The homological perturbation lemma and its applications to robust stabilization

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Abstract Within the lattice approach to analysis and synthesis problems, we show how standard results on robust stabilization can be obtained in a unified way and generalized when interpreted as a particular case of the so-called *homological perturbation lemma*. This lemma plays a significant role in algebraic topology, homological algebra, computer algebra, etc. Our results show that it is also central to robust control theory for (infinite-dimensional) linear systems.

Keywords: Robust control, H_∞ control, robust stabilization, homological perturbation lemma, models of perturbations, the lattice approach to analysis and synthesis problems.

1. THE FRACTIONAL REPRESENTATION APPROACH

In what follows, we consider the so-called *fractional representation approach* developed in the 80's by Vidyasagar, Desoer, Callier, Francis, etc (see Curtain et al. (1991); Desoer et al. (1980); Vidyasagar (1985) and the references therein). In this approach, the set of stable plants (in a sense to be defined afterwards) is considered to be an *integral domain* A, i.e., a commutative ring with no nonzero zero divisors. Examples of integral domains A usually encountered in the literature are:

- The *Hardy algebra* $H^\infty(\mathbb{C}_+)$ formed by all holomorphic functions in the *open right half-plane*
$$\mathbb{C}_+ := \{s \in \mathbb{C} \mid \Re(s) > 0\}$$
which are bounded with respect to the *sup norm*, i.e.:
$$\| h \|_\infty := \sup_{s \in \mathbb{C}_+} |h(s)|.$$
Let $\widehat{h}$ denote the Laplace transform of h and $H^2(\mathbb{C}_+) := \{\widehat{h} \mid h \in L^2(\mathbb{R}_+)\}$. If $\widehat{h} \in H^\infty(\mathbb{C}_+)$ then the input-output system $\widehat{y} = \widehat{h}\,\widehat{u}$ is $H^2(\mathbb{C}_+) - H^2(\mathbb{C}_+)$ stable (i.e., $\widehat{u} \in H^2(\mathbb{C}_+)$ yields $\widehat{y} \in H^2(\mathbb{C}_+)$), or equivalently $y = h \star u$ is the $L^2(\mathbb{R}_+) - L^2(\mathbb{R}_+)$ stable, where $\star$ denotes the standard convolution product.
- $RH_\infty := H^\infty(\mathbb{C}_+) \cap \mathbb{R}(s)$ the algebra of proper and stable rational transfer functions.
- The Wiener algebra defined by:
$$\widehat{\mathcal{A}} := \{\widehat{f} + \sum_{i=0}^{+\infty} a_i\, e^{-h_i s} \mid f \in L_1(\mathbb{R}_+),\, (a_i) \in l_1(\mathbb{N}),\; 0 = h_0 < h_1 < h_2 < \ldots\}.$$
If $\widehat{h} \in \widehat{\mathcal{A}}$, then the input-output system $y = h \star u$ is $L^\infty(\mathbb{R}_+) - L^\infty(\mathbb{R}_+)$ stable (BIBO stability).

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For more examples, see Curtain et al. (1991); Desoer et al. (1980); Quadrat (2006a); Vidyasagar (1985).

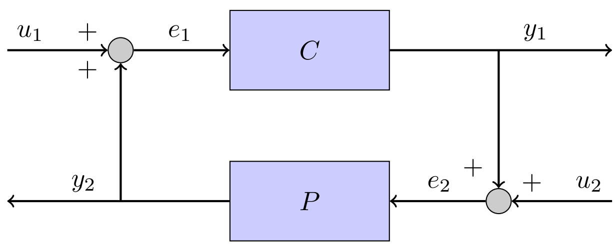
In what follows, $K := Q(A) = \left\{\frac{n}{d} \mid 0 \neq d,\, n \in A\right\}$ denotes the *quotient field* of A. Within the fractional representation approach, a transfer matrix is defined by $P \in K^{q \times r}$.

Definition 1. (Desoer et al. (1980); Vidyasagar (1985)). Let A be an integral domain of SISO stable transfer functions, $K := Q(A)$ and $P \in K^{q\times r}$. Then, P is *internally stabilizable* if there exists $C \in K^{r \times q}$ such that all entries of the following transfer matrix

$$\begin{aligned} H(P,\,C) &:= \begin{pmatrix} I_q & -P \\ -C & I_r \end{pmatrix}^{-1} \\ &= \begin{pmatrix} (I_q - P\,C)^{-1} & (I_q - P\,C)^{-1}\,P \\ C\,(I_q - P\,C)^{-1} & I_r + C\,(I_q - P\,C)^{-1}\,P \end{pmatrix} \\ &= \begin{pmatrix} I_q + P\,(I_r - C\,P)^{-1}\,C & P\,(I_r - C\,P)^{-1} \\ (I_r - C\,P)^{-1}\,C & (I_r - C\,P)^{-1} \end{pmatrix} \end{aligned} \quad (1)$$

belong to A, i.e., $H(P,\,C) \in A^{(q+r)\times(q+r)}$. Then, C is a *stabilizing controller* of P and we note $C \in \mathrm{Stab}(P)$.

With the notations of the following figure



we have:

$$\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = H(P,\,C) \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}.$$

The transfer matrix $H(P, C)$ connects the inputs u_1 and u_2 (references and perturbations) to e_1 and e_2. If we have $H(P, C) \in A^{(q+r)\times(q+r)}$, then all transfer matrices between two signals appearing in the above figure are stable. For more details, see, e.g., Desoer et al. (1980); Vidyasagar (1985). Since the context is clear, we shall only say "stabilizable" for "internally stabilizable".

Let us introduce standard transfer matrices:

- *Output sensitivity* transfer matrix $S_o := (I_q - P\,C)^{-1}$.
- *Input sensitivity* transfer matrix $S_i := (I_r - C\,P)^{-1}$.
- $U := C\,(I_q - P\,C)^{-1} = (I_r - C\,P)^{-1}\,C$.
- *Complementary input sensitivity* transfer matrix $T_i := U\,P$.
- *Complementary output sensitivity* transfer matrix $T_o := P\,U$.

Note that we have the relation $S_o\,P = P\,S_i$.

Let us introduce a few more definitions.

Definition 2. (Desoer et al. (1980)). Let $P \in K^{q\times r}$.

(1) A *fractional representation* of P is defined by
$$P = D^{-1}\,N = \widetilde{N}\,\widetilde{D}^{-1},$$
where $R := (D \quad -N) \in A^{q\times(q+r)}$, $\det D \neq 0$, $\widetilde{R} = (\widetilde{N}^T \quad \widetilde{D}^T)^T \in A^{(q+r)\times r}$ and $\det \widetilde{D} \neq 0$.

(2) A fractional representation $P = D^{-1}\,N$ is a *left coprime factorization* if there exist $X \in A^{q\times q}$ and $Y \in A^{r\times q}$ such that $D\,X - N\,Y = I_q$.

(3) A fractional representation $P = \widetilde{N}\,\widetilde{D}^{-1}$ is a *right coprime factorization* if there exist $\widetilde{X} \in A^{r\times r}$ and $\widetilde{Y} \in A^{r\times q}$ such that $-\widetilde{Y}\,\widetilde{N} + \widetilde{X}\,\widetilde{D} = I_r$.

(4) A fractional representation of $P = D^{-1}\,N = \widetilde{N}\,\widetilde{D}^{-1}$ is a *doubly coprime factorization* if $P = D^{-1}\,N$ is a left coprime factorization and $P = \widetilde{N}\,\widetilde{D}^{-1}$ is a right coprime factorization.

Remark 1. Any transfer matrix $P \in K^{q\times r}$ admits fractional representations (take, e.g., $D = d\,I_q$, $\widetilde{D} = d\,I_r$, where d is the product of the denominators of all the entries of P and $N := d\,P$ and $\widetilde{N} = P\,d$). But not all transfer matrices $P \in K^{q\times r}$ admit a left/right/doubly coprime factorization. For instance, see Quadrat (2006a,b).

2. THE LATTICE APPROACH

As we showed in Quadrat (2006a,b), the fractional representation approach to analysis and synthesis problems can be studied using the concept of the *lattice of a finite-dimensional K-vector space.* Before stating this definition again, let us introduce a few standard definitions.

Definition 3. (Rotman (2009)). Let A be an integral domain, $K := Q(A)$ and M a finitely generated A-module.

(1) The *rank* of M is the dimension of the K-vector space obtained by extending the coefficients of M from A to K, i.e., $\operatorname{rank}_A(M) := \dim_K(K \otimes_A M)$, where $\otimes_A$ denotes the *tensor product* of A-modules.

(2) If M and N are two A-modules, then $\hom_A(M, N)$ denotes the set of all the *A-homomorphisms* from M to N, i.e., $f \in \hom_A(M, N)$ satisfies
$$f(a_1\,m_1 + a_2\,m_2) = a_1\,f(m_1) + a_2\,f(m_2)$$
for all a_1, $a_2 \in A$ and for all m_1, $m_2 \in M$.

(3) M is *free* if M admits a basis or equivalently if M is isomorphic to direct sum of copies of A, i.e., $M \cong A^r$, where $\cong$ stands for an *isomorphism*, i.e., a homomorphism which is both injective and surjective.

(4) M is *projective* if there exist an A-module P and $r \in \mathbb{N}$ such that $M \oplus P \cong A^r$, where $\oplus$ denotes the *direct sum* of A-modules.

Definition 4. (Bourbaki (1989)). Let V be a finite-dimensional K-vector space. Then, an A-submodule M of V is a *lattice of* V if there exist two free A-submodules L_1 and L_2 of V such that $L_1 \subseteq M \subseteq L_2$ and $\operatorname{rank}_A(L_1) = \dim_K(V)$.

We have the following examples (Quadrat (2006a)).

Example 1. If $P \in K^{q\times r}$, then we have:

- $\mathcal{L} := (I_q \quad -P)\,A^{q+r}$ is a lattice of K^q.
- $\mathcal{M} := A^{1\times(q+r)}\,\begin{pmatrix}P^T & I_r^T\end{pmatrix}^T$ is a lattice of $K^{1\times r}$.

Definition 5. (Bourbaki (1989)). Let V be a finite-dimensional K-vector space and M a lattice of V. Then, $A : M$ is the A-submodule of $\hom_K(V, K) \cong V^\star$ formed by the K-linear maps $f : V \longrightarrow K$ which satisfy $f(M) \subseteq A$.

One can show that $A : M$ is a lattice of $\hom_K(V, K) \cong V^\star$.

We have the following examples (Quadrat (2006a)).

Example 2. With the notations of Example 1, we have:

- $A : \mathcal{L} = \{\lambda \in A^{1\times q} \mid \lambda\,P \in A^{1\times r}\}$ is a lattice of $K^{1\times q}$.
- $A : \mathcal{M} = \{\mu \in A^r \mid P\,\mu \in A^q\}$ is a lattice of K^r.

The next theorems give necessary and sufficient stabilization conditions (Quadrat (2006a)).

Theorem 1. With the notations of Example 1, the following assertions are equivalent:

(1) $P \in K^{q\times r}$ is stabilizable.

(2) There exists $L = (S_o^T \quad U^T)^T$, where $S_o \in A^{q\times q}$ and $U \in A^{r\times q}$, such that:
  (a) $L\,P = \begin{pmatrix} S_o\,P \\ U\,P \end{pmatrix} \in A^{(q+r)\times r}$.
  (b) $(I_q \quad -P)\,L = S_o - P\,U = I_q$.
  Then, we have $C := U\,S_o^{-1} \in \operatorname{Stab}(P)$ and:
$$S_o = (I_q - P\,C)^{-1}, \quad U = C\,(I_q - P\,C)^{-1}.$$

(3) $\mathcal{L}$ is a *projective lattice* of K^q, i.e., the lattice $\mathcal{L}$ of K^q is a finitely generated projective A-module of rank q.

Theorem 2. With the notations of Example 1, the following assertions are equivalent:

(1) $P \in K^{q\times r}$ is stabilizable.

(2) There exists $\widetilde{L} = (-U \quad S_i)$, where $U \in A^{r\times q}$ and $S_i \in A^{r\times r}$, such that:
  (a) $P\,\widetilde{L} = (-P\,U \quad P\,S_i) \in A^{q\times(q+r)}$.
  (b) $\widetilde{L}\,\begin{pmatrix} P \\ I_r \end{pmatrix} = -U\,P + S_i = I_r$.
  Then, we have $C := S_i^{-1}\,U \in \operatorname{Stab}(P)$ and:
$$S_i = (I_r - C\,P)^{-1}, \quad U = (I_r - C\,P)^{-1}\,C.$$

(3) $\mathcal{M}$ is a projective lattice of $K^{1\times r}$.

We have the following result (Quadrat (2006a)).

Corollary 3. (1) If $P = D^{-1}\,N$ is a left coprime factorization, $D\,X - N\,Y = I_q$, $X \in A^{q\times q}$, $Y \in A^{r\times q}$, then $\mathcal{L} = D^{-1}\,A^q \cong A^q$ is free and Theorem 1 holds with:
$$S_o = X\,D, \quad U = Y\,D.$$
Hence, P is stabilized by the controller $C := Y\,X^{-1}$.

(2) If $P = \widetilde{N}\,\widetilde{D}^{-1}$ is a right coprime factorization, $-\widetilde{Y}\,\widetilde{N} + \widetilde{X}\,\widetilde{D} = I_r$, $\widetilde{X} \in A^{r\times r}$, $\widetilde{Y} \in A^{r\times q}$, then $\mathcal{M} = A^{1\times r}\,\widetilde{D}^{-1} \cong A^{1\times r}$ is free and Theorem 2 holds with:
$$\widetilde{U} = \widetilde{D}\,\widetilde{Y}, \quad S_i = \widetilde{D}\,\widetilde{X}.$$
Hence, P is stabilized by the controller $C := \widetilde{X}^{-1}\,\widetilde{Y}$.

Let us introduce a few definitions of *homological algebra.*

Definition 6. (Rotman (2009)). Let $M = (M_i)_{i\in\mathbb{Z}}$ be a sequence of A-modules and $d = (d_i)_{i\in\mathbb{Z}}$ a sequence of A-homomorphisms, where $d_i \in \hom_A(M_i, M_{i-1})$ for $i \in \mathbb{Z}$.

(1) The sequence (M, d) is called a *complex* if $d_i \circ d_{i+1} = 0$ for all $i \in \mathbb{Z}$, i.e., if $\operatorname{im} d_{i+1} \subseteq \ker d_i$ for all $i \in \mathbb{Z}$. The complex (M, d) is simply denoted by:
$$\cdots \xrightarrow{d_{i+2}} M_{i+1} \xrightarrow{d_{i+1}} M_i \xrightarrow{d_i} M_{i-1} \xrightarrow{d_{i-1}} \cdots$$
(2) A complex (M, d) is an *exact sequence at* M_i if
$$\ker d_i = \operatorname{im} d_{i+1},$$
and is an *exact sequence* if it is exact at all the M_i's.
(3) A *short exact sequence* is an exact sequence of the form $0 \longrightarrow M_2 \xrightarrow{d_2} M_1 \xrightarrow{d_1} M_0 \longrightarrow 0$, i.e., d_2 is injective, d_1 is surjective and $\ker d_1 = \operatorname{im} d_2$.
(4) A *split exact sequence* is an exact sequence which is such that there exist $h_i \in \hom_A(M_{i+1}, M_i)$ satisfying:
$$\forall\, i \in \mathbb{Z}, \quad d_{i+1} \circ h_{i+1} + h_i \circ d_i = \mathrm{id}_{M_i}.$$
A split exact sequence is also called a *contractible complex* and $h = (h_i)_{i\in\mathbb{Z}}$ is a *contraction* (*homotopy*).

Remark 4. The indices of the d_i's and h_j's are usually dropped. The condition to be a complex (resp., contractible complex) becomes $d^2 = 0$ (resp., $d\circ h + h\circ d = \mathrm{id}$).

The next result is standard in homological algebra.

Lemma 5. (Rotman (2009)). A short exact sequence splits iff one of the following equivalent conditions is satisfied:

(1) There exists $h_1 \in \hom_A(M_0, M_1)$ such that:
$$d_1 \circ h_1 = \mathrm{id}_{M_0}.$$
(2) There exists $h_2 \in \hom_A(M_1, M_2)$ such that:
$$h_2 \circ d_2 = \mathrm{id}_{M_2}.$$
(3) There exist two homomorphisms $h_1 \in \hom_A(M_0, M_1)$ and $h_2 \in \hom_A(M_1, M_2)$ such that:
$$h_1 \circ d_1 + d_2 \circ h_2 = \mathrm{id}_{M_1}.$$
(4) $M_1 \cong M_0 \oplus M_2$.

Example 3. We have the following short exact sequence
$$0 \longrightarrow A : \mathcal{M} \xrightarrow{d_2} A^{q+r} \xrightarrow{d_1} \mathcal{L} \longrightarrow 0, \qquad (2)$$
where:
$$d_1 : A^{q+r} \longrightarrow \mathcal{L} \qquad\qquad d_2 : A : \mathcal{M} \longrightarrow A^{q+r}$$
$$\lambda \longmapsto (I_q \quad -P)\,\lambda, \qquad\qquad \mu \longmapsto \begin{pmatrix} P \\ I_r \end{pmatrix} \mu.$$

For a proof and more details, see Quadrat (2006a).

According to Theorem 1, P is stabilizable iff there exists $h_1 \in \hom_A(\mathcal{L}, A^{q+r})$ defined by $h_1(\nu) = L\,\nu$ for all $\nu \in \mathcal{L}$ which satisfies $d_1 \circ h_1 = \mathrm{id}_{\mathcal{L}}$. By 1 of Lemma 5, the short exact sequence (2) splits, i.e., we have $\mathcal{L} \oplus (A : \mathcal{M}) \cong A^{q+r}$.

Similarly, using Theorem 2, P is stabilizable iff there exists $h_2 \in \hom_A(A^{q+r}, A : \mathcal{M})$ defined by $h_2(\xi) = \widetilde{L}\,\xi$ for all $\xi \in A^{q+r}$ which satisfies $h_2 \circ d_2 = \mathrm{id}_{A:\mathcal{M}}$. By 2 of Lemma 5, the short exact sequence (2) splits, i.e., we have $\mathcal{L} \oplus (A : \mathcal{M}) \cong A^{q+r}$ and $\mathcal{L}$ is a projective A-module.

A standard result of homological asserts that a short exact sequence ending with a projective A-module splits (see, e.g., Rotman (2009)). An application of this result to the short exact sequence (2) yields 3 of Theorems 1 and 2.

Hence, if P is stabilizable and $C \in \mathrm{Stab}(P)$, then defining the matrices $S_o := (I_q - P\,C)^{-1}$, $U := C\,(I_q - P\,C)^{-1} = (I_r - C\,P)^{-1}\,C$ and $S_i := (I_r - C\,P)^{-1}$, we obtain the following split short exact sequence:
$$0 \longrightarrow A : \mathcal{M} \underset{(-U \quad S_i)}{\overset{\begin{pmatrix} P \\ I_r \end{pmatrix}}{\rightleftarrows}} A^{q+r} \underset{\begin{pmatrix} S_o \\ U \end{pmatrix}}{\overset{(I_q \quad -P)}{\rightleftarrows}} \mathcal{L} \longrightarrow 0.$$

The *homological perturbation lemma* is a technique of algebraic topology, homological algebra, algebraic geometry, computer algebra, etc. For more details, see Brown (1967); Gugenheim (1972); Crainic (2004); Sergeraert (1994).

Before stating the main result, let us introduce a definition.

Definition 7. (Crainic (2004)). Let (M, d) be a complex. We call *perturbation* δ of (M, d) a sequence $\delta = (\delta_i)_{i\in\mathbb{Z}}$, where $\delta_i \in \hom_A(M_i, M_{i-1})$ for all $i \in \mathbb{Z}$, which is such that $(M, d + \delta)$ is a complex, i.e., for all $i \in \mathbb{Z}$, we have
$$(d_i + \delta_i) \circ (d_{i+1} + \delta_{i+1}) = d_i \circ \delta_{i+1} + \delta_i \circ d_{i+1} + \delta_i \circ \delta_{i+1} = 0,$$
a condition which can be simply rewritten as follows:
$$(d + \delta)^2 = d \circ \delta + \delta \circ d + \delta^2 = 0.$$

Let us suppose that $\mathrm{id} + \delta \circ h$ is invertible. Then, so is $\mathrm{id} + h \circ \delta$ since $(\mathrm{id} + h \circ \delta)^{-1} = \mathrm{id} - h \circ (\mathrm{id} + \delta \circ h)^{-1} \circ \delta$:
$$\begin{aligned} &(\mathrm{id} + h \circ \delta) \circ (\mathrm{id} - h \circ (\mathrm{id} + \delta \circ h)^{-1} \circ \delta) \\ = {}& \mathrm{id} + h \circ (\mathrm{id} - (\mathrm{id} + \delta \circ h)^{-1} - \delta \circ h \circ (1 + \delta \circ h)^{-1}) \circ \delta \\ = {}& \mathrm{id} + h \circ (\mathrm{id} - (\mathrm{id} + \delta \circ h) \circ (1 + \delta \circ h)^{-1}) \circ \delta = \mathrm{id}, \\ &(\mathrm{id} - h \circ (\mathrm{id} + \delta \circ h)^{-1} \circ \delta) \circ (\mathrm{id} + h \circ \delta) \\ = {}& \mathrm{id} + h \circ (\mathrm{id} - (\mathrm{id} + \delta \circ h)^{-1} - (\mathrm{id} + \delta \circ h)^{-1} \circ \delta \circ h) \circ \delta \\ = {}& \mathrm{id} + h \circ (\mathrm{id} - (\mathrm{id} + \delta \circ h)^{-1} \circ (\mathrm{id} + \delta \circ h)) \circ \delta = \mathrm{id}. \end{aligned}$$

We shall only use a consequence of the homological perturbation lemma, i.e., the following *contractible case.*

Theorem 6. (Crainic (2004)). Let (M, d) be a contractible complex with contraction h and δ a perturbation of d such that $\mathrm{id} + \delta \circ h$ is invertible. Then, $(M, d + \delta)$ is still a contractible complex with the following contraction:
$$H := h \circ (\mathrm{id} + \delta \circ h)^{-1} = (\mathrm{id} + h \circ \delta)^{-1} \circ h. \qquad (3)$$

3. APPLICATIONS OF THE HOMOLOGICAL PERTURBATION LEMMA

The goal of this paper is to show that standard results on robust stabilization (see, e.g., Curtain et al. (1991); Vidyasagar (1985); Zhou et al. (1995)) can be found again as a particular application of Theorem 6 and generalized.

Let us consider the following A-homomorphisms
$$0 \longrightarrow A : \mathcal{M} \xrightarrow{\delta_2} A^{q+r} \xrightarrow{\delta_1} \mathcal{L} \longrightarrow 0,$$
where $\delta_1 \in \hom_A(A^{q+r}, \mathcal{L})$ and $\delta_2 \in \hom_A(A : \mathcal{M}, A^{q+r})$. Using the following isomorphisms (see Quadrat (2006a)),

$$\begin{cases} \hom_A(A^{q+r}, \mathcal{L}) \cong \mathcal{L}^{1\times(q+r)} = (I_q \quad -P)\, A^{(q+r)\times(q+r)}, \\ \hom_A(A : \mathcal{M}, A^{q+r}) \cong \mathcal{M}^{q+r} = A^{(q+r)\times(q+r)}\, (P^T \quad I_r^T)^T, \end{cases}$$

we obtain $\delta_1(\lambda) = (\Delta_1 \quad -\Delta_2)\,\lambda$ for all $\lambda \in A^{q+r}$ and $\delta_2(\mu) = (\Delta_3^T \quad \Delta_4^T)^T\,\mu$ for all $\mu \in A : \mathcal{M}$, where:

$$\begin{cases} (\Delta_1 \quad -\Delta_2) = (I_q \quad -P)\, V, \\ \begin{pmatrix} \Delta_3 \\ \Delta_4 \end{pmatrix} = W \begin{pmatrix} P \\ I_r \end{pmatrix}, \end{cases} \tag{4}$$

$$V = \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix}, \quad W = \begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix},$$

$$\begin{cases} V_{11} \in A^{q\times q},\ V_{12} \in A^{q\times r},\ V_{21} \in A^{r\times q},\ V_{22} \in A^{r\times r}, \\ W_{11} \in A^{q\times q},\ W_{12} \in A^{q\times r},\ W_{21} \in A^{r\times q},\ W_{22} \in A^{r\times r}. \end{cases} \tag{5}$$

Let us note:

$$\Pi_1 := \begin{pmatrix} S_o \\ U \end{pmatrix} (I_q \quad -P), \quad \Pi_2 := \begin{pmatrix} P \\ I_r \end{pmatrix} (-U \quad S_i). \tag{6}$$

Using Theorems 1 and 2, we can easily check that

$$\Pi_1^2 = \Pi_1 \in A^{(q+r)\times(q+r)}, \quad \Pi_2^2 = \Pi_2 \in A^{(q+r)\times(q+r)},$$

i.e., Π_1 and Π_2 are two *idempotents*. These idempotents play an important role in robust control.

A perturbation of (2) is then the complex defined by

$$0 \longrightarrow A : \mathcal{M} \xrightarrow{d_2+\delta_2} A^{q+r} \xrightarrow{d_1+\delta_1} \mathcal{L} \longrightarrow 0, \tag{7}$$

i.e., where δ_1 and δ_2 are such that

$$(d_1 + \delta_1) \circ (d_2 + \delta_2) = d_1 \circ \delta_2 + \delta_1 \circ d_2 + \delta_1 \circ \delta_2 = 0,$$

i.e., in terms of matrices, such that:

$$\begin{aligned} T &:= (I_q + \Delta_1 \quad -P - \Delta_2) \begin{pmatrix} P + \Delta_3 \\ I_r + \Delta_4 \end{pmatrix} \\ &= (I_q + \Delta_1)\,(P + \Delta_3) - (P + \Delta_2)\,(I_r + \Delta_4) = 0. \end{aligned}$$

Indeed, we must have $T\,\mu = 0$ for all $\mu \in A : \mathcal{M}$. Using the notations of Remark 1, we get $T\,d = 0$, $d \neq 0$, i.e., $T = 0$.

If $\det(I_q + \Delta_1) \neq 0$ and $\det(I_r + \Delta_4) \neq 0$, then we get

$$P' := (I_q + \Delta_1)^{-1}\,(P + \Delta_2) = (P + \Delta_3)\,(I_r + \Delta_4)^{-1}, \tag{8}$$

where the Δ_i's are defined by (4), i.e.

$$\begin{aligned} P' &= (I_q + V_{11} - P\,V_{21})^{-1}\,(P\,(I_r + V_{22}) - V_{12}) \\ &= ((I_q + W_{11})\,P + W_{12})\,(I_r + W_{22} + W_{21}\,P)^{-1}, \end{aligned} \tag{9}$$

where the matrices V_{ij}'s and W_{kl}'s are defined by (5).

Definition 8. The *general linear group of degree* r is defined by the group of the invertible matrices of $A^{r\times r}$, i.e.

$$\mathrm{GL}_r(A) := \{X \in A^{r\times r} \mid \exists\, Y \in A^{r\times r} :\ X\,Y = Y\,X = I_r\},$$

where I_r is the identity matrix. In particular, we have $\mathrm{GL}_1(A) = \mathrm{U}(A)$, where $\mathrm{U}(A)$ denotes the group of invertible elements of A.

According to Theorem 6, if $\mathrm{id} + \delta_1 \circ h_1$ and $\mathrm{id} + \delta_2 \circ h_2$ are both invertible, then (7) is again contractible, i.e., a split short exact sequence with a new contraction H defined by (3). Let us state the two above conditions. We have

$$\begin{cases} (\mathrm{id} + \delta_1 \circ h_1)(\nu) = (I_q + \Delta_1\,S_o - \Delta_2\,U)\,\nu, \\ (\mathrm{id} + \delta_2 \circ h_2)(\xi) = \begin{pmatrix} I_q - \Delta_3\,U & \Delta_3\,S_i \\ -\Delta_4\,U & I_r + \Delta_4\,S_i \end{pmatrix} \xi, \end{cases}$$

for all $\nu \in \mathcal{L}$ and $\xi \in A^{q+r}$. The last matrix belonging to $A^{(q+r)\times(q+r)}$, using (4) and (6), $\mathrm{id} + \delta_2 \circ h_2$ is invertible iff:

$$\begin{aligned} \begin{pmatrix} I_q - \Delta_3\,U & \Delta_3\,S_i \\ -\Delta_4\,U & I_r + \Delta_4\,S_i \end{pmatrix} &= I_{q+r} + \begin{pmatrix} \Delta_3 \\ \Delta_4 \end{pmatrix} (-U \quad S_i) \\ &= I_{q+r} + W\,\Pi_2 \in \mathrm{GL}_{q+r}(A). \end{aligned} \tag{10}$$

If $X \in K^{s\times t}$ and $Y \in K^{t\times s}$, then it is well-known that:

$$\det(I_s + X\,Y) = \det(I_t + Y\,X). \tag{11}$$

We note that $I_s + X\,Y \in \mathrm{GL}_s(A)$ is equivalent to:

$$\det(I_t + Y\,X) = \det(I_s + X\,Y) \in \mathrm{U}(A).$$

Hence, using (11), (10) is then also equivalent to:

$$\det(I_r - U\,\Delta_3 + S_i\,\Delta_4) = \det(I_{q+r} + W\,\Pi_2) \in \mathrm{U}(A). \tag{12}$$

Let us now study the invertibility of $\mathrm{id} + \delta_1 \circ h_1$. Since every element ν of $\mathcal{L}$ is of the form $\nu = (I_q \quad -P)\,\lambda$ for a certain $\lambda \in A^{q+r}$, (4) and (6) then yield

$$\begin{aligned} (\mathrm{id} + \delta_1 \circ h_1)(\nu) &= (I_q + \Delta_1\,S_o - \Delta_2\,U)\,\nu \\ &= \left(I_q + (\Delta_1 \quad -\Delta_2) \begin{pmatrix} S_o \\ U \end{pmatrix}\right) (I_q \quad -P)\,\lambda \\ &= (I_q \quad -P)\,\lambda + (I_q \quad -P)\,V\,\Pi_1\,\lambda \\ &= (I_q \quad -P)\,(I_{q+r} + V\,\Pi_1)\,\lambda, \end{aligned}$$

i.e., we obtain the following identity:

$$\begin{aligned} &(\mathrm{id} + \delta_1 \circ h_1)\,(I_q \quad -P) \\ &= (I_q + \Delta_1\,S_o - \Delta_2\,U)\,(I_q \quad -P) \\ &= (I_q \quad -P)\,(I_{q+r} + V\,\Pi_1). \end{aligned} \tag{13}$$

Using (13) and the following identity

$$\begin{aligned} &(I_{q+r} + V\,\Pi_1) \begin{pmatrix} P \\ I_r \end{pmatrix} \\ &= \left(I_{q+r} + V \begin{pmatrix} S_o \\ U \end{pmatrix} (I_q \quad -P)\right) \begin{pmatrix} P \\ I_r \end{pmatrix} = \begin{pmatrix} P \\ I_r \end{pmatrix}, \end{aligned}$$

we obtain the following commutative exact diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A : \mathcal{M} & \xrightarrow{\begin{pmatrix} P \\ I_r \end{pmatrix}} & A^{q+r} & \xrightarrow{(I_q \ \ -P)} & \mathcal{L} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle I_r} & & \downarrow{\scriptstyle I_{q+r} + V\,\Pi_1} & & \downarrow{\scriptstyle \mathrm{id} + \delta_1 \circ h_1} & & \\ 0 & \longrightarrow & A : \mathcal{M} & \xrightarrow{\begin{pmatrix} P \\ I_r \end{pmatrix}} & A^{q+r} & \xrightarrow{(I_q \ \ -P)} & \mathcal{L} & \longrightarrow & 0. \end{array}$$

Since $\mathrm{id}_{A:\mathcal{M}}$ is an isomorphism, the standard *snake lemma* (see, e.g., Rotman (2009)) then yields

$$\begin{aligned} \ker_{\mathcal{L}}(\mathrm{id} + \delta_1 \circ h_1) &\cong \ker_A(I_{q+r} + V\,\Pi_1), \\ \mathrm{coker}_{\mathcal{L}}(\mathrm{id} + \delta_1 \circ h_1) &\cong \mathrm{coker}_A(I_{q+r} + V\,\Pi_1), \end{aligned}$$

and thus we obtain that $\mathrm{id} + \delta_1 \circ h_1$ is invertible iff:

$$I_{q+r} + V\,\Pi_1 \in \mathrm{GL}_{q+r}(A). \tag{14}$$

Using (4) and (11), (14) is then equivalent to:

$$\det(I_q + \Delta_1\,S_o - \Delta_2\,U) = \det(I_{q+r} + \Pi_1\,V) \in \mathrm{U}(A). \tag{15}$$

Then, the identity (13) yields

$$(I_q \quad -P)\,(I_{q+r} + V\,\Pi_1)^{-1} = (I_q + \Delta_1\,S_o - \Delta_2\,U)^{-1}\,(I_q \quad -P) \tag{16}$$

which finally shows that $(\mathrm{id} + \delta_1 \circ h_1)^{-1}$ is defined by:

$$\begin{aligned} &(\mathrm{id} + \delta_1 \circ h_1)^{-1}((I_q \quad -P)\,\lambda) \\ &= (I_q + \Delta_1\,S_o - \Delta_2\,U)^{-1}\,((I_q \quad -P)\,\lambda), \\ &= (I_q \quad -P)\,(I_{q+r} + V\,\Pi_1)^{-1}\,\lambda. \end{aligned}$$

If the perturbation Δ_i's are defined by (4) and satisfy (10) and (14), then Theorem 6 shows that

$$0 \longrightarrow A : \mathcal{M} \underset{H_2}{\overset{d_2+\delta_2}{\rightleftarrows}} A^{q+r} \underset{H_1}{\overset{d_1+\delta_1}{\rightleftarrows}} \mathcal{L} \longrightarrow 0,$$

is a split short exact sequence with contractions

$$\begin{cases} H_1 := h_1 \circ (\mathrm{id} + \delta_1 \circ h_1)^{-1} = (\mathrm{id} + h_1 \circ \delta_1)^{-1} \circ h_1, \\ H_2 := h_2 \circ (\mathrm{id} + \delta_2 \circ h_2)^{-1} = (\mathrm{id} + h_2 \circ \delta_2)^{-1} \circ h_2, \end{cases}$$

i.e., where, for all $\nu \in \mathcal{L}$ and for all $\xi \in A^{q+r}$, we have:

$$\begin{cases} (h_1 \circ (\mathrm{id} + \delta_1 \circ h_1)^{-1})(\nu) \\ \qquad = \begin{pmatrix} S_o \\ U \end{pmatrix} (I_q + \Delta_1 S_o - \Delta_2 U)^{-1} \nu, \\ ((\mathrm{id} + h_2 \circ \delta_2)^{-1} \circ h_2)(\xi) \\ \qquad = (I_r - U \Delta_3 + S_i \Delta_4)^{-1} (-U \quad S_i)\, \xi. \end{cases}$$

Using Theorem 1, we obtain that the following controller

$$\begin{aligned} C' &:= (U (I_q + \Delta_1 S_o - \Delta_2 U)^{-1}) \\ &\qquad (S_o (I_q + \Delta_1 S_o - \Delta_2 U)^{-1})^{-1} \\ &= U S_o^{-1} = C, \end{aligned}$$

stabilizes P'. Similarly, using Theorem 2, we obtain that the following controller

$$\begin{aligned} C'' &:= ((I_r - U \Delta_3 + S_i \Delta_4)^{-1} S_i)^{-1} \\ &\qquad ((I_r - U \Delta_3 + S_i \Delta_4)^{-1} U) \\ &= S_i^{-1} U = C \end{aligned}$$

stabilizes P'.

Let us sum up the above results.

Theorem 7. Let P be a stabilizable plant, C a stabilizing controller of P,

$$\begin{cases} S_o := (I_q - P C)^{-1} \in A^{q \times q}, \\ U := C (I_q - P C)^{-1} = (I_r - C P)^{-1} C \in A^{r \times q}, \\ S_i := (I_r - C P)^{-1} \in A^{r \times r}, \end{cases}$$

and the matrices $\Pi_1, \Pi_2 \in A^{(q+r)\times(q+r)}$ defined by (6). Then, $C = U S_o^{-1} = S_i^{-1} U$ stabilizes the following plant

$$P' := (I_q + \Delta_1)^{-1} (P + \Delta_2) = (P + \Delta_3)(I_r + \Delta_4)^{-1},$$

for all perturbations Δ_n's of the form of (4) (where the V_{ij}'s and W_{kl}'s are given by (5)) which satisfy

$$\begin{cases} I_{q+r} + \Pi_1 V \in \mathrm{GL}_{q+r}(A), \\ I_{q+r} + W \Pi_2 \in \mathrm{GL}_{q+r}(A), \\ \det(I_q + V_{11} - P V_{21}) \neq 0, \\ \det(I_r + W_{22} + W_{21} P) \neq 0. \end{cases} \tag{17}$$

Remark 8. As shown in (12) and (15), the first two conditions of (17) are equivalent to the following conditions:

$$\begin{cases} \det(I_r - U \Delta_3 + S_i \Delta_4) \in \mathrm{U}(A), \\ \det(I_q + \Delta_1 S_o - \Delta_2 U) \in \mathrm{U}(A). \end{cases}$$

Remark 9. We point out that Theorem 7 holds for a *stabilizable plant P which does not necessarily admit a doubly coprime factorization.* Thus, Theorem 7 is an extension of the standard results developed in the literature. Moreover, as shown in Example 4 below, Theorem 7 yields the different standard models of perturbations at once.

Remark 10. Let us check again Theorem 7 by direct computations. We first have

$$\begin{aligned} I_q - P' C &= I_q - (I_q + \Delta_1)^{-1} (P + \Delta_2) U S_o^{-1} \\ &= (I_q + \Delta_1)^{-1} ((I_q + \Delta_1) S_o - (P + \Delta_2) U) S_o^{-1} \\ &= (I_q + \Delta_1)^{-1} (I_q + \Delta_1 S_o - \Delta_2 U) S_o^{-1} \end{aligned}$$

which yields:

$$\begin{cases} (I_q - P' C)^{-1} = S_o (I_q + \Delta_1 S_o - \Delta_2 U)^{-1} (I_q + \Delta_1), \\ C (I_q - P' C)^{-1} = U (I_q + \Delta_1 S_o - \Delta_2 U)^{-1} (I_q + \Delta_1), \\ (I_q - P' C)^{-1} P' = S_o (I_q + \Delta_1 S_o - \Delta_2 U)^{-1} (P + \Delta_2), \\ C (I_q - P' C)^{-1} P' = U (I_q + \Delta_1 S_o - \Delta_2 U)^{-1} (P + \Delta_2). \end{cases}$$

Using the above identities, we then obtain:

$$\begin{aligned} L(P', C) &:= \begin{pmatrix} (I_q - P' C)^{-1} & (I_q - P' C)^{-1} P' \\ C (I_q - P' C)^{-1} & C (I_q - P' C)^{-1} P' \end{pmatrix} \\ &= \begin{pmatrix} S_o \\ U \end{pmatrix} (I_q + \Delta_1 S_o - \Delta_2 U)^{-1} (I_q + \Delta_1 \quad P + \Delta_2). \end{aligned}$$

According to the definition of Δ_1 and Δ_2, i.e., (5), we have:

$$(I_q + \Delta_1 \quad P + \Delta_2) = (I_q \quad -P) \begin{pmatrix} I_q + V_{11} & V_{12} \\ V_{21} & -I_r + V_{22} \end{pmatrix}.$$

Combining the last two identities with (16), we obtain:

$$\begin{aligned} L(P', C) &= \begin{pmatrix} S_o \\ U \end{pmatrix} (I_q \quad -P) \\ &\qquad (I_{q+r} + V \Pi_1)^{-1} \begin{pmatrix} I_q + V_{11} & V_{12} \\ V_{21} & -I_r + V_{22} \end{pmatrix} \\ &= \Pi_1 (I_{q+r} + V \Pi_1)^{-1} \begin{pmatrix} I_q + V_{11} & V_{12} \\ V_{21} & -I_r + V_{22} \end{pmatrix}. \end{aligned}$$

Using Π_1, $V \in A^{(q+r)\times(q+r)}$, $I_{q+r} + V \Pi_1 \in \mathrm{GL}_{q+r}(A)$, we get $L(P', C) \in A^{(q+r)\times(q+r)}$, i.e., using (1), we obtain $H(P', C) \in A^{(q+r)\times(q+r)}$, i.e., C stabilizes P'.

Similar computations can be done with $C = S_i^{-1} U$ and:

$$P' = (P + \Delta_3)(I_r + \Delta_4)^{-1}.$$

We now show how Theorem 7 yields well-known conditions of robust stabilization for the different standard models of perturbations (i.e., additive, multiplicative, inverse additive, inverse multiplicative) (Zhou et al. (1995)).

Example 4. Using (9), we obtain the following results.

(1) If $V_{12} = 0$, $V_{21} = 0$ and $V_{22} = 0$, then Theorem 7 yields that C stabilizes $P' := (I_q + V_{11})^{-1} P$ if:

$$\begin{pmatrix} I_q + S_o V_{11} & 0 \\ U V_{11} & I_r \end{pmatrix} \in \mathrm{GL}_{q+r}(A)$$
$$\Leftrightarrow I_q + S_o V_{11} \in \mathrm{GL}_q(A).$$

(2) If $V_{11} = 0$, $V_{21} = 0$ and $V_{22} = 0$, then Theorem 7 yields that C stabilizes $P' := P - V_{12}$ if:

$$\begin{pmatrix} I_q & S_o V_{12} \\ 0 & I_r + U V_{12} \end{pmatrix} \in \mathrm{GL}_{q+r}(A)$$
$$\Leftrightarrow I_r + U V_{12} \in \mathrm{GL}_r(A).$$

(3) If $V_{11} = 0$, $V_{12} = 0$ and $V_{22} = 0$, then Theorem 7 yields that C stabilizes $P' := (I_q - P V_{21})^{-1} P$ if:

$$\begin{pmatrix} I_q - S_o P V_{21} & 0 \\ -U P V_{21} & I_r \end{pmatrix} \in \mathrm{GL}_{q+r}(A)$$
$$\Leftrightarrow I_q - S_o P V_{21} \in \mathrm{GL}_q(A).$$

(4) If $V_{11}=0$, $V_{12}=0$ and $V_{21}=0$, then Theorem 7 yields that C stabilizes $P' := P\,(I_r+V_{22})$ if:
$$\begin{pmatrix} I_q & -S_o\,P\,V_{22} \\ 0 & I_r - U\,P\,V_{22} \end{pmatrix} \in \mathrm{GL}_{q+r}(A)$$
$$\Leftrightarrow \; I_r - T_i\,V_{22} \in \mathrm{GL}_r(A).$$

(5) If $W_{12}=0$, $W_{21}=0$ and $W_{22}=0$, then Theorem 7 yields that C stabilizes $P' := (I_q+W_{11})\,P$ if:
$$\begin{pmatrix} I_q - W_{11}\,P\,U & W_{11}\,P\,S_i \\ 0 & I_r \end{pmatrix} \in \mathrm{GL}_{q+r}(A)$$
$$\Leftrightarrow \; I_q - W_{11}\,T_o \in \mathrm{GL}_q(A).$$

(6) If $W_{11}=0$, $W_{21}=0$ and $W_{22}=0$, then Theorem 7 yields that C stabilizes $P' := P + W_{12}$ if:
$$\begin{pmatrix} I_q - W_{12}\,U & W_{12}\,S_i \\ 0 & I_r \end{pmatrix} \in \mathrm{GL}_{q+r}(A)$$
$$\Leftrightarrow \; I_q - W_{12}\,U \in \mathrm{GL}_q(A).$$

(7) If $W_{11}=0$, $W_{12}=0$ and $W_{22}=0$, then Theorem 7 yields that C stabilizes $P' := P\,(I_r + W_{21}\,P)^{-1}$ if:
$$\begin{pmatrix} I_q & 0 \\ -W_{21}\,P\,U & I_r + W_{21}\,P\,S_i \end{pmatrix} \in \mathrm{GL}_{q+r}(A)$$
$$\Leftrightarrow \; I_r + W_{21}\,P\,S_i = I_r + W_{21}\,S_o\,P \in \mathrm{GL}_r(A).$$

(8) If $W_{11}=0$, $W_{12}=0$ and $W_{21}=0$, then Theorem 7 yields that C stabilizes $P' := P\,(I_r+W_{22})^{-1}$ if:
$$\begin{pmatrix} I_q & 0 \\ -W_{22}\,U & I_r + W_{22}\,S_i \end{pmatrix} \in \mathrm{GL}_{q+r}(A)$$
$$\Leftrightarrow \; I_r + W_{22}\,S_i \in \mathrm{GL}_r(A).$$

Using the standard *small gain theorem* (Curtain et al. (1991); Georgiou et al. (1992); Zhou et al. (1995)), we obtain the following result.

Corollary 11. Let $A := H^\infty(\mathbb{C}_+)$. If $V \in A^{(q+r)\times(q+r)}$ and $W \in A^{(q+r)\times(q+r)}$ are such that
$$\| V \|_\infty < \| \Pi_1 \|_\infty^{-1}, \quad \| W \|_\infty < \| \Pi_2 \|_\infty^{-1},$$
then the first two conditions of (17) are satisfied.

Let us now suppose that P admits a doubly coprime factorization $P = D^{-1}\,N = \widetilde{N}\,\widetilde{D}^{-1}$. Using Corollary 3, we obtain $\mathcal{L} = D^{-1}\,A^q$, $\mathcal{M} = A^{1\times r}\,\widetilde{D}^{-1}$, and thus $A : \mathcal{M} = \widetilde{D}\,A^r$ (see Quadrat (2006a)), $S_o = X\,D$, $U = Y\,D = \widetilde{D}\,\widetilde{Y}$ and $S_i = \widetilde{D}\,\widetilde{X}$. The split exact sequence (2) yields
$$0 \longrightarrow \widetilde{D}\,A^r \underset{\widetilde{D}\,(-\widetilde{Y}\;\;\widetilde{X})}{\overset{\begin{pmatrix} P \\ I_r \end{pmatrix}}{\rightleftarrows}} A^{q+r} \underset{\begin{pmatrix} X \\ Y \end{pmatrix} D}{\overset{(I_q\;\;-P)}{\rightleftarrows}} D^{-1}\,A^q \longrightarrow 0,$$
which yields the following split exact sequence:
$$0 \longrightarrow A^r \underset{(-Y\;\;\widetilde{X})}{\overset{\begin{pmatrix} \widetilde{N} \\ \widetilde{D} \end{pmatrix}}{\rightleftarrows}} A^{q+r} \underset{\begin{pmatrix} X \\ Y \end{pmatrix}}{\overset{(D\;\;-N)}{\rightleftarrows}} A^q \longrightarrow 0.$$

Now, note that we have
$$\begin{cases} \delta_1 \in \hom_A(A^{q+r}, \mathcal{L}) \cong \mathcal{L}^{1\times(q+r)} = D^{-1}\,A^{q\times(q+r)}, \\ \delta_2 \in \hom_A(A : \mathcal{M}, A^{q+r}) \cong \mathcal{M}^{q+r} = A^{(q+r)\times r}\,\widetilde{D}^{-1}, \end{cases}$$
i.e., we have:
$$(\Delta_1 \quad -\Delta_2) = D^{-1}\,(\Delta_D \quad -\Delta_N),$$
$$\begin{pmatrix} \Delta_3 \\ \Delta_4 \end{pmatrix} = \begin{pmatrix} \Delta_{\widetilde{N}} \\ \Delta_{\widetilde{D}} \end{pmatrix} \widetilde{D}^{-1},$$
$$\Delta_D \in A^{q\times q},\; \Delta_N \in A^{q\times r},\; \Delta_{\widetilde{N}} \in A^{q\times r},\; \Delta_{\widetilde{D}} \in A^{r\times r}. \tag{18}$$

Condition (15), i.e. $\det(I_q + \Delta_1\,S_o - \Delta_2\,U) \in \mathrm{U}(A)$, yields
$$\begin{aligned} & \det(I_q + D^{-1}\,(\Delta_D\,X - \Delta_N\,Y)\,D) \\ = \; & \det(D^{-1}\,(I_q + \Delta_D\,X - \Delta_N\,Y)\,D) \\ = \; & (\det D)^{-1}\,\det(I_q + \Delta_D\,X - \Delta_N\,Y)\,\det D \\ = \; & \det(I_q + \Delta_D\,X - \Delta_N\,Y) \in \mathrm{U}(A), \end{aligned}$$
and similarly with (12). Hence, if P admits a doubly coprime factorization $P = D^{-1}\,N = \widetilde{N}\,\widetilde{D}^{-1}$, then Theorem 7 yields that $C = Y\,X^{-1} = \widetilde{X}^{-1}\,\widetilde{Y}$ stabilizes the plant
$$P' = (D + \Delta_D)^{-1}\,(N + \Delta_N) = (\widetilde{N} + \Delta_{\widetilde{N}})\,(\widetilde{D} + \Delta_{\widetilde{D}})^{-1}$$
for all perturbations (18) satisfying the conditions:
$$\begin{cases} I_q + \Delta_D\,X - \Delta_N\,Y \in \mathrm{GL}_q(A), \\ I_r - \widetilde{Y}\,\Delta_{\widetilde{N}} + \widetilde{X}\,\Delta_{\widetilde{D}} \in \mathrm{GL}_r(A). \end{cases}$$
We have just found again the standard result of robust stabilization for perturbed doubly coprime factorizations (see, e.g., Curtain et al. (1991); Georgiou et al. (1992)).

REFERENCES

R. Brown. The twisted Eilenberg-Zilber theorem. *Celebrazioni Arch. Secolo XX, Simp. Top.*, 34–37, 1967.

N. Bourbaki. Commutative algebra, Chapters 1-7. Springer, 1989.

M. Crainic. On the perturbation lemma, and deformations. *arXiv:math/0403266*, `http://arxiv.org/abs/math/0403266v1`, 2004.

R. F. Curtain and H. J. Zwart. *An Introduction to Infinite-Dimensional Linear Systems Theory.* TAM 21, Springer-Verlag, 1991.

C. A. Desoer, R. W. Liu, J. Murray, and R. Saeks. Feedback system design: the fractional representation approach to analysis and synthesis. *IEEE Trans. Automat. Control*, 25:399–412, 1980.

T. T. Georgiou and M. C. Smith. Robust stabilization in the gap metric: controller design for distributed plants. *IEEE Trans. Automat. Control*, 37:1133-1143, 1992.

V. K. A. M. Gugenheim. On the chain complex of a fibration. *J. Math.*, 16:398–414, 1972.

A. Quadrat. A lattice approach to analysis and synthesis problems. *Math. Control Signals Systems*, 18:147–186, 2006.

A. Quadrat. On a generalization of the Youla-Kučera parametrization. Part II: The lattice approach to MIMO systems. *Math. Control, Signals, and Systems*, 18:199–235, 2006.

J. J. Rotman. *Introduction to Homological Algebra.* Springer, 2009.

F. Sergeraert. The computability problem in algebraic topology. *Advances in Mathematics*, 104:1–29, 1994.

M. Vidyasagar. *Control System Synthesis: A Factorization Approach.* MIT Press, 1985.

K. Zhou, J. C. Doyle, K. Glover. *Robust and Optimal Control.* Prentice Hall, 1995.