}+\left(x_2^2-x_3^2\right)y_{2,1}\quad y_{1,2}+\left(x_2^2-x_3^2\right)y_{2,2}\quad y_{1,3}+\left(x_2^2-x_3^2\right)y_{2,3}\quad y_{1,4}+\left(x_2^2-x_3^2\right)y_{2,4}\ \right]$$

```
> Row(Sol1[4][2],2);
```

$$[\,(-x_3\,x_1-x_3\,x_4)\;y_{3,1}+(x_2\,x_1+x_2\,x_4)\;y_{4,1}\quad(-x_3\,x_1-x_3\,x_4)\;y_{3,2}+(x_2\,x_1+x_2\,x_4)\;y_{4,2}$$
$$(-x_3\,x_1-x_3\,x_4)\;y_{3,3}+(x_2\,x_1+x_2\,x_4)\;y_{4,3}\quad(-x_3\,x_1-x_3\,x_4)\;y_{3,4}+(x_2\,x_1+x_2\,x_4)\;y_{4,4}\,]$$

```
> Row(Sol1[4][2],3);
```

$$[\,-y_{1,1}-\left(x_2^2-x_3^2\right)\,y_{2,1}+\left(x_2^2-x_3^2\right)\,\left(2\,\_y\,x_4\,y_{1,1}+\_y+2\,y_{2,1}\right)\quad -y_{1,2}-\left(x_2^2-x_3^2\right)\,y_{2,2}+\left(x_2^2-x_3^2\right)\,\left(2\,\_y\,x_4\,y_{1,2}+2\,y_{2,2}\right)$$
$$-y_{1,3}-\left(x_2^2-x_3^2\right)\,y_{2,3}+\left(x_2^2-x_3^2\right)\,\left(2\,\_y\,x_4\,y_{1,3}+2\,y_{2,3}\right)\quad -y_{1,4}-\left(x_2^2-x_3^2\right)\,y_{2,4}+\left(x_2^2-x_3^2\right)\,\left(2\,\_y\,x_4\,y_{1,4}+\_y+2\,y_{2,4}\right)\,]$$

```
> Row(Sol1[4][2],4);
```

$$[\,(x_2\,x_1+x_2\,x_4)\;y_{2,1}+(-x_3\,x_1-x_3\,x_4)\;y_{3,1}\quad(x_2\,x_1+x_2\,x_4)\;y_{2,2}+(-x_3\,x_1-x_3\,x_4)\;y_{3,2}$$
$$(x_2\,x_1+x_2\,x_4)\;y_{2,3}+(-x_3\,x_1-x_3\,x_4)\;y_{3,3}\quad(x_2\,x_1+x_2\,x_4)\;y_{2,4}+(-x_3\,x_1-x_3\,x_4)\;y_{3,4}\,]$$

Again, the $y_{i,j}$'s are arbitrary elements of $\mathbb{K}$. To check the correctness of the two solutions, these arbitrary parameters are added to the polynomial ring $\mathcal{R}$. The fifth output is a table with the two corresponding polynomial rings $\mathcal{R}[y_{i,j}]_{1\leq i,j\leq 4}[\_y]$, where $\_y$ is an extra variable to work in the localization $\mathcal{S}_{1,h_{1,1}}$ and $\mathcal{S}_{1,h_{1,2}}$ with the `IsSolution` function.

```
> Sol1[5][1];
```

$$\text{table}\,([1=[\ \mathit{Ore\_algebra}, [\text{“diff”},\text{“diff”},\text{“diff”},\text{“diff”},\text{“diff”}], [t_1,t_2,t_3,t_4,\_t], [x_1,x_2,x_3,x_4,\_y], [t_1, t_2,t_3,t_4,\_t], [y_{1,1},y_{1,2},y_{1,3},y_{1,4},y_{2,1},y_{2,2},y_{2,3},y_{2,4},y_{3,1},y_{3,2},y_{3,3},y_{3,4},y_{4,1},y_{4,2},y_{4,3},y_{4,4}, y_{5,1},y_{5,2},y_{5,3},y_{5,4},y_{6,1},y_{6,2},y_{6,3},y_{6,4}],0,[],[],[t_1,t_2,t_3,t_4,\_t],[],[],[\mathit{diff}=[x_1,t_1], \mathit{diff}=[x_2,t_2],\mathit{diff}=[x_3,t_3],\mathit{diff}=[x_4,t_4],\mathit{diff}=[\_y,\_t]],$$
$$\left[\_a\to\_a*x_1-\left(\frac{\partial}{\partial t_1}\_a\right),\_a\to\_a*x_2-\left(\frac{\partial}{\partial t_2}\_a\right),\_a\to\_a*x_3-\left(\frac{\partial}{\partial t_3}\_a\right), \_a\to\_a*x_4-\left(\frac{\partial}{\partial t_4}\_a\right),\_a\to\_a*\_y-\left(\frac{\partial}{\partial\_t}\_a\right)\right]\ ]])$$

```
> Sol1[5][2];
```

$$\text{table}\,([1=[\ \mathit{Ore\_algebra}, [\text{“diff”},\text{“diff”},\text{“diff”},\text{“diff”},\text{“diff”}], [t_1,t_2,t_3,t_4,\_t], [x_1,x_2,x_3,x_4,\_y], [t_1, t_2,t_3,t_4,\_t], [y_{1,1},y_{1,2},y_{1,3},y_{1,4},y_{2,1},y_{2,2},y_{2,3},y_{2,4},y_{3,1},y_{3,2},y_{3,3},y_{3,4},y_{4,1},y_{4,2},y_{4,3},y_{4,4}], 0,[],[],[t_1,t_2,t_3,t_4,\_t],[],[],[\mathit{diff}=[x_1,t_1], \mathit{diff}=[x_2,t_2],\mathit{diff}=[x_3,t_3],\mathit{diff}=[x_4,t_4],\mathit{diff}=[\_y,\_t]],$$
$$\left[\_a\to\_a*x_1-\left(\frac{\partial}{\partial t_1}\_a\right),\_a\to\_a*x_2-\left(\frac{\partial}{\partial t_2}\_a\right),\_a\to\_a*x_3-\left(\frac{\partial}{\partial t_3}\_a\right), \_a\to\_a*x_4-\left(\frac{\partial}{\partial t_4}\_a\right),\_a\to\_a*\_y-\left(\frac{\partial}{\partial\_t}\_a\right)\right]\ ]])$$

We can check again that the above expressions are solutions to the rank factorization problem:

```
> IsSolution(Sol1);
```

$$\text{table}\,([1=[\mathit{true}]\,,2=[\mathit{true}]])$$

Finally, we can add the "reduced" option to the `Solutions` function to get shorter outputs for the v-components of the solutions.

```
> Sol1bis := Solutions(M,[D1,D2,D3,D4],1,"reduced"):
> IsSolution(Sol1bis);
```

$$\text{table}\,([1=[\mathit{true}]\,,2=[\mathit{true}]])$$

Let us successively display the rows of the first solution:

```
> Row(Sol1bis[4][1],1);
```

$$\left[\ \_y+\left(x_2^2-x_3^2\right)\,y_{1,1}\quad\left(x_2^2-x_3^2\right)\,y_{1,2}\quad\left(x_2^2-x_3^2\right)\,y_{1,3}\quad\_y+\left(x_2^2-x_3^2\right)\,y_{1,4}\ \right]$$

```
> Row(Sol1bis[4][1],2);
```

$$[\,(-x_3\,x_1 - x_4\,x_3)\;y_{2,1} + (x_2\,x_1 + x_4\,x_2)\;y_{3,1} \quad (-x_3\,x_1 - x_4\,x_3)\;y_{2,2} + (x_2\,x_1 + x_4\,x_2)\;y_{3,2} \quad (-x_3\,x_1 - x_4\,x_3)\;y_{2,3} + (x_2\,x_1 + x_4\,x_2)\;y_{3,3} \quad (-x_3\,x_1 - x_4\,x_3)\;y_{2,4} + (x_2\,x_1 + x_4\,x_2)\;y_{3,4}\,]$$

```
> Row(Sol1bis[4][1],3);
```

$$\left[\ -\_y + \left(x_2^2 - x_3^2\right)\;y_{1,1} \quad \left(x_2^2 - x_3^2\right)\;y_{1,2} \quad \left(x_2^2 - x_3^2\right)\;y_{1,3} \quad -\_y + \left(x_2^2 - x_3^2\right)\;y_{1,4}\ \right]$$

```
> Row(Sol1bis[4][1],4);
```

$$[\,(x_2\,x_1 + x_4\,x_2)\;y_{2,1} + (-x_3\,x_1 - x_4\,x_3)\;y_{3,1} \quad (x_2\,x_1 + x_4\,x_2)\;y_{2,2} + (-x_3\,x_1 - x_4\,x_3)\;y_{3,2} \quad (x_2\,x_1 + x_4\,x_2)\;y_{2,3} + (-x_3\,x_1 - x_4\,x_3)\;y_{3,3} \quad (x_2\,x_1 + x_4\,x_2)\;y_{2,4} + (-x_3\,x_1 - x_4\,x_3)\;y_{3,4}\,]$$

Finally, let us successively display the rows of the second solution:

```
> Row(Sol1bis[4][2],1);
```

$$\left[\ \left(x_2^2 - x_3^2\right)\;y_{1,1} \quad \left(x_2^2 - x_3^2\right)\;y_{1,2} \quad \left(x_2^2 - x_3^2\right)\;y_{1,3} \quad \left(x_2^2 - x_3^2\right)\;y_{1,4}\ \right]$$

```
> Row(Sol1bis[4][2],2);
```

$$[\,(-x_1\,x_3 - x_3\,x_4)\;y_{2,1} + (x_1\,x_2 + x_2\,x_4)\;y_{3,1} \quad (-x_1\,x_3 - x_3 x_4)\;y_{2,2} + (x_1\,x_2 + x_2\,x_4)\;y_{3,2} \quad (-x_1\,x_3 - x_3\,x_4)\;y_{2,3} + (x_1\,x_2 + x_2\,x_4)\;y_{3,3} \quad (-x_1\,x_3 - x_3\,x_4)\;y_{2,4} + (x_1\,x_2 + x_2\,x_4)\;y_{3,4}\,]$$

```
> Row(Sol1bis[4][2],3);
```

$$\left[\,-\left(x_2^2 - x_3^2\right)\;y_{1,1} + \left(x_2^2 - x_3^2\right)\;(\_y + 2\,y_{1,1}) \quad \left(x_2^2 - x_3^2\right)\;y_{1,2} \quad \left(x_2^2 - x_3^2\right)\;y_{1,3} \quad -\left(x_2^2 - x_3^2\right)\;y_{1,4} + \left(x_2^2 - x_3^2\right)\;(\_y + 2\,y_{1,4})\,\right]$$

```
> Row(Sol1bis[4][2],4);
```

$$[\,(x_1\,x_2 + x_2\,x_4)\;y_{2,1} + (-x_1\,x_3 - x_3\,x_4)\;y_{3,1} \quad (x_1\,x_2 + x_2\,x_4)\;y_{2,2} + (-x_1\,x_3 - x_3\,x_4)\;y_{3,2} \quad (x_1\,x_2 + x_2\,x_4)\;y_{2,3} + (-x_1\,x_3 - x_3\,x_4)\;y_{3,3} \quad (x_1\,x_2 + x_2\,x_4)\;y_{2,4} + (-x_1\,x_3 - x_3\,x_4)\;y_{3,4}\,]$$

Hence, using the option "reduced", shorter expressions for the v-components of the solutions for $k = 1$ are obtained.

`RankFactorization`$(M, [D1, D2, D3, D4], 2)$ **&** `Solutions`$(M, [D1, D2, D3, D4], 2)$ Let us now apply `RankFactorization` with $k = 2$.

```
> RF2 := RankFactorization(M,[D1,D2,D3,D4],2):
```

We first obtain that the ideal $\mathcal{J}_2$ is generated by:

```
> RF2[1];
```

$$[x_3\,x_1 + x_3\,x_4,\; x_2^2 - x_3^2,\; x_2\,x_1 + x_2\,x_4]$$

Thus, we have $\mathcal{J}_2 = \langle x_3\,(x_1 + x_4),\, (x_2 - x_3)\,(x_2 + x_3),\, x_2\,(x_1 + x_4)\rangle$ and $\mathcal{S}_2 = \mathcal{R}/\mathcal{J}_2$.

The second output is the matrix K_2 defined by:

```
> RF2[2];
```

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -x_3 & x_2 & x_1 + x_4 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & x_3 & x_2 & 0 \\ 0 & x_2 & -x_3 & 0 & 0 & 0 & x_1 + x_4 \end{bmatrix}$$

Again, the third one is the matrix Y defined by:

```
> RF2[3];
```

$$\begin{bmatrix} 1 & 0 & 0 & 1 \end{bmatrix}$$

The fourth output is a list of the non-nilpotent elements of a set of generators $\{g_{2,i}\}_{i \in I_2}$ of $\mathcal{I}_2 = \mathrm{Fitt}_0(\mathcal{B}_2)$:

```
> RF2[4];
```

$$[x_1 - x_4,\ x_3\, x_4,\ x_2\, x_4]$$

We thus have $I_2 = \{1,\,2,\,3\}$, $g_{2,1} = x_1 - x_4$, $g_{2,2} = x_3\, x_4$, and $g_{2,3} = x_2\, x_4$. If we denote by $h_{2,i}$ the residue class of $g_{2,i}$ in $\mathcal{S}_2$ for $i = 1,\,2,\,3$, then three solutions exist respectively over the localization $\mathcal{S}_{2,h_{2,i}} = \mathcal{S}_2[y]/\langle y\, h_{2,i} - 1\rangle = \mathcal{S}_2\left[h_{2,i}^{-1}\right]$ for $i = 1,\,2,\,3$.

The next output is a right inverse of the matrix B_2 over respectively the ring $\mathcal{S}_{2,h_{2,i}}$ for $i = 1,\,2,\,3$:

```
> RF2[5];
```

$$\mathrm{table}\left(\left[1 = \begin{bmatrix} \_y \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, 2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \_y \\ 0 \\ 0 \end{bmatrix}, 3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \_y \\ 0 \end{bmatrix}\right]\right)$$

The sixth output is a table containing the right kernel of the matrix B_2 over respectively the ring $\mathcal{S}_{2,h_{2,i}}$, i.e., $\ker_{\mathcal{S}_{2,h_{2,i}}}(B_2.)$, for $i = 1,\,2,\,3$.

```
> RF2[6];
```

$$\mathrm{table}\,([$$

$$1 = \begin{bmatrix} x_3 & x_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & -x_3 & x_2 & x_1 + x_4 & 2\_y x_4 + 1 & -2\_y x_4 - 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & x_2 & -x_3 & 0 & 0 & 0 & x_1 + x_4 & 2\_y x_4 + 1 & -2\_y x_4 - 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$2 = \begin{bmatrix} x_3 & x_2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 2 & 0 & 2\_y x_4 & 0 & 0 & 0 & -x_3 & x_2 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & x_2 & -x_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$3 = \begin{bmatrix} x_3 & x_2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & -x_3 & x_2 & \_y x_3\, x_4 & 0 \\ 0 & 2 & 2\_y x_4 & 0 & 0 & 0 & x_2 & -x_3 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}])$$

The next output is the polynomial ring $\mathcal{R}[y]$.

```
> RF2[7];
```

$$[\mathit{Ore\_algebra}, [\text{“diff”}, \text{“diff”}, \text{“diff”}, \text{“diff”}, \text{“diff”}], [t_1, t_2, t_3, t_4, \_t], [x_1, x_2, x_3, x_4, \_y], [t_1, t_2, t_3, t_4, \_t], [],$$
$$0, [], [], [t_1, t_2, t_3, t_4, \_t], [], [], [\mathit{diff} = [x_1, t_1], \mathit{diff} = [x_2, t_2], \mathit{diff} = [x_3, t_3], \mathit{diff} = [x_4, t_4], \mathit{diff} = [\_y, \_t]],$$
$$\Bigg[\_a \to \_a * x_1 - \left(\frac{\partial}{\partial t_1}\_a\right), \_a \to \_a * x_2 - \left(\frac{\partial}{\partial t_2}\_a\right), \_a \to \_a * x_3 - \left(\frac{\partial}{\partial t_3}\_a\right),$$
$$\_a \to \_a * x_4 - \left(\frac{\partial}{\partial t_4}\_a\right), \_a \to \_a * \_y - \left(\frac{\partial}{\partial \_t}\_a\right)\Bigg]]$$

Again, the last output is the matrix $A = (D_1\, x\ \ldots\ D_4\, x)$, where $x = (x_1\ \ldots\ x_4)^T$, defined by:

```
> RF2[8];
```

$$\begin{bmatrix} x_1 & 0 & x_4 & 0 \\ 0 & x_2 & 0 & x_3 \\ 0 & -x_3 & 0 & -x_2 \\ -x_4 & 0 & -x_1 & 0 \end{bmatrix}$$

If the option "reduced" is added to the `RankFactorization`, then the same matrix K_2, returned in `RF2`[2], is obtained. However, the matrices $C_{h_{2,i}}$ defining $\ker_{\mathcal{S}_{2,h_{2,i}}}(B_2.)$ are then shorter:

```
> RF2bis := RankFactorization(M,[D1,D2,D3,D4],2,"reduced"):
> RF2bis[6];
```

$$\text{table}\left(\left[1 = \begin{bmatrix} x_3 & x_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, 2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 2\_y\,x_4 & 0 & 0 & 0 & x_2 & 0 \\ 0 & 0 & 0 & 0 & -x_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, 3 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \_y\,x_3\,x_4 & 0 \\ 2\_y\,x_4 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}\right]\right)$$

Hence, as we shall, the corresponding expressions for the solutions are shorter with the "reduced" option, i.e., fewer free parameters in the matrices Y' are needed (even if the expressions define the same sets of solutions). Let us now directly compute the solutions of the rank factorization for $k = 2$.

```
> Sol2 := Solutions(M,[D1,D2,D3,D4],2):
```

We first obtain that the ideal $\mathcal{J}_2$ is generated by:

```
> Sol2[1];
```

$$[x_3\,x_1 + x_3\,x_4,\ x_2^2 - x_3^2,\ x_2\,x_1 + x_2\,x_4]$$

i.e., $\mathcal{J}_2 = \langle x_3\,(x_1 + x_4),\ (x_2 - x_3)\,(x_2 + x_3),\ x_2\,(x_1 + x_4)\rangle$ and $\mathcal{S}_2 = \mathcal{R}/\mathcal{J}_2$. The second output

```
> Sol2[2];
```

$$[x_1 - x_4,\ x_3\,x_4,\ x_2\,x_4]$$

shows that three solutions can be found respectively over the localization $\mathcal{S}_{2,h_{2,1}}$ (resp., $\mathcal{S}_{2,h_{2,2}}$, $\mathcal{S}_{2,h_{2,3}}$) of the ring $\mathcal{S}_2$ with respect respectively to the multiplicatively closed set $\{h_{2,1}^k\}_{k\in\mathbb{Z}}$ (resp., $\{h_{2,2}^k\}_{k\in\mathbb{Z}}$, $\{h_{2,3}^k\}_{k\in\mathbb{Z}}$), where $h_{2,1}$ (resp., $h_{2,2}$, $h_{2,3}$) denotes the residue class of $g_{2,1} = x_1 - x_4$ (resp., $g_{2,2} = x_3\,x_4$, $g_{2,3} = x_2\,x_4$) in the ring $\mathcal{S}_{2,h_{2,1}}$ (resp., $\mathcal{S}_{2,h_{2,2}}$, $\mathcal{S}_{2,h_{2,3}}$). Thus, the u-component of the two solutions (u, v) respectively belongs to $\mathcal{V}(\mathcal{J}_2) \setminus \mathcal{V}(\langle g_{2,1}\rangle)$ $\mathcal{V}(\mathcal{J}_2) \setminus \mathcal{V}(\langle g_{2,2}\rangle)$, and $\mathcal{V}(\mathcal{J}_2) \setminus \mathcal{V}(\langle g_{2,3}\rangle)$.

The third output is the matrix $A = (D_1\,x\ \ldots\ D_4\,x)$, where $x = (x_1\ \ldots\ x_4)^T$:

```
> Sol2[3];
```

$$\begin{bmatrix} x_1 & 0 & x_4 & 0 \\ 0 & x_2 & 0 & x_3 \\ 0 & -x_3 & 0 & -x_2 \\ -x_4 & 0 & -x_1 & 0 \end{bmatrix}$$

```
> nops(op(Sol2[4]));
```

$$3$$

Thus, there are three families of solutions. For a better display, we successively show the rows of the three families of solutions.

Let us start with the first one

```
> Row(Sol2[4][1],1);
```

$$\begin{bmatrix} x_2\,y_{2,1} + x_3\,y_{1,1} + \_y & x_2\,y_{2,2} + x_3\,y_{1,2} & x_2\,y_{2,3} + x_3\,y_{1,3} & x_2\,y_{2,4} + x_3\,y_{1,4} + \_y \end{bmatrix}$$

```
> Row(Sol2[4][1],2);
```

$$[\,-x_3\,y_{3,1}+x_2\,y_{4,1}+(x_1+x_4)\;y_{5,1}\quad -x_3\,y_{3,2}+x_2\,y_{4,2}+(x_1+x_4)\;y_{5,2}\quad -x_3\,y_{3,3}+x_2\,y_{4,3}+(x_1+x_4)\;y_{5,3}\quad -x_3\,y_{3,4}+x_2\,y_{4,4}+(x_1+x_4)\;y_{5,4}\,]$$

```
>  Row(Sol2[4][1],3);
```

$$\begin{aligned}[\,&-x_2\,y_{2,1}-x_3\,y_{1,1}-\_y+x_3\;(2\,y_{1,1}-x_3\,y_{6,1}+x_2\,y_{7,1}+(x_1+x_4)\;y_{8,1}+(2\,\_y\,x_4+1)\;y_{9,1}+(-2\,\_y\,x_4-1)\;y_{10,1})\\ &+x_2\;(2\,y_{2,1}+x_2\,y_{6,1}-x_3\,y_{7,1}+(x_1+x_4)\;y_{11,1}+(2\,\_y\,x_4+1)\;y_{12,1}+(-2\,\_y\,x_4-1)\;y_{13,1})\\ &-x_2\,y_{2,2}-x_3\,y_{1,2}+x_3\;(2\,y_{1,2}-x_3\,y_{6,2}+x_2\,y_{7,2}+(x_1+x_4)\;y_{8,2}+(2\,\_y\,x_4+1)\;y_{9,2}+(-2\,\_y\,x_4-1)\;y_{10,2})\\ &+x_2\;(2\,y_{2,2}+x_2\,y_{6,2}-x_3\,y_{7,2}+(x_1+x_4)\;y_{11,2}+(2\,\_y\,x_4+1)\,y_{12,2}+(-2\,\_y\,x_4-1)\;y_{13,2})\\ &-x_2\,y_{2,3}-x_3\,y_{1,3}+x_3\;(2\,y_{1,3}-x_3\,y_{6,3}+x_2\,y_{7,3}+(x_1+x_4)\;y_{8,3}+(2\,\_y\,x_4+1)\;y_{9,3}+(-2\,\_y\,x_4-1)\;y_{10,3})\\ &+x_2\;(2\,y_{2,3}+x_2\,y_{6,3}-x_3\,y_{7,3}+(x_1+x_4)\;y_{11,3}+(2\,\_y\,x_4+1)\;y_{12,3}+(-2\,\_y\,x_4-1)\;y_{13,3})\\ &-x_2\,y_{2,4}-x_3\,y_{1,4}-\_y+x_3\;(2\,y_{1,4}-x_3\,y_{6,4}+x_2\,y_{7,4}+(x_1+x_4)\;y_{8,4}+(2\,\_y\,x_4+1)\;y_{9,4}+(-2\,\_y\,x_4-1)\;y_{10,4})\\ &+x_2\;(2\,y_{2,4}+x_2\,y_{6,4}-x_3\,y_{7,4}+(x_1+x_4)\;y_{11,4}+(2\,\_y\,x_4+1)\;y_{12,4}+(-2\,\_y\,x_4-1)\;y_{13,4})\,]\end{aligned}$$

```
>  Row(Sol2[4][1],4);
```

$$[\,x_2\,y_{3,1}-x_3\,y_{4,1}+(x_1+x_4)\;y_{14,1}\quad x_2\,y_{3,2}-x_3\,y_{4,2}+(x_1+x_4)\;y_{14,2}\quad x_2\,y_{3,3}-x_3\,y_{4,3}+(x_1+x_4)\;y_{14,3}\quad x_2\,y_{3,4}-x_3\,y_{4,4}+(x_1+x_4)\;y_{14,4}\,]$$

where the y's are arbitrary elements of $\mathbb{K}$.

Let us now show the v-component of the second family of solutions

```
>  Row(Sol2[4][2],1);
```

$$[\;x_2\,y_{2,1}+x_3\,y_{1,1}+y_{3,1}\quad x_2\,y_{2,2}+x_3\,y_{1,2}+y_{3,2}\quad x_2\,y_{2,3}+x_3\,y_{1,3}+y_{3,3}\quad x_2\,y_{2,4}+x_3\,y_{1,4}+y_{3,4}\;]$$

```
>  Row(Sol2[4][2],2);
```

$$[\,-x_3\,y_{4,1}+x_2\,y_{5,1}+(x_1+x_4)\;y_{6,1}\quad -x_3\,y_{4,2}+x_2\,y_{5,2}+(x_1+x_4)\;y_{6,2}\quad -x_3\,y_{4,3}+x_2\,y_{5,3}+(x_1+x_4)\;y_{6,3}\quad -x_3\,y_{4,4}+x_2\,y_{5,4}+(x_1+x_4)\;y_{6,4}\,]$$

```
>  Row(Sol2[4][2],3);
```

$$\begin{aligned}[\,&-x_2\,y_{2,1}-x_3\,y_{1,1}-y_{3,1}+x_3\;(2\,\_y\,x_4\,y_{3,1}+x_2\,y_{8,1}-x_3\,y_{7,1}+\_y+2\,y_{1,1})+x_2\;(x_2\,y_{7,1}-x_3\,y_{8,1}+2\,y_{2,1})\\ &-x_2\,y_{2,2}-x_3\,y_{1,2}-y_{3,2}+x_3\;(2\,\_y\,x_4\,y_{3,2}+x_2\,y_{8,2}-x_3\,y_{7,2}+2\,y_{1,2})+x_2\;(x_2\,y_{7,2}-x_3\,y_{8,2}+2\,y_{2,2})\\ &-x_2\,y_{2,3}-x_3\,y_{1,3}-y_{3,3}+x_3\;(2\,\_y\,x_4\,y_{3,3}+x_2\,y_{8,3}-x_3\,y_{7,3}+2\,y_{1,3})+x_2\;(x_2\,y_{7,3}-x_3\,y_{8,3}+2\,y_{2,3})\\ &-x_2\,y_{2,4}-x_3\,y_{1,4}-y_{3,4}+x_3\;(2\,\_y\,x_4\,y_{3,4}+x_2\,y_{8,4}-x_3\,y_{7,4}+\_y+2\,y_{1,4})+x_2\;(x_2\,y_{7,4}-x_3\,y_{8,4}+2\,y_{2,4})\,]\end{aligned}$$

```
>  Row(Sol2[4][2],4);
```

$$[\,x_2\,y_{4,1}-x_3\,y_{5,1}+(x_1+x_4)\;y_{9,1}\quad x_2\,y_{4,2}-x_3\,y_{5,2}+(x_1+x_4)\;y_{9,2}\quad x_2\,y_{4,3}-x_3\,y_{5,3}+(x_1+x_4)\;y_{9,3}\quad x_2\,y_{4,4}-x_3\,y_{5,4}+(x_1+x_4)\;y_{9,4}\,]$$

where the y's are arbitrary elements of $\mathbb{K}$.

Let us now show the v-component of the third family of solutions

```
>  Row(Sol2[4][3],1);
```

$$[\;x_2\,y_{2,1}+x_3\,y_{1,1}+y_{3,1}\quad x_2\,y_{2,2}+x_3\,y_{1,2}+y_{3,2}\quad x_2\,y_{2,3}+x_3\,y_{1,3}+y_{3,3}\quad x_2\,y_{2,4}+x_3\,y_{1,4}+y_{3,4}\;]$$

```
>  Row(Sol2[4][3],2);
```

$$[\,-x_3\,y_{4,1}+x_2\,y_{5,1}+(x_1+x_4)\;y_{6,1}\quad -x_3\,y_{4,2}+x_2\,y_{5,2}+(x_1+x_4)\;y_{6,2}\quad -x_3\,y_{4,3}+x_2\,y_{5,3}+(x_1+x_4)\;y_{6,3}\quad -x_3\,y_{4,4}+x_2\,y_{5,4}+(x_1+x_4)\;y_{6,4}\,]$$

```
>  Row(Sol2[4][3],3);
```

$$[\,-x_2\,y_{2,1}-x_3\,y_{1,1}-y_{3,1}+x_3\,\left(\_y\,x_3\,x_4\,y_{9,1}+x_2\,y_{8,1}-x_3\,y_{7,1}+2\,y_{1,1}\right)$$
$$+x_2\,\left(2\,\_y\,x_4\,y_{3,1}+x_2\,y_{7,1}-x_3\,y_{8,1}+\_y+2\,y_{2,1}-y_{9,1}\right)$$
$$-x_2\,y_{2,2}-x_3\,y_{1,2}-y_{3,2}+x_3\,\left(\_y\,x_3\,x_4\,y_{9,2}+x_2\,y_{8,2}-x_3\,y_{7,2}+2\,y_{1,2}\right)$$
$$+x_2\,\left(2\,\_y\,x_4\,y_{3,2}+x_2\,y_{7,2}-x_3\,y_{8,2}+2\,y_{2,2}-y_{9,2}\right)$$
$$-x_2\,y_{2,3}-x_3\,y_{1,3}-y_{3,3}+x_3\,\left(\_y\,x_3\,x_4\,y_{9,3}+x_2\,y_{8,3}-x_3\,y_{7,3}+2\,y_{1,3}\right)$$
$$+x_2\,\left(2\,\_y\,x_4\,y_{3,3}+x_2\,y_{7,3}-x_3\,y_{8,3}+2\,y_{2,3}-y_{9,3}\right)$$
$$-x_2\,y_{2,4}-x_3\,y_{1,4}-y_{3,4}+x_3\,\left(\_y\,x_3\,x_4\,y_{9,4}+x_2\,y_{8,4}-x_3\,y_{7,4}+2\,y_{1,4}\right)$$
$$+x_2\,\left(2\,\_y\,x_4\,y_{3,4}+x_2\,y_{7,4}-x_3\,y_{8,4}+\_y+2\,y_{2,4}-y_{9,4}\right)\,]$$

```
> Row(Sol2[4][3],4);
```

$$[\,x_2\,y_{4,1}-x_3\,y_{5,1}+(x_1+x_4)\;y_{10,1}\quad x_2\,y_{4,2}-x_3\,y_{5,2}+(x_1+x_4)\;y_{10,2}$$
$$x_2\,y_{4,3}-x_3\,y_{5,3}+(x_1+x_4)\;y_{10,3}\quad x_2\,y_{4,4}-x_3\,y_{5,4}+(x_1+x_4)\;y_{10,4}\,]$$

where the y's are arbitrary elements of $\mathbb{K}$.

The fifth output is a table with the two corresponding polynomial rings $\mathcal{R}[y]$, where $\_y$ is an extra variable to work in the localization $\mathcal{S}_{2,h_{2,1}}$, $\mathcal{S}_{2,h_{2,2}}$, and $\mathcal{S}_{2,h_{2,3}}$ with the `IsSolution` function.

```
> Sol2[5][1];
```

$$\text{table}\,([1=[\;\mathit{Ore\_algebra}, [\text{“diff”},\text{“diff”},\text{“diff”},\text{“diff”},\text{“diff”}], [t_1,t_2,t_3,t_4,\_t], [x_1,x_2,x_3,x_4,\_y], [t_1, t_2,t_3,t_4,\_t], [y_{1,1},y_{1,2},y_{1,3},y_{1,4},y_{2,1},y_{2,2},y_{2,3},y_{2,4},y_{3,1},y_{3,2},y_{3,3},y_{3,4},y_{4,1},y_{4,2},y_{4,3},y_{4,4},y_{5,1},y_{5,2},y_{5,3},y_{5,4}, y_{6,1},y_{6,2},y_{6,3},y_{6,4},y_{7,1},y_{7,2},y_{7,3},y_{7,4},y_{8,1},y_{8,2},y_{8,3},y_{8,4},y_{9,1},y_{9,2},y_{9,3},y_{9,4},y_{10,1},y_{10,2},y_{10,3},y_{10,4}, y_{11,1},y_{11,2},y_{11,3},y_{11,4},y_{12,1},y_{12,2},y_{12,3},y_{12,4},y_{13,1},y_{13,2},y_{13,3},y_{13,4},y_{14,1},y_{14,2},y_{14,3},y_{14,4}], 0, [], [], [t_1, t_2,t_3,t_4,\_t], [], [], [\mathit{diff}=[x_1,t_1], \mathit{diff}=[x_2,t_2], \mathit{diff}=[x_3,t_3], \mathit{diff}=[x_4,t_4], \mathit{diff}=[\_y,\_t]], \left[\_a\to\_a*x_1-\left(\frac{\partial}{\partial t_1}\_a\right), \_a\to\_a*x_2-\left(\frac{\partial}{\partial t_2}\_a\right), \_a\to\_a*x_3-\left(\frac{\partial}{\partial t_3}\_a\right), \_a\to\_a*x_4-\left(\frac{\partial}{\partial t_4}\_a\right), \_a\to\_a*\_y-\left(\frac{\partial}{\partial \_t}\_a\right)\right]\;]])$$

```
> Sol2[5][2];
```

$$\text{table}\,([1=[\;\mathit{Ore\_algebra}, [\text{“diff”},\text{“diff”},\text{“diff”},\text{“diff”},\text{“diff”}], [t_1,t_2,t_3,t_4,\_t], [x_1,x_2,x_3,x_4,\_y], [t_1, t_2,t_3,t_4,\_t], [y_{1,1},y_{1,2},y_{1,3},y_{1,4},y_{2,1},y_{2,2},y_{2,3},y_{2,4},y_{3,1},y_{3,2},y_{3,3},y_{3,4},y_{4,1},y_{4,2},y_{4,3},y_{4,4},y_{5,1},y_{5,2},y_{5,3},y_{5,4}, y_{6,1},y_{6,2},y_{6,3},y_{6,4},y_{7,1},y_{7,2},y_{7,3},y_{7,4},y_{8,1},y_{8,2},y_{8,3},y_{8,4},y_{9,1},y_{9,2},y_{9,3},y_{9,4}], 0, [], [], [t_1,t_2,t_3, t_4,\_t], [], [], [\mathit{diff}=[x_1,t_1], \mathit{diff}=[x_2,t_2], \mathit{diff}=[x_3,t_3], \mathit{diff}=[x_4,t_4], \mathit{diff}=[\_y,\_t]], \left[\_a\to\_a*x_1-\left(\frac{\partial}{\partial t_1}\_a\right), \_a\to\_a*x_2-\left(\frac{\partial}{\partial t_2}\_a\right), \_a\to\_a*x_3-\left(\frac{\partial}{\partial t_3}\_a\right), \_a\to\_a*x_4-\left(\frac{\partial}{\partial t_4}\_a\right), \_a\to\_a*\_y-\left(\frac{\partial}{\partial \_t}\_a\right)\right]\;]])$$

```
> Sol2[5][3];
```

$$\text{table}\,([1=[\;\mathit{Ore\_algebra}, [\text{“diff”},\text{“diff”},\text{“diff”},\text{“diff”},\text{“diff”}], [t_1,t_2,t_3,t_4,\_t], [x_1,x_2,x_3,x_4,\_y], [t_1, t_2,t_3,t_4,\_t], [y_{1,1},y_{1,2},y_{1,3},y_{1,4},y_{2,1},y_{2,2},y_{2,3},y_{2,4},y_{3,1},y_{3,2},y_{3,3},y_{3,4},y_{4,1},y_{4,2},y_{4,3},y_{4,4},y_{5,1},y_{5,2},y_{5,3},y_{5,4}, y_{6,1},y_{6,2},y_{6,3},y_{6,4},y_{7,1},y_{7,2},y_{7,3},y_{7,4},y_{8,1},y_{8,2},y_{8,3},y_{8,4},y_{9,1},y_{9,2},y_{9,3},y_{9,4},y_{10,1},y_{10,2},y_{10,3},y_{10,4}], 0, [], [], [t_1, t_2,t_3,t_4,\_t], [], [], [\mathit{diff}=[x_1,t_1], \mathit{diff}=[x_2,t_2], \mathit{diff}=[x_3,t_3], \mathit{diff}=[x_4,t_4], \mathit{diff}=[\_y,\_t]], \left[\_a\to\_a*x_1-\left(\frac{\partial}{\partial t_1}\_a\right), \_a\to\_a*x_2-\left(\frac{\partial}{\partial t_2}\_a\right), \_a\to\_a*x_3-\left(\frac{\partial}{\partial t_3}\_a\right), \_a\to\_a*x_4-\left(\frac{\partial}{\partial t_4}\_a\right), \_a\to\_a*\_y-\left(\frac{\partial}{\partial \_t}\_a\right)\right]\;]])$$

We can finally check again that the above expressions are solutions of the rank factorization problem:

```
> IsSolution(Sol2);
```

$$\text{table}\,([1=[\mathit{true}]\,,2=[\mathit{true}]\,,3=[\mathit{true}]])$$

Finally, we can add the “reduced” option to the `Solutions` function to get shorter outputs for the v-components of the solutions.

```
> Sol2bis := Solutions(M,[D1,D2,D3,D4],2,"reduced"):
> IsSolution(Sol2bis);
```

$$\text{table}\left([1=[true],2=[true]]\right)$$

Let us successively display the rows of the v-component of the first solution:

```
> Row(Sol2bis[4][1],1);
```

$$\left[\begin{array}{cccc} x_2\,y_{2,1}+x_3\,y_{1,1}+\_y & x_2\,y_{2,2}+x_3\,y_{1,2} & x_2\,y_{2,3}+x_3\,y_{1,3} & x_2\,y_{2,4}+x_3\,y_{1,4}+\_y \end{array}\right]$$

```
> Row(Sol2bis[4][1],2);
```

$$[-x_3\,y_{3,1}+x_2\,y_{4,1}+(x_1+x_4)\,y_{5,1} \quad -x_3\,y_{3,2}+x_2\,y_{4,2}+(x_1+x_4)\,y_{5,2}$$
$$-x_3\,y_{3,3}+x_2\,y_{4,3}+(x_1+x_4)\,y_{5,3} \quad -x_3\,y_{3,4}+x_2\,y_{4,4}+(x_1+x_4)\,y_{5,4}]$$

```
> Row(Sol2bis[4][1],3);
```

$$\left[\begin{array}{cccc} x_2\,y_{2,1}+x_3\,y_{1,1}-\_y & x_2\,y_{2,2}+x_3\,y_{1,2} & x_2\,y_{2,3}+x_3\,y_{1,3} & x_2\,y_{2,4}+x_3\,y_{1,4}-\_y \end{array}\right]$$

```
> Row(Sol2bis[4][1],4);
```

$$[x_2\,y_{3,1}-x_3\,y_{4,1}+(x_1+x_4)\,y_{6,1} \quad x_2\,y_{3,2}-x_3\,y_{4,2}+(x_1+x_4)\,y_{6,2}$$
$$x_2\,y_{3,3}-x_3\,y_{4,3}+(x_1+x_4)\,y_{6,3} \quad x_2\,y_{3,4}-x_3\,y_{4,4}+(x_1+x_4)\,y_{6,4}]$$

Let us successively display the rows of the v-component of the second solution:

```
> Row(Sol2bis[4][2],1);
```

$$\left[\begin{array}{cccc} y_{1,1} & y_{1,2} & y_{1,3} & y_{1,4} \end{array}\right]$$

```
> Row(Sol2bis[4][2],2);
```

$$[-x_3\,y_{2,1}+x_2\,y_{3,1}+(x_1+x_4)\,y_{4,1} \quad -x_3\,y_{2,2}+x_2\,y_{3,2}+(x_1+x_4)\,y_{4,2}$$
$$-x_3\,y_{2,3}+x_2\,y_{3,3}+(x_1+x_4)\,y_{4,3} \quad -x_3\,y_{2,4}+x_2\,y_{3,4}+(x_1+x_4)\,y_{4,4}]$$

```
> Row(Sol2bis[4][2],3);
```

$$[-y_{1,1}+x_3\,(2\_y\,x_4\,y_{1,1}+x_2\,y_{5,1}+\_y)-x_2\,x_3\,y_{5,1} \quad -y_{1,2}+x_3\,(2\,\_y\,x_4\,y_{1,2}+x_2\,y_{5,2})-x_2\,x_3\,y_{5,2}$$
$$-y_{1,3}+x_3\,(2\,\_y\,x_4\,y_{1,3}+x_2\,y_{5,3})-x_2\,x_3\,y_{5,3} \quad -y_{1,4}+x_3\,(2\,\_y\,x_4\,y_{1,4}+x_2\,y_{5,4}+\_y)-x_2\,x_3\,y_{5,4}]$$

```
> Row(Sol2bis[4][2],4);
```

$$[x_2\,y_{2,1}-x_3\,y_{3,1}+(x_1+x_4)\,y_{6,1} \quad x_2\,y_{2,2}-x_3\,y_{3,2}+(x_1+x_4)\,y_{6,2}$$
$$x_2\,y_{2,3}-x_3\,y_{3,3}+(x_1+x_4)\,y_{6,3} \quad x_2\,y_{2,4}-x_3\,y_{3,4}+(x_1+x_4)\,y_{6,4}]$$

Let us successively display the rows of the v-component of the third solution:

```
> Row(Sol2bis[4][3],1);
```

$$\left[\begin{array}{cccc} y_{1,1} & y_{1,2} & y_{1,3} & y_{1,4} \end{array}\right]$$

```
> Row(Sol2bis[4][3],2);
```

$$[-x_3\,y_{2,1}+x_2\,y_{3,1}+(x_1+x_4)\,y_{4,1} \quad -x_3\,y_{2,2}+x_2\,y_{3,2}+(x_1+x_4)\,y_{4,2}$$
$$-x_3\,y_{2,3}+x_2\,y_{3,3}+(x_1+x_4)\,y_{4,3} \quad -x_3\,y_{2,4}+x_2\,y_{3,4}+(x_1+x_4)\,y_{4,4}]$$

```
> Row(Sol2bis[4][3],3);
```

$$[-y_{1,1}+\_y\,x_3^2\,x_4\,y_{5,1}+x_2\,(2\,\_y\,x_4\,y_{1,1}+\_y-y_{5,1}) \quad -y_{1,2}+\_y\,x_3^2\,x_4\,y_{5,2}+x_2\,(2\,\_y\,x_4\,y_{1,2}-y_{5,2})$$
$$-y_{1,3}+\_y\,x_3^2\,x_4\,y_{5,3}+x_2\,(2\,\_y\,x_4\,y_{1,3}-y_{5,3}) \quad -y_{1,4}+\_y\,x_3^2\,x_4\,y_{5,4}+x_2\,(2\,\_y\,x_4\,y_{1,4}+\_y-y_{5,4})]$$

```
> Row(Sol2bis[4][3],4);
```

$$[x_2\,y_{2,1}-x_3\,y_{3,1}+(x_1+x_4)\,y_{6,1} \quad x_2\,y_{2,2}-x_3\,y_{3,2}+(x_1+x_4)\,y_{6,2}$$
$$x_2\,y_{2,3}-x_3\,y_{3,3}+(x_1+x_4)\,y_{6,3} \quad x_2\,y_{2,4}-x_3\,y_{3,4}+(x_1+x_4)\,y_{6,4}]$$

Using the option “reduced”, we get shorter expressions for the v-components of the solutions for $k=2$.

`RankFactorization`$(M,[D1,D2,D3,D4],3)$ & `Solutions`$(M,[D1,D2,D3,D4],3)$ Let us now apply `RankFactorization` with $k=3$.

```
>  RF3 := RankFactorization(M,[D1,D2,D3,D4],3):
```

We first obtain that the ideal $\mathcal{J}_3$ is generated by:

```
>  RF3[1];
```

$$[x_3, x_2, x_1 + x_4]$$

Thus, we have $\mathcal{J}_3 = \langle x_3,\, x_2,\, x_1 + x_4\rangle$ and $\mathcal{S}_3 = \mathcal{R}/\mathcal{J}_3$.

The second output is the matrix K_3 defined by:

```
>  RF3[2];
```

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Hence, we have $K_3 = I_4$. Again, the third one is the matrix Y defined by:

```
>  RF3[3];
```

$$\begin{bmatrix} 1 & 0 & 0 & 1 \end{bmatrix}$$

The fourth output is a list of the non-nilpotent elements of a set of generators $\{g_{3,i}\}_{i\in I_3}$ of $\mathcal{I}_3 = \mathrm{Fitt}_0(\mathcal{B}_3)$:

```
>  RF3[4];
```

$$[x_4]$$

We thus get $I_3 = \{1\}$ and $g_{3,1} = x_4$. In particular, a unique solution of the rank factorization problem eq. (4) can be found over the localization of $\mathcal{S}_3$ with respect to the multiplicative set $\{h_{3,1}^k\}_{k\in\mathbb{Z}}$, i.e., $\mathcal{S}_{3,h_{3,1}} = \mathcal{S}_0[y]/\langle y\, h_{3,1} - 1\rangle = \mathcal{S}_0\left[h_{3,1}^{-1}\right]$.

The next output is a right inverse of the matrix B_3 over the ring $\mathcal{S}_{3,h_{3,1}}$:

```
>  RF3[5];
```

$$\mathrm{table}\left(\left[1 = \begin{bmatrix} 0 \\ 0 \\ \_y \\ 0 \end{bmatrix}\right]\right)$$

The sixth output is the right kernel of the matrix B_3 over $\mathcal{S}_{3,h_{3,1}}$, i.e., $\ker_{\mathcal{S}_{3,h_{3,1}}}(B_3.)$.

```
>  RF3[6];
```

$$\mathrm{table}\left(\left[1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right]\right)$$

The next output is the polynomial ring $\mathcal{R}[y]$.

```
>  RF3[7];
```

$$[\mathit{Ore\_algebra}, [\text{“diff”}, \text{“diff”}, \text{“diff”}, \text{“diff”}, \text{“diff”}], [t_1, t_2, t_3, t_4, \_t], [x_1, x_2, x_3, x_4, \_y], [t_1, t_2, t_3, t_4, \_t], [],$$
$$0, [], [], [t_1, t_2, t_3, t_4, \_t], [], [], [\mathit{diff} = [x_1, t_1], \mathit{diff} = [x_2, t_2], \mathit{diff} = [x_3, t_3], \mathit{diff} = [x_4, t_4], \mathit{diff} = [\_y, \_t]],$$
$$\left[\_a \to \_a * x_1 - \left(\frac{\partial}{\partial t_1}\_a\right), \_a \to \_a * x_2 - \left(\frac{\partial}{\partial t_2}\_a\right), \_a \to \_a * x_3 - \left(\frac{\partial}{\partial t_3}\_a\right),\right.$$
$$\left.\_a \to \_a * x_4 - \left(\frac{\partial}{\partial t_4}\_a\right), \_a \to \_a * \_y - \left(\frac{\partial}{\partial \_t}\_a\right)\right]]$$

Again, the last output is the matrix $A = (D_1\, x \ \ldots \ D_4\, x)$, where $x = (x_1 \ \ldots \ x_4)^T$, defined by:

```
> RF3[8];
```

$$\begin{bmatrix} x_1 & 0 & x_4 & 0 \\ 0 & x_2 & 0 & x_3 \\ 0 & -x_3 & 0 & -x_2 \\ -x_4 & 0 & -x_1 & 0 \end{bmatrix}$$

If the "reduced" option is used, then the same outputs are obtained.

Let us now directly compute the solutions of the rank factorization for $k = 3$.

```
> Sol3 := Solutions(M,[D1,D2,D3,D4],3):
> for i from 1 to nops(Sol3) do print(Sol3[i]) od;
```

$$[x_3, x_2, x_1 + x_4], [x_4], \begin{bmatrix} x_1 & 0 & x_4 & 0 \\ 0 & x_2 & 0 & x_3 \\ 0 & -x_3 & 0 & -x_2 \\ -x_4 & 0 & -x_1 & 0 \end{bmatrix},$$

$$\text{table}\left(\left[1 = \begin{bmatrix} y_{1,1} & y_{1,2} & y_{1,3} & y_{1,4} \\ y_{2,1} & y_{2,2} & y_{2,3} & y_{2,4} \\ \_y + y_{1,1} & y_{1,2} & y_{1,3} & \_y + y_{1,4} \\ y_{3,1} & y_{3,2} & y_{3,3} & y_{3,4} \end{bmatrix}\right]\right)$$

$$\text{table}\,([1 = [\ \mathit{Ore\_algebra}, [\text{“diff”}, \text{“diff”}, \text{“diff”}, \text{“diff”}, \text{“diff”}], [t_1, t_2, t_3, t_4, \_t], [x_1, x_2, x_3, x_4, \_y], [t_1, t_2, t_3, t_4, \_t], [y_{1,1}, y_{1,2}, y_{1,3}, y_{1,4}, y_{2,1}, y_{2,2}, y_{2,3}, y_{2,4}, y_{3,1}, y_{3,2}, y_{3,3}, y_{3,4}], 0, [], [], [t_1, t_2, t_3, t_4, \_t], [], [], [\mathit{diff} = [x_1, t_1], \mathit{diff} = [x_2, t_2], \mathit{diff} = [x_3, t_3], \mathit{diff} = [x_4, t_4], \mathit{diff} = [\_y, \_t]],$$

$$\left[\_a \to \_a * x_1 - \left(\frac{\partial}{\partial t_1}\_a\right), \_a \to \_a * x_2 - \left(\frac{\partial}{\partial t_2}\_a\right), \_a \to \_a * x_3 - \left(\frac{\partial}{\partial t_3}\_a\right), \_a \to \_a * x_4 - \left(\frac{\partial}{\partial t_4}\_a\right), \_a \to \_a * \_y - \left(\frac{\partial}{\partial \_t}\_a\right)\right]\ ]])$$

Therefore, we have $\mathcal{J}_3 = \langle x_3,\, x_2,\, x_1 + x_4\rangle$, $\mathcal{S}_3 = \mathcal{R}/\mathcal{J}_3$, $\mathcal{I}_3 = \langle x_4\rangle_{\mathcal{S}_3}$, $u \in \mathcal{V}(\mathcal{J}_3) \setminus \mathcal{V}(\langle x_4\rangle)$ and the v-component of the corresponding solution $(u,\, v)$ of the rank factorization problem is given by the matrix defined in the above table.

We can finally check again that the above expressions are solutions of the rank factorization problem:

```
> IsSolution(Sol3);
```

$$\text{table}\,([1 = [\mathit{true}]])$$

As shown above, the "reduced" option does not simplify the solutions in the case of $k = 3$.

6.2.3 Computation of the solutions of the rank factorization for Example 2

Let us consider again Example 2, i.e., the rank factorization problem eq. (4) with the following matrices:

```
> M := Matrix([[1,0],[0,1]]);
```

$$M := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

```
> D1 := Matrix([[-1,-2],[1,2]]);
```

$$D1 := \begin{bmatrix} -1 & -2 \\ 1 & 2 \end{bmatrix}$$

```
> D2 := Matrix([[-3,-4],[3,4]]);
```

$$D2 := \begin{bmatrix} -3 & -4 \\ 3 & 4 \end{bmatrix}$$

Let us first compute the solutions of the corresponding rank factorization problem for $k = 0$.

```
> Sol0 := Solutions(M,[D1,D2],0);
```

"No solutions"

Thus, no solutions exist for $k = 0$.

Finally, let us compute the solutions of the corresponding rank factorization problem for $k = 1$.

```
> Sol1 := Solutions(M,[D1,D2],1);
```

"No solutions"

Therefore, as shown in Example 2, the rank factorization problem has no solution.

6.2.4 Computation of the solutions of the rank factorization for Example 3

Let us consider again Example 3, i.e., the rank factorization problem eq. (4) with the following matrices:

```
> M := Matrix([[15,14,13],[24,20,16]]);
```

$$M := \begin{bmatrix} 15 & 14 & 13 \\ 24 & 20 & 16 \end{bmatrix}$$

```
> D1 := Matrix([[1,-1],[1,1]]);
```

$$D1 := \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

```
> D2 := Matrix([[1,2],[-1,2]]);
```

$$D2 := \begin{bmatrix} 1 & 2 \\ -1 & 2 \end{bmatrix}$$

```
> D3 := Matrix([[1,3],[4,3]]);
```

$$D3 := \begin{bmatrix} 1 & 3 \\ 4 & 3 \end{bmatrix}$$

Let us first compute the solutions of the corresponding rank factorization problem for $k = 0$.

```
> Sol0 := Solutions(M,[D1,D2,D3],0):
> for i from 1 to 3 do print(Sol0[i]) od;
```

$$[x_2^2,\, x_2\, x_1,\, x_1^2], [x_2^2,\, x_2\, x_1,\, x_1^2], \begin{bmatrix} x_1 - x_2 & x_1 + 2x_2 & x_1 + 3x_2 \\ x_1 + x_2 & -x_1 + 2x_2 & 4x_1 + 3x_2 \end{bmatrix}$$

```
> nops(op(Sol0[4]));
```

3

Thus, we have three families of solutions.

Let us successively display the rows of the v-component of the first family of solutions

```
> Row(Sol0[4][1],1);
```

$$\Big[-\frac{15\_y\,(35\,x_1 + 194\,x_2)}{388} + \frac{6\_y(85\,x_1 + 194\,x_2)}{97} + \left(5\,x_1^2 + 12\,x_2 x_1\right)\, y_{1,1}$$
$$-\frac{7\_y\,(35\,x_1 + 194\,x_2)}{194} + \frac{5\_y\,(85\,x_1 + 194\,x_2)}{97} + \left(5\,x_1^2 + 12\,x_2\,x_1\right)\, y_{1,2}$$
$$-\frac{13\_y\,(35\,x_1 + 194\,x_2)}{388} + \frac{4\_y\,(85\,x_1 + 194\,x_2)}{97} + \left(5\,x_1^2 + 12\,x_2 x_1\right)\, y_{1,3} \Big]$$

```
> Row(Sol0[4][1],2);
```

$$\Big[\frac{15\_y\,(21\,x_1 + 31\,x_2)}{388} - \frac{6\_y\,(51\,x_1 - 91\,x_2)}{97} + \left(-3\,x_1^2 + 5\,x_2\,x_1 + 6\,x_2^2\right)\, y_{1,1}$$
$$\frac{7\_y\,(21\,x_1 + 31\,x_2)}{194} - \frac{5\_y\,(51\,x_1 - 91\,x_2)}{97} + \left(-3\,x_1^2 + 5\,x_2\,x_1 + 6\,x_2^2\right)\, y_{1,2}$$
$$\frac{13\_y\,(21\,x_1 + 31\,x_2)}{388} - \frac{4\_y\,(51\,x_1 - 91\,x_2)}{97} + \left(-3\,x_1^2 + 5\,x_2\,x_1 + 6\,x_2^2\right)\, y_{1,3} \Big]$$

```
> Row(Sol0[4][1],3);
```

$$[\ \frac{15\_y\,(7\,x_1+22\,x_2)}{194}-\frac{12\_y\,(17\,x_1-2\,x_2)}{97}+\left(-2\,x_1^2-4\,x_2^2\right)\,y_{1,1}$$
$$\frac{7\_y\,(7\,x_1+22\,x_2)}{97}-\frac{10\_y\,(17\,x_1-2\,x_2)}{97}+\left(-2\,x_1^2-4\,x_2^2\right)\,y_{1,2}$$
$$\frac{13\_y\,(7\,x_1+22\,x_2)}{194}-\frac{8\_y\,(17\,x_1-2\,x_2)}{97}+\left(-2\,x_1^2-4\,x_2^2\right)\,y_{1,3}\ ]$$

where the y's are arbitrary elements of $\mathbb{K}$.

Let us successively display the rows of the v-component of the second family of solutions

```
> Row(Sol0[4][2],1);
```

$$[\ -\frac{945\_y\,x_1}{194}+\left(5\,x_1^2+12\,x_2\,x_1\right)\,y_{1,1}+(5\,x_1+12\,x_2)\,y_{2,1}$$
$$-\frac{435\_y\,x_1}{97}+\left(5\,x_1^2+12\,x_2\,x_1\right)\,y_{1,2}+(5\,x_1+12\,x_2)\,y_{2,2}$$
$$-\frac{795\_y\,x_1}{194}+\left(5\,x_1^2+12\,x_2\,x_1\right)\,y_{1,3}+(5\,x_1+12\,x_2)\,y_{2,3}\ ]$$

```
> Row(Sol0[4][2],2);
```

$$[\ \frac{45\_y\,(11\,x_1+7\,x_2)}{194}+\frac{36\_y\,(x_1-17\,x_2)}{97}+\left(-3\,x_1^2+5\,x_2\,x_1+6\,x_2^2\right)\,y_{1,1}+\left(6\_y\,x_2^3-3\,x_1+5\,x_2\right)\,y_{2,1}$$
$$\frac{21\_y\,(11\,x_1+7\,x_2)}{97}+\frac{30\_y\,(x_1-17\,x_2)}{97}+\left(-3\,x_1^2+5\,x_2\,x_1+6\,x_2^2\right)\,y_{1,2}+\left(6\_y\,x_2^3-3\,x_1+5\,x_2\right)\,y_{2,2}$$
$$\frac{39\_y\,(11\,x_1+7\,x_2)}{194}+\frac{24\_y\,(x_1-17\,x_2)}{97}+\left(-3\,x_1^2+5\,x_2\,x_1+6\,x_2^2\right)\,y_{1,3}+\left(6\_y\,x_2^3-3\,x_1+5\,x_2\right)\,y_{2,3}\ ]$$

```
> Row(Sol0[4][2],3);
```

$$[\ \frac{15\_y\,(11\,x_1-7\,x_2)}{97}+\frac{24\_y\,(x_1+17\,x_2)}{97}+\left(-2\,x_1^2-4\,x_2^2\right)\,y_{1,1}+\left(-4\_y\,x_2^3-2\,x_1\right)\,y_{2,1}$$
$$\frac{14\_y\,(11\,x_1-7\,x_2)}{97}+\frac{20\_y\,(x_1+17\,x_2)}{97}+\left(-2\,x_1^2-4\,x_2^2\right)\,y_{1,2}+\left(-4\_y\,x_2^3-2\,x_1\right)\,y_{2,2}$$
$$\frac{13\_y\,(11\,x_1-7\,x_2)}{97}+\frac{16\_y\,(x_1+17\,x_2)}{97}+\left(-2\,x_1^2-4\,x_2^2\right)\,y_{1,3}+\left(-4\_y\,x_2^3-2\,x_1\right)\,y_{2,3}\ ]$$

where the y's are arbitrary elements of $\mathbb{K}$.

Finally, let us successively display the rows of the v-component of the third family of solutions

```
> Row(Sol0[4][3],1);
```

$$[\ \frac{1134\_y\,x_1}{97}+\left(5\,x_1^2+12\,x_2\,x_1\right)\,y_{1,1}+(12\_y\,x_1x_2+5)\,y_{2,1}+(5\,x_1+12\,x_2)\,y_{3,1}$$
$$\frac{1044\_y\,x_1}{97}+\left(5\,x_1^2+12\,x_2\,x_1\right)\,y_{1,2}+(12\_y\,x_1\,x_2+5)\,y_{2,2}+(5\,x_1+12\,x_2)\,y_{3,2}$$
$$\frac{954\_y\,x_1}{97}+\left(5\,x_1^2+12\,x_2\,x_1\right)\,y_{1,3}+(12\_y\,x_1\,x_2+5)\,y_{2,3}+(5\,x_1+12\,x_2)\,y_{3,3}\ ]$$

```
> Row(Sol0[4][3],2);
```

$$[\ \frac{15\_y\,(38\,x_1+33\,x_2)}{97}-\frac{24\_y\,(23\,x_1-3\,x_2)}{97}$$
$$+\left(-3\,x_1^2+5\,x_2\,x_1+6\,x_2^2\right)\,y_{1,1}+\left(5\_y\,x_1\,x_2+6\_y\,x_2^2-3\right)\,y_{2,1}+\left(6\_y\,x_1\,x_2^2-3\,x_1+5\,x_2\right)\,y_{3,1}$$
$$\frac{14\_y\,(38\,x_1+33\,x_2)}{97}-\frac{20\_y\,(23\,x_1-3\,x_2)}{97}+\left(-3\,x_1^2+5\,x_2\,x_1+6\,x_2^2\right)\,y_{1,2}$$
$$+\left(5\_y\,x_1\,x_2+6\_y\,x_2^2-3\right)\,y_{2,2}+\left(6\_y\,x_1\,x_2^2-3\,x_1+5\,x_2\right)\,y_{3,2}$$
$$\frac{13\_y\,(38\,x_1+33\,x_2)}{97}-\frac{16\_y\,(23\,x_1-3\,x_2)}{97}+\left(-3\,x_1^2+5\,x_2\,x_1+6\,x_2^2\right)\,y_{1,3}$$
$$+\left(5\_y\,x_1\,x_2+6\_y\,x_2^2-3\right)\,y_{2,3}+\left(6\_y\,x_1\,x_2^2-3\,x_1+5\,x_2\right)\,y_{3,3}\ ]$$

```
> Row(Sol0[4][3],3);
```

$$[-\frac{15\,\_y\,(7\,x_1+22\,x_2)}{97}+\frac{24\,\_y(17\,x_1-2\,x_2)}{97}+\left(-2\,x_1^2-4\,x_2^2\right)\,y_{1,1}+\left(-4\,\_y x_2^2-2\right)\,y_{2,1}+\left(-4\,\_y x_1\,x_2^2-2\,x_1\right)\,y_{3,1}$$
$$-\frac{14\,\_y\,(7\,x_1+22\,x_2)}{97}+\frac{20\,\_y\,(17\,x_1-2\,x_2)}{97}+\left(-2\,x_1^2-4\,x_2^2\right)\,y_{1,2}+\left(-4\,\_y x_2^2-2\right)\,y_{2,2}+\left(-4\,\_y x_1\,x_2^2-2\,x_1\right)\,y_{3,2}$$
$$-\frac{13\,\_y\,(7\,x_1+22\,x_2)}{97}+\frac{16\,\_y\,(17\,x_1-2\,x_2)}{97}+\left(-2\,x_1^2-4\,x_2^2\right)\,y_{1,3}+\left(-4\,\_y x_2^2-2\right)\,y_{2,3}+\left(-4\,\_y x_1\,x_2^2-2\,x_1\right)\,y_{3,3}\,]$$

where the y's are arbitrary elements of $\mathbb{K}$.

We can check again that the above expressions are solutions to the rank factorization problem:

```
> IsSolution(Sol0);
```

$$\mathrm{table}\left([1=[\mathit{true}]\,,2=[\mathit{true}]\,,3=[\mathit{true}]]\right)$$

Finally, let us compute the solutions of the corresponding rank factorization problem for $k = 1$.

```
> Sol1 := Solutions(M,[D1,D2,D3],1);
```

"No solutions"

Finally, we can add the "reduced" option to the `Solutions` function to get shorter outputs for the v-components of the solutions.

```
> Sol0bis := Solutions(M,[D1,D2,D3],0,"reduced"):
> for i from 1 to 3 do print(Sol0bis[i]) od;
```

$$[x_2^2,\,x_2\,x_1,\,x_1^2],[x_2^2,\,x_2\,x_1,\,x_1^2],\begin{bmatrix} x_1-x_2 & x_1+2\,x_2 & x_1+3\,x_2\\ x_1+x_2 & -x_1+2\,x_2 & 4\,x_1+3\,x_2\end{bmatrix}$$

```
> nops(op(Sol0bis[4]));
```

3

Thus, we have three families of solutions.

Let us successively display the rows of the v-component of the first family of solutions

```
> Row(Sol0bis[4][1],1);
```

$$[-\frac{15\,\_y(35\,x_1+194\,x_2)}{388}+\frac{6\,\_y\,(85\,x_1+194\,x_2)}{97}+\left(5\,x_1^2+12\,x_1\,x_2\right)\,y_{1,1}$$
$$-\frac{7\,\_y\,(35\,x_1+194\,x_2)}{194}+\frac{5\,\_y\,(85\,x_1+194\,x_2)}{97}+\left(5\,x_1^2+12\,x_1\,x_2\right)\,y_{1,2}$$
$$-\frac{13\,\_y\,(35\,x_1+194\,x_2)}{388}+\frac{4\,\_y\,(85\,x_1+194\,x_2)}{97}+\left(5\,x_1^2+12\,x_1\,x_2\right)\,y_{1,3}\,]$$

```
> Row(Sol0bis[4][1],2);
```

$$[\frac{15\,\_y\,(21\,x_1+31\,x_2)}{388}-\frac{6\,\_y\,(51\,x_1-91\,x_2)}{97}+\left(-3\,x_1^2+5\,x_1\,x_2+6\,x_2^2\right)\,y_{1,1}$$
$$\frac{7\,\_y\,(21\,x_1+31\,x_2)}{194}-\frac{5\,\_y\,(51\,x_1-91\,x_2)}{97}+\left(-3\,x_1^2+5\,x_1\,x_2+6\,x_2^2\right)\,y_{1,2}$$
$$\frac{13\,\_y\,(21\,x_1+31\,x_2)}{388}-\frac{4\,\_y\,(51\,x_1-91\,x_2)}{97}+\left(-3\,x_1^2+5\,x_1\,x_2+6\,x_2^2\right)\,y_{1,3}\,]$$

```
> Row(Sol0bis[4][1],3);
```

$$[\frac{15\,\_y\,(7\,x_1+22\,x_2)}{194}-\frac{12\,\_y\,(17\,x_1-2\,x_2)}{97}+\left(-2\,x_1^2-4\,x_2^2\right)\,y_{1,1}$$
$$\frac{7\,\_y\,(7\,x_1+22\,x_2)}{97}-\frac{10\,\_y(17\,x_1-2\,x_2)}{97}+\left(-2\,x_1^2-4\,x_2^2\right)\,y_{1,2}$$
$$\frac{13\,\_y\,(7\,x_1+22\,x_2)}{194}-\frac{8\,\_y\,(17\,x_1-2\,x_2)}{97}+\left(-2\,x_1^2-4\,x_2^2\right)\,y_{1,3}\,]$$

where the y's are arbitrary elements of $\mathbb{K}$.

Let us successively display the rows of the v-component of the second family of solutions

```
> Row(Sol0bis[4][2],1);
```

$$\left[\ -\frac{945\_y\,x_1}{194}+\left(5\,x_1^2+12\,x_1\,x_2\right)\,y_{1,1}\quad -\frac{435\_y\,x_1}{97}+\left(5\,x_1^2+12\,x_1\,x_2\right)\,y_{1,2}\quad -\frac{795\_y\,x_1}{194}+\left(5\,x_1^2+12\,x_1\,x_2\right)\,y_{1,3}\ \right]$$

```
> Row(SolObis[4][2],2);
```

$$[\ \frac{45\_y\,(11\,x_1+7\,x_2)}{194}+\frac{36\_y\,(x_1-17\,x_2)}{97}+\left(-3\,x_1^2+5\,x_1\,x_2+6\,x_2^2\right)\,y_{1,1}$$
$$\frac{21\_y\,(11\,x_1+7\,x_2)}{97}+\frac{30\_y\,(x_1-17\,x_2)}{97}+\left(-3\,x_1^2+5\,x_1\,x_2+6\,x_2^2\right)\,y_{1,2}$$
$$\frac{39\_y\,(11\,x_1+7\,x_2)}{194}+\frac{24\_y\,(x_1-17\,x_2)}{97}+\left(-3\,x_1^2+5\,x_1\,x_2+6\,x_2^2\right)\,y_{1,3}\ ]$$

```
> Row(SolObis[4][2],3);
```

$$[\ \frac{15\_y\,(11\,x_1-7\,x_2)}{97}+\frac{24\_y\,(x_1+17\,x_2)}{97}+\left(-2\,x_1^2-4\,x_2^2\right)\,y_{1,1}$$
$$\frac{14\_y\,(11\,x_1-7\,x_2)}{97}+\frac{20\_y\,(x_1+17\,x_2)}{97}+\left(-2\,x_1^2-4\,x_2^2\right)\,y_{1,2}$$
$$\frac{13\_y\,(11\,x_1-7\,x_2)}{97}+\frac{16\_y\,(x_1+17\,x_2)}{97}+\left(-2\,x_1^2-4\,x_2^2\right)\,y_{1,3}\ ]$$

where the y's are arbitrary elements of $\mathbb{K}$.

Finally, let us successively display the rows of the v-component of the third family of solutions

```
> Row(SolObis[4][3],1);
```

$$\left[\ \frac{1134\_y\,x_1}{97}+\left(5\,x_1^2+12\,x_1\,x_2\right)\,y_{1,1}\quad \frac{1044\_y\,x_1}{97}+\left(5\,x_1^2+12\,x_1\,x_2\right)\,y_{1,2}\quad \frac{954\_y\,x_1}{97}+\left(5\,x_1^2+12\,x_1\,x_2\right)\,y_{1,3}\ \right]$$

```
> Row(SolObis[4][3],2);
```

$$[\ \frac{15\_y\,(38\,x_1+33\,x_2)}{97}-\frac{24\_y\,(23\,x_1-3\,x_2)}{97}+\left(-3\,x_1^2+5\,x_1\,x_2+6\,x_2^2\right)\,y_{1,1}$$
$$\frac{14\_y\,(38\,x_1+33\,x_2)}{97}-\frac{20\_y\,(23\,x_1-3\,x_2)}{97}+\left(-3\,x_1^2+5\,x_1\,x_2+6\,x_2^2\right)\,y_{1,2}$$
$$\frac{13\_y\,(38\,x_1+33\,x_2)}{97}-\frac{16\_y\,(23\,x_1-3\,x_2)}{97}+\left(-3\,x_1^2+5\,x_1\,x_2+6\,x_2^2\right)\,y_{1,3}\ ]$$

```
> Row(SolObis[4][3],3);
```

$$[\ -\frac{15\_y\,(7\,x_1+22\,x_2)}{97}+\frac{24\_y\,(17\,x_1-2\,x_2)}{97}+\left(-2\,x_1^2-4\,x_2^2\right)\,y_{1,1}$$
$$-\frac{14\_y\,(7\,x_1+22\,x_2)}{97}+\frac{20\_y\,(17\,x_1-2\,x_2)}{97}+\left(-2\,x_1^2-4\,x_2^2\right)\,y_{1,2}$$
$$-\frac{13\_y\,(7\,x_1+22\,x_2)}{97}+\frac{16\_y\,(17\,x_1-2\,x_2)}{97}+\left(-2\,x_1^2-4\,x_2^2\right)\,y_{1,3}\ ]$$

where the y's are arbitrary elements of $\mathbb{K}$. Using the option "reduced", we thus get shorter expressions for the v-components of the solutions. We find again the solutions obtained in Example 2.

6.2.5 Computation of the solutions of the rank factorization for Example 4

Let us consider again Example 4, i.e., the rank factorization problem for the following matrices:

```
> M := Matrix([[30,0$2],[0$3],[12,0$2],[12,0$2]]);
```

$$\begin{bmatrix} 30 & 0 & 0 \\ 0 & 0 & 0 \\ 12 & 0 & 0 \\ 12 & 0 & 0 \end{bmatrix}$$

```
> D1 := Matrix([[0$3,2],[3,0$2,1],[0$4],[0$3,2]]);
```

$$\begin{bmatrix} 0 & 0 & 0 & 2 \\ 3 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

```
> D2 := Matrix([[5,3,0$2],[0$4],[0,5,2,0],[0,3,2,0]]);
```

$$\begin{bmatrix} 5 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 5 & 2 & 0 \\ 0 & 3 & 2 & 0 \end{bmatrix}$$

Let us compute the solutions of the corresponding rank factorization for $k = 0$:

```
> Sol0 := Solutions(M,[D1,D2],0,"reduced"):
> for i from 1 to 3 do print(Sol0[i]) od;
```

$$[2\,x_4\,x_1 - 3\,x_2\,x_4 - 2\,x_3\,x_4,\; 3\,x_1\,x_2 + x_2\,x_4,\; 2\,x_1^2 - 2\,x_1\,x_3 + x_2\,x_4,$$
$$9\,x_2^2\,x_4 + 6\,x_3\,x_2\,x_4 + 2\,x_2\,x_4^2], [5\,x_2\,x_4 + 2\,x_3\,x_4, 3\,x_1\,x_3 + x_3\,x_4],$$
$$\begin{bmatrix} 2\,x_4 & 5\,x_1 + 3\,x_2 \\ 3\,x_1 + x_4 & 0 \\ 0 & 5\,x_2 + 2\,x_3 \\ 2\,x_4 & 3\,x_2 + 2\,x_3 \end{bmatrix}$$

```
> nops(Sol0[4]);
```

$$2$$

Thus, we have two families of solutions. Let us display the v-component of the first family of solutions

```
> Sol0[4][1];
```

$$[\, 3\,x_2\,y_{1,1} + (2\,x_1 - 2\,x_3)\left(4\,\_y - y_{1,1} + \left(24\,\_y\,x_3\,x_4 + 20\,\_y\,x_4^2 + 18\right)\,y_{2,1}\right)$$
$$3\,x_2\,y_{1,2} + (2\,x_1 - 2\,x_3)\left(-y_{1,2} + \left(24\,\_y\,x_3\,x_4 + 20\,\_y\,x_4^2 + 18\right)\,y_{2,2}\right)$$
$$3\,x_2\,y_{1,3} + (2\,x_1 - 2\,x_3)\left(-y_{1,3} + \left(24\,\_y\,x_3\,x_4 + 20\,\_y\,x_4^2 + 18\right)\,y_{2,3}\right)\,]$$

where the y's are arbitrary elements of $\mathbb{K}$.

Let us now display the v-component of the second family of solutions

```
> Sol0[4][2];
```

$$[\, 3\,x_4\,y_{1,1} + 3\,x_4\,\left(4\,\_y - y_{1,1} + \left(24\,\_y\,x_3\,x_4 + 20\,\_y\,x_4^2 + 18\right)\,y_{2,1}\right) - 60\,(3\,x_1 + x_4)\,\_y\,x_4^2 y_{\,2,1}$$
$$3\,x_4\,y_{1,2} + 3\,x_4\,\left(-y_{1,2} + \left(24\,\_y\,x_3\,x_4 + 20\,\_y\,x_4^2 + 18\right)\,y_{2,2}\right) - 60\,(3\,x_1 + x_4)\,\_y\,x_4^2\,y_{2,2}$$
$$3\,x_4\,y_{1,3} + 3\,x_4\,\left(-y_{1,3} + \left(24\,\_y\,x_3\,x_4 + 20\,\_y\,x_4^2 + 18\right)\,y_{2,3}\right) - 60\,(3\,x_1 + x_4)\,\_y\,x_4^2\,y_{2,3}\,]$$

where the y's are arbitrary elements of $\mathbb{K}$.

The last element of `Sol0` is the ring $\mathcal{R}[y]$:

```
> Sol0[5];
```

$$\text{table}\,([1 = [\;\; \mathit{Ore\_algebra}, [\text{"diff"}, \text{"diff"}, \text{"diff"}, \text{"diff"}, \text{"diff"}], [t_1, t_2, t_3, t_4, \_t], [x_1, x_2, x_3, x_4, \_y], [t_1, t_2, t_3, t_4,$$
$$\_t], [y_{1,1}, y_{1,2}, y_{1,3}, y_{2,1}, y_{2,2}, y_{2,3}]\,, 0, [], [], [t_1, t_2, t_3, t_4, \_t], [], [], [\mathit{diff} = [x_1, t_1], \mathit{diff} = [x_2, t_2],$$
$$\mathit{diff} = [x_3, t_3], \mathit{diff} = [x_4, t_4], \mathit{diff} = [\_y, \_t]],$$
$$\left[\_a \to \_a * x_1 - \left(\frac{\partial}{\partial t_1}\_a\right), \_a \to \_a * x_2 - \left(\frac{\partial}{\partial t_2}\_a\right), \_a \to \_a * x_3 - \left(\frac{\partial}{\partial t_3}\_a\right),\right.$$
$$\left.\_a \to \_a * x_4 - \left(\frac{\partial}{\partial t_4}\_a\right), \_a \to \_a * \_y - \left(\frac{\partial}{\partial \_t}\_a\right)\right]\;\;]])$$

We can check again that the above two expressions are solutions to the rank factorization problem:

```
> IsSolution(Sol0);
```

$$\text{table}\,([1 = [\mathit{true}]\,, 2 = [\mathit{true}]])$$

Finally, let us compute the solutions for $k = 1$:

```
> Sol1 := Solutions(M,[D1,D2],1);
```

"No solutions"

No solutions thus exist for $k = 1$.

6.2.6 Computation of the solutions of the rank factorization for Example 7

Let us consider again Example 7 studying the rank factorization problem eq. (4) for the following matrices:

```
> M := Matrix([[0$3],[0$3]]);
```

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

```
> D1 := DiagonalMatrix([1$2]);
```

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

```
> D2 := Matrix([[0,1],[1,0]]);
```

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

```
> D3 := Matrix([[2,1],[1,2]]);
```

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

Let us compute a parametrization of the solutions of the rank factorization problem for $k = 0$:

```
> Sol0 := Solutions(M,[D1,D2,D3],0):
> for i from 1 to 4 do print(Sol0[i]) od;
```

$$[0], [0], \begin{bmatrix} x_1 & x_2 & 2\,x_1 + x_2 \\ x_2 & x_1 & x_1 + 2\,x_2 \end{bmatrix}, \text{table}\left(\left[1 = \begin{bmatrix} 2\,y_{1,1} & 2\,y_{1,2} & 2\,y_{1,3} \\ y_{1,1} & y_{1,2} & y_{1,3} \\ -y_{1,1} & -y_{1,2} & -y_{1,3} \end{bmatrix}\right]\right)$$

The y's are arbitrary elements of $\mathbb{K}$. `Sol0[5]` corresponds to the following polynomial ring:

```
> Sol0[5];
```

$$[\mathit{Ore\_algebra}, [\text{"diff"}, \text{"diff"}], [t_1, t_2, t_3], [x_1, x_2], [t_1, t_2], [y_{1,1}, y_{1,2}, y_{1,3}], 0, [], [], [t_1, t_2], [], [], [\mathit{diff} = [x_1, t_1],$$
$$\mathit{diff} = [x_2, t_2]], \left[\_a \to \_a * x_1 - \left(\frac{\partial}{\partial t_1}\_a\right), \_a \to \_a * x_2 - \left(\frac{\partial}{\partial t_2}\_a\right)\right]]$$

Let us now check again that `Sol0` defines a family of solutions to the rank factorization problem:

```
> IsSolution(Sol0);
```

$$\text{table}\left([1 = [\mathit{true}]]\right)$$

Let us now compute a parametrization of the solutions for $k = 1$:

```
> Sol1 := Solutions(M,[D1,D2,D3],1):
> for i from 1 to 3 do print(Sol1[i]) od;
```

$$[x_1^2 - x_2^2], [0], \begin{bmatrix} x_1 & x_2 & 2x_1 + x_2 \\ x_2 & x_1 & x_1 + 2x_2 \end{bmatrix}$$

```
> Sol1[4];
```

$$\text{table}\left(\left[1 = \begin{bmatrix} 2\,y_{1,1} & 2\,y_{1,2} & 2\,y_{1,3} \\ y_{1,1} - 3\,x_2\,y_{2,1} + (2\,x_1 + x_2)\;y_{3,1} & y_{1,2} - 3\,x_2\,y_{2,2} + (2\,x_1 + x_2)\;y_{3,2} & y_{1,3} - 3\,x_2\,y_{2,3} + (2\,x_1 + x_2)\;y_{3,3} \\ -y_{1,1} + (2\,x_1 - x_2)\;y_{2,1} - x_2\,y_{3,1} & -y_{1,2} + (2\,x_1 - x_2)\;y_{2,2} - x_2\,y_{3,2} & -y_{1,3} + (2\,x_1 - x_2)\;y_{2,3} - x_2\,y_{3,3} \end{bmatrix}\right]\right)$$

The y's are arbitrary elements of $\mathbb{K}$. `Sol1[5]` corresponds to the following polynomial ring:

```
> Sol1[5];
```

$$[\mathit{Ore\_algebra}, [\text{"diff"}, \text{"diff"}], [t_1, t_2, t_3], [x_1, x_2], [t_1, t_2], [y_{1,1}, y_{1,2}, y_{1,3}, y_{2,1}, y_{2,2}, y_{2,3}, y_{3,1}, y_{3,2}, y_{3,3}],$$
$$0, [], [], [t_1, t_2], [], [], [\mathit{diff} = [x_1, t_1], \mathit{diff} = [x_2, t_2]],$$
$$\left[\_a \to \_a * x_1 - \left(\frac{\partial}{\partial t_1}\_a\right), \_a \to \_a * x_2 - \left(\frac{\partial}{\partial t_2}\_a\right)\right]]$$

Let us now check again that `Sol1` defines a family of solutions to the rank factorization problem:

```
> IsSolution(Sol1);
```

$$\operatorname{table}([1 = [true]])$$

Finally, let us now compute a parametrization of the solutions for $k = 2$:

```
> Sol2 := Solutions(M,[D1,D2,D3],2):
> for i from 1 to 4 do print(Sol2[i]) od;
```

$$[x_2, x_1], [0], \begin{bmatrix} x_1 & x_2 & 2\,x_1 + x_2 \\ x_2 & x_1 & x_1 + 2\,x_2 \end{bmatrix}, \operatorname{table}\left(\left[1 = \begin{bmatrix} y_{1,1} & y_{1,2} & y_{1,3} \\ y_{2,1} & y_{2,2} & y_{2,3} \\ y_{3,1} & y_{3,2} & y_{3,3} \end{bmatrix}\right]\right)$$

```
> Sol2[5];
```

$$[Ore\_algebra, [\text{“diff”}, \text{“diff”}], [t_1, t_2, t_3], [x_1, x_2], [t_1, t_2], [y_{1,1}, y_{1,2}, y_{1,3}, y_{2,1}, y_{2,2}, y_{2,3}, y_{3,1}, y_{3,2}, y_{3,3}],$$
$$0, [], [], [t_1, t_2], [], [], [diff = [x_1, t_1], diff = [x_2, t_2]],$$
$$\left[\_a \to \_a * x_1 - \left(\frac{\partial}{\partial t_1}\_a\right), \_a \to \_a * x_2 - \left(\frac{\partial}{\partial t_2}\_a\right)\right]]$$

Let us now check again that `Sol2` defines a family of solutions to the rank factorization problem:

```
> IsSolution(Sol2);
```

$$\operatorname{table}([1 = [true]])$$

We find again the results given in Example 7.

6.3 Demodulation functions

We briefly describe a few more functions of the `RankFactorization` package which are useful for the study of the demodulation problems (which are associated with the rank factorization problem) listed in Table 3. For more details, see Section 1, [18], and the references therein.

The applications of the results obtained in this paper to the demodulation problems will be developed in a forthcoming paper. Therefore, more functions dedicated to the study of the demodulation problems will be added to the RankFactorization package in the future.

The `AntiDiagonal` function computes the antidiagonal matrix of a given size (see Section 1).

The `LeeMatrix` function defines a *Lee matrix* M of a given size n, namely, a matrix $M \in \mathbb{C}^{n\times n}$ that is *J-real*, i.e., $J_n\,\overline{M} = M$, where J_n is the antidiagonal matrix of size n. Lee matrices are used to define *Lee's transformations* which map sets of centrohermitian matrices to sets of real matrices. For more details, see Section 1, [18], and the references therein.

The `IsCentroHermitian` function tests whether or not a complex matrix $M \in \mathbb{C}^{m\times n}$ is *centroherrmitian*, namely, satisfies $\overline{M} = J_m\,M\,J_n$ (see Section 1).

The `CentroHermitian` function maps a matrix $M \in \mathbb{C}^{m\times n}$ to the centrohermitian $(M + J_m\,\overline{M}\,J_n)/2$.

Let us illustrate these functions with simple examples.

Let us first compute the antidiagonal matrix of sizes 1 and 4:

```
> AntiDiagonal(1);
```

$$\begin{bmatrix} 1 \end{bmatrix}$$

```
> AntiDiagonal(4);
```

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Now, let us define a Lee matrix of size 2:

```
> L := LeeMatrix(2);
```

$$\begin{bmatrix} 1 & I \\ 1 & -I \end{bmatrix}$$

Let us check whether or not the matrix L is centrohermitian:

```
> IsCentroHermitian(L);
```

$$false$$

The matrix L is not centrohermitian. We can then define the centrohermitian $(L + J_2\, L\, J_2)/2$ defined by:

```
> H := CentroHermitian(L);
```

$$H := \begin{bmatrix} \frac{1}{2} + \frac{I}{2} & \frac{1}{2} + \frac{I}{2} \\ \frac{1}{2} - \frac{I}{2} & \frac{1}{2} - \frac{I}{2} \end{bmatrix}$$

We can check again that the matrix H is centrohermitian:

```
> IsCentroHermitian(H);
```

$$true$$

Let us now define a Lee matrix of size 3:

```
> LeeMatrix(3);
```

$$\begin{bmatrix} 1 & 0 & I \\ 0 & 1 & 0 \\ 1 & 0 & -I \end{bmatrix}$$

Using the option "unitary", the `LeeMatrix` function then returns a unitary Lee matrix:

```
> M := LeeMatrix(3,"unitary");
```

$$M := \begin{bmatrix} \frac{\sqrt{2}}{2} & 0 & \frac{I}{2}\sqrt{2} \\ 0 & 1 & 0 \\ \frac{\sqrt{2}}{2} & 0 & -\frac{I}{2}\sqrt{2} \end{bmatrix}$$

We can check again that M is unitary, i.e., $M^\star\, M = I_3$:

```
> simplify(Transpose(conjugate(M)).M);
```

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

If we prefer to work with an algebraic expression for $\sqrt{2}$, i.e., if we want to use a symbol u satisfying the equation $u^2 = 2$, then we can use the option "unitary_symbolic":

```
> R := LeeMatrix(3,"unitary_symbolic",u);
```

$$R := \left[\begin{bmatrix} \frac{1}{u} & 0 & \frac{I}{u} \\ 0 & 1 & 0 \\ \frac{1}{u} & 0 & \frac{-I}{u} \end{bmatrix}, u, u^2 - 2 \right]$$

We can work algebraically with the symbol u as, for instance:

```
> assume(R[2],real):
> S := CentroHermitian(R[1]);
```

$$S := \begin{bmatrix} \frac{1}{2u} + \frac{\mathrm{I}}{2u} & 0 & \frac{1}{2u} + \frac{\mathrm{I}}{2u} \\ 0 & 1 & 0 \\ \frac{1}{2u} - \frac{\mathrm{I}}{2u} & 0 & \frac{1}{2u} - \frac{\mathrm{I}}{2u} \end{bmatrix}$$

```
> IsCentroHermitian(S);
```

$$true$$

Finally, the next two examples show how Lee's transformations can be used to bijectively transform centrohermitian matrices onto real matrices. Such transformations play a central role in the study of the demodulation problems as briefly explained in Section 1.

Let us first consider the following square centrohermitian matrix:

```
> M := Matrix([[9+18*I,-225,9+198*I],[0,0,0],[9-198*I,-225,9-18*I]]);
```

$$M := \begin{bmatrix} 9+18\,\mathrm{I} & -225 & 9+198\,\mathrm{I} \\ 0 & 0 & 0 \\ 9-198\,\mathrm{I} & -225 & 9-18\,\mathrm{I} \end{bmatrix}$$

We can check again that M is centrohermitian:

```
> IsCentroHermitian(S);
```

$$true$$

Let us now define a Lee Matrix of size 3

```
> U := LeeMatrix(3);
```

$$U := \begin{bmatrix} 1 & 0 & \mathrm{I} \\ 0 & 1 & 0 \\ 1 & 0 & -\mathrm{I} \end{bmatrix}$$

which is, by construction, invertible. Let us compute its inverse:

```
> U_inv := MatrixInverse(U);
```

$$U_{inv} := \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ -\frac{\mathrm{I}}{2} & 0 & \frac{\mathrm{I}}{2} \end{bmatrix}$$

We can now introduce the matrix $M_\rho = U^{-1}\, M\, U$ defined by:

```
> M_rho := U_inv.M.U;
```

$$M_{rho} := \begin{bmatrix} 18 & -225 & 180 \\ 0 & 0 & 0 \\ 216 & 0 & 0 \end{bmatrix}$$

We can check that $M_\rho \in \mathbb{R}^{3\times 3}$. Hence, the centrohermitian M is sent to the real matrix $M_\rho = U^{-1}\, M\, U$. Of course, this transformation is invertible:

```
> U.M_rho.U_inv;
```

$$\begin{bmatrix} 9+18\,\mathrm{I} & -225 & 9+198\,\mathrm{I} \\ 0 & 0 & 0 \\ 9-198\,\mathrm{I} & -225 & 9-18\,\mathrm{I} \end{bmatrix}$$

Therefore, the set $\mathrm{CH}_{3,3}$ of the 3×3 centrohermitian matrices is bijectively maps onto $\mathbb{R}^{3\times 3}$.

Let us consider the non-square centrohermitian matrix, i.e., the following 2×5 centrohermitian matrix:

```
> M := Matrix(2, 5, [[-29, 0, -26, -6*I, -56*I], [56*I, 6*I, -26, 0, -29]]);
```

$$M := \begin{bmatrix} -29 & 0 & -26 & -6\,\mathrm{I} & -56\,\mathrm{I} \\ 56\,\mathrm{I} & 6\,\mathrm{I} & -26 & 0 & -29 \end{bmatrix}$$

Defining a Lee matrix of size 2

```
> U := LeeMatrix(2);
```

$$U := \begin{bmatrix} 1 & \mathrm{I} \\ 1 & -\mathrm{I} \end{bmatrix}$$

and its inverse

```
> U_inv := MatrixInverse(U);
```

$$U_{inv} := \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{\mathrm{I}}{2} & \frac{\mathrm{I}}{2} \end{bmatrix}$$

as well as a Lee matrix of size 5

```
> V := LeeMatrix(5);
```

$$V := \begin{bmatrix} 1 & 0 & 0 & \mathrm{I} & 0 \\ 0 & 1 & 0 & 0 & \mathrm{I} \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -\mathrm{I} \\ 1 & 0 & 0 & -\mathrm{I} & 0 \end{bmatrix}$$

and its inverse

```
> V_inv := MatrixInverse(V);
```

$$V_{inv} := \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -\frac{\mathrm{I}}{2} & 0 & 0 & 0 & \frac{\mathrm{I}}{2} \\ 0 & -\frac{\mathrm{I}}{2} & 0 & \frac{\mathrm{I}}{2} & 0 \end{bmatrix}$$

then the matrix $M_\rho = U^{-1}\, M\, V$ is defined by:

```
> M_rho := U_inv.M.V;
```

$$M_{rho} := \begin{bmatrix} -29 & 0 & -26 & -56 & -6 \\ -56 & -6 & 0 & -29 & 0 \end{bmatrix}$$

Clearly, we have $M_\rho \in \mathbb{R}^{2\times 5}$. This transformation is invertible:

```
> U.M_rho.V_inv;
```

$$\begin{bmatrix} -29 & 0 & -26 & -6\,\mathrm{I} & -56\,\mathrm{I} \\ 56\,\mathrm{I} & 6\,\mathrm{I} & -26 & 0 & -29 \end{bmatrix}$$

More generally, the set $\mathrm{CH}_{2,5}$ of the 2×5 centrohermitian matrices is bijectively sent onto $\mathbb{R}^{2\times 3}$ by means of the transformation $M \longmapsto U^{-1}\, M\, V$.

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