Comments on "Calculs oummains reçus de Montréal", D1435

Alban Quadrat*

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1 Introduction

The purpose of this note is to give an interpretation of the content of [5]. In particular, we exhibit an integral transformation that satisfies the different identities listed in [5].

2 Reformulation of the problem

From the first part of the letter, we can understand that [5] deals with a *transformation*, that will be denoted by $\mathcal{T}$, which satisfies the following two properties:

1. $\mathcal{T}$ maps $e^{a\,t}$ to $\rho/(\rho - a)$.
2. $\mathcal{T}$ satisfies the following identity
$$\mathcal{T}(\theta_t\, f) = -\theta_\rho\, \mathcal{T}(f), \tag{1}$$
where θ_t and θ_ρ denote the *Eulerian operator*, namely:
$$\theta_t := t\,\frac{d}{dt}, \quad \theta_\rho = \rho\,\frac{d}{d\rho}.$$

Note that $\mathcal{T}(f)(\rho)$ corresponds to the notation $\phi(\rho)$ used in [5].

If we suppose that $\mathcal{T}$ satisfies that $\mathcal{T}(a\,f) = a\,\mathcal{T}(f)$ (e.g., $\mathcal{T}$ is a linear map), using Points 1 and 2, we then get:
$$\begin{aligned}\mathcal{T}\left(\theta_t\, e^{a\,t}\right) &= \mathcal{T}\left(a\,t\,e^{a\,t}\right) = a\,\mathcal{T}\left(t\,e^{a\,t}\right)\\ &= -\rho\,\frac{d}{d\rho}\left(\frac{\rho}{\rho-a}\right) = \frac{a\,\rho}{(\rho-a)^2}.\end{aligned}$$
If $a \neq 0$, then we obtain $\mathcal{T}\left(t\,e^{a\,t}\right) = \rho/(\rho-a)^2$ as shown in the second identity of [5].

Remark 1 If Point 1 holds for $a = 0$, then we have $\mathcal{T}(1) = 1$, and (1) yields:
$$\mathcal{T}(0) = \mathcal{T}(\theta_t\, 1) = -\theta_\rho\, 1 = 0.$$
The last sequence of [5] states that $\mathcal{T}(a) = a$ for all constants a, which is consistent with the fact that $\mathcal{T}(a) = a\,\mathcal{T}(1)$ and $\mathcal{T}(1) = 1$.

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*Inria Paris, Ouragan project-team, IMJ – PRG, Sorbonne University, France (alban.quadrat@inria.fr).

Using (1), we have

$$\mathcal{T}(\theta_t^2\, f) = -\theta_\rho\, \mathcal{T}(\theta_t\, f) = \theta_\rho^2\, \mathcal{T}(f) \;\Rightarrow\; \dots \;\Rightarrow \forall\, m \in \mathbb{N},\; \mathcal{T}(\theta_t^m\, f) = (-1)^m\, \theta_\rho^m\, \mathcal{T}(f),$$

where θ_t^m stands for the m^{th} composition $\theta_t \circ \ldots \circ \theta_t$ of the differential operator θ_t (and similarly for θ_ρ), which proves the identity in the middle of the first part of the letter.

From the second half of [5], $\mathcal{T}$ is supposed to satisfy the following identity

$$\mathcal{T}\left(f_c\right)(\rho) = \mathcal{T}(f)(c\,\rho), \tag{2}$$

where f_c denotes the function defined by $f_c(t) = f(t/c)$, and:

$$\int_0^{+\infty} \frac{f(t)}{t}\, dt = \int_0^{+\infty} \frac{\mathcal{T}(f)(\rho)}{\rho}\, d\rho. \tag{3}$$

3 Study of the problem

Doing an educated guess[1], let us search (if it exists) an integral transformation that satisfies Points 1 & 2, and (2) and (3). Let us write $\mathcal{T}$ as follows

$$\mathcal{T}(f)(\rho) = \int_0^{+\infty} K(t,\rho)\, f(t)\, dt, \tag{4}$$

where K is the so-called *kernel* of $\mathcal{T}$.

From Points 1 and 2, we get the following two conditions:

(a) $\mathcal{T}\left(e^{a\,t}\right)(\rho) = \displaystyle\int_0^{+\infty} K(t,\rho)\, e^{a\,t}\, dt = \frac{\rho}{\rho - a}.$

(b) $\mathcal{T}(\theta_t\, f)(\rho) = \displaystyle\int_0^{+\infty} K(t,\rho)\, t\, \dot{f}(t)\, dt = -\theta_\rho\, \mathcal{T}(f)(\rho).$

From (b), an integration by parts yields:

$$\begin{aligned} \mathcal{T}(\theta_t\, f)(\rho) &= -\int_0^{+\infty} \frac{\partial}{\partial t}\left(t\, K(t,\rho)\right)\, f(t)\, dt + \left[t\, K(t,\rho)\, f(t)\right]_0^{+\infty} \\ &= -\rho\, \frac{d}{d\rho} \int_0^{+\infty} K(t,\rho)\, f(t)\, dt = -\int_0^{+\infty} \rho\, \frac{\partial K(t,\rho)}{\partial \rho}\, f(t)\, dt. \end{aligned}$$

Let us now suppose that the above identities hold for all functions f in a certain *functional space* $\mathcal{F}$ which contains certain exponential functions $e^{a\ t}$ and is such that:

- $\left[t\, K(t,\rho)\, f(t)\right]_0^{+\infty} = 0$ for all $f \in \mathcal{F}$, i.e.:

$$\forall\, f \in \mathcal{F}, \quad \lim_{t \to +\infty} t\, K(t,\rho)\, f(t) = 0. \tag{5}$$

- The identity $\displaystyle\int_0^{+\infty} \left(\rho\, \frac{\partial K(t,\rho)}{\partial \rho} - t\, \frac{\partial K(t,\rho)}{\partial t} - K(t,\rho)\right) f(t)\, dt = 0$ for all $f \in \mathcal{F}$ yields:

$$\rho\, \frac{\partial K(t,\rho)}{\partial \rho} - t\, \frac{\partial K(t,\rho)}{\partial t} - K(t,\rho) = 0. \tag{6}$$

[1] In some textbooks, the notation $\Supset$ denotes the *Laplace transform*.

Using the standard *method of characteristic*, the general regular solution of (6) is then of the form

$$K(t,\rho) = \rho\, G(t\,\rho),$$

where G is a sufficiently regular arbitrary function[2]. Then, (4) becomes:

$$\mathcal{T}(f)(\rho) = \rho \int_0^{+\infty} G(t\,\rho)\, f(t)\, dt. \tag{7}$$

Now, combining (7) and (a), we obtain:

$$\mathcal{T}\left(e^{a\,t}\right)(\rho) = \rho \int_0^{+\infty} G(t\,\rho)\, e^{a\,t}\, dt = \frac{\rho}{\rho - a}.$$

If $\rho \neq 0$, then we get:

$$\mathcal{L}\left(e^{a\,t}\right)(\rho) := \int_0^{+\infty} G(t\,\rho)\, e^{a\,t}\, dt = \frac{1}{\rho - a}. \tag{8}$$

Now, we can check again that the following function

$$G(t\,\rho) = e^{-\rho\,t} \tag{9}$$

satisfies (8), i.e., $\mathcal{L}$ is the Laplace transform. Hence, (7) becomes:

$$\mathcal{T}(f)(\rho) = \rho\, \mathcal{L}(f)(\rho). \tag{10}$$

Remark 2 We note that (5), i.e., $\lim_{t\to+\infty} t\, G(t\,\rho)\, e^{a\,t} = 0$, is satisfied for $\Re(\rho - a) > 0$. ✓

Since the Laplace transform satisfies $\mathcal{L}(\dot{f})(\rho) = \rho\, \mathcal{L}(f)(\rho) - f(0)$, (10) is the Laplace transform of the following operator (acting, e.g., on the space of tempered distributions)

$$f \longmapsto \dot{f} + f(0)\,\delta,$$

where δ denotes the Dirac distribution.

4 Check of (2) and (3)

Using the change of variables $\tau = t/c$, where we suppose that $c > 0$, we get

$$\mathcal{T}\left(f_c\right)(\rho) = \rho \int_0^{+\infty} e^{-\rho\,t}\, f\left(\frac{t}{c}\right) dt = c\,\rho \int_0^{+\infty} e^{-c\,\rho\,\tau}\, f(\tau)\, d\tau = \mathcal{T}(f)(c\,\rho),$$

which proves (2).

[2] Alternatively, setting $G(t,\rho) := K(t,\rho)/\rho$, i.e., $K(t,\rho) = \rho\, G(t,\rho)$, (6) yields the following *transport equation* $\rho\left(\rho\, \frac{\partial G(t,\rho)}{\partial \rho} - t\, \frac{\partial G(t,\rho)}{\partial t}\right) = 0$, whose general regular solution is $G(t,\rho) = G(t\,\rho)$.

Remark 3 Using the change of variables $\tau = t/c$, (7) already satisfies the identity (2):

$$\mathcal{T}(f_c)(\rho) = \rho \int_0^{+\infty} G(t\,\rho)\, f\left(\frac{t}{c}\right)\, dt = c\,\rho \int_0^{+\infty} G(\tau\, c\,\rho)\, f(\tau)\, d\tau = \mathcal{T}(f)(c\,\rho).$$

(2) is in fact a direct consequence of (b), i.e., of (6) which can be rewritten as follows

$$(\rho\,\partial_\rho - t\,\partial_t)\, K(t,\rho) = K(t,\rho),$$

with the notations $\partial_\rho = \dfrac{\partial}{\partial\rho}$ and $\partial_t = \dfrac{\partial}{\partial t}$, and which shows that K is invariant under the *infinitesimal vector field* $\rho\,\partial_\rho - t\,\partial_t$. ✓

If we suppose that ρ is a non-negative real, $\mathcal{L}(f) = \mathcal{T}(f)/\rho$ is integrable and that the integrals can be interchanged, then we have

$$\begin{aligned}\int_0^{+\infty} \frac{\mathcal{T}(f)(\rho)}{\rho}\, d\rho &= \int_0^{+\infty} \left(\int_0^{+\infty} e^{-\rho\,t}\, f(t)\, dt\right) d\rho = \int_0^{+\infty} \left(\int_0^{+\infty} e^{-\rho\,t}\, d\rho\right) f(t)\, dt \\ &= -\int_0^{+\infty} \left[\frac{e^{-\rho\,t}}{t}\right]_0^{+\infty} f(t)\, dt = \int_0^{+\infty} \frac{f(t)}{t}\, dt,\end{aligned}$$

which finally proves (3).

Example 1 If we consider $f = t\, e^{a\,t}$, with $a < 0$, then $\mathcal{T}(\rho) = \rho/(\rho-a)^2$, which yields:

$$\int_0^{+\infty} \frac{\mathcal{T}(f)(\rho)}{\rho}\, d\rho = \int_0^{+\infty} \frac{d\rho}{(\rho-a)^2} = -\left[\frac{1}{\rho-a}\right]_0^{+\infty} = -\frac{1}{a}.$$

We also have:

$$\int_0^{+\infty} \frac{f(t)}{t}\, dt = \int_0^{+\infty} e^{a\,t}\, dt = \left[\frac{e^{a\,t}}{a}\right]_0^{+\infty} = -\frac{1}{a}.$$

5 A few comments

The Laplace transform

$$\mathcal{L}(f)(\rho) = \int_0^{+\infty} e^{-\rho\,t}\, f(t)\, dt$$

satisfies the following standard identity for $c > 0$:

$$\mathcal{L}(f_c)(\rho) = \int_0^{+\infty} e^{-\rho\,t}\, f\left(\frac{t}{c}\right)\, dt = c \int_0^{+\infty} e^{-c\,\rho\,\tau}\, f(\tau)\, d\tau = c\,\mathcal{L}(f)(c\,\rho).$$

Moreover, we have

$$\begin{aligned}\mathcal{L}(\theta_t\, f)(\rho) &= \int_0^{+\infty} e^{-\rho\,t}\, t\, \dot f(t)\, dt = -\frac{d}{d\rho} \int_0^{+\infty} e^{-\rho\,t}\, \dot f(t)\, dt \\ &= -\frac{d}{d\rho}\left(\rho \int_0^{+\infty} e^{-\rho\,t}\, f(t)\, dt + [e^{-\rho\,t}\, f(t)]_0^{+\infty}\right) \\ &= -\frac{d}{d\rho}\,(\rho\,\mathcal{L}(f)(\rho) - f(0)) = -\left(\rho\,\frac{d}{d\rho} + 1\right) \mathcal{L}(f)(\rho) = -(\theta_\rho + 1)\,\mathcal{L}(f)(\rho).\end{aligned}$$

Finally, we have:

$$\forall\, n \in \mathbb{N}, \quad \mathcal{L}(t^n)(\rho) = \frac{n!}{\rho^{n+1}}. \tag{11}$$

To our opinion, the above identities are less appealing than the following identities for $\mathcal{T}$:

$$\mathcal{T}\,(f_c)\,(\rho) = \mathcal{T}(f)(c\,\rho), \quad \mathcal{T}(\theta_t\, f) = -\theta_\rho\, \mathcal{T}(f), \quad \forall\, n \in \mathbb{N}, \quad \mathcal{T}(t^n)(\rho) = \frac{n!}{\rho^n}. \tag{12}$$

Remark 4 Note that $\{t^n\}_{n\in\mathbb{Z}}$ are eigenvectors of the Eulerian operator θ_t since $\theta_t\, t^n = n\, t^n$ for $n \in \mathbb{Z}$. Using (1), for $n \in \mathbb{N}$, we get $n\, \mathcal{T}(t^n) = \mathcal{T}(n\, t^n) = \mathcal{T}(\theta_t\, t^n) = -\theta_\rho \mathcal{T}(t^n)$, i.e., $\theta_\rho\, \mathcal{T}(t^n) = -n\, \mathcal{T}(t^n)$, i.e., $\mathcal{T}(t^n)$ is a eigenvector of θ_ρ associated with the eigenvalue $-n$. We find that (1) already yields $\mathcal{T}(t^n) = C_n\, \rho^{-n}$ for a certain constant C_n (which is now known to be equal to $n!$). Applying $\mathcal{T}$ to the following Taylor series of the entire function e^{at}, i.e.,

$$\sum_{n\in\mathbb{N}} \frac{(a\, t)^n}{n!},$$

and using (11), we finally obtain:

$$\mathcal{T}(e^{a\,t})(\rho) = \sum_{n\in\mathbb{N}} \mathcal{T}\left(\frac{(a\,t)^n}{n!}\right) = \sum_{n\in\mathbb{N}} \left(\frac{a}{\rho}\right)^n = \frac{1}{1-\frac{a}{\rho}} = \frac{\rho}{\rho - a}. \quad \checkmark$$

Addendum (16/12/2021)

I thank my brother, Arnaud Quadrat, who has recently found that the transformation considered in these notes, namely (10), is known as the so-called *Carson-Laplace transform* [1].

This transformation was found by J. R. Carson [1] in his search for the mathematical foundation of O. Heaviside's *operational calculus* (developed in a purely heuristic way). The Carson-Laplace transform transformation is a reminiscence of the *Laplace transform* [6].

In the middle of the 50's, the Carson-Laplace transform was well-known by the electrical engineer community (see, e.g., [2, 3, 4]) due to its important role in the study of circuits and linear systems. It has since been forgotten possibly due to the development of the *theory of distributions* [6] in the 50's, a mathematical framework which gives sense of *Heaviside and Dirac distributions* and Laplace transform for a class of distributions. Nowadays, electrical engineers and mathematicians learn Laplace transform within the theory of distributions.

Below, we shortly quote the books [2, 4] where the connections between Carson-Laplace transform and Laplace transform are emphasized, as well as certain interesting properties of the Carson-Laplace transform (see Section 5).

"1. **La transformée de Laplace.** – L'Analyse symbolique repose sur la transformation de Laplace qui fait correspondre à une fonction $f(t)$ de la variable réelle t, définie pour $t > 0$, une fonction $\varphi(s)$ par la relation

$$\varphi(s) = \int_0^\infty e^{-s\,t}\, f(t)\, dt,$$

s étant un paramètre réel ou complexe.

En fait, comme nous le verrons par la suite, les premiers travaux dus à Carson [1] effectués en vue de justifier la méthode de calcul d'Heaviside conduisaient à envisager plutôt la fonction $F(p)$ défine par l'intégrale

$$F(p) = p \int_0^\infty e^{-p\,t} f(t)\, dt.$$

La fonction $\varphi(p)$ ne diffère de $F(p)$ que par le facteur $\frac{1}{p}$:

$$\varphi(p) = \frac{F(p)}{p}."$$

[4], p. 1.

Moreover, in "Avertissement pour la 3e édition", p. 11–12 of [2], one can read:

"La transformation employée dans le présent cours est celle de Carson-Laplace

$$g(p) = p \int_0^\infty e^{-p\,t} h(t) dt$$

[. . .] et non pas celle de Laplace proprement dite:

$$f(p) = \int_0^\infty e^{-p\,t} h(t) dt.$$

Cette préférence se justifie de la manière suivante.

Au moyen de la transformation de Carson-Laplace :

1. La transformée d'une constante est une constance.
2. Les équations aux dimensions (donc l'homogénéité) sont conservées dans la transformation.
3. Les considérations de Heaviside sur la notion d'opérateur différentiel sont conservées, sans multiplication par p de la transformée.
4. L'emploi des méthodes matricielles et tensorielles est difficilement compatible, sans risques d'erreurs lors des mises en équation, avec la transformation de Laplace proprement dite, où il faut prendre $\frac{f(0)}{p}$ pour valeur initiale de $f(t)$.
5. Pour rester conformes à la philosophie de l'enseignement des sciences physiques dans l'état actuel de nos connaissances, il est nécessaire d'employer Carson-Laplace quand le point de vue est *énergétique* ; Laplace convient, au contraire, quand ce point de vue est *informationnel* (Cf. Eléments de Calcul informationnel)."

References

[1] J. R. Carson, Electric circuit theory and the operational calculs, Bell System Technical Journal, 1926, pp. 685–761. 5, 6

[2] M. Denis-Papin, A. Kaufmann, *Cours de calcul opérationnel appliqué (transformation de Carson-Laplace)*, Albin Michel, 1967, 5ème édition revue et corrigée, 1ère édition 1950. 5, 6

[3] M. Janet, *Précis de calcul matriciel et de calcul opérationnel*, Presses Universitaires de France, 1954. 5

[4] M. Parodi, *Introduction à l'étude de l'analyse symbolique*, Gauthiers-Villars, 1957. 5, 6

[5] Oummains, letter "Calculs oummains reçus de Montréal", D1435, T4-67, 69, received by Villagrassa, 30/12/1967, jointed to letter E12 (D143) (http://www.ummo-sciences.org/fr/data-E/E12.htm). 1, 2

[6] L. Schwartz, Théorie des distrubtions, Hermann, 1966. 5