Using morphism computations for factoring and decomposing general linear functional systems

Thomas Cluzeau and Alban Quadrat

Abstract— **Within a constructive homological algebra approach, we study the factorization and decomposition problems for general linear functional systems and, in particular, for multidimensional linear systems appearing in control theory. Using the concept of Ore algebras of functional operators (e.g., ordinary/partial differential operators, shift operators, time-delay operators), we first concentrate on the computation of morphisms from a finitely presented left module M over an Ore algebra to another one M', where M (resp., M') is a module intrinsically associated with the linear functional system $R\,y = 0$ (resp., $R'\,z = 0$). These morphisms define applications sending solutions of the system $R'\,z = 0$ to the ones of $R\,y = 0$. We explicitly characterize the kernel, image, cokernel and coimage of a general morphism. We then show that the existence of a non-injective endomorphism of the module M is equivalent to the existence of a non-trivial factorization $R = R_2\,R_1$ of the system matrix R. The corresponding system can then be integrated in cascade. Under certain conditions, we also show that the system $R\,y = 0$ is equivalent to a system $R'\,z = 0$, where R' is a block-triangular matrix. We show that the existence of projectors of the ring of endomorphisms of the module M allows us to reduce the integration of the system $R\,y = 0$ to the integration of two independent systems $R_1\,y_1 = 0$ and $R_2\,y_2 = 0$. Furthermore, we prove that, under certain conditions, idempotents provide decompositions of the system $R\,y = 0$, i.e., they allow us to compute an equivalent system $R'\,z = 0$, where R' is a block-diagonal matrix. Many applications of these results in mathematical physics and control theory are given. Finally, the different algorithms of the paper are implemented in a package** MORPHISMS **based on the library** OREMODULES.

Keywords— **Linear functional systems, factorization and decomposition problems, morphisms, equivalences of systems, Galois symmetries, r-pure autonomous observables, controllability, quadratic first integrals of motion, quadratic conservation laws, constructive homological algebra, module theory, symbolic computation.**

I. INTRODUCTION

Many systems coming from mathematical physics, applied mathematics and engineering sciences can be described by means of systems of ordinary or partial differential equations, difference equations, differential time-delay equations... If these systems are linear, they can then be defined by means of matrices with entries in non-commutative algebras of functional operators such as the rings of differential operators, shift operators, time-delay operators... An important class of such algebras is called *Ore algebras* ([12]). See also [14].

The methods of *algebraic analysis* give a way to intrinsically study a linear functional system by considering its associated finitely presented left module over an Ore algebra ([14], [28], [38], [40], [43], [61], [62]). This idea is natural as the structural properties of the linear functional systems can be studied by handling algebraic manipulations on the system matrix of functional operators, i.e., by performing linear algebra over a ring which is also called *module theory* ([29], [37], [56]). The tools of *homological algebra* have been developed in order to study the properties of modules ([56]), and thus, the structural properties of the corresponding systems. Using recent developments and implementations of Gröbner and Janet bases over Ore algebras ([12], [32]), it has been shown in [14], [43], [44], [45], [46], [47], [53], [54] how to make effective some of these tools as, for instance, free resolutions, parametrizations, projective dimensions, torsion-free degrees, Hilbert series, extension functors, classification of modules (torsion, torsion-free, reflexive, projective, stably free, free). Applications of these algorithms in multidimensional control theory have recently been given in [13], [14], [27], [40], [41], [43], [44], [45], [46], [47], [51], [52], [53], [54], [55], [61], [62], [63].

Continuing the development of *constructive homological algebra* for linear systems over Ore algebras and, in particular [48], [53], [52], the first part of the paper aims at computing effectively morphisms from a left D-module M, finitely presented by a matrix R with entries in an Ore algebra D, to a left D-module M' presented by a matrix R'. In particular, we show that a morphism from M to M' defines a transformation sending a solution of the system $R'\,z = 0$ into a solution of $R\,y = 0$. In the case where $R' = R$, the ring $\mathrm{end}_D(M)$ of endomorphisms of M corresponds to the "Galois symmetries" of the system $R\,y = 0$. In the case of 1-D linear systems, we explain how to find again classical results on the concept of *eigenring* developed in the system theory and symbolic computation literatures. Algorithms for computing morphisms are given in the cases where the underlying Ore algebra is commutative or non-commutative. As an application, we show how to use the computation of the morphisms from two modules in order to obtain quadratic first integrals of motion and conservation laws.

We then explicitly characterize the kernel, coimage, image and cokernel of a morphism from M to M' and deduce a method to check the equivalence of the corresponding

T. Cluzeau and A. Quadrat are with INRIA Sophia Antipolis, CAFE project, 2004 Route des Lucioles BP 93, 06902 Sophia Antipolis Cedex, France, Thomas.Cluzeau@sophia.inria.fr, Alban.Quadrat@sophia.inria.fr.

systems $R\,y=0$ and $R'\,z=0$. In Theorem 1, we prove that the existence of a non-injective endomorphism of a left D-module M, finitely presented by a matrix R with entries in an Ore algebra D, corresponds to a factorization of the form $R=R_2\,R_1$, where R_1 and R_2 are two matrices with entries in D. As a consequence, the integration of the system $R\,y=0$ is reduced to a cascade of integrations. In Theorem 2, under certain conditions on the morphism (freeness), we show that the system $R\,y=0$ is equivalent to a system of the form

$$\left(\begin{array}{cc} T_1 & T_2 \\ 0 & T_3 \end{array}\right)\left(\begin{array}{c} z_1 \\ z_2 \end{array}\right)=0, \qquad (1)$$

where T_1, T_2 and T_3 are three matrices with entries in D and such that (1) has the same dimensions as R. We finish the section by giving a way to constructively compute r-pure autonomous elements of a linear system ([43], [51]).

In the fourth part of the paper, we show how to effectively compute the projectors of $\mathrm{end}_D(M)$ and we prove in Theorem 3 that they allow us to decompose the system $R\,y=0$ into two decoupled systems $S_1\,y_1=0$ and $S_2\,y_2=0$, where S_1 and S_2 are two matrices with entries in D. Consequently, the integration of the system $R\,y=0$ is then equivalent to the integrations of the two independent systems $S_1\,y_1=0$ and $S_2\,y_2=0$. Then, under certain conditions on the projectors (e.g., idempotent, freeness), we prove in Theorem 4 that the system $R\,y=0$ is equivalent to a block diagonal system of the form

$$\left(\begin{array}{cc} T_1 & 0 \\ 0 & T_2 \end{array}\right)\left(\begin{array}{c} z_1 \\ z_2 \end{array}\right)=0, \qquad (2)$$

where T_1 and T_2 are two matrices with entries in D and such that (2) has the same dimensions as R. In particular, these conditions always hold in the case of a univariate Ore algebra over a field of coefficients (i.e., ordinary differential/difference systems over the field of rational functions) and in the case of a multivariate commutative Ore algebras due to the Quillen-Suslin theorem ([35], [56]) (e.g., linear system of partial differential equations with constant coefficients). Moreover, if some rank conditions on the projector are fulfilled, then, using a result due to Stafford ([37], [54]), we prove that a similar result also holds for the Weyl algebras $A_n(k)$ and $B_n(k)$ over a field k of characteristic 0 (i.e., linear system of partial differential equations with polynomial/rational coefficients). Using recent implementations of both Quillen-Suslin and Stafford results in the library OREMODULES ([13], [27], [54], [55]), we obtain a constructive way to compute the decomposition (2) of $R\,y=0$ when it exists.

We point out that, for all the above-mentioned results and, hence, for all the corresponding algorithms, no condition on the system $R\,y=0$ is required such as D-finite, determined, underdetermined, overdetermined, i.e., this approach handles general linear systems over an Ore algebra. To our knowledge, the problem of factoring or decomposing linear functional systems has been studied only for a few particular cases. For scalar linear differential operators or linear determined differential systems, we refer to [3], [7], [8], [11], [21], [23], [24], [25], [57], [59], [60]. Generalizations to linear determined difference and q-difference systems appear in [3], [9] and for D-finite partial differential systems (and finite-dimensional determined systems over a Ore algebra with rational coefficients), see [34], [64], [65]. A more general work in that direction is included in [33]. For similar cases where the base field is of positive characteristic and also for modular approaches, see [6], [15], [16], [17], [18], [22], [49].

All along the paper, we illustrate our results by considering some applications coming from mathematical physics (e.g., Galois symmetries of the linearized Euler equations, quadratic first integrals of motion and conservation laws, equivalence of systems appearing in linear elasticity) and control theory (controllability, r-autonomous elements, decoupling of the autonomous and controllable subsystems).

The different algorithms presented in the paper have been implemented in the package MORPHISMS based on the library OREMODULES ([13]). This package is available on the author's web pages and on the one of OREMODULES (see [13] for the precise address) with a library of examples which demonstrates the main results of the paper.

II. MORPHISMS OF LINEAR FUNCTIONAL SYSTEMS

A. Finitely presented modules and linear functional systems

In this paper, we consider linear functional systems defined by matrices with entries in an *Ore algebra* D and we study them by means of their associated left D-modules. In this first subsection, we gather many useful definitions and properties on these concepts.

Definition 1 ([12], [14]): Let A be a commutative ring, σ an *endomorphism* of A, namely,

$$\forall\, a,\, b\in A, \quad \left\{\begin{array}{l} \sigma(a+b)=\sigma(a)+\sigma(b), \\ \sigma(a\,b)=\sigma(a)\,\sigma(b), \end{array}\right.$$

and δ a *σ-derivation*, namely, $\delta : A\rightarrow A$ satisfies:

$$\forall\, a,\, b\in A, \quad \left\{\begin{array}{l} \delta(a+b)=\delta(a)+\delta(b), \\ \delta(a\,b)=\sigma(a)\,\delta(b)+\delta(a)\,b. \end{array}\right.$$

1) A (non-commutative) polynomial ring $A[\partial;\sigma,\delta]$ in ∂ is called *skew* if it satisfies the commutation rule:

$$\forall\, a\in A, \quad \partial\, a=\sigma(a)\,\partial+\delta(a). \qquad (3)$$

An element P of $A[\partial;\sigma,\delta]$ has the canonical form:

$$P=\sum_{i=0}^{r}a_i\,\partial^i, \quad r\in\mathbb{Z}_+, \quad \forall\, i\in\{1,\ldots,r\},\; a_i\in A.$$

If $a_r\neq 0$, then the *order* $\mathrm{ord}(P)$ of P is r.

2) Let k be a field and A be either k, the commutative polynomial ring $k[x_1,\ldots,x_n]$ or the commutative

ring of rational functions $k(x_1, \ldots, x_n)$. The skew polynomial ring

$$D = A[\partial_1; \sigma_1, \delta_1] \cdots [\partial_m; \sigma_m, \delta_m]$$

is then called an *Ore algebra* if the following conditions are fulfilled:

$$\begin{cases} \sigma_i\, \delta_j = \delta_j\, \sigma_i, & \forall\, 1 \leq i, j \leq m, \\ \sigma_i(\partial_j) = \partial_j, & \forall\, 1 \leq j < i \leq m, \\ \delta_i(\partial_j) = 0, & \forall\, 1 \leq j < i \leq m. \end{cases}$$

An element P of D has the canonical form

$$P = \sum_{0 \leq |\nu| \leq r} a_\nu\, \partial^\nu, \quad r \in \mathbb{Z}_+, \quad a_\nu \in A,$$

where $\nu = (\nu_1, \ldots, \nu_n) \in \mathbb{Z}_+^n$ denotes a multi-index of non-negative integers, $|\nu| = \nu_1 + \cdots + \nu_n$ its length, and $\partial^\nu = \partial_1^{\nu_1} \cdots \partial_n^{\nu_n}$.

We note that the commutation rule (3) must be understood as a generalization of the Leibniz rule for functional operators, namely, for an unknown y, we have:

$$\partial(a\, y) = \sigma(a)\, \partial\, y + \delta(a)\, y.$$

Let us give a few examples of skew polynomial rings and Ore algebras.

Example 1: 1) Let k be a field, $A = k$, $k[n]$ or $k(n)$, $\sigma : A \longrightarrow A$ the forward shift operator, namely, $\sigma(a)(n) = a(n+1)$, and $\delta = 0$. Then, the skew polynomial ring $A[\partial; \sigma, 0]$ is the ring of shift operators with coefficients in A (i.e., constant, polynomial or rational coefficients).

2) Let k be a field, $A = k$, $k[t]$ or $k(t)$, $\sigma = \mathrm{id}_A$ and $\delta : A \longrightarrow A$ the standard derivation $\frac{d}{dt}$. The skew polynomial ring $A\left[\partial; \mathrm{id}_A, \frac{d}{dt}\right]$ is then the ring of differential operators with coefficients in A (i.e., constant, polynomial or rational coefficients).

3) More generally, if k is a field and A is respectively k, $k[x_1, \ldots, x_n]$ or $k(x_1, \ldots, x_n)$, then we can consider $\sigma_i = id_{A[\partial_1; \sigma_1, \delta_1] \cdots [\partial_{i-1}; \sigma_{i-1}, \delta_{i-1}]}$ and $\delta_i(a) = \frac{\partial a}{\partial x_i}$ the standard derivation of $a \in A$ with respect to x_i. Then, the Ore algebra $A[\partial_1; \mathrm{id}, \delta_1] \cdots [\partial_n; \mathrm{id}, \delta_n]$ is the ring of differential operators with respectively constant, polynomial or rational coefficients. The last two algebras are called the *Weyl algebras* and they are respectively denoted by:

$$A_n(k) = k[x_1, \ldots, x_n][\partial_1; \mathrm{id}, \delta_1] \cdots [\partial_n; \mathrm{id}, \delta_n],$$

$$B_n(k) = k(x_1, \ldots, x_n)[\partial_1; \mathrm{id}, \delta_1] \cdots [\partial_n; \mathrm{id}, \delta_n].$$

4) Let k be a field, $A = k$, $k[t]$ or $k(t)$, and $A\left[\partial_1; \mathrm{id}_A, \frac{d}{dt}\right]$ the ring of differential operators with coefficients in A. Let $h \in \mathbb{R}_+$ be a positive real and let us denote by $\sigma_2(a) = a(t-h)$ the time-delay operator and $\delta_2(a) = 0$ for all $a \in A$. Then, $A\left[\partial_1; \mathrm{id}_A, \frac{d}{dt}\right][\partial_2; \sigma_2, 0]$ is the Ore algebra of differential time-delay operators with coefficients in the ring A.

We refer the reader to [12] for more examples of functional operators such as, for instance, difference, divided difference, q-difference, q-dilation operators and their applications in the study of special functions as well as in combinatorics.

We recall that a ring A is said to be *left noetherian* if every left ideal I of A is finitely generated as a left A-module, namely, if there exists a finite family $\{a_i\}_{i=1,\ldots,l(I)}$ of elements of A which satisfies $I = D\, a_1 + \cdots + D\, a_{l(I)}$. A similar definition exists for *right noetherian rings*.

Proposition 1 ([37]): If A is a left (resp., right) noetherian ring and σ is an automorphism of A, then the skew polynomial ring $D = A[\partial; \sigma, \delta]$ is a left (resp., right) noetherian.

The examples of Ore algebras given in Example 1 are left and right noetherian rings. Moreover, they are *domains*, namely, the product of non-zero elements is non-zero.

Proposition 2 ([12]): Let k be a computable field (e.g., $k = \mathbb{Q}$, $\mathbb{F}_p$), A be either k, $k[x_1, \ldots, x_n]$ or $k(x_1, \ldots, x_n)$ and $A[\partial_1; \sigma_1, \delta_1] \ldots [\partial_m; \sigma_m, \delta_m]$ an Ore algebra satisfying the following conditions

$$\begin{cases} \sigma_i(x_j) = a_{ij}\, x_j + b_{ij}, \\ \delta_i(x_j) = c_{ij}, \end{cases} \quad 1 \leq i \leq m, \quad 1 \leq j \leq n,$$

for certain $a_{ij} \in k \setminus \{0\}$, $b_{ij} \in k$, $c_{ij} \in A$. If the c_{ij} are of total degree at most 1 in the x_i's, then a non-commutative version of Buchberger's algorithm terminates for any monomial order on $x_1, \ldots, x_n$, $\partial_1, \ldots, \partial_m$, and its result is a Gröbner basis with respect to the given monomial order.

Proposition 2 holds for the examples of Ore algebras given in Example 1. In the rest of the paper, we shall only consider left noetherian domains which satisfy the hypotheses of Proposition 2.

In what follows, we shall assume that a linear functional system (LFS) is defined by means of a matrix of functional operators $R \in D^{q \times p}$, where D is an Ore algebra. Then, we consider the D-morphism of left D-modules (i.e., the left D-linear application) defined by:

$$\begin{array}{ccl} D^{1 \times q} & \xrightarrow{.R} & D^{1 \times p}, \\ (\lambda_1, \ldots, \lambda_q) & \longmapsto & (\lambda_1, \ldots, \lambda_q)\, R = \\ & & (\sum_{i=1}^q \lambda_i\, R_{i1}, \ldots, \sum_{i=1}^q \lambda_i\, R_{ip}). \end{array} \tag{4}$$

Generalizing an important idea coming from number theory and algebraic geometry, we shall consider the left D-module

$$M = D^{1\times p}/(D^{1\times q}\, R)$$

which is the cokernel of the D-morphism defined by (4).

This idea can be traced back to the work of B. Malgrange ([38]) on linear systems of PDEs with constant coefficients and it has been extended to the variable coefficients case by M. Kashiwara ([28]). We refer to [14] for the extension to linear functional systems.

Finally, we note that if k is a field, V a finite-dimensional k-vector space of dimension p, $E \in k^{p\times p}$ and $D = k[X]$, the D-module defined by

$$D^p/((X\, I_p - E)\, D^p) = D^{1\times p}/(D^{1\times p}\,(X\, I_p - E)^T)$$

plays a central role in the study of the reduction of the endomorphism E of V (see [10]).

Before explaining the main interest of the left D-module M, we first recall some basic concepts of homological algebra used in the sequel. We refer the reader to [56] for more details.

Definition 2: A sequence $(M_i, d_i)_{i\in\mathbb{Z}_+}$ of left D-modules M_i and D-morphisms $d_i : M_i \longrightarrow M_{i-1}$, with the convention that $M_{-1} = 0$, is said to be:

1) a *complex* if, for all $i \in \mathbb{Z}_+$, $d_i \circ d_{i+1} = 0$ or, equivalently, $\operatorname{im} d_{i+1} \subseteq \ker d_i$. The *defect of exactness at* M_i is then defined by $H(M_i) = \ker d_i/\operatorname{im} d_{i+1}$ and the complex $(M_i, d_i)_{i\in\mathbb{Z}_+}$ is denoted by:

$$\ldots \xrightarrow{d_{i+2}} M_{i+1} \xrightarrow{d_{i+1}} M_i \xrightarrow{d_i} M_{i-1} \xrightarrow{d_{i-1}} \ldots .$$

2) *exact* at M_i if $\ker d_i = \operatorname{im} d_{i+1}$, i.e., $H(M_i) = 0$.

3) *exact* if $\ker d_i = \operatorname{im} d_{i+1}$, for all $i \in \mathbb{Z}_+$.

4) *split exact* if it is exact and there further exist left D-morphisms $s_i : M_{i-1} \longrightarrow M_i$ satisfying the following conditions:

$$\forall\, i \in \mathbb{Z}_+, \quad \begin{cases} s_{i+1} \circ s_i = 0, \\ s_i \circ d_i + d_{i+1} \circ s_{i+1} = id_{M_i}. \end{cases}$$

The complex $(M_{i-1}, s_i)_{i\in\mathbb{Z}_+}$ is then exact.

Using (4), we obtain the exact sequence

$$D^{1\times q} \xrightarrow{.R} D^{1\times p} \xrightarrow{\pi} M = D^{1\times p}/(D^{1\times q}\, R) \longrightarrow 0, \quad (5)$$

where π denotes the canonical projection of $D^{1\times p}$ onto M that sends an element of $D^{1\times p}$ onto its residue class in M. The exact sequence (5) is called a *finite presentation* of M and M is said to be a *finitely presented* left D-module.

Let us describe M in terms of its generators and relations. Let $\{e_i\}_{1\le i\le p}$ (resp., $\{f_j\}_{1\le j\le q}$) be the standard basis of $D^{1\times p}$ (resp., $D^{1\times q}$), namely, the basis of $D^{1\times p}$ formed by the row vectors e_i defined by 1 at the i^{th} position and 0 elsewhere. We denote by y_i the residue class of e_i in M, i.e., $y_i = \pi(e_i)$. Then, $\{y_i\}_{1\le i\le p}$ is a set of generators of M as every element $m \in M$ is trivially of the form $\pi(\mu)$, where $\mu = (\mu_1, \ldots, \mu_p) \in D^{1\times p}$, and thus, we obtain $m = \pi(\mu) = \sum_{i=1}^p \mu_i\, \pi(e_i) = \sum_{i=1}^p \mu_i\, y_i$. The left D-module M is then said to be finitely generated. Now, for $j = 1, \ldots, q$, we have:

$$f_j\, R = (R_{j1}, \ldots, R_{jp}) \in (D^{1\times q}\, R) \Rightarrow \pi(f_j\, R) = 0.$$

Making explicit $\pi(f_j\, R)$, we obtain:

$$\pi(f_j\, R) = \sum_{k=1}^p R_{jk}\, \pi(e_k) = \sum_{k=1}^p R_{jk}\, y_k, \quad j = 1, \ldots, q.$$

Hence, the generators $\{y_i\}_{1\le i\le p}$ of M satisfy the relations $\sum_{k=1}^p R_{jk}\, y_k = 0$ for $j = 1, \ldots, q$, or, more compactly, $R\, y = 0$ where $y = (y_1, \ldots, y_p)^T$.

Example 2: Let us consider the equations of a fluid in a tank satisfying Saint-Venant's equations and subjected to a one dimensional horizontal move, developed in [26]:

$$\begin{cases} y_1(t - 2\,h) + y_2(t) - 2\,\dot{u}(t - h) = 0, \\ y_1(t) + y_2(t - 2\,h) - 2\,\dot{u}(t - h) = 0. \end{cases} \quad (6)$$

Let $D = \mathbb{Q}\left[\partial_1; 1, \frac{d}{dt}\right][\partial_2; \sigma_2, 0]$ be the Ore algebra of differential time-delay operators with coefficients in $\mathbb{Q}$ defined in 4) of Example 1 and let us consider the matrix:

$$R = \begin{pmatrix} \partial_2^2 & 1 & -2\,\partial_1\,\partial_2 \\ 1 & \partial_2^2 & -2\,\partial_1\,\partial_2 \end{pmatrix} \in D^{2\times 3}. \quad (7)$$

The D-module $M = D^{1\times 3}/(D^{1\times 2}\, R)$ is then defined by the following finite presentation:

$$0 \longrightarrow D^{1\times 2} \xrightarrow{.R} D^{1\times 3} \xrightarrow{\pi} M \longrightarrow 0.$$

To develop the relations between the properties of the finitely presented left D-module M in (5) and the solutions of the system $R\, y = 0$, we need to introduce a few more concepts of module theory (see [56] for details).

Definition 3: 1) Let N be a left D-module. We denote by $\hom_D(M, N)$ the abelian group of D-morphisms from M to N. If M has a D-D' bimodule structure, i.e., M is a right D'-module which satisfies $(a\, m)\, b = a\,(m\, b)$ for all a in D and b in D', then $\hom_D(M, N)$ inherits a right D'-module. In particular, if D is a commutative ring, then $\hom_D(M, N)$ inherits a D-module structure.

2) If $N = M$, then we denote the non-commutative ring of endomorphisms of M by $\operatorname{end}_D(M)$. Moreover, we denote by $\operatorname{iso}_D(M)$ the non-abelian group of isomorphisms of M, namely, the group of

injective and surjective D-morphisms from M to M.

3) A finitely generated left D-module is called *free* if M is isomorphic to a finite power of D, i.e., there exists an injective and surjective D-morphism from M to $D^{1\times r}$, where r is a non-negative integer.

4) A finitely generated left D-module M is called *projective* if there exist a left D-module N and a non-negative integer r such that that $M\oplus N \cong D^{1\times r}$, where $\oplus$ denotes the direct sum of left D-modules and $\cong$ an isomorphism. N is then also projective.

5) A projective resolution of a left D-module M is an exact sequence of the form

$$\ldots \xrightarrow{d_3} P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \longrightarrow 0, \qquad (8)$$

where the P_i are projective left D-modules. If all the P_i are free left D-modules, then (8) is called a *free resolution* of M. Finally, if there exists a non-negative integer s such that $P_r = 0$ for all $r \geq s$ and the P_i are finitely generated free left D-modules, then (8) is called a *finite free resolution* of M.

6) Let (8) be a projective resolution of a left D-module M. We call *truncated projective resolution of* M the complex defined by:

$$\ldots \xrightarrow{d_3} P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \longrightarrow 0.$$

Let us suppose that a finitely presented left D-module admits a finite free resolution (we note that it is always the case for the Ore algebras defined in Example 1 as it is proved in [14]):

$$0 \longrightarrow D^{1\times p_l} \xrightarrow{.R_l} \ldots \xrightarrow{.R_1} D^{1\times p_0} \xrightarrow{\pi} M \longrightarrow 0. \qquad (9)$$

Let $\mathcal{F}$ be a left D-module. Then, applying the functor $\hom(\cdot, \mathcal{F})$ to the truncated free resolution of M

$$0 \longrightarrow D^{1\times p_l} \xrightarrow{.R_l} \ldots \xrightarrow{.R_2} D^{1\times p_1} \xrightarrow{.R_1} D^{1\times p_0} \longrightarrow 0,$$

we get the following complex (see [14], [56])

$$0 \longleftarrow \mathcal{F}^{p_l} \xleftarrow{R_l.} \ldots \xleftarrow{R_2.} \mathcal{F}^{p_1} \xleftarrow{R_1.} \mathcal{F}^{p_0} \longleftarrow 0, \qquad (10)$$

where, for $i = 1, \ldots, l$, $R_i. : \mathcal{F}^{p_{i-1}} \longrightarrow \mathcal{F}^{p_i}$ is defined by $(R_i.)\,\zeta = R_i\,\zeta$ for all $\zeta = (\zeta_1, \ldots, \zeta_{p_{i-1}})^T \in \mathcal{F}^{p_{i-1}}$. We can prove that, up to isomorphisms, the defects of exactness of (10) only depend on M and $\mathcal{F}$ and not on the choice of the finite free resolution (9) of M. See [56] for more details. In particular, we note that these defects of exactness can be defined by using any projective resolution of M and not necessarily a finite free resolution of M as we have done for simplicity reasons. They are denoted by:

$$\begin{cases} \mathrm{ext}^0_D(M, \mathcal{F}) \cong \ker_{\mathcal{F}}(R_1.) = \{\eta \in \mathcal{F}^{p_0} \mid R_1\,\eta = 0\}, \\ \mathrm{ext}^i_D(M, \mathcal{F}) \cong \ker_{\mathcal{F}}(R_{i+1}.)/\mathrm{im}_{\mathcal{F}}\,(R_i.), \quad i \geq 1. \end{cases}$$

It is quite easy (see [56]) to show that

$$\mathrm{ext}^0_D(M, \mathcal{F}) = \hom_D(M, \mathcal{F}),$$

which proves that the abelian group $\ker_{\mathcal{F}}(R_1.)$ of $\mathcal{F}$-solutions of the linear functional system $R_1\,\eta = 0$ is isomorphic to $\hom_D(M, \mathcal{F})$. We refer to [14] for more details. The abelian group $\ker_{\mathcal{F}}(R_1.)$ is sometimes called the *behaviour* of the left D-module $M = D^{1\times p_0}/(D^{1\times p_1}\,R_1)$. Moreover, if we want to solve the inhomogeneous system $R_1\,\eta = \zeta$, where $\zeta \in \mathcal{F}^{p_1}$ is fixed, then, using the fact that (9) is exact, we obtain that a necessary condition for the existence of a solution $\eta \in \mathcal{F}^{p_0}$ is given by $R_2\,\zeta = 0$ as we have:

$$R_1\,\eta = \zeta \Rightarrow R_2(R_1\,\eta) = R_2\,\zeta \Rightarrow R_2\,\zeta = 0.$$

In order to understand if the compatibility condition $R_2\,\zeta = 0$ is also sufficient, we need to investigate the residue class of ζ in $\mathrm{ext}^1_D(M, \mathcal{F}) = \ker_{\mathcal{F}}(R_2.)/(R_1\,\mathcal{F}^{p_0})$. If its residue class is 0, then it means that $\zeta \in \mathcal{F}^{p_1}$ satisfying $R_2\,\zeta = 0$ is such that $\zeta \in (R_1\,\mathcal{F}^{p_0})$, i.e., there exists $\eta \in \mathcal{F}^{p_0}$ such that $R_1\,\eta = \zeta$. The solution η is generally not unique as we can add any element of $\ker_{\mathcal{F}}(R_1.) = \{\eta \in \mathcal{F}^{p_0} \mid R_1\,\eta = 0\}$ to it.

Definition 4 ([56]): 1) A left D-module $\mathcal{F}$ is called *injective* if, for every left D-module M, we have:

$$\mathrm{ext}^i_D(M, \mathcal{F}) = 0, \quad i \geq 1.$$

2) A left D-module $\mathcal{F}$ is called *cogenerator* if:

$$\hom_D(M, \mathcal{F}) = 0 \Rightarrow M = 0.$$

If $\mathcal{F}$ is an injective left D-module, then $R_2\,\zeta = 0$ is a necessary and sufficient condition for the existence of $\eta \in \mathcal{F}^{p_0}$ satisfying $R_1\,\eta = \zeta$. Moreover, if $\mathcal{F}$ is a cogenerator left D-module and M is not reduced to the trivial module 0, then $\hom_D(M, \mathcal{F}) \neq 0$, meaning that the system $R_1\,\eta = 0$ admits at least one solution in $\mathcal{F}^{p_0}$. Finally, if $\mathcal{F}$ is an injective cogenerator left D-module, then we can prove that any complex of the form (10) is exact if and only if the corresponding complex (9) is exact.

Proposition 3 ([56]): For every ring D, there exists an injective cogenerator left D-module $\mathcal{F}$.

In some interesting situations, explicit injective cogenerators are known. Let us give some examples.

Example 3: 1) If Ω is a convex open subset of $\mathbb{R}^n$, then the space $C^\infty(\Omega)$ (resp., $\mathcal{D}'(\Omega)$) of smooth functions (resp., distributions) on Ω is an injective cogenerator module over the commutative ring $\mathbb{R}[\partial_1; \mathrm{id}, \delta_1]\cdots[\partial_n; \mathrm{id}, \delta_n]$ (see [38]).

2) If $\mathcal{F}$ is the set of all functions that are smooth on $\mathbb{R}$ except for a finite number of points, then $\mathcal{F}$ is

an injective cogenerator left $\mathbb{R}(t)\left[\partial; \mathrm{id}_{\mathbb{R}(t)}, \frac{d}{dt}\right]$-module. See [63] for more details.

To finish, let us recall two classical results of homological algebra.

Proposition 4 ([56]): 1) Let us consider the following exact sequence of left D-modules:

$$0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0.$$

If M'' is a projective left D-module, then the previous exact sequence splits (see 4) of Definition 2).

2) Let $\mathcal{F}$ be a left D-module. Then, the functor $\hom_D(\cdot, \mathcal{F})$ transforms split exact sequences of left D-modules into split exact sequences of abelian groups.

B. Morphisms of finitely presented modules

1) Definitions and results: Let us first introduce a few definitions of homological algebra concerning morphisms of complexes. See [56] for more details.

Definition 5: 1) Let $(P_i, d_i)_{i\in\mathbb{Z}_+}$ and $(P'_i, d'_i)_{i\in\mathbb{Z}_+}$ be two complexes of left D-modules. A *morphism of complexes* $f : (P_i, d_i)_{i\in\mathbb{Z}_+} \longrightarrow (P'_i, d'_i)_{i\in\mathbb{Z}_+}$ is a set of D-morphisms $f_i : P_i \longrightarrow P'_i$ such that

$$\forall\, i \geq 1, \quad d'_i \circ f_i = f_{i-1} \circ d_i,$$

i.e., we have the following commutative diagram:

$$\begin{array}{ccccc} P_{i+1} & \xrightarrow{d_{i+1}} & P_i & \xrightarrow{d_i} & P_{i-1} \\ \downarrow f_{i+1} & & \downarrow f_i & & \downarrow f_{i-1} \\ P'_{i+1} & \xrightarrow{d'_{i+1}} & P'_i & \xrightarrow{d'_i} & P'_{i-1}. \end{array}$$

2) A morphism of complexes

$$f : (P_i, d_i)_{i\in\mathbb{Z}_+} \longrightarrow (P'_i, d'_i)_{i\in\mathbb{Z}_+}$$

is said to be *homotopic to zero* if there exist D-morphisms $s_i : P_i \longrightarrow P'_{i+1}$ such that:

$$\forall\, i \geq 1, \quad f_i = d'_{i+1} \circ s_i + s_{i-1} \circ d_i.$$

By extension, two morphisms of complexes

$$f, f' : (P_i, d_i)_{i\in\mathbb{Z}_+} \longrightarrow (P'_i, d'_i)_{i\in\mathbb{Z}_+}$$

are *homotopic* if $f - f'$ is homotopic to zero.

3) A morphism of complexes

$$f : (P_i, d_i)_{i\in\mathbb{Z}_+} \longrightarrow (P'_i, d'_i)_{i\in\mathbb{Z}_+}$$

is called a *homotopy equivalence* or a *homotopism* if there exists a morphism of complexes

$$g : (P'_i, d'_i)_{i\in\mathbb{Z}_+} \longrightarrow (P_i, d_i)_{i\in\mathbb{Z}_+}$$

such that $f \circ g - \mathrm{id}_{P'}$ and $g \circ f - \mathrm{id}_P$ are homotopic to zero, where $\mathrm{id}_P = (P_i, \mathrm{id}_{P_i})_{i\in\mathbb{Z}_+}$. The complexes $(P_i, d_i)_{i\in\mathbb{Z}_+}$ and $(P'_i, d'_i)_{i\in\mathbb{Z}_+}$ are then said to be *homotopy equivalent*.

We have the following important result. See [47], [56] for a proof.

Proposition 5 ([47], [56]): Let $(P_i, d_i)_{i\in\mathbb{Z}_+}$ (resp., $(P'_i, d'_i)_{i\in\mathbb{Z}_+}$) be a truncated projective resolution of M (resp., M'). Then, a morphism $f : M \longrightarrow M'$ induces a morphism of complexes $\tilde{f} : (P_i, d_i)_{i\in\mathbb{Z}_+} \longrightarrow (P'_i, d'_i)_{i\in\mathbb{Z}_+}$ defined uniquely up to a homotopy.

Conversely, a morphism of complexes

$$\tilde{f} : (P_i, d_i)_{i\in\mathbb{Z}_+} \longrightarrow (P'_i, d'_i)_{i\in\mathbb{Z}_+}$$

from a truncated projective resolution $(P_i, d_i)_{i\in\mathbb{Z}_+}$ of M to a truncated projective resolution $(P'_i, d'_i)_{i\in\mathbb{Z}_+}$ of M' induces a morphism $f : M \longrightarrow M'$.

We deduce the following interesting corollary.

Corollary 1: Let

$$D^{1\times q} \xrightarrow{.R} D^{1\times p} \xrightarrow{\pi} M \longrightarrow 0,$$

$$D^{1\times q'} \xrightarrow{.R'} D^{1\times p'} \xrightarrow{\pi'} M' \longrightarrow 0,$$

be a finite presentation of respectively M and M'.

1) The existence of a morphism $f : M \longrightarrow M'$ is equivalent to the existence of two matrices

$$P \in D^{p\times p'}, \quad Q \in D^{q\times q'}$$

satisfying the commutation relation:

$$R\,P = Q\,R'. \tag{11}$$

We then have the commutative exact diagram:

$$\begin{array}{ccccccc} D^{1\times q} & \xrightarrow{.R} & D^{1\times p} & \xrightarrow{\pi} & M & \longrightarrow 0 \\ \downarrow .Q & & \downarrow .P & & \downarrow f & \\ D^{1\times q'} & \xrightarrow{.R'} & D^{1\times p'} & \xrightarrow{\pi'} & M' & \longrightarrow 0. \end{array} \tag{12}$$

2) Moreover, if we denote by $R'_2 \in D^{r'\times q'}$ the matrix satisfying

$$\ker_D(.R') = D^{1\times r'}\,R_2,$$

then P and Q are defined up to a homotopy, i.e., the matrices

$$\begin{cases} \overline{P} = P + Z_1\,R', \\ \overline{Q} = Q + R\,Z_1 + Z_2\,R'_2, \end{cases}$$

where $Z_1 \in D^{p\times q'}$ and $Z_2 \in D^{q\times r'}$ are two arbitrary matrices, also satisfy the relation:

$$R\,\overline{P} = \overline{Q}\,R'.$$

3) Finally, for all $m \in M$, we have $f(m) = \pi'(\lambda\,P)$, where $\lambda \in D^{1\times p}$ is any element such that $m = \pi(\lambda)$.

In the particular case where $R' = R$, from Corollary 1, we obtain that the existence of an endomorphism f of M

is equivalent to the existence of two matrices $P \in D^{p \times p}$ and $Q \in D^{q \times q}$ satisfying the commutation relation:

$$R\,P = Q\,R. \tag{13}$$

Before illustrating Corollary 1, let us give a direct consequence of this corollary which shows one interest of computing morphisms between finitely presented left D-modules.

Corollary 2: With the same hypotheses and notations as in Corollary 1, if $\mathcal{F}$ is a left D-module, then the morphism $f^\star$ of abelian groups defined by

$$\forall\, \zeta \in \mathcal{F}^{p'}, \quad f^\star(\zeta) = P\,\zeta,$$

sends the elements of $\ker_{\mathcal{F}}(R'.)$ to elements $\ker_{\mathcal{F}}(R.)$, i.e., $\mathcal{F}$-solutions of the system $R'\,\zeta = 0$ to $\mathcal{F}$-solutions of the system $R\,\eta = 0$.

Proof: Applying the right-exact functor $\hom_D(\cdot, \mathcal{F})$ (see [56]) to the exact commutative exact diagram (12), we obtain the following exact commutative exact diagram:

$$\begin{array}{ccccccc}
 & & & & 0 & & \\
 & & & & \uparrow & & \\
 & & & & \mathrm{coker} f^\star & & \\
 & & & & \uparrow & & \\
\mathcal{F}^q & \xleftarrow{R.} & \mathcal{F}^p & \xleftarrow{\pi^\star} & \hom_D(M, \mathcal{F}) & \longleftarrow & 0 \\
\uparrow .Q & & \uparrow .P & & \uparrow f^\star & & \\
\mathcal{F}^{q'} & \xleftarrow{R'.} & \mathcal{F}^{p'} & \xleftarrow{(\pi')^\star} & \hom_D(M', \mathcal{F}) & \longleftarrow & 0. \\
 & & & & \uparrow & & \\
 & & & & \ker f^\star & & \\
 & & & & \uparrow & & \\
 & & & & 0 & &
\end{array}$$

Up to an isomorphism, we have seen at the end of the previous subsection that we can identify $\hom_D(M, \mathcal{F})$ (resp., $\hom_D(M', \mathcal{F})$) with $\ker_{\mathcal{F}}(R.)$ (resp., $\ker_{\mathcal{F}}(R'.)$). A chase in the previous exact diagram easily proves that, for all $\zeta \in \ker_{\mathcal{F}}(R'.)$, we have $f^\star(\zeta) = P\,\zeta \in \ker_{\mathcal{F}}(R.)$. ■

Remark 1: From Corollary 2, we see that the computation of morphisms from a finitely presented left D-module M to a finitely presented left D-module M' gives some kind of "Galois symmetries" which send solutions of the second system to solutions of the first one. This fact is particularly clear when we have $M = M'$: we then send a solution of the system to another one.

As an example, we now apply Corollary 1 to a particular case and recover in a unified way the so-called *eigenring* introduced in the literature (see [59], [3], [9], [15], [16], [22], [65]).

Example 4: Let $D = A[\partial; \sigma, \delta]$ be a skew polynomial ring over a commutative ring A and $E, F \in A^{p \times p}$. We consider the matrix $R = (\partial\, I_p - E) \in D^{p \times p}$ (resp., $R' = (\partial\, I_p - F) \in D^{p \times p}$) of functional operators and the finitely presented left D-module $M = D^{1 \times p}/(D^{1 \times p}\, R)$ (resp., $M' = D^{1 \times p}/(D^{1 \times p}\, R')$). Let π (resp., π') be the canonical projection of $D^{1 \times p}$ onto M (resp., M') and $\{e_i\}_{1 \leq i \leq p}$ the standard basis of $D^{1 \times p}$. As we have seen in Subsection II-A, $\{y_i = \pi(e_i)\}_{1 \leq i \leq p}$ and $\{z_i = \pi'(e_i)\}_{1 \leq i \leq p}$ satisfy:

$$\begin{array}{ll} \partial\, y_i = \sum_{j=1}^p E_{ij}\, y_j, & i = 1, \ldots, p, \\ \partial\, z_i = \sum_{j=1}^p F_{ij}\, z_j, & i = 1, \ldots, p. \end{array} \tag{14}$$

Let f be a morphism from M to M'. Then, there exist $P_{ij} \in D$ $(i, j = 1, \ldots, p)$ such that $f(y_i) = \sum_{j=1}^p P_{ij}\, z_j$. Using (14), we easily check that we can always suppose that all the P_{ij} belong to A, i.e., $P \in A^{p \times p}$. By Corollary 1, there exists $Q \in D^{p \times p}$ satisfying (11).

Clearly, f is the zero morphism if and only if there exists a matrix $Z \in D^{p \times p}$ satisfying $P = Z\, R'$. As the order of P is 0 and that of R' is 1, we obtain that $Z = 0$, i.e., $P = 0$ and $Q = 0$.

Now, let us suppose that P and Q are different from zero. As both the orders of $R\,P$ and R' in ∂ are 1, we deduce that the order of Q must be 0, i.e., $Q \in A^{p \times p}$. Then, we get:

$$\begin{array}{rl} (11) \;\Leftrightarrow & (\partial\, I_p - E)\, P = Q\, (\partial\, I_p - F) \\ \Leftrightarrow & \sigma(P)\, \partial + \delta(P) - E\, P = Q\, \partial - Q\, F \\ \Leftrightarrow & (\sigma(P) - Q)\, \partial + (\delta(P) - E\, P + Q\, F) = 0. \end{array} \tag{15}$$

The first order polynomial matrix in the left-hand side of Equation (15) must be equal to 0 so that:

$$(15) \Leftrightarrow \left\{ \begin{array}{l} Q = \sigma(P), \\ \delta(P) = E\, P - \sigma(P)\, F. \end{array} \right. \tag{16}$$

We then obtain the following commutative exact diagram:

$$\begin{array}{ccccccc}
0 \longrightarrow & D^{1 \times p} & \xrightarrow{.R} & D^{1 \times p} & \xrightarrow{\pi} & M & \longrightarrow 0 \\
 & \downarrow .\sigma(P) & & \downarrow .P & & \downarrow f & \\
0 \longrightarrow & D^{1 \times p} & \xrightarrow{.R'} & D^{1 \times p} & \xrightarrow{\pi'} & M' & \longrightarrow 0.
\end{array} \tag{17}$$

Conversely, if there exist $P \in A^{p \times p}$ and $Q \in A^{p \times p}$ which satisfy (16), we then check that we have (11), i.e., the commutative exact diagram (17) where the morphism $f : M \longrightarrow M'$ is defined by

$$\forall\, m \in M, \quad f(m) = \pi'(\lambda\, P),$$

where $\lambda \in D^{1 \times p}$ is any element such that $m = \pi(\lambda)$.

The previous results prove that we have:

$$\begin{array}{c} \hom_D(M, M') = \{f : M \longrightarrow M' \mid f(y_i) = \sum_{j=1}^p P_{ij}\, z_j, \\ i = 1, \ldots, p, \; P \in A^{p \times p}, \; \delta(P) = E\, P - \sigma(P)\, F\}, \\ \mathrm{end}_D(M) = \{f : M \longrightarrow M \mid f(y_i) = \sum_{j=1}^p P_{ij}\, y_j, \\ i = 1, \ldots, p, \; P \in A^{p \times p}, \; \delta(P) = E\, P - P\, E\}. \end{array}$$

For instance, if we consider the ring $A = k[t]$ or $k(t)$ and $D = A\left[\partial; \mathrm{id}_A, \frac{d}{dt}\right]$, then (16) becomes

$$\left\{ \begin{array}{l} Q(t) = P(t), \\ \dot{P}(t) = E(t)\, P(t) - P(t)\, F(t), \end{array} \right. \tag{18}$$

whereas, if we consider the ring $A = k[n]$ or $A = k(n)$ and $D = A[\partial; \sigma, 0]$ with $\sigma(a)(n) = a(n+1)$, then (16) gives:

$$\begin{cases} Q_n = \sigma(P_n) = P_{n+1}, \\ E_n\, P_n - \sigma(P_n)\, F_n = E_n\, P_n - P_{n+1}\, F_n = 0. \end{cases} \tag{19}$$

We find again in a unified way known results concerning the *eigenring* of a linear system (see [59], [3], [9], [15], [16], [22], [65]).

Finally, if $\mathcal{F}$ is a left D-module, then applying the functor $\hom_D(\cdot, \mathcal{F})$ to the commutative exact diagram (17), we obtain the following commutative exact diagram:

$$\begin{array}{ccccccc} \mathcal{F}^p & \xleftarrow{R.} & \mathcal{F}^p & \longleftarrow & \hom_D(M, \mathcal{F}) & \longleftarrow & 0 \\ \uparrow \sigma(P). & & \uparrow P. & & \uparrow f^\star & & \\ \mathcal{F}^p & \xleftarrow{R'.} & \mathcal{F}^p & \longleftarrow & \hom_D(M', \mathcal{F}) & \longleftarrow & 0. \end{array}$$

If $\eta \in \hom_D(M', \mathcal{F})$, i.e., $\eta \in \mathcal{F}^p$ is a solution of the system $\partial\, \eta = F\, \eta$, then the previous commutative exact diagram shows that $\zeta = P\, \eta$ is a solution of $\partial\, \zeta = E\, \zeta$, i.e., $\zeta = f^\star(\eta) \in \hom_D(M, \mathcal{F})$. Indeed, we have:

$$\begin{aligned} \partial\, \zeta - E\, \zeta &= \partial\, (P\, \eta) - E\, (P\, \eta) \\ &= \sigma(P)\, \partial\, \eta + \delta(P)\, \eta - (E\, P)\, \eta \\ &= \sigma(P)\, (\partial\, \eta - F\, \eta) = 0. \end{aligned}$$

For instance, if $D = A\left[\partial; \mathrm{id}_A, \frac{d}{dt}\right]$, using (18), we obtain:

$$\begin{aligned} \partial\, \zeta(t) - E(t)\, \zeta(t) &= \partial\, (P(t)\, \eta(t)) - (E(t)\, P(t))\, \eta(t) \\ &= P(t)\, \partial\, \eta(t) - \dot{P}(t)\, \eta(t) - (E\, P)\, \eta(t) \\ &= P(t)\, (\partial\, \eta(t) - F\, \eta(t)) = 0. \end{aligned}$$

If we now consider $D = A[\partial; \sigma, 0]$, using (19), we have:

$$\begin{aligned} \zeta_{n+1} - E_n\, \zeta_n &= P_{n+1}\, \eta_{n+1} - E_n\, P_n\, \eta_n \\ &= P_{n+1}\, (\eta_{n+1} - F_n\, \eta_n) = 0. \end{aligned}$$

2) Algorithms: Before giving two algorithms for the computation of morphisms between two finitely presented left modules, we first recall the notion of the *Kronecker product* of two matrices.

Definition 6: Let $E \in D^{q \times p}$ and $F \in D^{r \times s}$ be two matrices with entries in a ring D. The *Kronecker product* of E and F, denoted by $E \otimes F$, is the matrix defined by:

$$E \otimes F = \begin{pmatrix} E_{11}\, F & \ldots & E_{1p}\, F \\ \vdots & \vdots & \vdots \\ E_{q1}\, F & \ldots & E_{qp}\, F \end{pmatrix} \in D^{(q\, r) \times (p\, s)}.$$

The next result is very classical.

Lemma 1: Let D be a commutative ring, $E \in D^{r \times q}$, $F \in D^{q \times p}$ and $G \in D^{p \times m}$ three matrices. If we denote by $\mathrm{row}(F) = (F_{1\bullet}, \ldots, F_{q\bullet}) \in D^{1 \times q\, p}$ the row vector obtained by stacking the rows of F one after the other, then the product of the three matrices can be obtained by:

$$E\, F\, G = \mathrm{row}(F)\, (E^T \otimes G).$$

We point out that Lemma 1 is only valid for commutative rings. Let us consider a commutative ring D and the matrices $R \in D^{q \times p}$, $R' \in D^{q' \times p'}$, $P \in D^{p \times p'}$ and $Q \in D^{q \times q'}$. Then, from the previous lemma, we have

$$\begin{cases} R\, P = R\, P\, I_{p'} = \mathrm{row}(P)\, (R^T \otimes I_{p'}), \\ Q\, R' = I_q\, Q\, R' = \mathrm{row}(Q)\, (I_q \otimes R'), \end{cases}$$

which implies that (11) is equivalent to:

$$(\mathrm{row}(P) \quad \mathrm{row}(Q)) \begin{pmatrix} R^T \otimes I_{p'} \\ -I_q \otimes R' \end{pmatrix} = 0.$$

This leads to an algorithm for computing matrices $P \in D^{p \times p'}$ and $Q \in D^{q \times q'}$ satisfying (11) in the case where the Ore algebra D is commutative.

Algorithm 1:

- **Input**: A commutative Ore algebra D and two matrices $R \in D^{q \times p}$ and $R' \in D^{q' \times p'}$.
- **Output**: A finite family of generators $\{f_i\}_{i \in I}$ of the D-module $\hom_D(M, M')$, where

  $$M = D^{1 \times p}/(D^{1 \times q}\, R), \quad M' = D^{1 \times p'}/(D^{1 \times q'}\, R'),$$

  and each f_i is defined by means of two matrices $\overline{P_i}$ and $\overline{Q_i}$ satisfying the commutation relation (11), i.e.:

  $$\forall\, \lambda \in D^{1 \times p}: \quad f_i(\pi(\lambda)) = \pi'(\lambda\, \overline{P_i}), \quad i \in I.$$

1) Form the following matrix with entries in D:

$$K = \begin{pmatrix} R^T \otimes I_{p'} \\ -I_q \otimes R' \end{pmatrix} \in D^{(p\, p' + q\, q') \times q\, p'}.$$

2) Compute $\ker_D(.K)$, i.e., the first syzygy left D-module of $D^{1 \times (p\, p' + q\, q')}\, K$, by means of a computation of a Gröbner basis for an elimination order (see [14]). We obtain a matrix $L \in D^{s \times (p\, p' + q\, q')}$ satisfying:

$$\ker_D(.K) = D^{1 \times s}\, L.$$

3) For $i = 1, \ldots, s$, construct the following matrices

$$\begin{cases} P_i(j, k) = r_i(L)(1, (j-1)\, p' + k), \\ Q_i(l, m) = r_i(L)(1, p\, p' + (l-1)\, q' + m), \end{cases}$$

where $r_i(L)$ denotes the i^{th} row of L, $E(i, j)$ the $i \times j$ entry of the matrix E, $j = 1, \ldots, p$, $k = 1, \ldots, p'$, $l = 1, \ldots, q$ and $m = 1, \ldots, q'$. We then have:

$$R\, P_i = Q_i\, R', \quad i = 1, \ldots, s.$$

4) Compute a Gröbner basis G of the rows of R' for a total order.

5) For $i = 1, \ldots, s$, reduce the rows of P_i with respect to G by computing their normal forms with respect to G. We obtain the matrices $\overline{P}_i$ which satisfy

$$\overline{P}_i = P_i + Z_i\, R',$$

where $Z_i \in D^{p \times q'}$ are certain matrices which can be easily obtained by means of a factorization (see

[13] for details).

6) For $i = 1, \ldots, s$, define the following matrices
$$\overline{Q}_i = Q_i + R\, Z_i.$$
The pair $(\overline{P_i}, \overline{Q_i})$ then satisfies the relation:
$$R\,\overline{P}_i = \overline{Q_i}\, R'.$$

Remark 2: If we denote by $R_2' \in D^{r' \times q'}$ a matrix satisfying $\ker_D(.R) = D^{1\times r'}\, R_2$, we then note that any matrix of the form
$$\overline{Q}_i = Q_i + R\, Z_i + Z_i'\, R_2',$$
where $Z_i' \in D^{q' \times r'}$ is an arbitrary matrix, also satisfies the relation $R\,\overline{P}_i = \overline{Q_i}\, R'$.

Remark 3: As D is a commutative ring, we know that $\hom_D(M, M')$ has a D-module structure. In particular, if $\{f_i\}_{i\in I}$ is a family of generators of $\hom_D(M, M')$ defined the pairs of matrices $(\overline{P}_i, \overline{Q}_i)$, then any element $f \in \hom_D(M, M')$ has the form $f = \sum_{i\in I} \alpha_i\, f_i$, where $\alpha_i \in D$ for $i \in I$, and f can then be defined by the pair:
$$\left(\sum_{i\in I} \alpha_i\, \overline{P}_i,\ \sum_{i\in I} \alpha_i\, \overline{Q}_i\right).$$

Example 5: Let us consider again Example 2. Applying Algorithm 1 to the matrix R defined by (7), we obtain that the D-endomorphisms of M are generated by the matrices
$$P_\alpha = \left(\begin{array}{ccc} \alpha_1 & \alpha_2 & 2\,\alpha_3\,\partial_1\,\partial_2 \\ \alpha_2 + 2\,\alpha_4\,\partial_1 + 2\,\alpha_5\,\partial_1\,\partial_2 & \alpha_1 - 2\,\alpha_4\,\partial_1 - 2\,\alpha_5\,\partial_1\,\partial_2 & 2\,\alpha_3\,\partial_1\,\partial_2 \\ \alpha_4\,\partial_2 + \alpha_5 & -\alpha_4\,\partial_2 - \alpha_5 & \alpha_1 + \alpha_2 + \alpha_3\,(\partial_2^2 + 1) \end{array}\right),$$
and
$$Q_\alpha = \left(\begin{array}{cc} \alpha_1 - 2\,\alpha_4\,\partial_1 & \alpha_2 + 2\,\alpha_4\,\partial_1 \\ \alpha_2 + 2\,\alpha_5\,\partial_1\,\partial_2 & \alpha_1 - 2\,\alpha_5\,\partial_1\,\partial_2 \end{array}\right),$$
where $\alpha_1, \ldots, \alpha_5$ are arbitrary elements of D, i.e.,
$$\forall\, \lambda \in D^{1\times 3}, \quad f_\alpha(\pi(\lambda)) = \pi(\lambda\, P_\alpha).$$

As it is noticed in 1) of Definition 3, if D is a non-commutative ring, then $\hom_D(M, M')$ is an abelian group or an infinite-dimensional k-vector space. Hence, the only possibility to access to $\hom_D(M, M')$ is to use a certain filtration, i.e., to only consider morphisms of $\hom_D(M, M')$ which can be defined by means of a matrix P with a fixed total order in the functional operators ∂_i, and a fixed degree in x_i for the numerators and denominators of the polynomial/rational coefficients. We obtain the following algorithm:

Algorithm 2:
- **Input**: An Ore algebra D, two matrices $R \in D^{q\times p}$, $R' \in D^{q'\times p'}$ and three non-negative integers α, β, γ.
- **Output**: A family of pairs $(\overline{P}_i, \overline{Q}_i)_{i\in I}$ satisfying:
$$\left\{\begin{array}{l} R\,\overline{P}_i = \overline{Q}_i\, R', \\ \mathrm{ord}_\partial(\overline{P}_i) \leq \alpha,\ i.e.,\ \overline{P}_i = \sum_{0\leq|\nu|\leq\alpha} a_\nu^{(i)}\,\partial^\nu, \\ \text{and}\ \forall\, 0 \leq |\nu| \leq \alpha,\ a_\nu^{(i)} \in A\ \text{satisfies}: \\ \deg_x(\mathrm{num}(a_\nu^{(i)})) \leq \beta, \\ \deg_x(\mathrm{denom}(a_\nu^{(i)})) \leq \gamma, \end{array}\right.$$
where $\mathrm{ord}_\partial(\overline{P}_i)$ denotes the maximal of the total orders of the entries of $\overline{P}_i$, $\deg_x(\mathrm{num}(a_\nu^{(i)}))$ (resp., $\deg_x(\mathrm{denom}(a_\nu^{(i)}))$) the degree of the numerator (resp., denominator) of $a_\nu^{(i)}$. For all $i \in I$, the morphisms f_i are then defined by:
$$\forall\, \lambda \in D^{1\times p}: \quad f_i(\pi(\lambda)) = \pi'(\lambda\,\overline{P}_i).$$

1) Take an ansatz for P satisfying the input
$$P(i,j) = \sum_{0\leq|\nu|\leq\alpha} a_\nu^{(i,j)}\partial^\nu, \quad 1 \leq i \leq p,\ 1 \leq j \leq p',$$
where $a_\nu^{(i,j)}$ is a rational function whose numerator (resp., denominator) has a total degree β (resp., γ).

2) Compute $R\,P$ and denote the result by F.

3) Compute a Gröbner basis G of the rows of R'.

4) Reduce the rows of F with respect to G by computing their normal forms with respect to G.

5) Solve the system for the coefficients of $a_\nu^{(i,j)}$ so that all the normal forms vanish.

6) Substitute the solutions into the matrix P. Denote the set of solutions by $\{P_i\}_{i\in I}$.

7) For $i \in I$, reduce the rows of P_i with respect to G by computing their normal forms with respect to G. We obtain $\overline{P}_i$ for $i \in I$.

8) Using $r_j(R\,\overline{P}_i) \in (D^{1\times q'}\, R')$, $j = 1, \ldots, q$, where $r_j(R\,\overline{P}_i)$ denotes the j^{th} row of $R\,\overline{P}_i$, compute a matrix $\overline{Q}_i \in D^{q\times q'}$ satisfying $R\,\overline{P}_i = \overline{Q}_i\, R$, $i \in I$.

Let us illustrate Algorithm 2 by means of an example.

Example 6: We consider the so-called *Euler-Tricomi equation* $\partial_1^2\, u(x_1, x_2) - x_1\,\partial_2^2\, u(x_1, x_2) = 0$ which appears in the study of transonic flow. Let $D = A_2(\mathbb{Q})$ denote the Weyl algebra, $R = (\partial_1^2 - x_1\,\partial_2^2) \in D$ and $M = D/(D\,R)$ be the associated left D-module. Using Algorithm 2, we can compute the endomorphisms of M defined by $P \in D$ with given total order in ∂_i and total degree in x_i. We denote by $\mathrm{end}_D(M)_{\alpha,\beta}$ the $\mathbb{Q}$-vector space of all the elements of $\mathrm{end}_D(M)$ defined by a differential operator $P_{\alpha,\beta}$ which total order (resp., degree) in ∂_i (resp., x_i) is less or equal to α (resp., β), where α and β are two non-negative integers. Below is a list of some of these vector spaces obtained by means of Algorithm 2:

- $\mathrm{end}_D(M)_{0,0}$ is defined by $P = Q = a$, $a \in \mathbb{Q}$.
- $\mathrm{end}_D(M)_{1,1}$ is defined by

$$\begin{cases} P = a_1 + a_2\,\partial_2 + \frac{3}{2}\,a_3\,x_2\,\partial_2 + a_3\,x_1\,\partial_1, \\ Q = (a_1 + 2\,a_3) + a_2\,\partial_2 + \frac{3}{2}\,a_3\,x_2\,\partial_2 + a_3\,x_1\,\partial_1, \end{cases}$$

where a_1, a_2 and $a_3 \in \mathbb{Q}$.

- $\mathrm{end}_D(M)_{2,0}$ is defined by:

$$P = Q = a_1 + a_2\,\partial_2 + a_3\,\partial_2^2, \quad a_1, a_2, a_3 \in \mathbb{Q}.$$

- $\mathrm{end}_D(M)_{2,1}$ is defined by $(a_1, \ldots, a_5 \in \mathbb{Q})$:

$$\begin{cases} P = a_1 + a_2\,\partial_2 + \frac{3}{2}\,a_3\,x_2\,\partial_2 + a_3\,x_1\,\partial_1 \\ \qquad + a_4\,\partial_2^2 + \frac{3}{2}\,a_5\,x_2\,\partial_2^2 + a_5\,x_1\,\partial_1\,\partial_2, \\ Q = (a_1 + 2\,a_3) + a_2\,\partial_2 + \frac{3}{2}\,a_3\,x_2\,\partial_2 + a_3\,x_1\,\partial_1 \\ \qquad + a_4\,\partial_2^2 + a_5\,x_1\,\partial_1\,\partial_2 + 2\,a_5\,\partial_2 + \frac{3}{2}\,a_5\,x_2\,\partial_2^2. \end{cases}$$

Remark 4: If D is a non-commutative ring, we then note that $\hom_D(M, M')$ is generally an infinite-dimensional k-vector space and an abelian group. In particular, $\hom_D(M, M')$ has no non-trivial module structure, a fact implying that there does not exist a finite family of generators of $\hom_D(M, M')$ as a left or right D-module.

However, if M and M' are two finite-dimensional k-vector spaces (e.g., the linear systems defined in Example 4, connections, D-finite modules [12]), we can then compute a basis of the finite-dimensional k-vector space $\hom_D(M, M')$. In order to do that, we need to know some bounds on the orders and degrees of the entries of solutions of (11) so that we can know whether or not Algorithm 2 finds a k-basis of the morphisms. In some cases, such bounds are known. Let us recall some known results.

In Example 4, we saw that if $D = A[\partial; \sigma, \delta]$ was a skew polynomial ring over a commutative ring A, $E, F \in A^{p\times p}$ and $R = (\partial\, I_p - E)$, $R' = (\partial\, I_p - F)$, the morphisms from $M = D^{1\times p}/(D^{1\times p}\,R)$ to $M' = D^{1\times p}/(D^{1\times p}\,R)$ are defined by means of matrices $P \in A^{p\times p}$ satisfying:

$$\delta(P) = E\,P - \sigma(P)\,F. \tag{20}$$

Hence, we need to solve (20). There are two main cases:

1) If $A = k[t]$ or $k(t)$ and $D = A\left[\partial; \mathrm{id}_A, \frac{d}{dt}\right]$, then (20) becomes $\dot{P}(t) = E(t)\,P(t) - P(t)\,F(t)$. A direct method to solve the previous linear system of ODEs is developed in [7]. Another method, based on the fact that the entries of the matrices E, F and P belong to a commutative ring A, uses the equivalent of the previous system with the following first order linear system of ODEs

$$\delta(\mathrm{row}(P)) = \mathrm{row}(P)\,((E^T \otimes I_p) - (I_p \otimes F)), \tag{21}$$

where $\otimes$ denotes the Kronecker product (see Definition 6). Hence, computing $\hom_D(M, M')$ is equivalent to computing the A-solutions of an auxiliary linear differential system (21) (see for example [7], [15], [16], [22], [59]). Consequently, we can use the bounds appearing in [2], [4] on the degrees of numerators (and denominators) of polynomial (rational) solutions to deduce bounds on the entries of P. We note that in that case, the matrices P and Q have necessarily 0 order. We may precise that these bounds depend only on the valuations and degrees of the entries of the two matrices E and F.

2) If we consider the ring $A = k[n]$ or $A = k(n)$ and $D = A[\partial; \sigma, 0]$ with $\sigma(a)(n) = a(n+1)$, then (20) becomes $P_{n+1}\,F_n = E_n\,P_n$. A direct method to solve the previous linear difference system is developed in [5]. Another one, based again on the fact that the entries of the matrices E_n, F_n and P_n belong to a commutative ring A, uses the equivalent of the previous system with the following first order linear discrete system:

$$\mathrm{row}(P_{n+1})\,(E_n^T \otimes I_p) = \mathrm{row}(P_n)\,(I_p \otimes F_n). \tag{22}$$

Moreover, if $E \in \mathrm{GL}_p(A)$, i.e., the matrix E is invertible, then (22) becomes

$$\mathrm{row}(P_{n+1}) = \mathrm{row}(P_n)\,((I_p \otimes F)\,(E_n^T \otimes I_p)^{-1}).$$

As in the differential case, some bounds exist on the degrees of numerators (and denominators) of polynomial (rational) solutions of the previous system (see [1], [5]), and thus, for the matrices P and Q.

Finding bounds in more general situations is a subject for future researches.

3) Applications: quadratic first integrals of motion and conservation laws: We give two applications which illustrate the interest of the computation of morphisms in the search of quadratic first integrals of motion of linear systems of ODEs and quadratic conservation laws of linear systems of PDEss.

We consider the Ore algebra $D = A\left[\partial; \mathrm{id}_A, \frac{d}{dt}\right]$ of ordinary differential operators with coefficients in the k-algebra A (e.g., $A = k[t]$, $k(t)$), where k is a field, and the matrix $R = (\partial\, I_p - E) \in D^{p\times p}$. Using (18), we easily check that any solution $P \in A^{p\times p}$ of the following *Liapunov equation*

$$\dot{P}(t) + E^T(t)\,P(t) + P(t)\,E(t) = 0$$

defines a morphism from the left D-module $M' = D^{1\times p}/(D^{1\times p}\,\widetilde{R})$ to the left D-module $M = D^{1\times p}/(D^{1\times p}\,R)$, where $\widetilde{R} = (\partial\, I_p + E^T) \in D^{p\times p}$ denotes, up to a sign, the *formal adjoint* of R.

We recall that the formal adjoint $\widetilde{R}$ of a matrix R of differential operators is obtained by contracting the column vector $R\,\eta$ by a row vector λ and integrating the result by parts (see [44], [45], [46]). Hence, there exists a bilinear application Φ which satisfies:

$$< \lambda, R\,\eta > = < \widetilde{R}\,\lambda, \eta > + \partial\,(\Phi(\lambda, \eta)). \quad (23)$$

In particular, in our case, we have:

$$\lambda^T\,(\partial\eta - E\,\eta) = -\eta^T\,(\partial\lambda + E^T\,\lambda) + \partial\,(\eta^T\,\lambda). \quad (24)$$

If $\mathcal{F}$ is a left D-module and $\eta \in \mathcal{F}^p$ satisfies the system $\partial\eta - E\,\eta = 0$, then, following the results obtained in Example 4, $\lambda = P\,\eta$ is a solution of $\partial\,\lambda + E^T\,\lambda = 0$. Hence, using (24), we then get

$$\partial\,(\eta^T\,\lambda) = \partial(\eta^T\,P\,\eta) = 0,$$

which proves that the quadratic form $V = \eta^T\,P\,\eta$ is a first integral of the motion of the system $\partial\eta - E\,\eta = 0$. Hence, we obtain that there exists a one-to-one correspondence between the quadratic first integrals of the motion of the form $V = \eta^T\,P\,\eta$, where $P \in A^{q\times p}$, of the system $\partial\,\eta - E\,\eta = 0$ and the morphisms between the left D-modules $\widetilde{N}$ and M, i.e., the elements of $\hom_D(\widetilde{N}, M)$.

Let us illustrate the previous result.

Example 7: Let us consider the example of a linear system of ODEs defined in p. 117 of [30]. In order to do that, let us introduce the following matrix of coefficients

$$E = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -\omega^2 & 0 & \alpha & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\omega^2 & \alpha \end{pmatrix} \in \mathbb{Q}(\omega, \alpha)^{4\times 4},$$

the ring $D = \mathbb{Q}(\omega, \alpha)\left[\partial; \mathrm{id}, \frac{d}{dt}\right]$ of differential operators, the matrix $R = (\partial\,I_4 - E) \in D^{4\times 4}$ of differential operators and the D-module $M = D^{1\times 4}/(D^{1\times 4}\,R)$. Then, we have $\widetilde{R} = (\partial\,I_4 + E^T)$ and $\widetilde{N} = D^{1\times 4}/(D^{1\times 4}\,\widetilde{R})$. Using Algorithm 1, we obtain that an element of the D-module $\hom_D(\widetilde{N}, M)$ can be defined by means of the matrix

$$P = \begin{pmatrix} c_1\,\omega^4 & c_2\,\omega^2 & -\omega^2\,(c_1\,\alpha + c_2) & c_1\,\omega^2 \\ -c_2\,\omega^2 & c_1\,\omega^2 & -c_1\,\omega^2 + c_2\,\alpha & -c_2 \\ -\omega^2\,(c_1\,\alpha - c_2) & -c_1\,\omega^2 - c_2\,\alpha & c_1\,(\alpha^2 + \omega^2) & -c_1\,\alpha + c_2 \\ c_1\,\omega^2 & c_2 & -c_1\,\alpha - c_2 & c_1 \end{pmatrix},$$

where c_1 and c_2 are two constants, which leads to the quadratic first integral $V(x) = x^T\,P\,x$, i.e.:

$$\begin{aligned} V(x) &= c_1\,\omega^4\,x_1(t)^2 - 2\,c_1\,\alpha\,\omega^2\,x_1(t)\,x_3(t) \\ &\quad + 2\,c_1\,\omega^2\,x_1(t)\,x_4(t) + c_1\,\omega^2\,x_2(t)^2 \\ &\quad - 2\,c_1\,\omega^2\,x_2(t)\,x_3(t) + c_1\,\alpha^2\,x_3(t)^2 \\ &\quad + c_1\,\omega^2\,x_3(t)^2 - 2\,c_1\,\alpha\,x_3(t)\,x_4(t) \\ &\quad + c_1\,x_4(t)^2. \end{aligned}$$

More generally, let us consider a matrix $R \in D^{q\times p}$ of differential operators, $\widetilde{R} \in D^{p\times q}$ its formal adjoint and the finitely presented left D-modules $M = D^{1\times p}/(D^{1\times q}\,R)$ and $\widetilde{N} = D^{1\times q}/(D^{1\times p}\,\widetilde{R})$. Let us suppose that there exists a morphism f from $\widetilde{N}$ to M defined by $P \in D^{q\times p}$ and $Q \in D^{p\times q}$, i.e., we have the commutative exact diagram:

$$\begin{array}{ccccccc} D^{1\times p} & \xrightarrow{.\widetilde{R}} & D^{1\times q} & \xrightarrow{\pi'} & \widetilde{N} & \longrightarrow 0 \\ \downarrow .Q & & \downarrow .P & & \downarrow f & \\ D^{1\times q} & \xrightarrow{.R} & D^{1\times p} & \xrightarrow{\pi} & M & \longrightarrow 0. \end{array}$$

Applying the right exact functor $\hom_D(\cdot, \mathcal{F})$ to the previous commutative exact diagram, we then obtain the following commutative exact diagram

$$\begin{array}{ccccccc} \mathcal{F}^p & \xleftarrow{\widetilde{R}.} & \mathcal{F}^q & \longleftarrow & \ker_{\mathcal{F}}(\widetilde{R}.) & \longleftarrow 0 \\ \uparrow Q. & & \uparrow P. & & \uparrow f^\star & \\ \mathcal{F}^q & \xleftarrow{R.} & \mathcal{F}^p & \longleftarrow & \ker_{\mathcal{F}}(R.) & \longleftarrow 0, \end{array}$$

where $f^\star(\eta) = P\,\eta$. Hence, if $\eta \in \mathcal{F}^p$ is a solution of $R\,\eta = 0$, then $\lambda = P\,\eta$ is a solution of $\widetilde{R}\,\lambda = 0$ as:

$$\widetilde{R}\,(P\,\eta) = Q\,(R\,\eta) = 0.$$

Therefore, using (23), we obtain that $V = \Phi(P\,\eta, \eta)$ is a quadratic first integral of the motion of system $R\,\eta = 0$, i.e., V satisfies $\partial\,V = 0$.

An extension of the previous ideas exists for the computation of quadratic conservation laws of linear system of PDEs, namely, a vector $\Phi = (\Phi_1, \ldots, \Phi_n)^T$ of quadratic functions of the system variables and their derivatives which satisfies $\mathrm{div}\,\Phi = \sum_{i=1}^n \partial_i\,\Phi_i = 0$, where n denotes the number of independent variables. Let us give a simple example as the general theory follows exactly the same lines.

Example 8: Consider the PDE $\Delta\,y(x_1, x_2) = 0$, where $\Delta = \partial_1^2 + \partial_2^2 \in D = \mathbb{Q}\left[\partial_1; \mathrm{id}, \frac{\partial}{\partial x_1}\right]\left[\partial_2; \mathrm{id}, \frac{\partial}{\partial x_2}\right]$ is the Laplacian operator. Multiplying $\Delta\,y(x_1, x_2)$ by a function $\lambda(x_1, x_2)$ and integrating the result by parts, we obtain:

$$\begin{aligned} &\lambda\,(\Delta\,y) - (\Delta\,\lambda)\,y \\ &= \partial_1\,(\lambda\,(\partial_1\,y) - (\partial_1\,\lambda)\,y) + \partial_2\,(\lambda\,(\partial_2\,y) - (\partial_2\,\lambda)\,y). \end{aligned} \quad (25)$$

Using the fact that $R = \Delta$ is a differential operator with constant coefficients and $\widetilde{R} = R$, we then obtain $\hom_D(\widetilde{N}, M) = \mathrm{end}_D(M) = D$. Hence, if $\mathcal{F}$ is a D-module (e.g., $C^\infty(\Omega)$), then, for all $\alpha \in D$ and $y \in \mathcal{F}$ satisfying $\Delta\,y = 0$, $\lambda = \alpha\,y$ is then a solution of $\Delta\,\lambda = 0$. Substituting $\lambda = \alpha\,y$ in (25), we finally obtain

$$\mathrm{div}\,\Phi = \partial_1\,\Phi_1 + \partial_2\,\Phi_2 = 0,$$

with the notation:

$$\Phi = \begin{pmatrix} (\alpha\,y)\,(\partial_1\,y) - y\,(\partial_1\,\alpha\,y) \\ (\alpha\,y)\,(\partial_2\,y) - y\,(\partial_2\,\alpha\,y) \end{pmatrix}.$$

III. Reducible modules and factorizations

A. Modules associated with a morphism and equivalences

Let $f : M \longrightarrow M'$ be a morphism between two left D-modules. Then, we can define the following left D-modules:

$$\begin{cases} \ker f = \{m \in M \,|\, f(m) = 0\}, \\ \operatorname{im} f = \{m' \in M' \,|\, \exists\, m \in M : \; m' = f(m)\}, \\ \operatorname{coim} f = M/\ker f, \\ \operatorname{coker} f = M'/\operatorname{im} f. \end{cases}$$

Let us explicitly characterize the above-mentioned kernel, image, coimage and cokernel of a morphism $f : M \longrightarrow M'$ between two finitely presented left D-modules M and M'.

Proposition 6: Let $R \in D^{q \times p}$, $R' \in D^{q' \times p'}$,

$$M = D^{1\times p}/(D^{1\times q}\, R), \quad M' = D^{1\times p'}/(D^{1\times q'}\, R').$$

Let $f : M \longrightarrow M'$ be a morphism defined by two matrices $P \in D^{p\times p'}$ and $Q \in D^{q \times q'}$ satisfying (11). Then, we have:

1) $\ker f = (D^{1\times r}\, S)/(D^{1\times q}\, R)$, where $S \in D^{r\times p}$ is the matrix defined by:

$$\ker_D \left(. \begin{pmatrix} P \\ R' \end{pmatrix} \right) = D^{1\times r}\, (S \quad -T), \quad T \in D^{r\times q'}. \tag{26}$$

2) $\operatorname{coim} f = D^{1\times p}/(D^{1\times r}\, S)$,

3) $\operatorname{im} f = \left(D^{1\times (p+q')} \begin{pmatrix} P \\ R' \end{pmatrix} \right) /(D^{1\times q'}\, R')$,

4) $\operatorname{coker} f = D^{1\times p'} / \left(D^{1\times (p+q')} \begin{pmatrix} P \\ R' \end{pmatrix} \right)$.

Proof: 1. Let $m \in \ker f$ and write $m = \pi(\lambda)$ for a certain $\lambda \in D^{1\times p}$. Then, $f(m) = \pi'(\lambda\, P) = 0$ implies that $\lambda\, P \in (D^{1\times q'}\, R')$, i.e., there exists $\mu \in D^{1\times q'}$ satisfying $\lambda\, P = \mu\, R'$. Hence, $m = \pi(\lambda) \in \ker f$ implies that there exists $\mu \in D^{1\times q'}$ such that $\lambda\, P = \mu\, R'$. Conversely, we easily check that any element of

$$(\lambda \quad -\mu) \in \ker_D \left(. \begin{pmatrix} P \\ R' \end{pmatrix} \right)$$

gives $m = \pi(\lambda) \in \ker f$, which proves the result.

2. Using the canonical short exact sequence

$$0 \longrightarrow \ker f \xrightarrow{i} M \xrightarrow{\rho} \operatorname{coim} f \longrightarrow 0,$$

where i (resp., ρ) denotes the canonical injection (resp., surjection), and the fact that $M = D^{1\times p}/(D^{1\times q}\, R)$ and $\ker f = (D^{1\times r}\, S)/(D^{1\times q}\, R)$, we obtain the following exact sequence

$$0 \longrightarrow (D^{1\times r}\, S)/(D^{1\times q}\, R) \xrightarrow{i} D^{1\times p}/(D^{1\times q}\, R) \xrightarrow{\rho} \operatorname{coim} f \longrightarrow 0,$$

which proves that $\operatorname{coim} f = D^{1\times p}/(D^{1\times r}\, S)$ (see [56]).

3. For all $\lambda \in D^{1\times p}$, we have $f(\pi(\lambda)) = \pi'(\lambda\, P)$, which clearly proves that we have:

$$\operatorname{im} f = \left(D^{1\times (p+q')} \begin{pmatrix} P \\ R' \end{pmatrix} \right) /(D^{1\times q'}\, R').$$

4. Using the canonical short exact sequence

$$0 \longrightarrow \operatorname{im} f \xrightarrow{j} M' \xrightarrow{\sigma} \operatorname{coker} f \longrightarrow 0,$$

where j (resp., σ) denotes the canonical injection (resp., surjection), and the fact that $M' = D^{1\times p'}/(D^{1\times q'}\, R')$ and $\operatorname{im} f = (D^{1\times p}\, P + D^{1\times q'}\, R')/(D^{1\times q'}\, R')$, we then obtain the following exact sequence

$$0 \longrightarrow \left(D^{1\times (p+q')} \begin{pmatrix} P \\ R' \end{pmatrix} \right) /(D^{1\times q'}\, R') \xrightarrow{j} D^{1\times p'}/(D^{1\times q'}\, R') \xrightarrow{\sigma} \operatorname{coker} f \longrightarrow 0,$$

which proves that:

$$\operatorname{coker} f = D^{1\times p'} / \left(D^{1\times (p+q')} \begin{pmatrix} P \\ R' \end{pmatrix} \right).$$

■

Let us state the first main result of the paper.

Theorem 1: With the notations of Proposition 6, any non-injective morphism $f : M \longrightarrow M'$ leads to a non-trivial factorization of $R \in D^{q\times p}$ of the form $R = L\, S$, where $L \in D^{q\times r}$ and $S \in D^{r \times p}$.

Proof: Using (26) and the fact that $R\, P = Q\, R'$, i.e.,

$$(R \quad -Q) \begin{pmatrix} P \\ R' \end{pmatrix} = 0,$$

we obtain that $(D^{1\times q}\, (R \quad -Q)) \subseteq (D^{1\times r}\, (S \quad -T))$, and thus, there exists a matrix $L \in D^{q\times r}$ satisfying:

$$\begin{cases} R = L\, S, \\ Q = L\, T. \end{cases} \tag{27}$$

We then obtain the following commutative exact diagram

$$\begin{array}{ccccccc} & & & & 0 & & \\ & & & & \downarrow & & \\ & & 0 & & \ker f & & \\ & & \downarrow & & \downarrow & & \\ D^{1\times q} & \xrightarrow{.R} & D^{1\times p} & \xrightarrow{\pi} & M & \longrightarrow 0 & \\ \downarrow .L & & \| & & \downarrow \rho & & \\ D^{1\times r} & \xrightarrow{.S} & D^{1\times p} & \xrightarrow{\kappa} & \operatorname{coim} f & \longrightarrow 0, & \\ & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & \end{array} \tag{28}$$

where $\rho : M \longrightarrow \operatorname{coim} f$ denotes the canonical projection.

■

Let us illustrate Theorem 1 by means of an example.

Example 9: We consider the linearized Euler equations for an incompressible fluid (p. 519 of [36])

$$\left\{ \begin{array}{l} \operatorname{div} \vec{v}(x,t) = 0, \\ \dfrac{\partial \vec{v}(x,t)}{\partial t} + \operatorname{grad} p(x,t) = 0, \end{array} \right. \tag{29}$$

where $\vec{v} = (v_1, v_2, v_3)^T$ (resp., p) denotes the perturbations of the speed (resp., pressure) around a steady-state position and $x = (x_1, x_2, x_3)$. If we denote by D the Ore algebra

$$\mathbb{Q}\left[\partial_1; \mathrm{id}, \frac{\partial}{\partial x_1}\right]\left[\partial_2; \mathrm{id}, \frac{\partial}{\partial x_2}\right]\left[\partial_3; \mathrm{id}, \frac{\partial}{\partial x_3}\right]\left[\partial_t; \mathrm{id}, \frac{\partial}{\partial t}\right]$$

of differential operators with rational constant coefficients, the system matrix corresponding to (29) can be defined by:

$$R = \begin{pmatrix} \partial_1 & \partial_2 & \partial_3 & 0 \\ \partial_t & 0 & 0 & \partial_1 \\ 0 & \partial_t & 0 & \partial_2 \\ 0 & 0 & \partial_t & \partial_3 \end{pmatrix} \in D^{4\times 4}.$$

Let $M = D^{1\times 4}/(D^{1\times 4} R)$ be the left D-module associated with the system (29). An endomorphism f of M is defined by the matrices:

$$P = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \partial_3^2 & -\partial_2\,\partial_3 & 0 \\ 0 & -\partial_2\,\partial_3 & \partial_2^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$Q = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \partial_3^2 & -\partial_2\,\partial_3 \\ 0 & 0 & -\partial_2\,\partial_3 & \partial_2^2 \end{pmatrix}.$$

We then obtain the following factorization $R = L\,S$ where:

$$S = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \partial_2 & \partial_3 & 0 \\ 0 & -\partial_t & 0 & 0 \\ 0 & 0 & \partial_t & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$L = \begin{pmatrix} \partial_1 & 1 & 0 & 0 & 0 \\ \partial_t & 0 & 0 & 0 & \partial_1 \\ 0 & 0 & -1 & 0 & \partial_2 \\ 0 & 0 & 0 & 1 & \partial_3 \end{pmatrix}.$$

We can check that $\ker f = (D^{1\times 5} S)/(D^{1\times 4} R) \neq 0$, which shows that $R = L\,S$ is a non-trivial factorization of R. The solutions of the system $S\,\eta = 0$ are in particular solutions of $R\,\eta = 0$. If we consider $\mathcal{F} = C^\infty(\Omega)$, where Ω is an open convex subset of $\mathbb{R}^4$, we easily check that all $\mathcal{F}$-solutions of $S\,\eta = 0$ are given by

$$\eta = \left(0, \quad -\frac{\partial \xi(x)}{\partial x_3}, \quad \frac{\partial \xi(x)}{\partial x_2}, \quad 0\right)^T, \tag{30}$$

where ξ is any function of $C^\infty(\Omega \cap \mathbb{R}^3)$. In other words, (30) gives a family of stationary solutions of (29).

Let us state a useful lemma.

Lemma 2: Let $R \in D^{q\times p}$, $R' \in D^{q'\times p}$, $R'' \in D^{q\times q'}$ be three matrices satisfying the relation $R = R''\,R'$ and let $T' \in D^{r'\times q'}$ be such that $\ker_D(.R') = D^{1\times r'}\,T'$. Let us also consider the following canonical projections:

$$\pi_1 : (D^{1\times q'}\,R') \longrightarrow M_1 = (D^{1\times q'}\,R')/(D^{1\times q}\,R),$$

$$\pi_2 : D^{1\times q'} \longrightarrow M_2 = D^{1\times q'}/(D^{1\times q}\,R'' + D^{1\times r'}\,T').$$

Then, the morphism ψ defined by

$$\begin{array}{rcl} \psi : M_2 & \longrightarrow & M_1 \\ m_2 = \pi_2(\lambda) & \longmapsto & \psi(m_2) = \pi_1(\lambda\,R'), \end{array}$$

is an isomorphism and its inverse ϕ is defined by:

$$\begin{array}{rcl} \phi : M_1 & \longrightarrow & M_2 \\ m_1 = \pi_1(\lambda\,R') & \longmapsto & \phi(m_1) = \pi_2(\lambda). \end{array}$$

In other words, we have the following isomorphism:

$$(D^{1\times q'}\,R')/(D^{1\times q}\,R) \cong D^{1\times q'}/(D^{1\times q}\,R'' + D^{1\times r'}\,T'). \tag{31}$$

Proof: Let us first prove that ψ is a well-defined morphism. We assume that we have $m_2 = \pi_2(\lambda) = \pi_2(\lambda')$, where $\lambda,\ \lambda' \in D^{1\times q'}$. Then, we have $\pi_2(\lambda - \lambda') = 0$, i.e., $\lambda - \lambda' \in (D^{1\times q}\,R'' + D^{1\times r'}\,T')$ so that there exist $\mu \in D^{1\times q}$ and $\nu \in D^{1\times r'}$ such that $\lambda - \lambda' = \mu\,R'' + \nu\,T'$. We then have:

$$(\lambda - \lambda')\,R' = (\mu\,R'' + \nu\,T')\,R' = \mu\,R$$

$$\Rightarrow \pi_1((\lambda - \lambda')\,R') = \pi_1(\mu\,R) = 0$$

$$\Rightarrow \pi_1(\lambda'\,R') = \pi_1(\lambda\,R') = \psi(m_2).$$

Now, let us prove that the morphism ϕ is also well-defined. Let us suppose that:

$$m_1 = \pi_1(\lambda\,R') = \pi_1(\lambda'\,R'), \quad \lambda,\ \lambda' \in D^{1\times q'}.$$

We have $\pi_1(\lambda\,R') - \pi_1(\lambda'\,R') = \pi_1((\lambda - \lambda')\,R') = 0$, and thus, $(\lambda - \lambda')\,R' \in (D^{1\times q}\,R)$, i.e., there exists $\mu \in D^{1\times q}$ such that $(\lambda - \lambda')\,R' = \mu\,R$. Now, using the factorization $R = R''\,R'$, we then get $(\lambda - \lambda' - \mu\,R'')\,R' = 0$ so that we have $\lambda - \lambda' - \mu\,R'' \in \ker_D(.R') = (D^{1\times r'}\,T')$. Therefore, there exists $\nu \in D^{1\times r'}$ such that $\lambda - \lambda' = \mu\,R'' + \nu\,T'$ and then:

$$\pi_2(\lambda) - \pi_2(\lambda') = \pi_2(\lambda - \lambda') = \pi_2(\mu\,R'' + \nu\,T') = 0.$$

Finally, for all

$$m_1 = \pi_1(\lambda\,R') \in M_1, \quad m_2 = \pi_2(\lambda) \in M_2,$$

where $\lambda \in D^{1\times q'}$, we have

$$\left\{ \begin{array}{l} (\psi \circ \phi)(m_1) = \psi(\pi_2(\lambda)) = \pi_1(\lambda\,R') = m_1, \\ (\phi \circ \psi)(m_2) = \phi(\pi_1(\lambda\,R')) = \pi_2(\lambda) = m_2, \end{array} \right.$$

which proves that $\psi \circ \phi = \mathrm{id}_{M_1}$, $\phi \circ \psi = \mathrm{id}_{M_2}$ and we thus have (31).

■

We deduce the following corollary of Lemma 2 and Proposition 6.

Corollary 3: With the notations of Proposition 6:

1) If $L \in D^{q\times r}$ denotes a matrix satisfying $R = L\,S$ and $\ker_D(.S) = D^{1\times r_2}\,S_2$, where $S_2 \in D^{r_2\times r}$, we then have:

$$\ker f \cong D^{1\times r}/\left(D^{1\times(q+r_2)}\begin{pmatrix} L \\ S_2 \end{pmatrix}\right).$$

2) We have $\operatorname{im} f \cong \operatorname{coim} f$.

Proof: 1. It is a straightforward application of the isomorphism (31) to this particular case.

2. Using the following two facts

$$\begin{cases} R' = (0 \quad I_{q'})\begin{pmatrix} P \\ R' \end{pmatrix}, \\ \ker_D\left(.\begin{pmatrix} P \\ R' \end{pmatrix}\right) = (D^{1\times r}\,(S \quad -T)), \end{cases}$$

where $S \in D^{r\times p}$ and $T \in D^{r\times q}$, applying Lemma 2 to 3) of Proposition 6, we get:

$$\begin{array}{rcl} \operatorname{im} f & \cong & D^{1\times(p+q)}/\left(D^{1\times(q'+r)}\begin{pmatrix} 0 & I_{q'} \\ S & -T \end{pmatrix}\right) \\ & \cong & D^{1\times p}/(D^{1\times r}\,S) = \operatorname{coim} f. \end{array}$$

■

We give a corollary of Proposition 6 and Corollary 3.

Corollary 4: With the notations of Corollary 3 and Proposition 6, a morphism $f : M \longrightarrow M'$ is:

1) the *zero morphism* ($f = 0$) if and only if one of the following conditions holds:

   a) There exists a matrix $Z \in D^{p\times q'}$ such that:
   $$P = Z\,R'.$$
   In this case, there exists a matrix $Z' \in D^{q\times q'_2}$ such that
   $$Q = R\,Z + Z'\,R'_2,$$
   where $\ker_D(.R') = (D^{1\times q'_2}\,R'_2)$.

   b) The matrix S admits a left-inverse.

2) *injective* if and only if one of the following conditions holds:

   a) There exists a matrix $F \in D^{r\times q}$ such that:
   $$S = F\,R.$$

   b) The matrix $(L^T \quad S_2^T)^T$ admits a left-inverse.

3) *surjective* if and only if $(P^T \quad R'^T)^T$ admits a left-inverse.

4) an *isomorphism* ($f \in \operatorname{iso}(M)$) if the matrices $(L^T \quad S_2^T)^T$ and $(P^T \quad R'^T)^T$ admit left-inverses.

Proof: 1. Using 3) of Proposition 6, $\operatorname{im} f = 0$ if and only if we have

$$(D^{1\times p}\,P) + (D^{1\times q'}\,R') = (D^{1\times q'}\,R'),$$

that is, if and only if $(D^{1\times p}\,P) \subseteq (D^{1\times q'}\,R')$ which is equivalent to the existence of a matrix $Z \in D^{p\times q'}$ such that $P = Z\,R'$. Now, substituting $P = Z\,R'$ into (11), we then get:

$$R\,Z\,R' = Q\,R' \Rightarrow (Q - R\,Z)\,R' = 0.$$

Thus, there exists $Z' \in D^{q\times q'_2}$ satisfying

$$Q - R\,Z = Z'\,R'_2,$$

which proves the result. We note also that 1.a) is a trivial consequence of Corollary 1.

Let us prove 1.b). Using the standard isomorphism

$$\epsilon : \operatorname{coim} f \longrightarrow \operatorname{im} f, \quad \forall\, m \in M : \quad \epsilon(\sigma(m)) = f(m),$$

where $\sigma : M \longrightarrow \operatorname{coim} f$ denotes the canonical projection, we obtain that $\operatorname{im} f = 0$ if and only if

$$\operatorname{coim} f = D^{1\times p}/(D^{1\times r}\,S) = 0 \Leftrightarrow (D^{1\times r}\,S) = D^{1\times p},$$

i.e., if and only if S admits a left-inverse.

2. From 1) of Proposition 6, $\ker f = 0$ if and only if $(D^{1\times r}\,S) = (D^{1\times q}\,R)$, i.e., if and only if there exists $F \in D^{r\times q}$ satisfying $S = F\,R$.

Moreover, using Corollary 3, we have $\ker f = 0$ if and only if $(D^{1\times q}\,L) + (D^{1\times r_2}\,S_2) = D^{1\times r}$, i.e., if and only if the matrix $(L^T \quad S_2^T)^T$ admits a left-inverse.

3. f is surjective if and only if $\operatorname{coker} f = 0$, i.e., from 4) of Proposition 6, if and only if $(D^{1\times p}\,P) + (D^{1\times q'}\,R') = D^{1\times p}$ which is equivalent to the fact that the matrix $(P^T \quad R'^T)^T$ admits a left-inverse.

4. The result is a direct consequence of 2.b) and 3).

■

Let us see how to apply the previous results in order to check the equivalence between two modules, and thus, between two systems.

Example 10: We consider two systems of PDEs appearing in the theory of linear elasticity (see [43]): one half of the so-called *Killing operator*, namely, the *Lie derivative* of the euclidean metric defined by $\omega_{ij} = 1$ for $i = j$ and 0 otherwise ($1 \le i, j \le 2$) and *the Spencer operator* of the

Killing operator:

$$\left\{\begin{array}{l} d_1\,\xi_1 = 0,\\ \frac{1}{2}\,(d_2\,\xi_1 + d_1\,\xi_2) = 0,\\ d_2\,\xi_2 = 0, \end{array}\right. \qquad \left\{\begin{array}{l} d_1\,z_1 = 0,\\ d_2\,z_1 - z_2 = 0,\\ d_1\,z_2 = 0,\\ d_1\,z_3 + z_2 = 0,\\ d_2\,z_3 = 0,\\ d_2\,z_2 = 0. \end{array}\right.$$

Let $D = \mathbb{Q}\left[\partial_1; \mathrm{id}, \frac{\partial}{\partial x_1}\right]\left[\partial_2; \mathrm{id}, \frac{\partial}{\partial x_2}\right]$ be the ring of differential operators and let us define the following matrices

$$R = \left(\begin{array}{cc} \partial_1 & 0\\ \frac{1}{2}\,\partial_2 & \frac{1}{2}\,\partial_1\\ 0 & \partial_2 \end{array}\right), \quad R' = \left(\begin{array}{ccc} \partial_1 & 0 & 0\\ \partial_2 & -1 & 0\\ 0 & \partial_1 & 0\\ 0 & 1 & \partial_1\\ 0 & 0 & \partial_2\\ 0 & \partial_2 & 0 \end{array}\right),$$

and the associated finitely presented D-modules:

$$M = D^{1\times 2}/(D^{1\times 3}\,R), \quad M' = D^{1\times 3}/(D^{1\times 6}\,R').$$

Using Algorithm 1, we find that the matrices

$$P = \left(\begin{array}{ccc} 1 & 0 & 0\\ 0 & 0 & 1 \end{array}\right), \; Q = \frac{1}{2}\left(\begin{array}{cccccc} 2 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 2 & 0 \end{array}\right),$$

satisfy the relation $R\,P = Q\,R'$, i.e., they define a morphism $f : M \longrightarrow M'$ by:

$$\left\{\begin{array}{l} f(\xi_1) = z_1,\\ f(\xi_2) = z_3. \end{array}\right.$$

The morphism f is injective as the matrix S (with the same notations as in Corollary 4) defined by

$$S = \left(\begin{array}{cccc} \partial_2 & \partial_1 & \partial_2^2 & 0\\ \partial_1 & 0 & 0 & \partial_2 \end{array}\right)^T$$

satisfies the relation $S = F\,R$, where:

$$F = \left(\begin{array}{ccc} 0 & 2 & 0\\ 1 & 0 & 0\\ 0 & 2\,\partial_2 & -\partial_1\\ 0 & 0 & 1 \end{array}\right).$$

Moreover, f is surjective as the matrix $(P^T \quad R'^T)^T$ admits the following left-inverse:

$$\left(\begin{array}{cccccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & -\partial_1 & 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

This proves that f is an isomorphism and $M \cong M'$.

To finish this section, we show an important application of Lemma 2. In order to simplify the exposition, we only consider a commutative Ore algebra of partial differential operators but the extension to non-commutative one can be easily obtained by using the concept of formal adjoint instead of the simple transposition ([14], [43], [44], [45]).

Let M be a D-module defined by a finite free resolution of the form (9). If we consider (10) with $\mathcal{F} = D$, we then obtain the D-modules:

$$\begin{array}{rcl} \mathrm{ext}^i_D(M, D) & \cong & \ker_D(R_{i+1}.)/(R_i\,D^{p_{i-1}}), \quad i \geq 1,\\ & \cong & \ker_D(.R^T_{i+1})/(D^{1\times p_{i-1}}\,R_i^T), \quad i \geq 1. \end{array}$$

Computing the first syzygy module of $\ker_D(.R^T_{i+1})$, we obtain a matrix $Q_i^T \in D^{p'_{i-1}\times p_i}$ such that:

$$\ker_D(.R^T_{i+1}) = (D^{1\times p'_{i-1}}\,Q_i^T).$$

Therefore, we obtain:

$$\mathrm{ext}^i_D(M, D) \cong (D^{1\times p'_{i-1}}\,Q_i^T)/(D^{1\times p_{i-1}}\,R_i^T).$$

Using Lemma 2, we obtain

$$\mathrm{ext}^i_D(M, D) \cong D^{1\times p'_{i-1}}/((D^{1\times p_{i-1}}\,F_i^T) + (D^{1\times p'_{i-2}}\,P_i^T)), \tag{32}$$

where $F_i^T \in D^{p_{i-1}\times p'_{i-1}}$ and $P_i^T \in D^{p'_{i-2}\times p'_{i-1}}$ satisfy:

$$\left\{\begin{array}{l} R_i^T = F_i^T\,Q_i^T,\\ \ker_D(.Q_i^T) = (D^{1\times p'_{i-2}}\,P_i^T). \end{array}\right.$$

The isomorphism (32) is useful for computation of

$$\mathrm{ext}^j_D(\mathrm{ext}^i_D(M, D)), \quad 1 \leq i,\, j \leq n,$$

which play a crucial role in the study of *r-pure differential modules* as it is explained in [43], [51].

Example 11: Let us consider the linear system of PDEs:

$$\left\{\begin{array}{l} \dfrac{\partial^2 y}{\partial x_2^2} = 0,\\ \dfrac{\partial^2 y}{\partial x_1\,\partial x_2} = 0. \end{array}\right. \tag{33}$$

We easily check that we have:

$$\left\{\begin{array}{l} z_1 = \dfrac{\partial y}{\partial x_2},\\ \dfrac{\partial z_1}{\partial x_1} = 0,\\ \dfrac{\partial z_1}{\partial x_2} = 0, \end{array}\right. \qquad \left\{\begin{array}{l} z_2 = \dfrac{\partial y}{\partial x_1},\\ \dfrac{\partial z_2}{\partial x_2} = 0, \end{array}\right.$$

We obtain that z_1 is an arbitrary constant, i.e., the Krull dimension of z_1 is 0, whereas z_2 is an arbitrary function of x_1, i.e., the Krull dimension of z_2 is 1. An important issue in system theory is to be able to classify the *observables* of a system of PDEs, namely, the differential linear combinations of the system variables ([14], [43]), in terms of their Krull dimensions. As it was explained in [43], we need to be able to compute $\mathrm{ext}^j_D(\mathrm{ext}^i_D(M, D))$, $1 \leq i,\, j \leq n$, in order to achieve this classification. Let us illustrate these computations of the system (33).

Let $D = \mathbb{Q}\left[\partial_1; \mathrm{id}, \frac{\partial}{\partial x_1}\right]\left[\partial_2; \mathrm{id}, \frac{\partial}{\partial x_2}\right]$ be the ring of differential operators with constant coefficients, the matrix

$R = (d_2^2 \quad d_1\, d_2)^T$ and the D-module $M = D/(D^{1\times 2}\, R)$. Let us compute $\mathrm{ext}_D^j(\mathrm{ext}_D^i(M, D))$, $1 \leq i,\, j \leq 2$.

We have the following finite free resolution of M

$$0 \longrightarrow D \xrightarrow{.R_2} D^{1\times 2} \xrightarrow{.R} D \xrightarrow{\pi} M \longrightarrow 0,$$

where $R_2 = (d_1 \quad -d_2)$. The defects of exactness of the complex $0 \longleftarrow D \xleftarrow{.R_2^T} D^{1\times 2} \xleftarrow{.R^T} D \longleftarrow 0$ are:

$$\begin{cases} \mathrm{ext}_D^0(M, D) \cong \ker_D(.R^T) = 0, \\ \mathrm{ext}_D^1(M, D) \cong \ker_D(.R_2^T)/(D\, R^T), \\ \mathrm{ext}_D^2(M, D) \cong D/(D^{1\times 2}\, R_2^T). \end{cases}$$

Using the finite free resolution of $\mathrm{ext}_D^2(M, D)$

$$0 \longrightarrow D \xrightarrow{.L} D^{1\times 2} \xrightarrow{.R_2^T} D \longrightarrow \mathrm{ext}_D^2(M, D) \longrightarrow 0,$$

where $L = (d_2 \quad d_1)$, the defects of exactness of the complex $0 \longleftarrow D \xleftarrow{.L^T} D^{1\times 2} \xleftarrow{.R_2} D \longleftarrow 0$ are:

$$\begin{cases} \mathrm{ext}_D^0(\mathrm{ext}_D^2(M, D), D) \cong \ker_D(.R_2) = 0, \\ \mathrm{ext}_D^1(\mathrm{ext}_D^2(M, D), D) \cong \ker_D(.L^T)/(D\, R_2), \\ \mathrm{ext}_D^2(\mathrm{ext}_D^2(M, D), D) \cong D/(D^{1\times 2}\, L^T). \end{cases}$$

We easily check that $\ker_D(.L^T) = (D\, R_2)$, which proves:

$$\mathrm{ext}_D^1(\mathrm{ext}_D^2(M, D), D) = 0.$$

We check that we have $\ker_D(.R_2^T) = (D\, L)$, which shows that $\mathrm{ext}_D^1(M, D) = (D\, L)/(D\, R^T)$. Using Lemma 2, we then have

$$\mathrm{ext}_D^1(M, D) \cong (D\, L)/(D\, R^T) \cong D/(D\, d_2),$$

as $R^T = d_2\, L$ and $\ker_D(.L) = 0$. Using the following finite free resolution of $\mathrm{ext}_D^1(M, D) \cong D/(D\, d_2)$

$$0 \longrightarrow D \xrightarrow{.d_2} D \longrightarrow \mathrm{ext}_D^1(M, D) \longrightarrow 0,$$

the defects of exactness of the complex

$$0 \longleftarrow D \xleftarrow{.d_2} D \longleftarrow 0,$$

are then defined:

$$\begin{cases} \mathrm{ext}_D^0(\mathrm{ext}_D^1(M, D), D) \cong \ker_D(.d_2) = 0, \\ \mathrm{ext}_D^1(\mathrm{ext}_D^1(M, D), D) \cong D/(D\, d_2). \end{cases}$$

If we denote by $t_r(M) = \{m \in M \mid \dim(D\, m) \leq 1 - r\}$, $r = 0,\, 1$, the D-submodule M formed by the elements of M of Krull dimension less or equal to $1 - r$, $t_0(M) = M$ and $t_2(M) = 0$, we then have ([43]):

$$0 \longrightarrow t_r(M) \longrightarrow t_{r-1}(M) \longrightarrow \mathrm{ext}_D^r(\mathrm{ext}_D^r(M, D), D) \longrightarrow 0.$$

Hence, we obtain that

$$\begin{cases} t_1(M) = \mathrm{ext}_D^2(\mathrm{ext}_D^2(M, D), D) \cong D/(D^{1\times 2}\, L^T), \\ M/t_1(M) \cong D/(D\, d_2). \end{cases}$$

Finally, using the fact that $\mathrm{ext}_D^i(\mathrm{ext}_D^i(M, D), D)$ is a pure D-module of Krull dimension $n - i$ ([43]), from the first equality, we find again that the Krull dimension of the residue class z_1 of 1 in $t_1(M)$ is 0, which was easy to find directly on the simple example (33) but could be much more difficult on more general linear systems.

B. Reducible modules and block-triangular matrices

The next proposition will play an important role in what follows.

Proposition 7: Let us consider a matrix $P \in D^{p\times p}$. The following assertions are equivalent:

1) The left D-modules $\ker_D(.P)$ and $\mathrm{coim}_D(.P)$ are free of rank respectively m and $p - m$.

2) There exists a unimodular matrix $U \in D^{p\times p}$, i.e., $U \in \mathrm{GL}_p(D)$, and a matrix $J \in D^{p\times p}$ of the form

$$J = \begin{pmatrix} 0 & 0 \\ J_1 & J_2 \end{pmatrix},$$

where $J_1 \in D^{(p-m)\times m}$ and $J_2 \in D^{(p-m)\times(p-m)}$, $(J_1 \quad J_2)$ has full row rank, satisfying the relation:

$$U\, P = J\, U. \tag{34}$$

The matrix U has then the form

$$U = \begin{pmatrix} U_1 \\ U_2 \end{pmatrix}, \tag{35}$$

where the full row rank matrix $U_1 \in D^{m\times p}$ is defined by $\ker_D(.P) = (D^{1\times m}\, U_1)$ and $U_2 \in D^{(p-m)\times p}$ is any matrix such that U is invertible over D.

In particular, we have the relations:

$$\begin{cases} U_1\, P = 0, \\ U_2\, P = J_1\, U_1 + J_2\, U_2. \end{cases}$$

Proof: $(1 \Rightarrow 2)$. Let us suppose that $\ker_D(.P)$ and $\mathrm{coim}_D(.P)$ are two free left D-modules of rank respectively m and $p - m$.

Let $U_1 \in D^{m\times p}$ be a basis of $\ker_D(.P)$, i.e., the full row rank matrix U_1 satisfies $\ker_D(.P) = (D^{1\times m}\, U_1)$. Using the fact that we have the exact sequence

$$0 \longrightarrow \ker_D(.P) \longrightarrow D^{1\times p} \xrightarrow{\kappa} \mathrm{coim}_D\,(.P) \longrightarrow 0$$

and $\ker_D(.P) = (D^{1\times m}\, U_1)$, we then obtain the following exact sequence:

$$0 \longrightarrow D^{1\times m} \xrightarrow{.U_1} D^{1\times p} \xrightarrow{\kappa} \mathrm{coim}_D\,(.P) \longrightarrow 0.$$

If we denote by $L = D^{1\times p}/(D^{1\times m}\, U_1)$, then we get:

$$\mathrm{coim}\,(.P) = D^{1\times p}/\ker_D(.P) = L.$$

Using the fact that L is a free left D-module of rank $p-m$ and if we denote by $\phi : L \longrightarrow D^{1\times(p-m)}$ the previous isomorphism, by $\kappa : D^{1\times p} \longrightarrow L$ the canonical projection and by $W_2 \in D^{p\times(p-m)}$ the matrix corresponding to the

D-morphism $\phi \circ \kappa$ in the canonical bases of $D^{1\times p}$ and $D^{1\times(p-m)}$, we then obtain the exact exact sequence:

$$0 \longrightarrow D^{1\times m} \xrightarrow{.U_1} D^{1\times p} \xrightarrow{.W_2} D^{1\times(p-m)} \longrightarrow 0.$$

Using the fact that $D^{1\times(p-m)}$ is a free left D-module, by 1) of Proposition 4, the previous short exact sequence splits, and thus, there exist two matrices $W_1 \in D^{p\times m}$ and $U_2 \in D^{(p-m)\times p}$ such that we have the Bézout identities:

$$\left\{ \begin{array}{l} \left(\begin{array}{c} U_1 \\ U_2 \end{array} \right) (W_1 \quad W_2) = I_p, \\ (W_1 \quad W_2) \left(\begin{array}{c} U_1 \\ U_2 \end{array} \right) = I_p. \end{array} \right.$$

Using the fact that $U^{-1} = (W_1 \quad W_2) \in D^{p\times p}$, we have

$$\begin{array}{rl} U\,P & = \left(\begin{array}{c} U_1 \\ U_2 \end{array} \right) P = \left(\begin{array}{c} U_1\,P \\ U_2\,P \end{array} \right) \\ & = \left(\begin{array}{c} 0 \\ (U_2\,P\,U^{-1})\,U \end{array} \right) = \left(\begin{array}{c} 0 \\ U_2\,P\,U^{-1} \end{array} \right) U, \end{array}$$

which proves the result with the notation:

$$J = \left(\begin{array}{c} 0 \\ U_2\,P\,U^{-1} \end{array} \right) \in D^{p\times p}.$$

Finally, if $\lambda \in \ker_D(.(U_2\,P\,U^{-1}))$, we then have

$$\begin{array}{c} \lambda\,(U_2\,P\,U^{-1}) = 0 \Leftrightarrow (\lambda\,U_2)\,P = 0 \\ \Leftrightarrow \lambda\,U_2 \in \ker_D(.P) = (D^{1\times m}\,U_1) \\ \Leftrightarrow \exists\,\mu \in D^{1\times m} :\ \lambda\,U_2 = \mu\,U_1 \\ \Leftrightarrow \exists\,\mu \in D^{1\times m} :\ (\mu, \quad \lambda) \in \ker_D(.U) = 0, \end{array} \quad (36)$$

which proves that $\lambda = 0$, i.e., $\ker_D(.(U_2\,P\,U^{-1})) = 0$, and the matrix $(J_1 \quad J_2)$ has full row rank.

$(2 \Rightarrow 1)$. Using the relation (34) and the fact that U is a unimodular matrix, we have the commutative exact diagram:

$$\begin{array}{ccccc} & & 0 & & 0 \\ & & \uparrow & & \uparrow \\ 0 \longrightarrow \ker_D(.P) & \longrightarrow & D^{1\times p} & \xrightarrow{.P} & D^{1\times p} \\ & & \uparrow .U & & \uparrow .U \\ 0 \longrightarrow \ker_D(.J) & \longrightarrow & D^{1\times p} & \xrightarrow{.J} & D^{1\times p}, \\ & & \uparrow & & \uparrow \\ & & 0 & & 0 \end{array}$$

which shows that $\ker_D(.P) \cong \ker_D(.J)$ (more precisely, $\ker_D(.P) = (\ker_D(.J))\,U$). Let us characterize $\ker_D(.J)$. Let us consider $(\lambda_1, \quad \lambda_2) \in \ker_D(.J)$. We then have $\lambda_2\,(J_1 \quad J_2) = 0$ and using the fact that $(J_1 \quad J_2)$ has full row rank, we obtain that $\lambda_2 = 0$ and λ_1 is any arbitrary element of $D^{1\times m}$, which proves that $\ker_D(.J) = D^{1\times m}$ and $\ker_D(.P)$ is a free left D-module of rank m.

Similarly, we have $\mathrm{im}_D(.P) = (\mathrm{im}_D(.J))\,U$ as U is a unimodular matrix and:

$$\forall\ \lambda,\, \mu \in D^{1\times p}, \quad \left\{ \begin{array}{l} \lambda\,P = ((\lambda\,U^{-1})\,J)\,U, \\ (\mu\,J)\,U = (\mu\,U)\,P. \end{array} \right.$$

Therefore, we have:

$$\mathrm{im}_D(.P) \cong \mathrm{im}_D(.J) = (D^{1\times(p-m)}\,(J_1 \quad J_2)).$$

Using the fact that the matrix $(J_1 \quad J_2)$ has full row rank, we obtain that $(D^{1\times(p-m)}\,(J_1 \quad J_2)) \cong D^{1\times(p-m)}$, which proves that $\mathrm{coim}_D(.P) \cong \mathrm{im}_D(.P)$ (see 2 of Corollary 3) is a free left D-module of rank $p-m$.

■

Remark 5: We note that (34) is equivalent to

$$P = U^{-1}\,J\,U,$$

which means that the two matrices P and J are similar.

We shall need the next two lemmas.

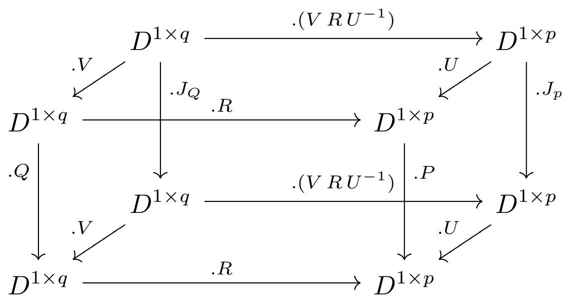
Lemma 3: Let $R \in D^{q\times p}$, $P \in D^{p\times p}$ and $Q \in D^{q\times q}$ be three matrices satisfying (11). Assume further that there exist $U \in \mathrm{GL}_p(D)$ and $V \in \mathrm{GL}_q(D)$ such that

$$\left\{ \begin{array}{l} U\,P = J_P\,U, \\ V\,Q = J_Q\,V, \end{array} \right. \quad (37)$$

for certain matrices $J_P \in D^{p\times p}$ and $J_Q \in D^{q\times q}$. Then, we have the following equality:

$$(V\,R\,U^{-1})\,J_P = J_Q\,(V\,R\,U^{-1}). \quad (38)$$

Proof: We easily check that we have the following commutative diagram



from which we obtain (38). Let us give the corresponding explicit computations. Starting with the second equation of (37) and multiplying it on the right by R and using (11), we obtain:

$$J_Q\,V\,R = V\,Q\,R = V\,R\,P = (V\,R\,U^{-1})\,(U\,P).$$

Now, using the first equation of (37), we get

$$J_Q\,V\,R = (V\,R\,U^{-1})\,(J_P\,U),$$

and multiplying the previous equality by U^{-1} on the right, we finally have $J_Q\,(V\,R\,U^{-1}) = (V\,R\,U^{-1})\,J_P$, which proves (38).

■

Lemma 4: Let us consider two matrices of the form

$$\left\{ \begin{array}{l} J_P = \begin{pmatrix} 0 & 0 \\ J_1 & J_2 \end{pmatrix}, \\ \\ J_Q = \begin{pmatrix} 0 & 0 \\ J_3 & J_4 \end{pmatrix}, \end{array} \right. \quad (39)$$

with the notations

$$J_1 \in D^{(p-m)\times m}, \quad J_2 \in D^{(p-m)\times(p-m)},$$
$$J_3 \in D^{(q-l)\times l}, \quad J_4 \in D^{(q-l)\times(q-l)},$$

and $1 \leq m \leq p$, $1 \leq l \leq q$. Moreover, let us suppose that the matrix $(J_1 \quad J_2)$ has full row rank. If the matrix $\overline{R} \in D^{q\times p}$ satisfies the relation

$$\overline{R}\, J_P = J_Q\, \overline{R},$$

then there exist three matrices

$$\overline{R}_1 \in D^{l\times m}, \quad \overline{R_2} \in D^{l\times(p-m)}, \quad \overline{R_3} \in D^{(q-l)\times(p-m)},$$

such that:

$$\overline{R} = \begin{pmatrix} \overline{R}_1 & 0 \\ \overline{R}_2 & \overline{R}_3 \end{pmatrix}. \quad (40)$$

Proof: Let us write

$$\overline{R} = \begin{pmatrix} \overline{R}_{11} & \overline{R}_{12} \\ \overline{R}_{21} & \overline{R}_{22} \end{pmatrix},$$

where $\overline{R}_{11} \in D^{l\times m}$, $\overline{R}_{12} \in D^{l\times(p-m)}$, $\overline{R}_{21} \in D^{(q-l)\times m}$, $\overline{R}_{22} \in D^{(q-l)\times(p-m)}$, then, we have:

$$\overline{R}\, J_P = \begin{pmatrix} \overline{R}_{12}\, J_1 & \overline{R}_{12}\, J_2 \\ \overline{R}_{22}\, J_1 & \overline{R}_{22}\, J_2 \end{pmatrix},$$

$$J_Q\, \overline{R} = \begin{pmatrix} 0 & 0 \\ J_3\, \overline{R}_{11} + J_4\, \overline{R}_{21} & J_3\, \overline{R}_{12} + J_4\, \overline{R}_{21} \end{pmatrix}.$$

Therefore, we obtain $\overline{R}_{12}\,(J_1 \quad J_2) = 0$. Using the fact that $(J_1 \quad J_2)$ has full row rank, we then get $\overline{R}_{12} = 0$, which proves the result.

■

Let us state the second main result of the paper (the first fairy's theorem).

Theorem 2: Let $R \in D^{q\times p}$ and $M = D^{1\times p}/(D^{1\times q}\, R)$. Let $f : M \longrightarrow M$ be an endomorphism defined by two matrices $P \in D^{p\times p}$ and $Q \in D^{q\times q}$ satisfying (11). If the left D-modules $\ker_D(.P)$, $\mathrm{coim}_D(.P)$, $\ker_D(.Q)$, $\mathrm{coim}_D(.Q)$ are free of rank respectively m, $p-m$, l and $q-l$ (for some $1 \leq m \leq p$ and $1 \leq l \leq q$), then the following results hold:

1) There exist $U \in \mathrm{GL}_p(D)$ and $V \in \mathrm{GL}_q(D)$ satisfying the relations

$$\left\{ \begin{array}{l} P = U^{-1}\, J_P\, U, \\ Q = V^{-1}\, J_Q\, V, \end{array} \right.$$

where J_P and J_Q are the matrices defined by (39). In particular, the matrices U and V are defined by

$$\left\{ \begin{array}{ll} U = \begin{pmatrix} U_1 \\ U_2 \end{pmatrix}, & U_1 \in D^{m\times p}, \quad U_2 \in D^{(p-m)\times p}, \\ V = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix}, & V_1 \in D^{l\times q}, \quad V_2 \in D^{(q-l)\times q}, \end{array} \right.$$

where the matrices U_1 and V_1 respectively define the bases of the free left D-modules $\ker_D(.P)$ and $\ker_D(.Q)$, i.e.,

$$\left\{ \begin{array}{l} \ker_D(.P) = D^{1\times m}\, U_1, \\ \ker_D(.Q) = D^{1\times l}\, V_1, \end{array} \right.$$

and U_2, and V_2 are any matrices such that:

$$\left\{ \begin{array}{l} U = (U_1^T \quad U_2^T)^T \in \mathrm{GL}_p(D), \\ V = (V_1^T \quad V_2^T)^T \in \mathrm{GL}_q(D). \end{array} \right.$$

2) The matrix R is equivalent to $\overline{R} = V\, R\, U^{-1}$.

3) If we denote by
$U^{-1} = (W_1 \quad W_2)$, $W_1 \in D^{p\times m}$, $W_2 \in D^{p\times(p-m)}$,
we then have:

$$\overline{R} = \begin{pmatrix} V_1\, R\, W_1 & 0 \\ V_2\, R\, W_1 & V_2\, R\, W_2 \end{pmatrix} \in D^{q\times p}.$$

Proof: 1. The result directly follows from 2) of Proposition 7.

2. Using the fact that the matrices U and V are unimodular, we obtain $R = V^{-1}\, \overline{R}\, U$, which proves the result.

3. From Lemma 3, the matrix $\overline{R} = V\, R\, U^{-1}$ satisfies Relation (38). Then, applying Lemma 4 to $\overline{R}$, we obtain that $\overline{R}$ has the triangular form (40), where $\overline{R}_1 \in D^{l\times m}$, $\overline{R}_2 \in D^{l\times(p-m)}$ and $\overline{R}_3 \in D^{(q-l)\times(p-m)}$. We have

$$\overline{R} = V\, R\, U^{-1} = \begin{pmatrix} V_1\, R\, W_1 & V_1\, R\, W_2 \\ V_2\, R\, W_1 & V_2\, R\, W_2 \end{pmatrix} \in D^{q\times p},$$

where $V_1\, R\, W_1 \in D^{l\times m}$, $V_2\, R\, W_1 \in D^{(p-l)\times m}$ and $V_1\, R\, W_2 \in D^{l\times(p-m)}$, $V_2\, R\, W_2 \in D^{(p-l)\times(p-m)}$, which finally proves the result.

■

We refer to Remark 10 of Section IV-B for more details on the way that we can constructively obtain the unimodular matrices U and V defined in Theorem 2 by computing bases of free modules over different classes of skew polynomial rings and Ore algebras.

Example 12: Let us consider the linearized equations of a bipendulum subjected to a horizontal move described in

$$\left\{ \begin{array}{l} \ddot{y}_1 + \dfrac{g}{l_1}\, y_1 - \dfrac{g}{l_1}\, u = 0, \\ \ddot{y}_2 + \dfrac{g}{l_2}\, y_2 - \dfrac{g}{l_2}\, u = 0, \end{array} \right.$$

where l_1 and l_2 are the length of the two pendulum and g is gravity. For more details, see [13] and the references therein. Let us define the ring $D = \mathbb{Q}(g, l_1, l_2)\left[\partial; \mathrm{id}, \frac{d}{dt}\right]$ of differential operators with constant coefficients and the system matrix

$$R = \begin{pmatrix} \partial^2 + \frac{g}{l_1} & 0 & -\frac{g}{l_1} \\ 0 & \partial^2 + \frac{g}{l_2} & -\frac{g}{l_2} \end{pmatrix} \in D^{2\times 3},$$

and the D-module $M = D^{1\times 3}/(D^{1\times 2}\, R)$.

Using Algorithm 1, we obtain that an endomorphism f of M is defined by the matrices

$$P = \begin{pmatrix} 0 & 0 & g\, l_2 \\ 0 & g\,(l_2 - l_1) & g\, l_1 \\ 0 & 0 & l_1\, l_2\, \partial^2 + g\, l_2 \end{pmatrix},$$

$$Q = \begin{pmatrix} 0 & 0 \\ 0 & g\,(l_2 - l_1) \end{pmatrix}.$$

Using algorithms developed in [14], we obtain that $\ker_D(.P)$, $\mathrm{im}_D(.P)$, $\ker_D(.Q)$ and $\mathrm{im}_D(.Q)$ are free D-modules of rank respectively 1, 2, 1 and 1. We can easily compute the bases of $\ker_D(.P)$, $\mathrm{coim}_D\,(.P)$, $\ker_D(.Q)$ and $\mathrm{coim}\,(.Q)$, which are defined by means of the following matrices:

$$U_1 = (l_1\, \partial^2 + g \quad 0 \quad -g),$$

$$U_2 = \begin{pmatrix} \frac{1}{g} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

$$V_1 = (1 \quad 0),$$

$$V_2 = (0 \quad 1).$$

We can check that the matrices $U = (U_1^T \quad U_2^T)^T \in D^{3\times 3}$ and $V = (V_1^T \quad V_2^T)^T \in D^{2\times 2}$ are unimodular and:

$$J_P = U\, P\, U^{-1} = \begin{pmatrix} 0 & 0 & 0 \\ -\frac{l_2}{g} & l_2\,(l_1\, \partial^2 + g) & 0 \\ -l_1 & g\, l_1\,(l_1\, \partial^2 + g) & g\,(l_2 - l_1) \end{pmatrix},$$

$$J_Q = V\, Q\, V^{-1} = \begin{pmatrix} 0 & 0 \\ 0 & g\,(l_2 - l_1) \end{pmatrix}.$$

Finally, we obtain that R is similar to the following triangular matrix:

$$\overline{R} = V\, R\, U^{-1} = \begin{pmatrix} \frac{1}{l_1} & 0 & 0 \\ \frac{1}{l_2} & \frac{g}{l_2}\,(l_1\, \partial^2 + g) & \partial^2 + \frac{g}{l_2} \end{pmatrix}.$$

Remark 6: If $D = A[\partial; \sigma, \delta]$ is a skew polynomial ring over a commutative ring A, $R = (\partial\, I_p - E) \in D^{p\times p}$ and $M = D^{1\times p}/(D^{1\times q}\, R)$ the left D-module associated with the linear functional system $\partial y = E\, y$, using the results proved in Example 4, we then know that any endomorphism f can always be defined by means of two matrices $P \in A^{p\times p}$ and $Q \in A^{q\times q}$. Hence, if A is a field (e.g., $A = k(t)$, $k(n)$), then we can do linear algebra in order to compute the bases of the A-vector spaces $\ker_A(.P)$, $\mathrm{coim}_A(.P)$, $\ker_A(.Q)$ and $\mathrm{coim}_A(.Q)$, i.e., compute the matrices $U_1 \in A^{m\times p}$, $U_2 \in A^{(p-m)\times p}$, $V_1 \in A^{l\times q}$ and $V_2 \in A^{(q-l)\times q}$ defined in Theorem 2 as we then have

$$\begin{cases} \ker_D(.P) = D \otimes_A \ker_A(.P), \\ \mathrm{coim}_D(.P) = D \otimes_A \mathrm{coim}_A(.P), \end{cases}$$

and similarly for $\ker_D(.Q) = D \otimes_A \ker_A(.Q)$ and $\mathrm{coim}_D(.Q) = D \otimes_A \mathrm{coim}_A(.Q)$.

IV. Projectors, idempotents and decompositions

A. Projectors of $\mathrm{end}_D(M)$ and decompositions

We start this section by a lemma which characterizes the projectors of $\mathrm{end}_D(M)$ and we deduce an algorithm for computing them.

Lemma 5: Let us consider a finite free resolution of M

$$D^{1\times q_2} \xrightarrow{.R_2} D^{1\times q} \xrightarrow{.R} D^{1\times p} \xrightarrow{\pi} M \longrightarrow 0,$$

and a morphism $f : M \longrightarrow M$ defined by two matrices $P \in D^{p\times p}$ and $Q \in D^{q\times q}$ satisfying (13). Then, f is a *projector* of $\mathrm{end}_D(M)$, i.e., $f^2 = f$, if and only if there exists a matrix $Z \in D^{p\times q}$ satisfying:

$$P^2 = P + Z\, R. \tag{41}$$

Then, there exists $Z' \in D^{q\times q_2}$ such that:

$$Q^2 = Q + R\, Z + Z'\, R_2. \tag{42}$$

In particular, if $R \in D^{q\times p}$ has *full row rank*, namely, $R_2 = 0$, we then have:

$$Q^2 = Q + R\, Z. \tag{43}$$

Proof: Multiplying (13) on the right by P, we obtain $R\, P^2 = Q\, R\, P$ and using again (13), we get

$$R\, P^2 = Q^2\, R,$$

which shows that $f^2 : M \longrightarrow M$ can be defined by the matrices P^2 and Q^2. From 1) of Corollary 4, the morphism $f^2 - f$ is 0 if and only if there exists a matrix $Z \in D^{p\times q}$ satisfying (41). Then, there also exists a matrix $Z' \in D^{q\times q_2}$ such that (42) holds (see also Corollary 1). The end of the lemma is straightforward.

■

From this lemma, we deduce an algorithm which computes projectors of $\mathrm{end}_D(M)$.

Algorithm 3:
- **Input**: An Ore algebra D, a matrix $R \in D^{q\times p}$ and the output of Algorithm 2 for fixed α, β and γ.

- **Output**: A family of pairs $(\overline{P}_i, \overline{Q}_i)_{i\in I}$ and a set of matrices $\{Z_i\}_{i\in I}$ satisfying

$$\left\{\begin{array}{l} R\,\overline{P}_i = \overline{Q}_i\,R, \\ \overline{P}_i^2 = \overline{P}_i + \overline{Z}_i\,R, \text{ for } Z_i \in D^{p\times q}, \\ \mathrm{ord}_\partial(\overline{P}_i) \leq \alpha, \ i.e., \ \overline{P}_i = \sum_{0\leq|\nu|\leq\alpha} a_\nu^{(i)}\,\partial^\nu, \\ \text{and } \forall\, 0 \leq |\nu| \leq \alpha, \ a_\nu^{(i)} \in A \text{ satisfies :} \\ \deg_x(\mathrm{num}(a_\nu^{(i)})) \leq \beta, \\ \deg_x(\mathrm{denom}(a_\nu^{(i)})) \leq \gamma, \end{array}\right.$$

where $\mathrm{ord}_\partial(\overline{P}_i)$ denotes the maximal of the total orders of the entries of $\overline{P}_i$, $\deg_x(\mathrm{num}(a_\nu^{(i)}))$ (resp., $\deg_x(\mathrm{denom}(a_\nu^{(i)}))$) the degree of the numerator (resp., denominator) of $a_\nu^{(i)}$. The morphisms f_i are then defined by:

$$\forall\,\lambda \in D^{1\times p}: \quad f_i(\pi(\lambda)) = \pi'(\lambda\,\overline{P}_i), \quad i \in I.$$

1) Consider a generic element P of the outpout of Algorithm 2 for fixed α, β and γ.
2) Compute $P^2 - P$ and denote the result by F.
3) Compute a Gröbner basis G of the rows of R.
4) Reduce the rows of F with respect to G by computing their normal forms with respect to G.
5) Solve the system on the coefficients of $a_\nu^{(i,j)}$ so that all the normal forms vanish.
6) Substitute the solutions into the matrix P. Denote the set of solutions by $\{P_i\}_{i\in I}$.
7) For $i \in I$, reduce the rows of P_i with respect to G by computing their normal forms with respect to G. We obtain $\overline{P}_i$ for $i \in I$.
8) Using $r_j(\overline{P}_i^2 - \overline{P}_i) \in (D^{1\times q}\,R)$, $j = 1, \ldots, p$, where $r_j(\overline{P}_i^2 - \overline{P}_i)$ denotes the j^{th} row of $\overline{P}_i^2 - \overline{P}_i$, compute a matrix $\overline{Z}_i \in D^{q\times q'}$ satisfying $\overline{P}_i^2 - \overline{P}_i = \overline{Z}_i\,R$, for $i \in I$.

We are now going to show how projectors can be used to decompose the system $R\,y = 0$ into decoupled (independent) systems $S_1\,y_1 = 0$ and $S_2\,y_2 = 0$ or, in other words, to decompose the left D-module M into two direct summands. We start with a first lemma.

Lemma 6: Let $R \in D^{q\times p}$ and $M = D^{1\times p}/(D^{1\times q}\,R)$. Let $f \in \mathrm{end}_D(M)$ be a projector, i.e., $f^2 = f$.

1) We have the following split exact sequence

$$0 \longrightarrow \ker f \ \underset{\overleftarrow{\ id_M - f\ }}{\overset{i}{\longrightarrow}} \ M \ \underset{\overleftarrow{\ f^\sharp\ }}{\overset{\rho}{\longrightarrow}} \ \mathrm{coim}\, f \longrightarrow 0,$$

where $f^\sharp : \mathrm{coim}\, f \longrightarrow M$ is defined by:

$$\forall\, m \in M, \quad f^\sharp(\rho(m)) = f(m). \qquad (44)$$

2) We have the following isomorphism

$$\begin{array}{rcl} \varphi : \ker f & \longrightarrow & \mathrm{coker}\, f \\ m & \longmapsto & \sigma(m), \end{array}$$

whose inverse is defined by

$$\begin{array}{rcl} \psi : \mathrm{coker}\, f & \longrightarrow & \ker f \\ \sigma(m) & \longmapsto & m - f(m), \end{array}$$

where $\sigma : M \longrightarrow \mathrm{coker}\, f$ denotes the canonical projection.

Proof: 1. For all $\rho(m) \in \mathrm{coim}\, f$, we have

$$((\mathrm{id}_M - f) \circ f^\sharp)(\rho(m)) = f(m) - f^2(m) = 0,$$

i.e., $(\mathrm{id}_M - f) \circ f^\sharp = 0$. Moreover, we easily check that $(\mathrm{id}_M - f) \circ i = \mathrm{id}_{\ker f}$. Now, for all $m \in M$, we have

$$(i \circ (\mathrm{id}_M - f) + f^\sharp \circ \rho)(m) = m - f(m) + f(m) = m,$$

i.e., $(i \circ (\mathrm{id}_M - f)) + f^\sharp \circ \rho = \mathrm{id}_M$. Multiplying the last identity by ρ on the left and using the fact that $\rho \circ i = 0$, we get $\rho \circ f^\sharp \circ \rho = \rho$ which proves $\rho \circ f^\sharp = \mathrm{id}_{\mathrm{coim}\, f}$ and ends the proof of 1).

2. We check that ψ is well-defined as $m - f(m) \in \ker f$. For all $m \in \ker f$, we have $(\psi \circ \varphi)(m) = m - f(m) = m$, i.e., $\psi \circ \varphi = \mathrm{id}_{\ker f}$.

On the other hand, for all $\sigma(m) \in \mathrm{coker}\, f$, we have

$$(\varphi \circ \psi)(\sigma(m)) = \varphi(m - f(m)) = \sigma(m),$$

and thus, $\varphi \circ \psi = \mathrm{id}_{\mathrm{coker}\, f}$, which proves the result. ■

The next proposition gives a necessary and sufficient condition for the existence of projector f of $\mathrm{end}_D(M)$, i.e., for the existence of a direct summand of the finitely presented left D-module M.

Proposition 8: Let $R \in D^{q\times p}$, $M = D^{1\times p}/(D^{1\times q}\,R)$. With the notations of Proposition 6, if $f : M \longrightarrow M$ is an endomorphism of M, the following results are equivalent:

1) f is a projector of $\mathrm{end}_D(M)$, namely, $f^2 = f$.
2) There exists $X \in D^{p\times r}$ satisfying:

$$P = I_p - X\,S. \qquad (45)$$

Then, we have the following commutative exact diagram

$$\begin{array}{ccccccc} & & & & 0 & & \\ & \xleftarrow{.X} & & & \downarrow & & \\ D^{1\times r} & \xrightarrow{.S} & D^{1\times p} & \xrightarrow{\kappa} & \mathrm{coim}\, f & \longrightarrow 0 & \\ \downarrow .T & & \downarrow .P & & \downarrow f^\sharp & & \\ D^{1\times q} & \xrightarrow{.R} & D^{1\times p} & \xrightarrow{\pi} & M & \longrightarrow 0, & \\ & & & & \downarrow & & \\ & & & & \ker f & & \\ & & & & \downarrow & & \\ & & & & 0 & & \end{array}$$

where $f^\sharp$ is defined by (44).

Proof: $(1 \Rightarrow 2)$. By 1) of Lemma 6, the morphism $f^\sharp$ defined by (44) satisfies the relation $\rho \circ f^\sharp = \mathrm{id}_{\mathrm{coim}\, f}$, and thus, we have $M = i(\ker f) \oplus f^\sharp(\mathrm{coim}\, f)$. Using the relation $S\,P = T\,R$, we obtain that $f^\sharp$ induces the following morphism of complexes:

$$\begin{array}{ccccccc} D^{1\times r} & \xrightarrow{.S} & D^{1\times p} & \xrightarrow{\kappa} & \mathrm{coim}\, f & \longrightarrow 0 \\ \downarrow .T & & \downarrow .P & & \downarrow f^\sharp & \\ D^{1\times q} & \xrightarrow{.R} & D^{1\times p} & \xrightarrow{\pi} & M & \longrightarrow 0. \end{array}$$

Composing the morphisms of complexes corresponding to ρ (see Theorem 1) and $f^\sharp$, we obtain that the morphism $\mathrm{id} - \rho \circ f^\sharp = 0$ is defined by the following morphism of complexes

$$\begin{array}{ccccc} D^{1\times r_2} & \xrightarrow{.S_2} & D^{1\times r} & \xrightarrow{.S} & D^{1\times p} \\ & & \downarrow .(I_q - L\,T) & & \downarrow .(I_p - P) \\ D^{1\times r_2} & \xrightarrow{.S_2} & D^{1\times r} & \xrightarrow{.S} & D^{1\times p} \end{array}$$

which must be homotopic to zero. Thus, there exist a matrix $X \in D^{p\times r}$ and $X_2 \in D^{r\times r_2}$ such that:

$$\left\{ \begin{array}{l} I_p - P = X\,S, \\ I_q - L\,T = S\,X + X_2\,S_2. \end{array} \right.$$

We note that, using the relation $Q = L\,T$, the previous system leads to (45) and $Q = I_q - S\,X - X_2\,S_2$.

$(2 \Rightarrow 1)$. Using (45) and $S\,P = T\,R$, we obtain

$$\begin{array}{ll} P^2 & = (I_p - X\,S)\,P \\ & = P - X\,S\,P \\ & = P - (X\,T)\,R, \end{array}$$

which proves that f is a projector by Lemma 5. ∎

We remark that, substituting (45) into $S\,P = T\,R$, we obtain:

$$S\,(I_p - X\,S) = T\,R \Leftrightarrow S - S\,X\,S = T\,R.$$

We now give a necessary and sufficient condition for a module to be a direct summand of another one.

Proposition 9: Let $R \in D^{q\times p}$ and $S \in D^{r\times p}$ be two matrices satisfying $(D^{1\times q}\,R) \subseteq (D^{1\times r}\,S)$. Then, the left D-module $M' = D^{1\times p}/(D^{1\times r}\,S)$ is isomorphic to a direct summand of $M = D^{1\times p}/(D^{1\times q}\,R)$, i.e., we have

$$M \cong M' \oplus \ker \rho, \tag{46}$$

where $\rho : M \longrightarrow M'$ is defined by

$$\forall\, \lambda \in D^{1\times p}, \quad \rho(\pi(\lambda)) = \kappa(\lambda),$$

and $\kappa : D^{1\times p} \longrightarrow M'$ denotes the canonical projection, if and only if there exist two matrices $X \in D^{p\times r}$ and $T \in D^{r\times q}$ satisfying the following relation:

$$S - S\,X\,S = T\,R. \tag{47}$$

Proof: $(\Rightarrow)$. The isomorphism (46) is equivalent to the existence of a morphism $g : M' \longrightarrow M$ which satisfies $\rho \circ g = \mathrm{id}_{M'}$ (see [14], [56]). Following the same techniques as the ones used in the proof of Proposition 8, (46) is then equivalent to the existence of $P \in D^{p\times p}$, $T \in D^{r\times q}$ and $X \in D^{p\times r}$ satisfying:

$$\left\{ \begin{array}{l} S\,P = T\,R, \\ I_p - P = X\,S, \end{array} \right. \Rightarrow S - S\,X\,S = T\,R.$$

$(\Leftarrow)$. From (47), we obtain $S\,(I_p - X\,S) = T\,R$, and, if we set $P = I_p - X\,S$, then we have the following commutative diagram

$$\begin{array}{ccccccc} D^{1\times r} & \xrightarrow{.S} & D^{1\times p} & \xrightarrow{\kappa} & M' & \longrightarrow 0 \\ \downarrow .T & & \downarrow .P & & & \\ D^{1\times q} & \xrightarrow{.R} & D^{1\times p} & \xrightarrow{\pi} & M & \longrightarrow 0, \end{array}$$

which induces a morphism $g : M' \longrightarrow M$ defined by:

$$\forall\, \lambda \in D^{1\times p}, \quad g(\kappa(\lambda)) = \pi(\lambda\,P).$$

Using $\kappa = \rho \circ \pi$, for all $\lambda \in D^{1\times p}$, we obtain:

$$\begin{array}{ll} (\rho \circ g)(\kappa(\lambda)) & = \rho(\pi(\lambda\,P)) = \kappa(\lambda\,P) \\ & = \kappa(\lambda) - \kappa((\lambda\,X)\,S) = \kappa(\lambda). \end{array}$$

We then have $\rho \circ g = \mathrm{id}_{M'}$, which shows that the exact sequence $0 \longrightarrow \ker \rho \xrightarrow{i} M \xrightarrow{\rho} M' \longrightarrow 0$ splits, and thus, we finally obtain $M = \ker \rho \oplus g(M')$. ∎

Remark 7: If S has full row rank, i.e., $\ker_D(.S) = 0$, using the factorization $R = L\,S$, (47) becomes:

$$(I_r - S\,X - T\,L)\,S = 0 \Rightarrow S\,X + T\,L = I_r. \tag{48}$$

Hence, we obtain that the matrix $(X^T \quad L^T)^T$ admits a left-inverse. Note that (48) is nothing else than the generalization for matrices and non-commutative rings of the classical decomposition of a commutative polynomial into coprime factors. Indeed, if R belongs to a commutative polynomial ring $D = k[x_1, \ldots, x_n]$, where k is a field, then (48) becomes $X\,S + T\,L = 1$ (Bézout identity), i.e., the ideal of D generated by S and L is the whole ring D and we obtain that $R = L\,S$ is a factorization of R into coprime factors L and S.

We have the following corollary of Proposition 8.

Corollary 5: With the hypotheses and notations of Proposition 8, we have the equality:

$$(D^{1\times r}\,S) = \left(D^{1\times (p+q)} \left(\begin{array}{c} I_p - P \\ R \end{array} \right) \right).$$

Proof: Using the factorization $R = L\,S$ and (45), we obtain the following equality

$$\left(\begin{array}{c} I_p - P \\ R \end{array} \right) = \left(\begin{array}{c} X \\ L \end{array} \right) S,$$

which proves the first inclusion. The second inclusion is a direct consequence of (47) as we have $X\,S = I_p - P$ and:

$$S = S\,X\,S + T\,R = (S \quad T) \begin{pmatrix} X\,S \\ R \end{pmatrix}.$$

■

Let us state the third main result of the paper.

Theorem 3: Let $R \in D^{q\times p}$ and let us assume that the finitely presented left D-module $M = D^{1\times p}/(D^{1\times q}\,R)$ admits a decomposition of the form $M \cong \ker f \oplus \operatorname{im} f$, where $f \in \operatorname{end}_D(M)$. Moreover, let us suppose that $\mathcal{F}$ is an injective left D-module. Then, with the notations previously introduced in this section, we obtain that a solution $\eta \in \mathcal{F}^p$ of $R\,\eta = 0$ has the form $\eta = \zeta + X\,\tau$, where $\zeta \in \mathcal{F}^p$ is a fundamental solution of $S\,\zeta = 0$ and $\tau \in \mathcal{F}^r$ is a fundamental solution of the system:

$$\begin{cases} L\,\tau = 0, \\ S_2\,\tau = 0. \end{cases} \tag{49}$$

Hence, the integration of the system $R\,\eta = 0$ is equivalent to the integration of the two independent systems $S\,\zeta = 0$ and (49).

Proof: Applying the functor $\hom_D(\cdot, \mathcal{F})$ to the commutative exact diagram (28), we obtain the following commutative exact diagram:

$$\begin{array}{ccccccccc} & & \mathcal{F}^q & \xleftarrow{R.} & \mathcal{F}^p & \longleftarrow & \ker_{\mathcal{F}}(R.) & \longleftarrow & 0 \\ & & \uparrow L. & & \| & & \uparrow \rho^\star & & \\ \mathcal{F}^{r_2} & \xleftarrow{S_2.} & \mathcal{F}^r & \xleftarrow{S.} & \mathcal{F}^p & \longleftarrow & \ker_{\mathcal{F}}(S.) & \longleftarrow & 0. \end{array}$$

Let us first prove that an element of the form

$$\eta = \zeta + X\,\tau,$$

where $\zeta \in \mathcal{F}^p$ (resp., $\tau \in \mathcal{F}^r$) satisfies $S\,\zeta = 0$ (resp., (49)) is a solution of the system $R\,\eta = 0$. Using the factorization $R = L\,S$ and $S\,\zeta = 0$, we get:

$$\begin{aligned} R\,\eta &= R\,\zeta + R\,(X\,\tau) = L\,(S\,\zeta) + L\,(S\,(X\,\tau)) \\ &= L\,(S\,(X\,\tau)). \end{aligned}$$

Using the fact that τ satisfies the second equation of (49) and the exactness of the last horizontal exact sequence of the previous commutative exact diagram, there exists $\overline{\eta} \in \mathcal{F}^p$ satisfying $\tau = S\,\overline{\eta}$. Substituting this relation into the first equation of (49), we obtain:

$$L\,\tau = L\,(S\,\overline{\eta}) = R\,\overline{\eta} = 0.$$

Then, using (47), we obtain:

$$\begin{gathered} S\,\overline{\eta} - S\,(X\,(S\,\overline{\eta})) = T\,(R\,\overline{\eta}) = 0 \\ \Rightarrow S\,(X\,\tau) = S\,\overline{\eta} \Rightarrow L\,(S\,(X\,\tau)) = L\,(S\,\overline{\eta}) = R\,\overline{\eta} = 0. \end{gathered}$$

This last result proves that $R\,\eta = 0$, and thus, $\eta = \zeta + X\,\tau$ is a solution of the system $R\,\eta = 0$.

Conversely, let us prove that any solution $\eta \in \mathcal{F}^p$ of $R\,\eta = 0$ has the form of $\eta = \zeta + X\,\tau$, where $\zeta \in \mathcal{F}^p$ satisfies $S\,\zeta = 0$ and $\tau \in \mathcal{F}^r$ satisfies (49). Let us consider $\eta \in \mathcal{F}^p$ satisfying $R\,\eta = 0$, i.e., $(L\,S)\,\eta = 0$. Using the previous commutative exact diagram, we obtain that the element $\tau \in \mathcal{F}^r$ defined by $\tau = S\,\eta \in \mathcal{F}^r$ satisfies (49). Then, from (47), we obtain:

$$S\,\eta - S\,(X\,(S\,\eta)) = T\,(R\,\eta) = 0 \Rightarrow S\,(X\,\tau) = \tau.$$

All the solutions of the inhomogeneous system $S\,\eta = \tau$ are defined by the sum of the general solution of $S\,\zeta = 0$ and a particular solution of $S\,\overline{\eta} = \tau$, i.e., we have $\eta = \zeta + X\,\tau$, which ends the proof.

■

We note that the previous result has already been obtained in [52] in the particular case where

$$M \cong t(M) \oplus (M/t(M)),$$

where the torsion submodule $t(M)$ is defined by

$$t(M) = \{m \in M \mid \exists\, 0 \neq P \in D : \; P\,m = 0\},$$

i.e., in the control theoretical language, when a behaviour $\hom_D(M, \mathcal{F})$ can be split into the autonomous behaviour $\hom_D(t(M), \mathcal{F})$ and the controllable behaviour $\hom_D(M/t(M), \mathcal{F})$. We refer the reader to [52], [53] for more details and examples.

Let us illustrate Theorem 3 by means of an example.

Example 13: Let D be the Weyl algebra $A_1(k)$, namely, $D = k[t]\left[\partial; \mathrm{id}_{k[t]}, \frac{d}{dt}\right]$, where k is a field of characteristic 0 and let us consider the matrix of differential operators

$$R = \begin{pmatrix} \partial & -t & t & \partial \\ \partial & t\,\partial - t & \partial & -1 \\ \partial & -t & \partial + t & \partial - 1 \\ \partial & \partial - t & t & \partial \end{pmatrix} \in D^{4\times 4}, \tag{50}$$

and the left D-module $M = D^{1\times 4}/(D^{1\times 4}\,R)$ associated with the linear system $R\,y = 0$. We can easily check that an endomorphism f of M can be defined by means of the following two matrices

$$P = Q = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in k^{4\times 4}, \tag{51}$$

i.e., we have $R\,P = P\,R$. With the notations used in this section, we obtain the following matrices:

$$S = \begin{pmatrix} \partial & -t & 0 & 0 \\ 0 & \partial & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$L = \begin{pmatrix} 1 & 0 & t & \partial \\ 1 & t & \partial & -1 \\ 1 & 0 & \partial + t & \partial - 1 \\ 1 & 1 & t & \partial \end{pmatrix}.$$

Moreover, we easily check that $P^2 = P$, i.e., P is an *idempotent* of $D^{3\times 3}$. Then, using (45), we obtain:

$$X = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

We can also verify that $\ker_D(.S) = 0$ which implies $S_2 = 0$ (with the notations of this section). Theorem 3 then asserts that the integration of $R\,\eta = 0$ is equivalent to both the integration of $S\,\zeta = 0$, which easily gives

$$\zeta_1 = \frac{1}{2} C_1\, t + C_2, \quad \zeta_2 = C_1, \quad \zeta_3 = 0, \quad \zeta_4 = 0,$$

where C_1 and C_2 are two constants, and the integration of $L\,\tau = 0$, which can be seen to be equivalent to:

$$\begin{cases} \tau_1 = 0, \\ \tau_2 = 0, \\ t\,\tau_3 + \partial\,\tau_4 = 0, \\ \partial\,\tau_3 - \tau_4 = 0. \end{cases} \Leftrightarrow \begin{cases} \tau_1 = 0, \\ \tau_2 = 0, \\ \partial^2\,\tau_3 + t\,\tau_3 = 0, \\ \tau_4 = \partial\,\tau_3. \end{cases}$$

The third equation can be integrated by means of the Airy functions Ai and Bi which are the two independent solutions of $\partial^2\,y(t) - t\,y(t) = 0$ (see [31]). We then have

$$\begin{cases} \tau_1 = 0, \\ \tau_2 = 0, \\ \tau_3(t) = C_3\,\mathrm{Ai}(t) + C_4\,\mathrm{Bi}(t), \\ \tau_4(t) = C_3\,\partial\,\mathrm{Ai}(t) + C_4\,\partial\,\mathrm{Bi}(t), \end{cases}$$

where C_3 and C_4 are two constants. The general solution of $R\,\eta = 0$ is then given by

$$\eta = \zeta + X\,\tau = \begin{pmatrix} \frac{1}{2}\,C_1\,t + C_2 \\ C_1 \\ C_3\,\mathrm{Ai}(t) + C_4\,\mathrm{Bi}(t) \\ C_3\,\partial\,\mathrm{Ai}(t) + C_4\,\partial\,\mathrm{Bi}(t) \end{pmatrix}, \qquad (52)$$

where C_1, C_2, C_3 and C_4 are four arbitrary constants.

B. Idempotents of $D^{p\times p}$ and decompositions

We are now going further by proving that, under certain conditions, the existence of idempotents P of $D^{p\times p}$ allows us to obtain a system $\overline{R}\,\overline{y} = 0$ equivalent to $R\,y = 0$, where $\overline{R}$ is a block-diagonal matrix of the same size than R. We shall need the following lemmas.

Lemma 7: Let $R \in D^{q\times p}$ be a full row rank matrix, i.e., $\ker_D(.R) = 0$, and $P \in D^{p\times p}$, $Q \in D^{q\times q}$ be two matrices satisfying (13). Then, if P is an idempotent, namely $P^2 = P$, so is Q, i.e., $Q^2 = Q$.

Proof: Multiplying (13) on the right by P, we obtain $R\,P^2 = Q\,R\,P$. Using again (13), we get $R\,P^2 = Q^2\,R$. Then, the relation $P^2 = P$ implies $R\,P = Q^2\,R$, and using again (13), we obtain $Q^2\,R = Q\,R$, i.e., $(Q^2 - Q)\,R = 0$. Finally, the fact that R has full row rank implies $Q^2 = Q$. ∎

Lemma 8: Let $R \in D^{q\times p}$ be a full row rank matrix, i.e., $\ker_D(.R) = 0$, and $M = D^{1\times p}/(D^{1\times q}\,R)$. Let us consider a projector $f : M \longrightarrow M$ defined by two matrices $P \in D^{p\times p}$ and $Q \in D^{q\times q}$ satisfying (13), $P^2 = P + Z\,R$ and $Q^2 = Q + R\,Z$ (see Lemma 5). If there exists a solution $\Lambda \in D^{p\times q}$ of the following *Riccati equation*

$$\Lambda\,R\,\Lambda + (P - I_p)\,\Lambda + \Lambda\,Q + Z = 0, \qquad (53)$$

then the matrices

$$\begin{cases} \overline{P} = P + \Lambda\,R, \\ \overline{Q} = Q + R\,\Lambda, \end{cases} \qquad (54)$$

satisfy $R\,\overline{P} = \overline{Q}\,R$, i.e., they define an endomorphism of M and are idempotents, i.e., we have:

$$\overline{P}^2 = \overline{P}, \quad \overline{Q}^2 = \overline{Q}.$$

Proof: By hypothesis, the matrices P and Q satisfy (41) and (43). Let us define $\overline{P} = P + \Lambda\,R$ for a certain matrix $\Lambda \in D^{p\times q}$. Then, we have:

$$\begin{aligned} \overline{P}^2 &= (P + \Lambda\,R)\,(P + \Lambda\,R) \\ &= P^2 + P\,\Lambda\,R + \Lambda\,R\,P + \Lambda\,R\,\Lambda\,R. \end{aligned}$$

Using (13) and $P^2 = P + Z\,R$, we then get:

$$\overline{P}^2 = P^2 + (P\,\Lambda + \Lambda\,Q + \Lambda\,R\,\Lambda)\,R.$$

Then, from (41) and $\overline{P} = P + \Lambda\,R$, we finally obtain:

$$\overline{P}^2 = \overline{P} + (Z - \Lambda + P\,\Lambda + \Lambda\,Q + \Lambda\,R\,\Lambda)\,R.$$

Hence, we have $\overline{P}^2 = \overline{P}$ if and only if Λ satisfies the following equation

$$(Z - \Lambda + P\,\Lambda + \Lambda\,Q + \Lambda\,R\,\Lambda)\,R = 0,$$

i.e., satisfies (53) since R has full row rank.

Finally, we have:

$$\begin{aligned} \overline{Q}^2 &= (Q + R\,\Lambda)\,(Q + R\,\Lambda) \\ &= Q^2 + Q\,R\,\Lambda + R\,\Lambda\,Q + R\,\Lambda\,R\,\Lambda. \end{aligned}$$

Using (11), we get

$$\overline{Q}^2 = Q^2 + R\,(P\,\Lambda + \Lambda\,Q + \Lambda\,R\,\Lambda),$$

and using (43) and $\overline{Q} = Q + R\,\Lambda$, we then obtain:

$$\overline{Q}^2 = \overline{Q} + R\,(Z - \Lambda + P\,\Lambda + \Lambda\,Q + \Lambda\,R\,\Lambda) = \overline{Q}.$$

∎

Remark 8: We are currently not able to understand when the Riccati equation (53) admits a solution. This problem will be studied with care in the future. However, we can always try to compute a solution Λ of (53) with fixed order and fixed degrees for numerators and denominators by substituting an ansatz in (53) and solving the corresponding system obtained on the coefficients.

Example 14: Let D be the Weyl algebra $A_1(\mathbb{Q})$, i.e., $D = \mathbb{Q}[t]\left[\partial; \mathrm{id}, \frac{d}{dt}\right]$, the matrix $R = \begin{pmatrix} \frac{d^2}{dt^2} & -t\,\frac{d}{dt} - 1 \end{pmatrix} \in D^{1\times 2}$ and the finitely presented left D-module $M = D^{1\times 2}/(D\,R)$. Searching for projectors of total order 1 and total degree 2, Algorithm 3 gets $P_1 = 0$, $P_2 = I_2$ and

$$\left\{\begin{array}{l} P_3 = \left(\begin{array}{cc} -(t+a)\,\partial + 1 & t^2 + a\,t \\ 0 & 1 \end{array}\right), \\ Q_3 = -((t+a)\,\partial + 1), \end{array}\right.$$

$$\left\{\begin{array}{l} P_4 = \left(\begin{array}{cc} (t-a)\,\partial & -t^2 + a\,t \\ 0 & 0 \end{array}\right), \\ Q_4 = (t-a)\,\partial + 2, \end{array}\right.$$

where a is an arbitrary constant of $\mathbb{Q}$. We can check that $P_i^2 = P_i + Z_i\,R$, $i = 3, 4$, where:

$$Z_3 = ((t+a)^2 \quad 0)^T, \quad Z_4 = ((t-a)^2 \quad 0)^T.$$

We obtain that (53) admits respectively the solution:

$$\Lambda_3 = (a\,t \quad a\,\partial - 1)^T, \quad \Lambda_4 = (a\,t \quad a\,\partial + 1)^T.$$

The matrices (54) are then defined by

$$\overline{P}_3 = \left(\begin{array}{cc} a\,t\,\partial^2 - (t+a)\,\partial + 1 & t^2\,(1 - a\,\partial) \\ (a\,\partial - 1)\,\partial^2 & -a\,t\,\partial^2 + (t - 2\,a)\,\partial + 2 \end{array}\right),$$

$$\overline{Q}_3 = 0,$$

$$\overline{P}_4 = \left(\begin{array}{cc} a\,t\,\partial^2 + (t-a)\,\partial & -t^2\,(1 + a\,\partial) \\ (a\,\partial + 1)\,\partial^2 & -a\,t\,\partial^2 - (t + 2\,a)\,\partial - 1 \end{array}\right),$$

$$\overline{Q}_4 = 1,$$

and we can easily check that we have:

$$\overline{P}_i^2 = \overline{P}_i, \quad \overline{Q}_i^2 = \overline{Q}_i, \quad i = 3, 4.$$

The next lemma characterizes the kernel and the image of an idempotent P of $D^{p\times p}$ in terms of module theory.

Lemma 9: Let $P \in D^{p\times p}$ be an idempotent, i.e., $P^2 = P$. Then, we have the following results:

1) $\ker_D(.P)$ and $\mathrm{im}_D(.P)$ are projective left D-modules of rank respectively m and $p - m$, with $0 \leq m \leq p$.

2) We have the following equalities:

$$\left\{\begin{array}{l} \mathrm{im}_D(.P) = \ker_D(.(I_p - P)), \\ \mathrm{im}_D(.(I_p - P)) = \ker_D(.P). \end{array}\right.$$

Proof: 1. We have the following short exact sequence:

$$0 \longrightarrow \ker_D(.P) \longrightarrow D^{1\times p} \xrightarrow{.P} \mathrm{im}_D(.P) \longrightarrow 0.$$

Let us define the D-morphism $i : \mathrm{im}_D(.P) \longrightarrow D^{1\times p}$ by $i(m) = m$, for all $m \in \mathrm{im}_D(.P)$. Now, for every element $m \in \mathrm{im}_D(.P)$, there exists $\lambda \in D^{1\times p}$ such that $m = \lambda\,P$. Therefore, we have $((.P)\circ i)(m) = m\,P = \lambda\,P^2$ and using the fact that $P^2 = P$, we get $((.P)\circ i)(m) = \lambda\,P = m$, i.e., $((.P)\circ i) = \mathrm{id}_{\mathrm{im}_D(.P)}$, which shows that the previous short exact sequence splits, and thus, we obtain:

$$D^{1\times p} = \ker_D(.P) \oplus \mathrm{im}_D(.P). \qquad (55)$$

This proves that $\ker_D(.P)$ and $\mathrm{im}_D(.P)$ are two finitely generated projective left D-modules. Finally, we have

$$\mathrm{rank}_D(D^{1\times p}) = \mathrm{rank}_D(\ker_D(.P)) + \mathrm{rank}_D(\mathrm{im}_D(.P)),$$

and using the fact that, by hypothesis, D is a left noetherian ring, and thus, D has *the Invariant Basis Number* (IBN) ([29]), we finally get $\mathrm{rank}_D(D^{1\times p}) = p$, which proves the first result.

2. The fact that $P^2 = P$ implies that $P\,(I_p - P) = 0$, which shows that $\mathrm{im}_D(.P) \subseteq \ker_D(.(I_p - P))$. Now, let $\lambda \in \ker_D(.(I_p - P))$ and let us prove that $\lambda \in \mathrm{im}_D(.P)$. Applying λ on the left of the identity $I_p = P + (I_p - P)$, we obtain $\lambda = \lambda\,P$, which proves the equality.

The second result can be proved similarly. ■

We note that if $P = 0$ (resp., $P = I_p$) is the trivial idempotent, then we have $\ker_D(.P) = D^{1\times p}$ and $\mathrm{im}_D(.P) = 0$ (resp., $\ker_D(.P) = 0$, $\mathrm{im}_D(.P) = D^{1\times p}$), i.e., $\ker_D(.P)$ and $\mathrm{im}_D(.P)$ are two trivial free left D-modules. We are going to show that the case where $\ker_D(.P)$ and $\mathrm{im}_D(.P)$ are two non-trivial free left D-modules plays an important role in the decomposition problem.

The next proposition will play in important role in what follows.

Proposition 10: Let $P \in D^{p\times p}$ be an idempotent, i.e., $P^2 = P$. The following assertions are equivalent:

1) The left D-modules $\ker_D(.P)$ and $\mathrm{im}_D(.P)$ are free of rank respectively m and $p - m$.

2) There exists a unimodular matrix $U \in D^{p\times p}$, i.e., $U \in \mathrm{GL}_p(D)$, and a matrix $J_P \in D^{p\times p}$ of the form

$$J_P = \left(\begin{array}{cc} 0 & 0 \\ 0 & I_{p-m} \end{array}\right),$$

which satisfy the relation:

$$U\,P = J_P\,U. \qquad (56)$$

The matrix U has then the form

$$U = \left(\begin{array}{c} U_1 \\ U_2 \end{array}\right), \qquad (57)$$

where the matrices $U_1 \in D^{m\times p}$ and $U_2 \in D^{(p-m)\times p}$ have full row ranks and satisfy the conditions:

$$\left\{\begin{array}{l} \ker_D(.P) = D^{1\times m}\,U_1, \\ \mathrm{im}_D(.P) = D^{1\times (p-m)}\,U_2. \end{array}\right. \qquad (58)$$

In particular, we have $U_1\,P=0$ and $U_2\,P=U_2$.

Proof: ($1 \Rightarrow 2$). Let us suppose that $\ker_D(.P)$ (resp., $\mathrm{im}_D(.P)$) is a free left D-module of rank m (resp., $p-m$) and let $U_1 \in D^{m\times p}$ (resp., $U_2 \in D^{(p-m)\times p}$) be a basis of $\ker_D(.P)$ (resp., $\mathrm{im}_D(.P)$) , i.e., (58) holds. Let us form the matrix U defined by (57).

Now, using (55), for all $\lambda \in D^{1\times p}$, there exist unique $\lambda_1 \in \ker_D(.P)$ and $\lambda_2 \in \mathrm{im}(.P)$ such that $\lambda = \lambda_1 + \lambda_2$. Then, there exist unique $\mu_1 \in D^{1\times m}$ and $\mu_2 \in D^{1\times(p-m)}$ such that $\lambda_1 = \mu_1\,U_1$ and $\lambda_2 = \mu_2\,U_2$, and thus, a unique $\mu = (\mu_1,\ \mu_2) \in D^{1\times p}$ satisfying $\lambda = \mu\,U$. Hence, using the standard basis $\{e_i\}_{1\le i\le p}$ of $D^{1\times p}$, for $i=1,\ldots,p$, there exists a unique $V_i \in D^{1\times p}$ such that $e_i = V_i\,U$. The matrix $V = (V_1^T,\ldots,V_p^T)^T$ is thus a left-inverse of U. By hypothesis, D is a left noetherian ring, and thus, D is *stably finite* ([29]), which implies that we then have $U\,V = I_p$, i.e., $U \in \mathrm{GL}_p(D)$.

Finally, for all $\mu \in D^{1\times p}$, we have $\mu\,U_2 \in \mathrm{im}_D(.P)$, and thus, there exists $\nu \in D^{1\times p}$ such that $\mu\,U_2 = \nu\,P$. Using the fact that $P^2 = P$, we get:

$$\mu\,U_2\,P = \nu\,P^2 = \nu\,P = \mu\,U_2.$$

In particular, we have $e_i\,(U_2\,P) = e_i\,U_2$, for $i=1,\ldots,p$, which proves that $U_2\,P = U_2$. Using $U_1\,P=0$, we obtain:

$$\begin{aligned} U\,P &= \begin{pmatrix} U_1 \\ U_2 \end{pmatrix} P = \begin{pmatrix} U_1\,P \\ U_2\,P \end{pmatrix} = \begin{pmatrix} 0 \\ U_2 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ 0 & I_{p-m} \end{pmatrix} U. \end{aligned}$$

($2 \Rightarrow 1$). Using the relation (56) and the fact that U is a unimodular matrix, we have the commutative exact diagram:

$$\begin{array}{ccccc} & & 0 & & 0 \\ & & \uparrow & & \uparrow \\ 0 \longrightarrow \ker_D(.P) \longrightarrow & & D^{1\times p} & \xrightarrow{.P} & D^{1\times p} \\ & & \uparrow .U & & \uparrow .U \\ 0 \longrightarrow \ker_D(.J_P) \longrightarrow & & D^{1\times p} & \xrightarrow{.J_P} & D^{1\times p}, \\ & & \uparrow & & \uparrow \\ & & 0 & & 0 \end{array}$$

which shows that $\ker_D(.P) \cong \ker_D(.J_P)$ (more precisely, $\ker_D(.P) = \ker_D(.J_P)\,U$). Using the fact that we have trivially $\ker_D(.J_P) = D^{1\times m}$, we obtain that $\ker_D(.P)$ is a free left D-module of rank m. Similarly, we have $\mathrm{im}_D(.P) = \mathrm{im}_D(.J_P)\,U$ as U is a unimodular matrix and:

$$\forall\ \lambda,\ \mu \in D^{1\times p}, \quad \left\{ \begin{array}{l} \lambda\,P = ((\lambda\,U^{-1})\,J_p)\,U, \\ (\mu\,J_P)\,U = (\mu\,U)\,P. \end{array} \right.$$

Therefore, we have $\mathrm{im}_D(.P) \cong \mathrm{im}_D(.J_P)$. We now easily check that $\mathrm{im}_D(.J_P) = D^{1\times(p-m)}$, which proves that $\mathrm{im}_D(.P)$ is a free left D-module of rank $p-m$.

■

Remark 9: We note that (56) is equivalent to

$$P = U^{-1}\,J_P\,U,$$

which means that the two matrices P and J_P are similar.

We shall need the next lemma.

Lemma 10: Let us consider the following two matrices

$$\left\{ \begin{array}{l} J_P = \begin{pmatrix} 0 & 0 \\ 0 & I_{p-m} \end{pmatrix} \in D^{p\times p}, \\ J_Q = \begin{pmatrix} 0 & 0 \\ 0 & I_{q-l} \end{pmatrix} \in D^{q\times q}, \end{array} \right. \tag{59}$$

where $1 \le m \le p$ and $1 \le l \le q$ and a matrix $\overline{R} \in D^{q\times p}$ satisfying the following relation:

$$\overline{R}\,J_P = J_Q\,\overline{R}. \tag{60}$$

Then, there exist $\overline{R}_1 \in D^{l\times m}$ and $\overline{R}_2 \in D^{(q-l)\times(p-m)}$ such that:

$$\overline{R} = \begin{pmatrix} \overline{R}_1 & 0 \\ 0 & \overline{R}_2 \end{pmatrix}. \tag{61}$$

Proof: If we write

$$\overline{R} = \begin{pmatrix} \overline{R}_{11} & \overline{R}_{12} \\ \overline{R}_{21} & \overline{R}_{22} \end{pmatrix},$$

where $\overline{R}_{11} \in D^{l\times m}$, $\overline{R}_{12} \in D^{l\times(p-m)}$, $\overline{R}_{21} \in D^{(q-l)\times m}$, $\overline{R}_{22} \in D^{(q-l)\times(p-m)}$, then, we have:

$$\overline{R}\,J_P = \begin{pmatrix} \overline{R}_{11} & \overline{R}_{12} \\ \overline{R}_{21} & \overline{R}_{22} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & I_{p-m} \end{pmatrix} = \begin{pmatrix} 0 & \overline{R}_{12} \\ 0 & \overline{R}_{22} \end{pmatrix},$$

$$J_Q\,\overline{R} = \begin{pmatrix} 0 & 0 \\ 0 & I_{q-l} \end{pmatrix} \begin{pmatrix} \overline{R}_{11} & \overline{R}_{12} \\ \overline{R}_{21} & \overline{R}_{22} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \overline{R}_{21} & \overline{R}_{22} \end{pmatrix}.$$

Therefore, (60) implies that $\overline{R}_{12} = 0$ and $\overline{R}_{21} = 0$, which proves the result.

■

We are now in position to state the last main result of the paper (the second fairy's theorem).

Theorem 4: Let $R \in D^{q\times p}$ and $M = D^{1\times p}/(D^{1\times q}\,R)$. Let $f : M \longrightarrow M$ be a projector defined by two matrices $P \in D^{p\times p}$ and $Q \in D^{q\times q}$ satisfying (11) and let us assume that:

$$P^2 = P, \quad Q^2 = Q.$$

If the left D-modules $\ker_D(.P)$, $\mathrm{im}_D(.P)$, $\ker_D(.Q)$, $\mathrm{im}_D(.Q)$ are free of rank respectively m, $p-m$, l and $q-l$ (for some $1 \le m \le p$ and $1 \le l \le q$), then the following results hold:

1) There exist $U \in \mathrm{GL}_p(D)$ and $V \in \mathrm{GL}_q(D)$ satisfying the relations

$$\left\{ \begin{array}{l} P = U^{-1}\,J_P\,U, \\ Q = V^{-1}\,J_Q\,V, \end{array} \right.$$

where J_P and J_Q are the matrices defined by (59). In particular, the matrices U and V are defined by

$$\left\{ \begin{array}{lll} U = \begin{pmatrix} U_1 \\ U_2 \end{pmatrix}, & U_1 \in D^{m\times p}, & U_2 \in D^{(p-m)\times p}, \\ V = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix}, & V_1 \in D^{l\times q}, & V_2 \in D^{(q-l)\times q}, \end{array} \right.$$

where the matrices U_1, U_2, V_1 and V_2 respectively define the bases of the corresponding free left D-modules, i.e., we have:

$$\left\{\begin{array}{l} \ker_D(.P) = D^{1\times m}\, U_1, \\ \operatorname{im}_D(.P) = D^{1\times (p-m)}\, U_2, \\ \ker_D(.Q) = D^{1\times l}\, V_1, \\ \operatorname{im}_D(.Q) = D^{1\times (q-l)}\, V_2. \end{array}\right.$$

2) The matrix R is equivalent to $\overline{R} = V\,R\,U^{-1}$.

3) If we denote by

$$U^{-1} = (W_1 \quad W_2),\ W_1 \in D^{p\times m},\ W_2 \in D^{p\times (p-m)},$$

we then have:

$$\overline{R} = \left(\begin{array}{cc} V_1\,R\,W_1 & 0 \\ 0 & V_2\,R\,W_2 \end{array}\right) \in D^{q\times p}. \quad (62)$$

Proof: 1. The result directly follows from 2) of Proposition 10.

2. Using the fact that the matrices U and V are unimodular, we obtain $R = V^{-1}\,\overline{R}\,U$, which proves the result.

3. From Lemma 3, the matrix $\overline{R} = V\,R\,U^{-1}$ satisfies the relation (60). Then, applying Lemma 10 to $\overline{R}$, we obtain that $\overline{R}$ has the block diagonal form (61), where $\overline{R}_1 \in D^{l\times m}$ and $\overline{R}_2 \in D^{(q-l)\times (p-m)}$. Finally, we have

$$\overline{R} = V\,R\,U^{-1} = \left(\begin{array}{cc} V_1\,R\,W_1 & V_1\,R\,W_2 \\ V_2\,R\,W_1 & V_2\,R\,W_2 \end{array}\right) \in D^{q\times p},$$

where $V_1\,R\,W_1 \in D^{l\times m}$, $V_2\,R\,W_1 \in D^{(p-l)\times m}$ and $V_1\,R\,W_2 \in D^{l\times (p-m)}$, $V_2\,R\,W_2 \in D^{(p-l)\times (p-m)}$, which proves the result.

■

Example 15: Let us consider again system (6) defined in Example 2. We can easily check that the matrices

$$P = \frac{1}{2}\left(\begin{array}{ccc} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{array}\right), \quad Q = \frac{1}{2}\left(\begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array}\right),$$

define a projector $f \in \operatorname{end}_D(M)$ and satisfy $P^2 = P$ and $Q^2 = Q$. As P and Q are two matrices with rational coefficients, we obtain that $\ker_D(.P)$, $\operatorname{im}_D(.P)$, $\ker_D(.Q)$ and $\operatorname{im}_D(.Q)$ are trivially free D-modules since we have

$$\left\{\begin{array}{l} \ker_D(.P) = D \otimes_D \ker_{\mathbb{Q}}(.P), \\ \operatorname{im}_D(.P) = D \otimes_D \operatorname{im}_{\mathbb{Q}}(.P), \end{array}\right.$$

and similarly with $\ker_D(.Q)$ and $\operatorname{im}_D(.Q)$. We get

$$\left\{\begin{array}{l} U_1 = \ker_{\mathbb{Q}}(.P) = \left(\begin{array}{ccc} 1 & -1 & 0 \end{array}\right), \\ U_2 = \operatorname{im}_{\mathbb{Q}}(.P) = \left(\begin{array}{ccc} 1 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right), \\ V_1 = \ker_{\mathbb{Q}}(.Q) = \left(\begin{array}{cc} 1 & -1 \end{array}\right), \\ V_2 = \operatorname{im}_{\mathbb{Q}}(.Q) = \left(\begin{array}{cc} 1 & 1 \end{array}\right), \end{array}\right.$$

and thus, we obtain the following unimodular matrices:

$$U = \left(\begin{array}{ccc} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right), \quad V = \left(\begin{array}{cc} 1 & -1 \\ 1 & 1 \end{array}\right).$$

We finally verify that:

$$\overline{R} = V\,R\,U^{-1} = \left(\begin{array}{ccc} \partial_2^2 - 1 & 0 & 0 \\ 0 & 1 + \partial_2^2 & -4\,\partial_1\,\partial_2 \end{array}\right).$$

We note that the first scalar diagonal block corresponds to the autonomous (uncontrollable) subsystem

$$\left\{\begin{array}{l} z_1(t) = y_1(t) - y_2(t), \\ z_1(t - 2\,h) - z_1(t) = 0, \end{array}\right.$$

i.e., z_1 is a $2\,h$-periodic function, whereas the second diagonal block corresponds to the controllable subsystem

$$\left\{\begin{array}{l} z_2(t) = y_1(t) + y_2(t), \\ v(t) = u(t), \\ z_2(t) + z_2(t - 2\,h) - 4\,\dot{v}(t - h) = 0, \end{array}\right.$$

of the system $R\,(y_1, y_2, u)^T = 0$. Finally, the previous decomposition can be seen as a generalization of the classical Kalman decomposition of state-space control systems for multidimensional linear systems.

We have the following important corollary of Theorem 4.

Corollary 6: Let us consider $R \in D^{q\times p}$ and the left D-module $M = D^{1\times p}/(D^{1\times q}\,R)$. Let $f : M \longrightarrow M$ be a projector defined by two matrices $P \in D^{p\times p}$ and $Q \in D^{q\times q}$ satisfying (11) and let us suppose that:

$$P^2 = P, \quad Q^2 = Q.$$

Assume further that one of the following condition holds:

1) $D = A[\partial; \sigma, \delta]$ is a skew polynomial ring over a division ring A (e.g., A is a field) and σ is injective, as, e.g., the ring $D = k(t)\left[\partial; \operatorname{id}_{k(t)}, \frac{d}{dt}\right]$ of differential operators with rational coefficients or the ring $D = k(n)[\partial; \sigma, 0]$ of shift operators with rational coefficients ($\sigma(a)(n) = a(n+1)$),

2) $D = k[\partial_1; \sigma_1, \delta_1] \ldots [\partial_n; \sigma_n, \delta_n]$ is a commutative Ore algebra where k is a field as, e.g., the ring of differential operators with constant coefficients,

3) $D = A[\partial_1; \operatorname{id}, \delta_1] \ldots [\partial_n; \operatorname{id}, \delta_n]$ is a Weyl algebra ($\forall\ a \in A$, $\delta_i(a) = \partial a/\partial x_i$, $1 \leq i \leq n$), where

$A = k[x_1, \ldots, x_n]$ or $k(x_1, \ldots, x_n)$ and k is a field of characteristic 0, and:

$$\begin{cases} \mathrm{rank}_D(\ker_D(.P)) \geq 2, \\ \mathrm{rank}_D(\mathrm{im}_D\,(.P)) \geq 2, \\ \mathrm{rank}_D(\ker_D(.Q)) \geq 2, \\ \mathrm{rank}_D(\mathrm{im}_D\,(.Q)) \geq 2. \end{cases}$$

Then, there exist $U \in \mathrm{GL}_p(D)$ and $V \in \mathrm{GL}_q(D)$ such that $\overline{R} = V\,R\,U^{-1}$ is a block diagonal matrix of the form

$$\overline{R} = \begin{pmatrix} \overline{R}_1 & 0 \\ 0 & \overline{R}_2 \end{pmatrix} \in D^{q \times p},$$

where $\overline{R}_1 \in D^{l \times m}$, $\overline{R}_2 \in D^{(p-l)\times(p-m)}$ and:

$$m = \mathrm{rank}_D(\ker_D(.P)), \quad l = \mathrm{rank}_D(\ker_D(.Q)).$$

Proof: 1. By Lemma 9, we know that $\ker_D(.P)$, $\ker_D(.Q)$, $\mathrm{im}_D\,(.P)$ and $\mathrm{im}_D\,(.Q)$ are projective D-modules. By ii) of Theorem 1.2.9 of [37], D is a left principal ideal domain. Therefore, $\ker_D(.P)$, $\ker_D(.Q)$, $\mathrm{im}_D\,(.P)$ and $\mathrm{im}_D\,(.Q)$ are free left D-modules of rank respectively m, l, $p-m$ and $q-l$ (see [14], [37], [56]). The result directly follows from Theorem 4.

2. By Lemma 9, we obtain that $\ker_D(.P)$, $\ker_D(.Q)$, $\mathrm{im}_D\,(.P)$ and $\mathrm{im}_D\,(.Q)$ are projective D-modules. As D is a commutative polynomial ring over a field k, by the famous Quillen-Suslin theorem, we know that they are free D-modules of rank respectively m, l, $p-m$ and $q-l$. See [27], [56] for more details. Then, the result directly follows from Theorem 4.

3. By Lemma 9, we obtain that $\ker_D(.P)$, $\ker_D(.Q)$, $\mathrm{im}_D\,(.P)$ and $\mathrm{im}_D\,(.Q)$ are projective left D-modules. A result of J. T. Stafford asserts that projective modules of rank at least 2 over a Weyl algebra with a field k of characteristic 0 are free. For more details, we refer to [54], [55], [58]. The result directly follows from Theorem 4. ■

Remark 10: In order to constructively obtain the unimodular matrices U and V defined in Corollary 6, we need to compute bases of the free left D-modules $\ker_D(.P)$ and $\mathrm{im}_D\,(.P)$, $\ker_D(.Q)$ and $\mathrm{im}_D\,(.Q)$. In the first case of Corollary 6, we can use Smith or Jacobson forms in order to compute bases of these modules over $D = A[\partial; \sigma, \delta]$ (see [37], [42]). In the second case of Corollary 6, we can use constructive versions of the famous Quillen-Suslin theorem of Serre's conjecture ([56]). For more details, we refer to [35] and references therein. See also [27] for an implementation. In the last case of Corollary 6, we can use the constructive algorithm recently obtained in [54], [55] and its implementation developed in the package STAFFORD of OREMODULES available in [13].

Remark 11: Let $D = A[\partial; \sigma, \delta]$ be a skew polynomial ring over a ring A, $E \in A^{p \times p}$, $R = (\partial\,I_p - E) \in D^{p \times p}$ and $M = D^{1 \times p}/(D^{1 \times p}\,R)$ the left D-module associated with the linear functional system $\partial\,y = E\,y$. In Example 4, we proved that we can always suppose with any restriction that $f \in \mathrm{end}_D(M)$ is defined by $P \in A^{p \times p}$ and $Q \in A^{q \times q}$ satisfying (16) where $F = E$. By Lemma 5, we obtain that any projector f of $\mathrm{end}_D(M)$ is defined by a matrix $P \in A^{p \times p}$ satisfying $P^2 = P + Z\,R$, where $Z \in D^{p \times q}$. Using the fact that R is a first order matrix and P is a zero order matrix, we obtain that $Z = 0$, i.e., $P^2 = P$. Now, the fact that R has full row rank, i.e., $\ker_D(.R) = 0$, by Lemma 7, we obtain that $Q^2 = Q$. Hence, if A is division ring and σ is injective, then the hypotheses of 1) of Corollary 6 are satisfied, and thus, there exist $U \in \mathrm{GL}_p(D)$ and $V \in \mathrm{GL}_q(D)$ such that the matrix $\overline{R} = V\,R\,U^{-1}$ is block diagonal. We can then consider again each of the blocks separately. If A is a field, then the matrices U and V can easily be obtained by doing linear algebra as we have $\ker_A(.P) = (A^{m \times p}\,U_1)$, $\mathrm{im}_A(.P) = (A^{(p-m)\times p}\,U_2)$, $\ker_A(.Q) = (A^{l \times q}\,V_1)$, $\mathrm{im}_A(.P) = (A^{(q-l)\times q}\,V_2)$ and $U = (U_1^T \quad U_2^T)^T$, $V = (V_1^T \quad V_2^T)^T$.

Example 16: Let us consider again Example 13, i.e., let us consider the Weyl algebra $D = A_1(\mathbb{Q})$, the matrix $R \in D^{4 \times 4}$ of differential operator defined by (50) and the left D-module $M = D^{1 \times 4}/(D^{1 \times 4}\,R)$. Using the algorithm for computing projectors of $\mathrm{end}_D(M)$, we obtain that the matrix $P = Q$ defined by (51) generates a projector f, which proves that M is decomposable. Moreover, we easily check that $P^2 = P$, i.e., P is an idempotent of $D^{3 \times 3}$. Now, using the fact that the entries of P belong to the field k, we can easily compute bases of $\ker_k(.P)$ and $\mathrm{im}_k(.P) = \ker_k(.(I_4 - P))$. This way, we obtain that the following unimodular matrices (see Theorem 4):

$$U = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad V = U.$$

We then obtain that R is equivalent to the following block diagonal matrix:

$$\overline{R} = V\,R\,U^{-1} = \begin{pmatrix} \partial & -1 & 0 & 0 \\ t & \partial & 0 & 0 \\ 0 & 0 & \partial & -t \\ 0 & 0 & 0 & \partial \end{pmatrix}.$$

The diagonal blocks of the matrix $\overline{R}$ are equivalent to the two systems that we had to solve in Example 13 in order to integrate the solutions of $R\,\eta = 0$. Hence, we find again that the general solution of $R\,\eta = 0$ is given by (52).

We note that $\mathrm{rank}_D(\ker_D(.P)) = 2$ and $\mathrm{rank}_D(\mathrm{im}_D(.P)) = 2$. Hence, if we had some other idempotents P' and Q' over D, then we could have used the constructive algorithm for the computation of bases over D developed in [54], [55] in order to compute the corresponding decomposition of M.

Example 17: If we consider the idempotent $\overline{P}_3 \in D^{2\times 2}$ defined in Example 14, where $D = A_1(\mathbb{Q})$, we have $\mathrm{rank}_D(\ker_D(.\overline{P}_3)) = 1$ and $\mathrm{rank}_D(\mathrm{im}_D(.\overline{P}_3)) = 1$. Hence, we cannot use Corollary 6 in order to conclude that $R = (\partial^2 \quad -t\,\partial - 1)$ is equivalent to $\overline{R} = (\alpha \quad 0)$, $\alpha \in D$, by means of unimodular matrices over D. Indeed, we easily prove that $\ker_D(.\overline{P}_3)) = D\,(\partial \quad -t)$, which implies that $\ker_D(.\overline{P}_3))$ is a free left D-module of rank 1. However, we have $\mathrm{im}_D(.\overline{P}_3) \cong D^{1\times 2}/(D\,(\partial \quad -t))$ and it was proved in [54] that the last left D-module was not free. A similar comment holds for $\overline{P}_4$ as we have $\ker_D(.\overline{P}_4) \cong D^{1\times 2}/(D\,(\partial \quad -t))$. Of course, if we consider the Weyl algebra $B_1(\mathbb{Q})$ instead of D, namely, $B_1(\mathbb{Q}) = \mathbb{Q}(t)\left[\partial; \mathrm{id}, \frac{d}{dt}\right]$, using a computation of a Jacobson form, we can easily prove that R is equivalent to the matrix $\overline{R} = (\partial \quad 0)$. However, we point out that some singularities appear in the matrices U and V defined in Theorem 4.

Example 18: Let us consider the differential time-delay model of a flexible rod with a torque developed in [39]:

$$\begin{cases} \dot{y}_1(t) - \dot{y}_2(t-1) - u(t) = 0, \\ 2\,\dot{y}_1(t-1) - \dot{y}_2(t) - \dot{y}_2(t-2) = 0. \end{cases} \tag{63}$$

Let us define the Ore algebra $D = \mathbb{Q}\left[\partial_1; 1, \frac{d}{dt}\right][\partial_2; \sigma_2, 0]$ of differential time-delay operators with rational constant coefficients defined in 4) of Example 1 and the corresponding matrix of the system (63) defined by:

$$R = \begin{pmatrix} \partial_1 & -\partial_1\,\partial_2 & -1 \\ 2\,\partial_1\,\partial_2 & -\partial_1\,\partial_2^2 - \partial_1 & 0 \end{pmatrix}.$$

Let $M = D^{1\times 3}/(D^{1\times 2}\,R)$ be the left D-module associated with (63). Using Algorithm 3, we obtain that the following matrices

$$P = \begin{pmatrix} 1+\partial_2^2 & -\frac{1}{2}\,\partial_2^2\,(1+\partial_2) & 0 \\ 2\,\partial_2 & -\partial_2^2 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$Q = \begin{pmatrix} 1 & -\frac{1}{2}\,\partial_2 \\ 0 & 0 \end{pmatrix},$$

define a projector $f \in \mathrm{end}_D(M)$. Moreover, we can check that $P^2 = P$ and $Q^2 = Q$, i.e., P and Q are idempotents. Then, using 2) of Corollary 6, we obtain that R is equivalent to a block diagonal matrix. Let us compute it. Using the implementation of the Quillen-Suslin theorem developed in [27] or the heuristics given in [14], we obtain the following unimodular matrices:

$$U = \begin{pmatrix} -2\,\partial_2 & \partial_2^2+1 & 0 \\ -2 & \partial_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad V = \begin{pmatrix} 0 & -1 \\ 2 & -\partial_2 \end{pmatrix}.$$

Using the fact that the inverse of U is then defined by

$$U^{-1} = \begin{pmatrix} -\frac{1}{2}\,\partial_2 & -\frac{1}{2}\,(\partial_2^2+1) & 0 \\ 1 & -\partial_2 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

we finally obtain:

$$\overline{R} = V\,R\,U^{-1} = \begin{pmatrix} \partial_1 & 0 & 0 \\ 0 & \partial_1\,(\partial_2^2 - 1) & -2 \end{pmatrix}.$$

As in Example 15 for the tank model, we obtain that the first scalar diagonal block corresponds to the autonomous (uncontrollable) subsystem, whereas the second diagonal block defines the controllable subsystem.

More examples of decomposable modules coming from mathematical physics can be given. For instance, we refer the interested reader to [53] for examples of PDEs such as $R = \mathrm{grad} \circ \mathrm{div}$ or $R = (\Delta^2 \quad -\Delta)$, $\Delta = \partial_1^2 + \partial_2^3$, which appear in linear elasticity.

V. Conclusion and implementation

Within a constructive homogical algebra approach developed in this paper, we have obtained new and general results on the factorization and decomposition problems of linear systems over Ore algebras. We point out that no particular assumption on the linear functional systems was required. Hence, the different results of the paper can be applied to underdetermined or overdetermined as well as D-finite ([12]) or general determined linear systems. In particular, we have shown how some classical results of the literature of the factorization and decomposition problems such as the ones using the concept of the eigenring ([3], [9], [59], [33], [22], [15]) can be seen as particular cases of Theorems 1, 2, 3 and 4.

Moreover, we have shown how our results could be applied in mathematical physics (e.g., Galois symmetries of the linearized Euler equations, quadratic first integrals of motion, quadratic conservation laws, equivalence of linear systems appearing in linear elasticity) and in control theory (controllability, autonomous elements, decoupling the autonomous and the controllable subsystems of a tank and a flexible rod). More details and applications will be developed in [20].

Finally, all the algorithms presented in the paper have been implemented in the package Morphisms ([19]) of OreModules (see [13]). This package is available on the authors' web pages as well as the ones of OreModules (see [13] for the precise address). A library of examples, including the ones of the paper, is also available and it illustrates the main results obtained in this paper and the main functions of Morphisms.

REFERENCES

[1] S. A. Abramov, M. A. Barkatou. Rational solutions of first order linear difference systems. In *Proceedings of ISSAC'98*, ACM Press, 1998.

[2] S. A. Abramov, M. Bronstein. On solutions of linear functional systems. In *Proceedings of ISSAC'01*, 1-6, ACM Press, 2001.

[3] M. A. Barkatou. On the reduction of matrix pseudo-linear equations. *Technical Report RR 1040*, Rapport de Recherche de l'Institut IMAG, 2001.

[4] M. A. Barkatou. On rational solutions of systems of linear differential equations. In *J. Symbolic computation*, 28: 547-567, 1999.

[5] M. A. Barkatou. Rational solutions of matrix difference equations: the problem of equivalence and factorization. In *Proceedings of ISSAC'90*, ACM Press, 1999.

[6] M. A. Barkatou, T. Cluzeau, J.-A. Weil. Factoring partial differential systems in positive characteristic. In *Differential Equations with Symbolic Computation (DESC Book)*, Editor D. Wang, Birkhaüser, 2005.

[7] M. A. Barkatou, E. Pflügel. On the equivalence problem of linear differential systems and its application for factoring completely reducible systems. In *Proceedings of ISSAC'98*, 268-275, ACM Press, 1998.

[8] E. Beke. Die irreduzibilität der homogenen linearen differentialgleichungen. In *Math. Ann.*, 45: 278-294, 1894.

[9] R. Bomboy. Réductibilité et résolubilité des équations aux différences finies. *PhD thesis*, University of Nice-Sophia Antipolis (France), 2001.

[10] N. Bourbaki. Éléments de mathématique, Algèbre, Chapitre 7: Modules sur les anneaux principaux. Hermann, 1964.

[11] M. Bronstein. An improved algorithm for factoring linear ordinary differential operators. In *Proceedings of ISSAC'94*, 336-340, ACM Press, 1994.

[12] F. Chyzak, B. Salvy. Non-commutative elimination in Ore algebras proves multivariate identities. In *J. Symbolic Computation*, 26: 187-227, `http://algo.inria.fr/chyzak/mgfun.html`, 1998.

[13] F. Chyzak, A. Quadrat, D. Robertz. OREMODULES project, `http://wwwb.math.rwth-aachen.de/OreModules`.

[14] F. Chyzak, A. Quadrat, D. Robertz. Effective algorithms for parametrizing linear control systems over Ore algebras. In *Appl. Algebra Engrg. Comm. Comput.*, 16: 319-376, 2005.

[15] T. Cluzeau. Algorithmique modulaire des équations différentielles linéaires. *PhD thesis*, University of Limoges (France), 2004.

[16] T. Cluzeau. Factorization of differential systems in characteristic p. In *Proceedings of ISSAC'03*, 58-65, ACM Press, 2003.

[17] T. Cluzeau, M. van Hoeij. A modular algorithm to compute the exponential solutions of a linear differential operator. In *J. Symbolic Computation*, 38(3): 1043-1076, 2004.

[18] T. Cluzeau, M. van Hoeij. Computing hypergeometric solutions of linear difference equations. To appear in *Appl. Algebra Engrg. Comm. Comput.*, 2006.

[19] T. Cluzeau, A. Quadrat. MORPHISMS project.

[20] T. Cluzeau, A. Quadrat. Morphisms, equivalences and symmetries of linear functional systems. In preparation, 2006.

[21] M. Giesbrecht. Factoring in skew-polynomial rings over finite fields. In *J. Symbolic Computation*, 24(5): 463-486, 1998.

[22] M. Giesbrecht, Y. Zhang. Factoring and decomposing Ore polynomials over $\mathbb{F}_p(t)$. In *Proceedings of ISSAC'03*, 127-134, ACM Press, 2003.

[23] D. Y. Grigoriev. Complexity of irreducibility testing for a system of linear ordinary differential equations. In *Proceedings of ISSAC'90*, 225-230, ACM Press, 1990.

[24] M. van Hoeij. Factorization of differential operators with rational functions coefficients. *J. Symbolic Computation*, 24:537-561, 1997.

[25] N. Jacobson. Pseudo-linear transformations. In *Annals of Mathematics*, 38(2): 484-507, 1937.

[26] F. Dubois, N. Petit, P. Rouchon. Motion planning and nonlinear simulations for a tank containing a fluid. In *Proceedings of the* 5^{th} *European Control Conference*, Karlsruhe (Germany), 1999.

[27] A. Fabiańska, A. Quadrat. Flat multidimensional linear systems with constant coefficients are algebraically equivalent to controllable 1-D linear systems. To appear in *Proceedings of MTNS 2006*, Kyoto (Japan) (24-28/07/06).

[28] M. Kashiwara. Algebraic study of systems of partial differential equations. *PhD Thesis*, Tokyo Univ. 1970, Mémoires de la Société Mathématiques de France 63 (1995) (english translation).

[29] T. Y. Lam. Lectures on modules and rings. *Graduate Texts in Mathematics*, 189, Springer, 1999.

[30] L. Landau, L. Lifschitz. *Physique théorique*, Tome 1: Mécanique 4^{th} edition, MIR.

[31] N. N. Lebedev. *Special Functions & Their Applications*. Dower, 1972.

[32] V. Levandovskyy, H. Schönemann. Plural - a computer algebra system for non-commutative polynomial algebras. In *Proceedings of ISSAC'03*,176-183,ACM Press, 2003.

[33] D. Grigoriev, F. Schwarz. Generalized Loewy-decomposition of D-modules. In *Proceedings of ISSAC'05*, ACM Press, 2005.

[34] Z. Li, F. Schwarz, S. Tsarëv. Factoring systems of PDE's with finite-dimensional solution space. In *J. Symbolic Computation*, 36: 443-471, 2003.

[35] A. Logar, B. Sturmfels. Algorithms for the Quillen-Suslin Theorem. *Journal of Algebra*, 145: 231-239, 1992.

[36] C. C. Lin, L. A. Segel. Mathematics applied to deterministic problems in the natural sciences. SIAM, Vol. I, 1988.

[37] J. C. McConnell, J. C. Robson. Noncommutative Noetherian rings. American Mathematical Society, 2000.

[38] B. Malgrange. Ideals of differential functions. Oxford University Press, 1966.

[39] H. Mounier, J. Rudolph, M. Petitot, M. Fliess. A flexible rod as a linear delay system. In *Proceedings of the* 3^{rd} *European Control Conference*, Roma (Italy), 1995.

[40] U. Oberst. Multidimensional constant linear systems. In *Acta Appl. Math.*, 20: 1-175, 1990.

[41] H. K. Pillai, S. Shankar, "A behavioural approach to control of distributed systems", *SIAM Journal on Control and Optimization*, 37 (1999), 388-408.

[42] J. W. Polderman, J. C. Willems. Introduction to mathematical systems theory. A behavioral approach. TAM 26, Springer, 1998.

[43] J.-F. Pommaret. *Partial Differential Control Theory* Kluwer, 2001.

[44] J.-F. Pommaret, A. Quadrat. Generalized Bezout identity. In *Appl. Algebra Engrg. Comm. Comput.*, 9: 91-116, 1998.

[45] J.-F. Pommaret, A. Quadrat. Localization and parametrization of linear multidimensional control systems. In *Systems Control Lett.* 37: 247-260, 1999.

[46] J.-F. Pommaret, A. Quadrat. Algebraic analysis of linear multidimensional control systems. In *IMA J. Control and Optimization*, 16: 275-297, 1999.

[47] J.-F. Pommaret, A. Quadrat. Equivalences of linear control systems. In *Proceedings of the* 14^{th} *Mathematical Theory of Networks and Systems*, Perpignan (France), 2000.

[48] J.- F. Pommaret, A. Quadrat. A functorial approach to the behaviour of multidimensional control systems. In *Int. J. Appl. Math. Comput. Sci.*, 13: 7-13, 2003.

[49] M. van der Put. Differential equations in characteristic p. *Compositio Mathematica*, 97:227-251, 1995.

[50] M. van der Put, M. F. Singer. Galois theory of linear differential equations. In *Grundlehren der mathematischen Wissenschaften*, 328, Springer, 2003.

[51] A. Quadrat, "Extended Bézout identities", *Proceedings of ECC*, Porto (Portugal), 2001.

[52] A. Quadrat, D. Robertz. Parametrizing all solutions of uncontrollable multidimensional linear systems. In *Proceedings of the* 16^{th} *IFAC World Congress*, Prague (Czech Republic), 2005.

[53] A. Quadrat, D. Robertz. On the Monge problem and multidimensional optimal control. To appear in the *Proceedings of MTNS 2006*, Kyoto (Japan) (24-28/07/06).

[54] A. Quadrat, D. Robertz. Constructive computation of bases of free modules over the Weyl algebras. In *INRIA report RR-5786* (`http://www.inria.fr/rrrt/rr-5786.html`), submitted for publication, 2005.

[55] A. Quadrat, D. Robertz. Constructive computation of flat outputs of a class of multidimensional linear systems with variable coefficients. To appear in the *Proceedings of MTNS 2006*, Kyoto (Japan) (24-28/07/06).

[56] J. J. Rotman. An introduction to homological algebra. Academic Press, 1979.

[57] F. Schwarz. A factorization algorihm for linear ordinary differential operators. In *Proceedings of ISSAC'89*, 17-25, ACM Press, 1989.

[58] J. T. Stafford. Module structure of Weyl algebras. In *Journal of London Mathematical Society* 18: 429-442, 1978.

[59] M. F. Singer. Testing reducibility of linear differential operators: a group theoretic perspective. In *Appl. Algebra Engrg. Comm. Comput.*, 7: 77-104, 1996.

[60] S. P. Tsarëv. Some problems that arise in the factorization of linear ordinary differential operators. In *Programmirovanie*, 1: 45-48, 1994. Translation in *Programming and Comput. Software*, 20(1): 27-29, 1994.

[61] J. Wood. Modules and behaviours in nD systems theory. In *Multidimens. Systems Signal Process.*, 11: 11-48, 2000.

[62] E. Zerz. Topics in multidimensional linear systems theory. In *Lecture Notes in Control and Information Sciences*, 256, Springer, 2000.

[63] E. Zerz. An algebraic analysis approach to linear time-varying systems. To appear in *IMA J. Math. Control Inform.*, 2006.

[64] M. Wu. On the factorization of differential modules. In *Differential Equations with Symbolic Computation (DESC Book)*, Editor D. Wang, Birkhaüser, 2005.

[65] M. Wu. On solutions of linear functional systems and factorization of modules over Laurent-Ore algebras. *PhD Thesis*, University of Nice-Sophia Antipolis (France), 2005