# Noncommutative geometric structures on stabilizable infinite-dimensional linear systems

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***Abstract*— This paper aims at showing that noncommutative geometric structures such as connections and curvatures exist on internally stabilizable infinite-dimensional linear systems and on their stabilizing controllers. To see this new geometry, using the noncommutative geometry developed by Connes, we have to replace the standard differential calculus by the quantized differential calculus and classical vector bundles by projective modules. We give an explicit description of the connections on an internally stabilizable system and on its stabilizing controllers in terms of the projectors of the closed-loop system classically used in robust control. These connections aim at studying the variations of the signals in the closed-loop system in response to a disturbance or a change of the reference. We also compute the curvatures of these connections.**

## I. STABILIZABILITY

In what follows, we shall consider the *fractional representation approach to analysis and synthesis problems* developed by Vidyasagar, Desoer, Callier, … in the eighties. See [3], [4], [13]. Within this approach, an *integral domain* $A$ (i.e., a commutative ring with no non-zero divisors) of stable transfer functions is considered and the set of SISO transfer functions is defined by the *field of fractions* of $A$:

$$K := Q(A) = \left\{ \frac{n}{d} \mid 0 \neq d,\, n \in A \right\}.$$

Hence, a transfer function $p \in K$ is $A$*-stable* if $p \in A$ and $A$*-unstable* if $p \in K \setminus A$. Standard rings $A$ of stable transfer functions are $RH_\infty$, $H^\infty(\mathbb{C}_+)$, $\widehat{\mathcal{A}}$, $W_+$, $A(\mathbb{D})$ [3], [4], [13].

*Definition 1:* Let $A$ be an integral domain of stable SISO plants, $K := Q(A)$ and $P \in K^{q \times r}$ a transfer matrix. Then, the plant $P$ is *internally stabilizable* if there exists a controller $C \in K^{r \times q}$ such that all the entries of the matrix

$$\begin{aligned} H(P,\, C) &:= \begin{pmatrix} I_q & P \\ C & I_r \end{pmatrix}^{-1} \\ &= \begin{pmatrix} (I_q - P\,C)^{-1} & -(I_q - P\,C)^{-1}\,P \\ -C\,(I_q - P\,C)^{-1} & I_r + C\,(I_q - P\,C)^{-1}\,P \end{pmatrix} \\ &= \begin{pmatrix} I_q + P\,(I_r - C\,P)^{-1}\,C & -P\,(I_r - C\,P)^{-1} \\ -(I_r - C\,P)^{-1}\,C & (I_r - C\,P)^{-1} \end{pmatrix} \end{aligned} \quad (1)$$

belong to $A$. Then, $C$ is called a *stabilizing controller* of $P$, which is denoted by $C \in \mathrm{Stab}(P)$.

With the notations of Figure 1, we have:

$$\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = H(P,\, C) \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}.$$

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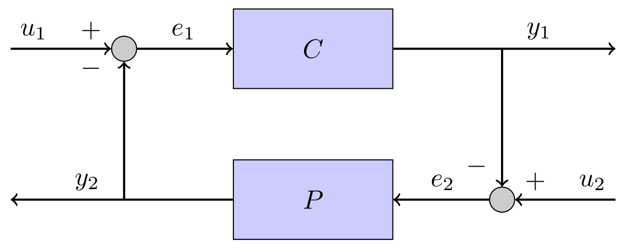


Fig. 1. Closed-loop system

It can be shown that internal stabilization, simply called *stabilization*, implies that any transfer matrix of Figure 1 is $A$-stable, i.e., all its entries belong to $A$. See [4], [13].

Let us introduce the following transfer matrices:

- *Output sensitivity* transfer matrix $S_o := (I_q - P\,C)^{-1}$.
- *Input sensitivity* transfer matrix $S_i := (I_r - C\,P)^{-1}$.
- $U := C\,(I_q - P\,C)^{-1} = (I_r - C\,P)^{-1}\,C$.
- *Complementary input sensitivity* transfer matrix $T_i := U\,P$.
- *Complementary output sensitivity* transfer matrix $T_o := P\,U$.

*Lemma 1:* The following assertions are equivalent:

1) $C \in K^{r \times p}$ stabilizes $P \in K^{p \times r}$.
2) $\Pi_C := (S_o^T \quad U^T)^T\,(I_q \quad -P)$ satisfies:

$$\Pi_C^2 = \Pi_C \in A^{(q+r) \times (q+r)}.$$

3) $\Pi_P := (P^T \quad I_r^T)^T\,(-S_i \quad U)$ satisfies:

$$\Pi_P^2 = \Pi_P \in A^{(q+r) \times (q+r)}.$$

*Proof:* The controller $C \in K^{r \times p}$ stabilizes the plant $P \in K^{p \times r}$ iff $H(P, C) \in A^{(q+r) \times (q+r)}$, where $H(P, C)$ is the transfer matrix defined by (1) (see Definition 1), and thus iff $\Pi_C \in A^{(q+r) \times (q+r)}$, or iff $\Pi_P \in A^{(q+r) \times (q+r)}$. Using $S_o - P\,U = I_q$ and $U - S_i\,P = I_r$, we can check that $\Pi_C^2 = \Pi_C$, $\Pi_P^2 = \Pi_P$ and $\Pi_C + \Pi_P = I_{q+r}$, i.e., $\Pi_C$ and $\Pi_P$ are *complementary idempotents* of the ring $A^{(q+r) \times (q+r)}$. ∎

With the notations of Figure 1, we note that we have

$$\begin{aligned} \begin{pmatrix} e_1 \\ y_1 \end{pmatrix} &= \begin{pmatrix} (I_q - P\,C)^{-1} & -(I_q - P\,C)^{-1}\,P \\ C\,(I_q - P\,C)^{-1} & -C\,(I_q - P\,C)^{-1}\,P \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \\ &= \Pi_C \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \end{aligned} \quad (2)$$

$$\begin{pmatrix} y_2 \\ e_2 \end{pmatrix} = \begin{pmatrix} -P\,(I_r - C\,P)^{-1}\,C & P\,(I_r - C\,P)^{-1} \\ -(I_r - C\,P)^{-1}\,C & (I_r - C\,P)^{-1} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \Pi_P \begin{pmatrix} u_1 \\ u_2 \end{pmatrix},$$

i.e., $\Pi_C$ (resp., $\Pi_P$) projects the disturbance and the reference $(u_1^T \quad u_2^T)^T$ onto the graph $(e_1^T \quad y_1^T)^T$ (resp., $(y_2^T \quad e_2^T)^T$) of the controller $C$ (resp., the plant $P$).

The two projectors $\Pi_C$ and $\Pi_P$ play a fundamental role in robust control theory and particularly the two quantities $b_{P,C} :=\| \Pi_P \|_A^{-1}$ and $b_{C,P} :=\| \Pi_C \|_A^{-1}$. In particular, if $A = RH_\infty$ or $H^\infty(\mathbb{C}_+)$, the *optimal robust radius* [7] is:

$$b_{\text{opt}} := \sup_{C \in \text{Stab}(P)} b_{P,C} = (\inf_{C \in \text{Stab}(P)} \| \Pi_C \|_\infty)^{-1} = (\inf_{C \in \text{Stab}(P)} \| \Pi_P \|_\infty)^{-1}.$$

In module theory, a finitely generated $A$-module $M$ is *projective* if there exist $r \in \mathbb{Z}_{\geq 0} := \{0, 1, \ldots\}$ and an $A$-module $P$ such that $M \oplus P \cong A^r$, where $\oplus$ (resp., $\cong$) denotes the *direct sum* (resp., *isomorphic* modules, i.e. the existence of a bijective $A$-*homomorphism* (i.e., $A$-linear map)) [11].

The idempotents $\Pi_C$ and $\Pi_P$ of the ring $A^{(q+r)\times(q+r)}$ define the projective modules $M_C := \Pi_C\, A^{(q+r)\times 1}$ and $M_P := A^{1\times(q+r)}\,\Pi_P$ of rank respectively $q$ and $r$ since:

$$M_C \oplus \ker_A(\Pi_C .) = A^{(q+r)\times 1}, \; M_P \oplus \ker_A(.\Pi_P) = A^{1\times(q+r)},$$

$$\begin{cases} \ker_A(\Pi_C .) = \{\xi \in A^{(q+r)\times 1} \mid \Pi_C\, \xi = 0\} = \Pi_P\, A^{(q+r)\times 1}, \\ \ker_A(.\Pi_P) = \{\mu \in A^{1\times(q+r)} \mid \mu\, \Pi_P = 0\} = A^{1\times(q+r)}\, \Pi_C. \end{cases}$$

Thus, $P$ is stabilizable, i.e., is stabilized by a controller $C$, iff the two finitely generated $A$-modules $M_C := \Pi_C\, A^{(q+r)\times 1}$ and $M_P := A^{1\times(q+r)}\,\Pi_P$ are projective of rank $q$ and $r$ [8].

Let us now consider the following two $A$-modules:

$$\mathcal{L} := (I_q \quad -P)\, A^{(q+r)\times 1}, \; \mathcal{M} := A^{1\times(q+r)}\,(P^T \quad I_r^T)^T.$$

$\mathcal{L}$ (resp., $\mathcal{M}$) is a finitely generated $A$-submodule of $K^q$ (resp., $K^{1\times r}$) called a *lattice* of $K^q$ (resp., $K^{1\times r}$). See [8].

Let us consider the following two $A$-homomorphisms:

$$\begin{array}{rcl} \mathcal{L} = (I_q \quad -P)\, A^{(q+r)\times 1} & \xrightarrow{\iota_1} & M_C = \Pi_C\, A^{(q+r)\times 1} \\ \eta_1 = (I_q \quad -P)\, \xi & \longmapsto & (S_o^T \quad U^T)^T\, \eta_1 = \Pi_C\, \xi, \\ \mathcal{M} = A^{1\times(q+r)}\,(P^T \quad I_r^T)^T & \xrightarrow{\iota_2} & M_P = A^{1\times(q+r)}\,\Pi_P \\ \eta_2 = \mu\,(P^T \quad I_r^T)^T & \longmapsto & \eta_2\,(-S_i \quad U) = \mu\,\Pi_P. \end{array} \tag{3}$$

Since $S_o - P\,U = I_q$ and $U - S_i\,P = I_r$, we get

$$(I_q \quad -P)\, \iota_1(\eta_1) = \eta_1, \quad \iota_2(\eta_2)\,(P^T \quad I_r^T)^T = \eta_2,$$

which yield that $\iota_1$ and $\iota_2$ are injective $A$-homomorphisms, and thus $P$ is stabilizable iff $\mathcal{L} \cong \iota_1(\mathcal{L}) = \Pi_C\, A^{(q+r)\times 1}$ is a projective $A$-module of rank $q$, or equivalently iff $\mathcal{M} \cong \iota_2(\mathcal{M}) = A^{1\times(q+r)}\,\Pi_P$ is a projective $A$-module of rank $r$.

Let us note $F := (I_q \quad -P)$, $f_j := F_{j\bullet}$ the $j^{\text{th}}$ column of $F$, $G := (S_o^T \quad U^T)^T$, $g_k := G_{\bullet k}$ the $k^{\text{th}}$ row of $G$. Then, $C \in K^{r\times q}$ stabilizes $P \in K^{q\times r}$ iff $\Pi_C \in A^{(q+r)\times(q+r)}$, i.e., $(S_o^T \quad U^T)^T\,(I_q \quad -P)\, A^{(q+r)\times 1} \subseteq A^{(q+r)\times 1}$, and thus iff $g_k\, \eta_1 \in A$ for all $\eta_1 \in \mathcal{L} = (I_q \quad -P)\, A^{q+r}$ and $k = 1, \ldots, q+r$, i.e., iff the following $A$-homomorphisms hold

$$\begin{array}{rcl} \alpha_k : \mathcal{L} = (I_q \quad -P)\, A^{q+r} & \longrightarrow & A \\ \eta_1 & \longmapsto & g_k\, \eta_1, \end{array} \tag{4}$$

for $k = 1, \ldots, q+r$. Now, $F\,G = S_o - P\,U = I_q$ yields

$$\eta_1 = F\,G\,\eta_1 = \sum_{j=1}^{q+r} f_j\,(g_j\,\eta_1) = \sum_{j=1}^{q+r} f_j\,\alpha_j(\eta_1),$$

which shows that every element of $\mathcal{L} = \sum_{j=1}^{q+r} f_j\, A$ has the form of $\eta_1 = \sum_{j=1}^{q+r} f_j\, \alpha_j(\eta_1)$, where $\alpha_j$ is the *form of* $\mathcal{L}$ defined by (4). The pair $S := (\{f_j\}_{j=1,\ldots,q+r}, \{\alpha_j\}_{j=1,\ldots,q+r})$ formed by the *set of generators* $\{f_j\}_{j=1,\ldots,q+r}$ of $\mathcal{L}$ and the forms $\{\alpha_j\}_{j=1,\ldots,q+r}$ is called a *projective basis* of $\mathcal{L}$ [11].

*Proposition 1:* $P \in K^{q\times r}$ is stabilizable iff the finitely generated $A$-module $\mathcal{L} := (I_q \quad -P)\, A^{(q+r)\times 1}$ admits a projective basis, i.e., iff $\mathcal{L} \cong \Pi_C\, A^{(q+r)\times 1}$ is a projective $A$-module of rank $q$. Similarly for $\mathcal{M} \cong A^{1\times(q+r)}\,\Pi_P$.

We point out that a projective basis of $\mathcal{L}$ is explicitly defined by means of the stabilizing controller $C \in K^{r\times q}$, i.e., the existence of a stabilizing controller $C$ of $P$ is equivalent to the existence of an embedding of $\mathcal{L} = (I_q \; -P)\, A^{(q+r)\times 1}$ into $A^{(q+r)\times 1}$. In algebra, projective bases play the role of a *system of coordinates* in differential geometry: the "variety" $\mathcal{L}$ can be embedded into the "affine space" $A^{q+r}$. Only the stabilizable plants have this important property. This fact will play a crucial role in Section III where this embedding is used to develop a differential geometric approach to stabilizable plants based on a *differential calculus on* $A$ and the concept of *parallel transport*, i.e., of *connections* [12] (connections play a fundamental role in modern physics). Finally, the links between algebra and differential geometry are at the core of mathematical results connecting these two realms: the *Serre-Swan theorem* states that the *category* of finitely generated projective $A = C^\infty(X)$-modules is equivalent to the category of *vector bundles* over the manifold $X$ [12]. See also [1], [10]. Prototypical examples of vector bundles are the tangent or cotangent vector bundles.

A particular instance of Serre-Swan theorem is well-known in control theory: it is the equivalence of *Kalman's criterion of controllability* of $\dot{x} = A\,x + B\,u$ in terms of $\text{rank}_{\mathbb{R}}(B \;\; A\,B \;\; A^2\,B \;\; \ldots \;\; A^{n-1}\,B) = n$ and *Hautus' test* $\text{rank}_{s\in\mathbb{C}}(s\,I_n - A \quad -B) = n$. Indeed, on the one hand, Kalman's test is equivalent to the existence of a right inverse $S := (X(s)^T \quad Y(s)^T)^T \in \mathbb{R}[s]^{(n+m)\times n}$ of the polynomial matrix $R := (s\,I_n - A \quad -B) \in \mathbb{R}[s]^{n\times(n+m)}$, and thus of a projector $\Pi := S\,R$ of $\mathbb{R}[s]^{(n+m)\times(n+m)}$ (since $R\,S = I_n$ implies that $\Pi^2 = \Pi$), i.e., and thus is equivalent to the fact that the $\mathbb{R}[s]$-module $\Pi\,\mathbb{R}[s]^{(n+m)\times 1} \cong R\,\mathbb{R}[s]^{(n+m)\times 1}$ is a projective $\mathbb{R}[s]$-module of rank $n$. On the other hand, Hautus' test asserts that the following family of vector $\mathbb{C}$-spaces

$$\begin{array}{rcl} E : \mathbb{C} & \longrightarrow & \mathbb{C}^{n\times 1} \\ s & \longmapsto & E_s := (s\,I_n - A \quad -B)\,\mathbb{C}^{(n+m)\times 1} \end{array}$$

forms a vector bundle over $\mathbb{C}$ of rank $n$ [12].

## II. QUANTIZED CALCULUS

"One way to quantify how sensitive $T$ is to variations in $P$ is to take the limiting ratio of a relative perturbation in $T$ (i.e., $\Delta T/T$) to a relative perturbation in $P$ (i.e., $\Delta P/P$).

$$\dots \quad \lim_{\Delta P\to 0} \frac{\Delta T/T}{\Delta P/P} = \frac{dT}{dP}\,\frac{P}{T} = S.$$

In this way, $S$ is the sensitivity of the closed-loop transfer function $T$ to an infinitesimal perturbation in $P$." p. 40 of [5]. Following Connes' theory [1], the aim of this section is to give a precise meaning of the above statement by mathematically characterizing the variations, i.e., the *differential* $dT$ of $T \in L^\infty(i\,\mathbb{R})$ or $T \in H^\infty(\mathbb{C}_+)$. We show that the *1-dimensional quantized calculus* [1] gives a complete answer by interpreting these differentials as certain bounded operators on the *separable Hilbert space* $\mathcal{H} := L^2(i\,\mathbb{R})$ [14]. The quantized differential calculus then inherits a noncommutative structure from the noncommutative $C^\star$-*algebra* $\mathcal{L}(\mathcal{H})$ *of bounded operators on* $\mathcal{H}$ [10], [14].

In what follows, $k$ will always denote a commutative ring containing $\mathbb{Q}$ and $A$ a *unital* $k$-*algebra*, i.e., a $k$-algebra $A$ with a unit $1 \in A$, i.e., $1\,a = a\,1$ for all $a \in A$.

Let us give a general definition of a *differential calculus*.

*Definition 2 ([1], [6]):* 1) A $k$-algebra $A$ is said to be *graded* if there exists a family of $k$-submodules $(A_i)_{i\in\mathbb{Z}_{\geq 0}}$ of $A$ satisfying:

a) $A = \bigoplus_{i\in\mathbb{Z}_{\geq 0}} A_i$.

b) $\forall\, i, j \in \mathbb{Z}_{\geq 0}$, $A_i\, A_j \subseteq A_{i+j}$, i.e.:

$$\forall\, a_i \in A_i, \quad \forall\, a_j \in A_j, \quad a_i\, a_j \in A_{i+j}.$$

2) A graded algebra $A$ is *graded-commutative* if:

$$\forall\, a_i \in A_i, \quad \forall\, a_j \in A_j, \quad a_i\, a_j = (-1)^{ij}\, a_j\, a_i.$$

3) A graded algebra $A$ is a *differential graded algebra* if there exist $k$-homomorphisms $d_i : A_i \longrightarrow A_{i+1}$ for $i \in \mathbb{Z}_{\geq 0}$ satisfying the two following conditions:

a) $d_{i+1} \circ d_i = 0$ for all $i \in \mathbb{Z}_{\geq 0}$.

b) For $a_i \in A_i$ and $a_j \in A_j$, we have:

$$d_{i+j}(a_i\, a_j) = (d_i a_i)\, a_j + (-1)^i\, a_i\, (d_j a_j). \quad (5)$$

We simply note $d = d_i$ for all $i \in \mathbb{Z}_{\geq 0}$. Then, the identity $d_{i+1} \circ d_i = 0$ yields $d^2 = 0$ and (5) becomes:

$$\forall\, a_i \in A_i,\ \forall\, a_j \in A_j,\ d(a_i\, a_j) = (da_i)\, a_j + (-1)^i\, a_i\, (da_j).$$

4) A *differential calculus* on a $k$-algebra $A$ is a graded differential algebra $(\Omega_A^\bullet = \bigoplus_{i\in\mathbb{N}} \Omega_A^i, d)$ with:

$$\Omega_A^0 = A.$$

If $\omega \in \Omega_A^i$, then $i$ is called the *degree* of $\omega$, which is denoted by $\deg(\omega)$ or simply by $|\omega|$.

*Example 1:* Let $X$ be a smooth manifold of dimension $n$, $A = C^\infty(X)$ the ring of smooth $k$-valued functions on $X$ ($k = \mathbb{R}$, $\mathbb{C}$), and $\Omega_A^i = \Omega^i(X)$ the $A$-module of the *differential* $i$-*forms* on $X$ [12]. Then, the $A$-module $\Omega_A^\bullet = \bigoplus_{i\in\mathbb{Z}_{\geq 0}} \Omega_A^i$ equipped with the *wedge product* $\wedge$ of differential forms and the *exterior derivative* $d$ defined in a local coordinate system $x = (x^1, \dots, x^n)$ by

$$d\left(f_I\, dx^I\right) = \sum_{i=1}^n \frac{\partial f_I}{\partial x_i}\, dx^i \wedge dx^I, \quad I = (i_1, \dots, i_k),$$

where $1 \leq i_1 < i_2 < \cdots < i_k \leq n$, $dx^I = dx^{i_1} \wedge \ldots \wedge dx^{i_k}$, is a differential graded algebra over $A$. Moreover, $(\Omega_A^\bullet, d)$ is graded-commutative:

$$\forall\, \omega_i \in \Omega_A^i, \quad \forall\, \omega_j \in \Omega_A^j, \quad \omega_i \wedge \omega_j = (-1)^{ij}\, \omega_j \wedge \omega_i.$$

Finally, note that $\Omega_A^i = 0$ for $i > n$, i.e., $\Omega_A^\bullet = \bigoplus_{i=0}^n \Omega_A^i$.

We introduce the *1-dimensional quantized calculus* developed in [1] which plays a key role in the rest of the paper.

*Example 2:* [1] Let $A = L^\infty(\mathbb{T})$ be the space of *Lebesgue measurable* $\mathbb{C}$-*valued functions* on the unit circle

$$\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$$

which are *essentially bounded* [14], i.e.:

$$\forall\, a \in A, \quad \|\, a \,\|_\infty := \operatorname{ess\,sup}_{z\in\mathbb{T}} |a(z)| < +\infty.$$

The complex *Banach space* $(A, \|\cdot\|_\infty)$ is a commutative *von Neumann algebra*. Let $\mathcal{H} = L^2(\mathbb{T})$ be the Hilbert space of Lebesgue measurable $\mathbb{C}$-valued functions on $\mathbb{T}$ which are square-integrable, i.e., such that $\|\, h \,\|_2 = \sqrt{< h, h >}$ is finite, where the inner product $< \cdot\,, \cdot >$ of $\mathcal{H}$ is defined by:

$$< g, h > := \frac{1}{2\pi} \int_0^{2\pi} \overline{g(e^{i\theta})}\, h(e^{i\theta})\, d\theta = \frac{1}{2\pi i} \oint_{\mathbb{T}} \overline{g(z)}\, h(z)\, \frac{dz}{z}.$$

We note that $h \in \mathcal{H}$ has a unique expression

$$h = \sum_{n\in\mathbb{Z}} h_n\, e^{in\theta} = \sum_{n\in\mathbb{Z}} h_n\, z^n$$

in the orthogonal basis $\left(z^n = e^{in\theta}\right)_{n\in\mathbb{Z}}$ of $\mathcal{H}$, where:

$$h_n = < h, e^{in\theta} > = \frac{1}{2\pi} \int_0^{2\pi} h(e^{i\theta})\, e^{-in\theta}\, d\theta.$$

Let $\mathcal{L}(\mathcal{H})$ be the noncommutative $C^\star$-algebra of the bounded operators on $\mathcal{H}$. Since $\mathcal{H}$ is an $A$-module, we have the following *representation* $\chi : A \longrightarrow \mathcal{L}(\mathcal{H})$ of $A$ on $\mathcal{H}$:

$$\begin{array}{rcl} \underline{a} := \chi(a) : \mathcal{H} & \longrightarrow & \mathcal{H} \\ h & \longmapsto & a\, h. \end{array}$$

We have $\|\, \underline{a} \,\|_{\mathcal{L}(\mathcal{H})} = \|\, a \,\|_\infty$ ([3], [14]), which shows that the *involutive representation* $\chi$, i.e., $\chi(a^\star) = \chi(a)^\star$ for all $a \in A$, where $a^\star(z) = \overline{a(\overline{z})} = \overline{a(z^{-1})}$, is an *isometry* from $A$ to $\chi(A)$. In particular, $\chi$ is a *faithful representation*.

Let $\operatorname{sign} : \mathbb{Z} \longrightarrow \mathbb{Z}$ be the *sign function*, i.e., $\operatorname{sign}(n) = n$ for $n \in \mathbb{Z}_{\geq 0} = \{0, 1, \ldots\}$ and $-n$ for $n \in \mathbb{Z} \setminus \mathbb{N}$, and the self-adjoint bounded operator $F$ on $\mathcal{H}$ defined by:

$$\forall\, n \in \mathbb{Z}, \quad F\left(e^{in\theta}\right) = \operatorname{sign}(n)\, e^{in\theta}.$$

Then, we can define the *differential* $da$ of $a \in A$ as:

$$\forall\, a \in A, \quad da := [F, \underline{a}] = F \circ \underline{a} - \underline{a} \circ F \in \mathcal{L}(\mathcal{H}). \tag{6}$$

Using $\underline{a} \circ \underline{b} = \underline{a\, b}$, the *Leibniz rule* holds for $d$ since

$$\begin{aligned} d(a\, b) &= F \circ \underline{a\, b} - \underline{a\, b} \circ F \\ &= (F \circ \underline{a} - \underline{a} \circ F) \circ \underline{b} + \underline{a} \circ (F \circ \underline{b} - \underline{b} \circ F) \\ &= da \circ \underline{b} + \underline{a} \circ db, \end{aligned} \tag{7}$$

for all $a,\, b \in A$. The $\mathbb{C}$-vector space $\Omega^1_A$ of the 1-forms on $A$ is then defined by:

$$\Omega^1_A = \left\{ \sum_{i=1}^{r} \underline{a_0} \circ da_i \mid a_0, \ldots, a_n \in A \right\}.$$

The trivial identity $\underline{b} \circ (\underline{a_0} \circ da_1) = (\underline{b\, a_0}) \circ da_1$ for all $b \in A$ shows that $\Omega^1_A$ has a *left $A$-module* structure [11]. $\Omega^1_A$ also has a *right $A$-module* [11] since (7) yields:

$$\forall\, b \in A, \quad da \circ \underline{b} = d(a\, b) - \underline{a} \circ db \in \Omega^1_A. \tag{8}$$

Using the associativity of the composition of operators of $\mathcal{L}(\mathcal{H})$, we get $(\underline{c} \circ da) \circ \underline{b} = \underline{c} \circ (da \circ \underline{b})$ for all $a, b, c \in A$, which shows that $\Omega^1_A$ is an $A-A$-*bimodule* [11].

The $\mathbb{C}$-vector space $\Omega^i_A$ of the $i$-forms on $A$ is defined by the $\mathbb{C}$-linear span of bounded operators of the form

$$\underline{a_0} \circ da_1 \circ \cdots \circ da_i = \underline{a_0} \circ [F, \underline{a_1}] \circ \cdots \circ [F, \underline{a_i}] \in \mathcal{L}(\mathcal{H}),$$

where $a_0, \ldots, a_i \in A$. The product of forms is the composition of bounded operators:

$$\begin{array}{ccc} \Omega^i_A \times \Omega^j_A & \longrightarrow & \Omega^{i+j}_A \\ (\omega_i,\, \omega_j) & \longmapsto & \omega_i \circ \omega_j. \end{array}$$

The *differential* $d : \Omega^i_A \longrightarrow \Omega^{i+1}_A$ is then defined by:

$$\forall\, \omega_i \in \Omega^i_A : \quad d\omega_i = F \circ \omega_i - (-1)^i\, \omega_i \circ F. \tag{9}$$

Using $F^2 = I$, we get $d^2 = 0$ since:

$$\begin{aligned} d^2(\omega_i) &= F \circ (F \circ \omega_i - (-1)^i\, \omega_i \circ F) \\ &\quad -(-1)^{i+1}\, (F \circ \omega_i - (-1)^i\, \omega_i \circ F) \circ F \\ &= \omega_i - (-1)^i\, F \circ \omega_i \circ F + (-1)^i\, F \circ \omega_i \circ F - \omega_i = 0. \end{aligned}$$

Now, using $F^\star = F$, we get:

$$\begin{aligned} (da)^\star &= (F \circ \underline{a} - \underline{a} \circ F)^\star = \underline{a^\star} \circ F^\star - F^\star \circ \underline{a^\star} \\ &= \underline{a^\star} \circ F - F \circ \underline{a^\star} = -[F, \underline{a^\star}] = -da^\star. \end{aligned}$$

Then, the involution $\star$ of $A$ can be extended to $\Omega^i_A$ by:

$$(\underline{a_0} \circ da_1 \circ \cdots \circ da_i)^\star := (-1)^i\, da_i^\star \circ \cdots \circ da_1^\star \circ \underline{a_0}^\star.$$

Then, $(\Omega^\bullet_A = \bigoplus \Omega^i_A, d)$ is a differential calculus on $A$.

Let us introduce the following bounded operators on $\mathcal{H}$:

$$P_+ := \frac{1}{2}\, (I + F), \quad P_- := I - P_+ = \frac{1}{2}\, (I - F). \tag{10}$$

Since $F^2 = I$, $P_+$ and $P_-$ are two complementary projectors of $\mathcal{L}(\mathcal{H})$, i.e., $P_+^2 = P_+$, $P_-^2 = P_-$ and $P_+ + P_- = I$. The Hilbert space $\mathcal{H}$ can then be decomposed as follows:

$$\mathcal{H} = \operatorname{im} P_+ \oplus \ker P_+ = \operatorname{im} P_+ \oplus \operatorname{im} P_-.$$

Let $\mathcal{H}_+ = P_+\, \mathcal{H} = H^2(\mathbb{D})$ and $\mathcal{H}_- = P_-\, \mathcal{H} = H^2(\mathbb{D})^\perp$,

$$\left\{ \begin{array}{l} H^2(\mathbb{D}) := \left\{ h \in L^2(\mathbb{T}) \mid h = \sum_{n \in \mathbb{N}} h_n\, z^n \right\}, \\ H^2(\mathbb{D})^\perp := \left\{ h \in L^2(\mathbb{T}) \mid h = \sum_{n \leq -1} h_n\, z^n \right\}, \end{array} \right.$$

where $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$ is the open unit disc. The *Hardy space* $H^2(\mathbb{D})$ is the Hilbert space formed by holomorphic functions in $\mathbb{D}$ which are square-integrable, i.e.,

$$\|\, f\, \|_{H^2(\mathbb{D})} := \sup_{0 \leq |r| < 1} \left( \frac{1}{2\pi} \int_0^{2\pi} |f(r\, e^{i\theta})|^2\, d\theta \right)^{1/2}.$$

Then, $F_{|\mathcal{H}_+} = I$ and $F_{|\mathcal{H}_+} = -I$. Since $da$ anti-commutes with $F$, i.e., $F \circ da = -da \circ F$, we get:

$$da + F \circ da = da - da \circ F \;\Leftrightarrow\; P_+ \circ da = da \circ P_-.$$

Post-multiplying the last equality by $P_+$, pre-multiplying it by $P_-$, and using $P_- \circ P_+ = 0$, we obtain:

$$\left\{ \begin{array}{l} P_+ \circ da \circ P_+ = da \circ P_- \circ P_+ = 0, \\ P_- \circ da \circ P_- = P_- \circ P_+ \circ da = 0. \end{array} \right. \tag{11}$$

Let us now decompose $da$ in $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$, namely:

$$\begin{array}{rcl} da : \mathcal{H}_+ \oplus \mathcal{H}_- & \longrightarrow & \mathcal{H}_+ \oplus \mathcal{H}_- \\ h = \left( \begin{array}{c} h_+ \\ h_- \end{array} \right) & \longmapsto & \left( \begin{array}{cc} X & Y \\ U & V \end{array} \right) \left( \begin{array}{c} h_+ \\ h_- \end{array} \right). \end{array}$$

Then, (11) implies $X = 0$ and $V = 0$ since:

$$\begin{aligned} (P_+ \circ da \circ P_+)(\mathcal{H}) &= (P_+ \circ da)(\mathcal{H}_+) = P_+(da(\mathcal{H}_+)) = 0, \\ (P_- \circ da \circ P_-)(\mathcal{H}) &= (P_- \circ da)(\mathcal{H}_-) = P_-(da(\mathcal{H}_-)) = 0. \end{aligned}$$

Let us now explicitly characterize the operators $Y$ and $U$. Let $a \in A$ and $h_+ \in \mathcal{H}_+$. Since $F(h_+) = h_+$, we then get

$$da(h_+) = F(a\, h_+) - a\, F(h_+) = F(a\, h_+) - a\, h_+ = -2\, P_-\, (a\, h_+),$$

i.e., $U = -2\, P_- \circ \underline{a}$. If $h_- \in \mathcal{H}_-$, then $F(h_-) = -h_-$ yields

$$da(h_-) = F(a\, h_-) - a\, F(h_-) = F(a\, h_-) + a\, h_- = 2\, P_+\, (a\, h_-),$$

i.e., $Y = 2\, P_+ \circ \underline{a}$. Thus, for $a \in A$, $da$ is defined by:

$$\begin{array}{rcl} da : \mathcal{H}_+ \oplus \mathcal{H}_- & \longrightarrow & \mathcal{H}_+ \oplus \mathcal{H}_- \\ h = \left( \begin{array}{c} h_+ \\ h_- \end{array} \right) & \longmapsto & \left( \begin{array}{cc} 0 & 2\, P_+ \circ \underline{a} \\ -2\, P_- \circ \underline{a} & 0 \end{array} \right) \left( \begin{array}{c} h_+ \\ h_- \end{array} \right). \end{array} \tag{12}$$

Let us now study the differential of an element of the subalgebra $H^\infty(\mathbb{D})$ of $A$ formed by the holomorphic functions in the unit disc $\mathbb{D}$ which are bounded for the norm $\|\, a\, \|_\infty = \sup_{z \in \mathbb{D}} |a(z)|$. It is well-known that $\mathcal{H}_+ = H^2(\mathbb{D})$ is a $H^\infty(\mathbb{D})$-module, i.e., $a\, h_+ \in \mathcal{H}_+$ for all $a \in H^\infty(\mathbb{D})$ and $h_+ \in \mathcal{H}_+$ ([3], [14]), which yields $da(h_+) = 0$ for all $a \in H^\infty(\mathbb{D})$ and $h_+ \in \mathcal{H}_+$, i.e., $U = 0$. Hence, if $a \in H^\infty(\mathbb{D})$, then $da$ reduces to the *Hankel operator* [3], [7], [14] with *symbol* $2\, a$, i.e.:

$$\begin{array}{rcl} da : \mathcal{H}_- & \longrightarrow & \mathcal{H}_+ \\ h_- & \longmapsto & P_+(2\, a\, h_-). \end{array} \tag{13}$$

If $H^\infty(\mathbb{C} \setminus \overline{\mathbb{D}})$ is the Banach algebra of bounded holomorphic functions in $\mathbb{C} \setminus \overline{\mathbb{D}} = \{z \in \mathbb{C} \mid |z| > 1\}$, then

$H^\infty(\mathbb{C}\setminus\overline{\mathbb{D}}) \subseteq L^\infty(\mathbb{T})$, and $H^2(\mathbb{D})^\perp$ is a $H^\infty(\mathbb{C}\setminus\overline{\mathbb{D}})$-module [14], and thus, $P_+(\underline{a}(h_-)) = 0$ for all $a \in H^\infty(\mathbb{C}\setminus\overline{\mathbb{D}})$ and for all $h_- \in \mathcal{H}-$. If $a \in H^\infty(\mathbb{C}\setminus\overline{\mathbb{D}})$, then $da$ is defined by:

$$\begin{array}{rcl} da : \mathcal{H}_+ & \longrightarrow & \mathcal{H}_- \\ h_+ & \longmapsto & P_-(-2\, a\, h_+). \end{array} \tag{14}$$

If $a_1,\, a_2 \in A$, then $\omega = da_1 \circ da_2 \in \Omega_A^2$ is defined by

$$-4 \left( \begin{array}{cc} P_+ \circ \underline{a_1} \circ P_- \circ \underline{a_2} & 0 \\ 0 & P_- \circ \underline{a_1} \circ P_+ \circ \underline{a_2} \end{array} \right)$$

in the decomposition $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$. If $a_1,\, a_2 \in H^\infty(\mathbb{D})$, then $\omega = 0$ since $P_-(a_2\, h_+) = 0$ and $P_+(a_2\, h_-) \in \mathcal{H}_+ \Rightarrow g_+ := a_1\, P_+(a_2\, h_-) \in \mathcal{H}_+ \Rightarrow P_-(g_+) = 0$. Therefore, any 2-form over $H^\infty(\mathbb{D})$ or $H^\infty(\mathbb{D})^\perp$ vanishes.

*Example 3:* We can similarly consider the quantized calculus on the real axis $\mathbb{R}$. It is defined by $A := L^\infty(\mathbb{R})$, $\mathcal{H} := L^2(\mathbb{R})$ and $F$ is the *Hilbert transform*

$$F(a) = \text{p.v.}\, \frac{1}{\pi\, i} \int_{-\infty}^{+\infty} \frac{a(t)}{t - x}\, dt,$$

for all $a \in A$, $x \in \mathbb{R}$, where p.v. is the *Cauchy principal value* of the convolution of $i/(\pi\, x)$ by $a$, i.e.:

$$\begin{aligned} F(a)(x) &= \lim_{\varepsilon \to 0_+} \frac{1}{\pi\, i} \int_{\mathbb{R}\setminus[-\varepsilon, +\varepsilon]} \frac{a(t)}{t - x}\, dt \\ &= \lim_{\varepsilon \to 0_+} \frac{1}{\pi\, i} \int_{\varepsilon}^{+\infty} \frac{a(x+t) - a(x-t)}{t}\, dt. \end{aligned}$$

For more details, see [1]. Then, we can check that we have:

$$\forall\, h \in \mathcal{H},\ (da(h))(x) = \text{p.v.}\, \frac{1}{\pi\, i} \int_{-\infty}^{+\infty} \frac{a(x) - a(t)}{x - t}\, h(t)\, dt.$$

## III. CONNECTIONS ON STABILIZABLE PLANTS

In what follows, we shall consider a differential calculus $(\Omega_A^\bullet = \bigoplus_{i\in\mathbb{N}} \Omega_A^i, d)$ on a $k$-algebra $A$.

Let us introduce the fundamental concept of a connection.

*Definition 3 ([1], [6]):* A *connection* of a right $A$-module $M$ is a $k$-linear map $\nabla : M \longrightarrow M \otimes_A \Omega_A^1$ satisfying

$$\forall\, a \in A, \quad \forall\, m \in M, \quad \nabla(m\, a) = \nabla(m)\, a + m \otimes da, \tag{15}$$

where $\otimes_A$ denotes the *tensor product* of the right $A$-module $M$ and the left $A$-module $\Omega_A^1$ [11].

*Remark 1:* If $\nabla$ and $\nabla'$ are two connections on $M$, then (15) yields $(\nabla - \nabla')(m\, a) = (\nabla - \nabla')(m)\, a$ for all $a \in A$, which shows that $\nabla - \nabla'$ is a right $A$-homomorphism from $M$ to $M \otimes_A \Omega_A^1$, denoted by $\nabla - \nabla' \in \hom_A(M, M \otimes_A \Omega_A^1)$ [11]. Thus the space of all the connections on $M$ is an affine space over the $\mathbb{C}$-vector space $\hom_A(M, M \otimes_A \Omega_A^1)$.

*Theorem 1 ([2]):* A left/right $A$-module $M$ admits a connection iff $M$ is a finitely generated projective module.

The results obtained in Section I then yields the result.

*Corollary 1:* Let $A$ be a domain, $K := Q(A)$ and a plant $P \in K^{q\times r}$. Then, the lattice $\mathcal{L} := (I_q \quad -P)\, A^{(q+r)\times 1}$ of $K^r$ admits a connection iff $P$ is a stabilizable plant.

*Proposition 2 ([11]):* If $M$ is a finitely generated projective right $A$-module and $N$ a right $A$-module, then:

$$\hom_A(M, N) \cong N \otimes_A \hom_A(M, A).$$

If $M$ is a finitely generated projective left/right $A$-module and if we denote the projective right/left $A$-module $\hom_A(M, A)$ by $M^\star$, then Proposition 2 yields:

$$\forall\, i \in \mathbb{Z}_{\geq 0},\ \hom_A(M, M \otimes_A \Omega_A^i) \cong M \otimes_A \Omega_A^i \otimes_A M^\star. \tag{16}$$

*Example 4:* If $M = A^r$ is a finitely generated free right $A$-module, then $M \otimes_A \Omega_A^1 \cong (\Omega_A^1)^r$. Let us consider the following $k$-linear map:

$$\begin{array}{rcl} d : M & \longrightarrow & M \otimes_A \Omega_A^1 \\ m = (a_1\ \ldots\ a_r)^T & \longmapsto & dm = (da_1\ \ldots\ da_r)^T. \end{array}$$

We have $d(m\, a) = (dm)\, a + m\, da$ for all $m \in M$ and $a \in A$, which shows that $d$ is a connection on $M$. If $\nabla$ is another connection on $M$, then Remark 1 and (16) show that:

$$\nabla - d \in \hom_A(M, M \otimes_A \Omega_A^1) \cong A^r \otimes \Omega_A^1 \otimes_A A^{1\times r} \cong (\Omega_A^1)^{r\times r}.$$

Let $\Lambda \in (\Omega_A^1)^{r\times r}$ be such that $\nabla - d = \gamma$, where $\gamma$ is the left $A$-homomorphism defined by:

$$\begin{array}{rcl} \gamma : M & \longrightarrow & M \otimes_A \Omega_A^1 \\ m & \longmapsto & \Lambda\, m. \end{array}$$

Thus, we get that $\nabla(m) = dm + \Lambda\, m$ for all $m \in M$.

Let us now suppose that $A$ is an integral domain of SISO stable plants, $K = Q(A)$ and $P \in K^{q\times r}$ a stabilizable plant. Using the results of Section I, the finitely generated $A$-module $\mathcal{L} := (I_q \quad -P)\, A^{(q+r)\times 1}$, i.e., the lattice of $K^q$, is projective. Using the embedding $\iota_1$ defined by (3), we get $\mathcal{L} \cong \iota_1(\mathcal{L}) = M_C := \Pi_C\, A^{(q+r)\times 1}$, where $\Pi_C$ is the projector of $A^{(q+r)\times(q+r)}$, i.e., $\Pi_C^2 = \Pi_C \in A^{(q+r)\times(q+r)}$, defined in Lemma 1. Hence, without loss of generality, we can consider the projective $A$-module $M_C$ of rank $q$. Using the differential calculus $(\Omega_A^\bullet = \bigoplus_{i\in\mathbb{N}} \Omega_A^i, d)$ on $A$, let us define the so-called *Levi-Civita* or *Grassmann connection* on $M_C$. Let us consider the following right $A$-homomorphisms

$$\begin{array}{ccccc} M_C & \xrightarrow{i} & A^{(q+r)\times 1} & \xrightarrow{d} & A^{(q+r)\times 1} \otimes_A \Omega_A^1 \\ \xi & \longmapsto & \xi & \longmapsto & \sum_{j=1}^{q+r} e_j \otimes d\xi_j, \end{array}$$

where $i$ is the embedding $M_C \subseteq A^{(q+r)\times 1}$, $e_j$ is the column vector of $A^{(q+r)\times 1}$ defined by 1 at the $j^{\text{th}}$ position and 0 elsewhere, $\{e_j\}_{j=1,\ldots,q+r}$ the *standard basis* of the free $A$-module $A^{(q+r)\times 1}$ of rank $q+r$, and $\xi = (\xi_1\ \ldots\ \xi_{q+r})^T$. We note that $\sum_{j=1}^{q+r} e_j \otimes d\xi_j$ corresponds to $d\xi \in (\Omega_A^1)^{(q+r)\times 1}$ written in the standard basis of $A^{(q+r)\times 1}$. Let us also consider the following right $A$-homomorphism:

$$\begin{array}{ccc} A^{(q+r)\times 1} \otimes_A \Omega_A^1 & \xrightarrow{\Pi_C \otimes \mathrm{id}_{\Omega_A^1}} & M_C \otimes_A \Omega_A^1 \\ \sum_{j=1}^{q+r} e_j \otimes d\xi_j & \longmapsto & \sum_{j=1}^{q+r} \Pi_C\, e_j \otimes d\xi_j. \end{array}$$

Using the embedding $i : M_C \longrightarrow A^{(q+r)\times 1}$, we can identify $M_C \otimes_A \Omega_A^1$ with its image in $(\Omega_A^1)^{(q+r)\times 1}$, i.e., we can identity $\sum_{j=1}^{q+r} \Pi_C\, e_j \otimes d\xi_j$ with $\Pi_C\, d\xi = \Pi_C\, (d\xi_1\ \ldots\ d\xi_{q+r})^T$.

*Definition 4:* The *Levi-Civita/Grassmann connection* on the projective $A$-module $M_C = \Pi_C\, A^{(q+r)\times 1}$ is defined by:

$$\begin{array}{rcl} \nabla : M_C & \longrightarrow & M_C \otimes_A \Omega_A^1 \\ \xi = \Pi_C\, \eta & \longmapsto & \nabla\, \xi = \Pi_C\, d\xi. \end{array} \tag{17}$$

Let us interpret the Levi-Civita/Grassmann connection (17). With the notations of Figure 1, using (2), we have:

$$\begin{array}{rcl} \nabla : M_C & \longrightarrow & M_C \otimes_A \Omega_A^1 \\ \begin{pmatrix} e_1 \\ y_1 \end{pmatrix} & \longmapsto & \Pi_C \begin{pmatrix} de_1 \\ dy_1 \end{pmatrix}. \end{array} \tag{18}$$

For instance, if $A = H^\infty(\mathbb{D})$, then $de_1$ and $dy_1$ have to be interpreted as in Example 2, i.e., as two Hankel operators.

Using the identity $\xi = \Pi_C\, \xi$ for all $\xi \in M_C$, we get $d\xi = d(\Pi_C\, \xi) = d\Pi_C\, \xi + \Pi_C\, d\xi$, which yields:

$$\forall\, \xi \in M_C, \quad \nabla\, \xi = d\xi - d\Pi_C\, \xi. \tag{19}$$

Since $\xi \in M_C = \Pi_C\, A^{(q+r)\times 1}$ is of the form of $\xi = \Pi_C\, \eta$ for a certain $\eta \in A^{(q+r)\times 1}$, (17) and $\Pi_C^2 = \Pi_C$ then yield:

$$\begin{aligned} \nabla\, \Pi_C\, \eta &= \Pi_C\, d(\Pi_C\, \eta) = \Pi_C\, (\Pi_C\, d\eta + d\Pi_C\, \eta) \\ &= \Pi_C\, d\eta + \Pi_C\, d\Pi_C\, \eta = \Pi_C\, (d\eta + d\Pi_C\, \eta). \end{aligned}$$

With the notations of Figure 1, using (2) again, we have:

$$\begin{array}{rcl} \nabla : M_C & \longrightarrow & M_C \otimes_A \Omega_A^1 \\ \Pi_C \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} & \longmapsto & \Pi_C \left( \begin{pmatrix} du_1 \\ du_2 \end{pmatrix} + d\Pi_C \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \right). \end{array}$$

Let us characterize all the connections on $M_C$. Applying Proposition 2 to the projective $A$-module $M_C$, we get:

$$\begin{aligned} &\hom_A(M_C, M_C \otimes_A \Omega_A^1) \cong M_C \otimes_A \Omega_A^1 \otimes_A M_C^\star \\ &\cong \Pi_C\, A^{(q+r)\times 1} \otimes_A \Omega_A^1 \otimes_A A^{1\times(q+r)}\, \Pi_C \\ &= \Pi_C\, (\Omega_A^1)^{(q+r)\times(q+r)}\, \Pi_C. \end{aligned}$$

By Remark 1, all the connections on $M_C$ are of the form of

$$\forall\, \xi \in M_C,\ \nabla' \xi = (\Pi_C\, d + \Pi_C\, \Gamma\, \Pi_C)\, \xi = \Pi_C\, (d + \Gamma)\, \Pi_C\, \xi,$$

where $\Gamma \in (\Omega_A^1)^{(q+r)\times(q+r)}$ is any matrix of 1-differential forms. The term $\Pi_C\, \Gamma\, \Pi_C$ added to the connection (17) is a so-called *gauge potential*.

Similar results can be obtained for the finitely generated projective $A$-module $M_P = A^{1\times(q+r)}\, \Pi_P \cong \mathcal{M}$ [9].

## IV. CURVATURES

Let us extend the definition of a connection.

*Proposition 3 ([1], [6]):* If $\nabla : M \longrightarrow M \otimes_A \Omega_A^1$ is a connection on a right $A$-module $M$, then $\nabla$ admits a unique extension to $\widetilde{\nabla} : M \otimes_A \Omega_A^\bullet \longrightarrow M \otimes_A \Omega_A^\bullet$ satisfying

$$\widetilde{\nabla}(\omega_i \otimes \omega_j) = \widetilde{\nabla}(\omega_i) \otimes \omega_j + (-1)^i\, \omega_i \otimes d\omega_j, \tag{20}$$

for all $\omega_i \in M \otimes_A \Omega_A^i$ and $\omega_j \in \Omega^j(A)$.

A connection has a *curvature*. Let us define this concept.

*Definition 5:* The *curvature* of the connection $\nabla$ is defined by $\nabla^2 = \nabla \circ \nabla : M \longrightarrow M \otimes_A \Omega_A^2$.

Let $m \in M$ and $a \in A$. Then, using (20), we obtain

$$\begin{aligned} \nabla^2(m\, a) &= \nabla((\nabla\, m)\, a + m \otimes da) \\ &= (\nabla^2\, m)\, a - \nabla\, m \otimes da + \nabla\, m \otimes da + m \otimes d^2 a \\ &= \nabla^2\, m\, a, \end{aligned} \tag{21}$$

i.e., $\nabla^2 \in \hom_A(M, M \otimes_A \Omega_A^2) \cong \hom_A(M, M) \otimes_A \Omega_A^2$.

Let $M_C = \Pi_C\, A^{(q+r)\times 1}$ be the finitely generated projective $A$-module and $\nabla : M_C \longrightarrow M_C \otimes_A \Omega_A^1$ a connection on $M_C$ defined by $\nabla\, \xi = \Pi_C\, d\xi + \Lambda\, \xi$, where $\Lambda = \Pi_C\, \Gamma\, \Pi_C$ for a certain $\Gamma \in (\Omega_A^1)^{(q+r)\times(q+r)}$. Using (20), the curvature $\nabla^2$ of $\nabla$ is then defined by:

$$\begin{aligned} \nabla^2\, \xi &= (\Pi_C\, d + \Lambda)(\Pi_C\, d\xi + \Lambda\, \xi) \\ &= \Pi_C\, d(\Pi_C\, d\xi) + \Pi_C\ d(\Lambda\, \xi) + \Lambda\, \Pi_C\, d\xi + \Lambda^2\, \xi \\ &= \Pi_C\, d\Pi_C\, d\xi + \Pi_C^2\, d^2\xi + \Pi_C\, d\Lambda\, \xi - \Pi_C\, \Lambda\, d\xi \\ &\quad + \Lambda\, \Pi_C\, d\xi + \Lambda^2\, \xi \\ &= \Pi_C\, d\Pi_C\, d\xi + \Pi_C\, d\Lambda\, \xi + \Lambda^2\, \xi. \end{aligned}$$

From (21), $\nabla^2$ is a right $A$-homomorphism. It means that the term $\Pi_C\, d\Pi_C\, d\xi$ can be rewritten as a sum of products of matrices of 2-forms multiplied by $\xi$. To do that, we first note that $\xi = \Pi_C\, \xi$ for all $\xi \in M_C$, which yields $\Pi_C\, d\Pi_C\, d\xi = \Pi_C\, d\Pi_C\, d(\Pi_C\, \xi) = \Pi_C\, d\Pi_C\, d\Pi_C\, \xi + \Pi_C\, d\Pi_C\, \Pi_C\, d\xi$. Now, $\Pi_C^2 = \Pi_C$ gives $d\Pi_C\, \Pi_C + \Pi_C\, d\Pi_C = d\Pi_C$, i.e., $d\Pi_C\, \Pi_C = (I_{q+r} - \Pi_C)\, d\Pi_C$, and thus $\Pi_C\, d\Pi_C\, \Pi_C = (\Pi_C\, (I_{q+r} - \Pi_C))\, d\Pi_C = 0$, which shows that $\Pi_C\, d\Pi_C\, d\xi = \Pi_C\, d\Pi_C\, d\Pi_C\, \xi$ and finally proves that:

$$\forall\, \xi \in M_C,\ \nabla^2\, \xi = (\Pi_C\, (d\Pi_C)^2 + \Pi_C\, d\Lambda + \Lambda^2)\, \xi.$$

If $M = A^r$ is a free right $A$-module (i.e., a "trivial vector bundle"), then $\Pi_C = I_{q+r}$ and $\nabla^2\, \xi = (d\Lambda + \Lambda^2)\, \xi$.

Similar results can be obtained for the finitely generated projective $A$-module $M_P = A^{1\times(q+r)}\, \Pi_P \cong \mathcal{M}$ [9].

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