On the Effective Computation of Stabilizing Controllers of 2D Systems

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Abstract. In this paper, we show how stabilizing controllers for 2D systems can effectively be computed based on computer algebra methods dedicated to polynomial systems, module theory and homological algebra. The complete chain of algorithms for the computation of stabilizing controllers, implemented in `Maple`, is illustrated with an explicit example.

Keywords: Multidimensional systems theory · 2D systems · Stability analysis · Stabilization · Computation of stabilizing controllers · Polynomial systems · Module theory · Homological algebra

1 Introduction

In the eighties, *the fractional representation approach to analysis and synthesis problems* was introduced by Vidyasagar, Desoer, etc., to unify different problems studied in the control theory community (e.g., internal/strong/simultaneous/optimal/robust stabilizability) within a unique mathematical framework [12,27]. Within this approach, different classes of linear systems (e.g., discrete, continuous, finite-dimensional systems, infinite-dimensional systems, multidimensional systems) can be studied by means of a common mathematical formulation.

The main idea of this approach is to reformulate the concept of *stability* – central in control theory – as a *membership problem*. More precisely, a *single-input single-output (SISO) linear system*, also called *plant*, is defined as an element p of the *quotient field* (*field of fractions*) $Q(A) = \left\{ \frac{n}{d} \mid 0 \neq d,\, n \in A \right\}$ of an integral domain A of SISO stable plants [27]. Hence, if $p \in A$, then p is A-*stable* (simply *stable* when the reference to A is clear) and unstable if $p \in Q(A) \setminus A$. More generally, a *multi-input multi-output (MIMO)* plant can be defined by a matrix $P \in Q(A)^{q \times r}$. Hence, it is stable if $P \in A^{q \times r}$, unstable otherwise.

Different integral domains A of SISO stable plants are considered in the control theory literature depending on the class of systems which is studied. For

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instance, the *Hardy (Banach) algebra* $H^\infty(\mathbb{C}_+)$ formed by all the holomorphic functions in the open-right half plane $\mathbb{C}_+ = \{s \in \mathbb{C} \mid \Re(s) > 0\}$ which are bounded for the norm $\|f\|_\infty = \sup_{s\in\mathbb{C}_+} |f(s)|$ plays a fundamental role in stabilization problems of infinite-dimensional linear time-invariant systems (e.g., differential time-delay systems, partial differential systems) since its elements can be interpreted as the *Laplace transform* of $L^2(\mathbb{R}_+) - L^2(\mathbb{R}_+)$-stable plant (i.e., any input u of the system in $L^2(\mathbb{R}_+)$ yields an output y in $L^2(\mathbb{R}_+)$) [10]. Similarly, the integral domain RH_∞ of proper and stable rational functions, i.e., the ring of all rational functions in $H^\infty(\mathbb{C}_+)$, corresponds to the ring of *exponentially stable* finite-dimensional linear time-invariant systems (i.e., exponentially stable ordinary differential systems with constant coefficients) [27].

In this paper, we shall focus on the class of *discrete multidimensional systems* which are defined by multivariate recurrence relations with constant coefficients or, using the standard $\mathcal{Z}$*-transform*, by elements of the field $\mathbb{R}(z_1, \ldots, z_n)$ of real rational functions in $z_1, \ldots, z_n$. The latter is the field of fractions $Q(A)$ of the integral domain A of *SISO structurally stable plants* defined by

$$\mathbb{R}(z_1, \ldots, z_n)_S := \left\{ \frac{n}{d} \mid 0 \neq d,\, n \in B,\ \gcd(d, n) = 1,\ V(\langle d\rangle) \cap \mathbb{U}^n = \emptyset \right\},$$

where $B := \mathbb{R}[z_1, \ldots, z_n]$ denotes the polynomial ring in $z_1, \ldots, z_n$ with coefficients in $\mathbb{R}$, $\gcd(d, n)$ the *greatest common divisor* of d, $n \in B$, and

$$V(\langle d\rangle) = \{z = (z_1, \ldots, z_n) \in \mathbb{C}^n \mid d(z) = 0\}$$

the *affine algebraic set* defined by $d \in B$, i.e., the complex zeros of d, and finally

$$\mathbb{U}^n = \{z = (z_1, \ldots, z_n) \in \mathbb{C}^n \mid |z_i| \leq 1,\ i = 1, \ldots, n\}$$

the *closed unit polydisc* of $\mathbb{C}^n$. It can be shown that $p \in \mathbb{R}(z_1, \ldots, z_n)_S$ implies that the plant p is *bounded-input bounded-output stable* in the sense that an input u in $l^\infty(\mathbb{Z}^n_+)$ yields an output y in $l^\infty(\mathbb{Z}^n_+)$. See, e.g., [15,16].

Despite the simplicity of the main idea of the fractional representation approach, i.e., to express stability as a membership problem, many problems studied in control theory were reformulated as algebraic (analysis) problems. For instance, internal/strong/simultaneous/optimal/robust stabilizability can be reformulated within this mathematical approach and solved for particular integral domains A such as RH_∞ [27]. But these problems are still open for the class of infinite-dimensional systems [10] and multidimensional systems [15,16].

The goal of this article is to combine results obtained in [2,4–6,20,21] to obtain a complete algorithmic approach to the computation of stabilizing controllers for 2D stabilizable MIMO systems. In [5], the problem was solved for SISO systems. To handle the class of MIMO systems, we use the module-theoretic approach to the fractional representation approach [20,21]. More precisely, in [6], the main steps towards an algorithmic computation of stabilizing controllers for general nD systems are explained based on computer algebra methods. In this paper, we focus on 2D MIMO systems for which the so-called *Polydisk Nullstellensatz* [7] has received an effective version in [5] (which is not the case for general nD systems), which yields a complete algorithmic approach to the computation of stabilizing controllers for 2D stabilizable MIMO systems.

Our algorithms were implemented in the computer algebra system `Maple`, based on both the package `nDStab` – dedicated to stability and stabilizability of nD systems – and on the OreModules package [9] which aims to study linear systems theory based on effective module theory and homological algebra.

2 The Fractional Representation Approach

In what follows, we shall use the following notations. A will denote an integral domain of SISO plants, $K := Q(A) = \left\{ \frac{n}{d} \mid 0 \neq d,\, n \in A \right\}$ its quotient field, $P \in K^{q \times r}$ a plant, $C \in K^{r \times q}$ a controller, and $p = q + r$.

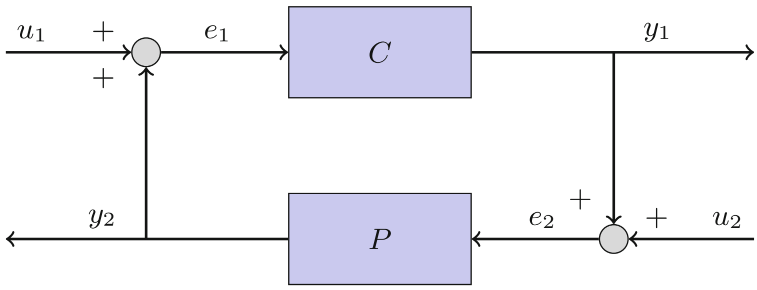

Fig. 1. Closed-loop system

With the notations of Fig. 1 defining the closed-loop system formed by the plant P and the controller C, we get $(e_1^T \quad e_2^T)^T = H(P, C)\,(u_1^T \quad u_2^T)^T$, where the transfer matrix $H(P, C) \in K^{p \times p}$ is defined by:

$$H(P,\, C) := \begin{pmatrix} I_q & -P \\ -C & I_r \end{pmatrix}^{-1} = \begin{pmatrix} (I_q - P\,C)^{-1} & (I_q - P\,C)^{-1}\,P \\ C\,(I_q - P\,C)^{-1} & I_r + C\,(I_q - P\,C)^{-1}\,P \end{pmatrix}$$
$$= \begin{pmatrix} I_q + P\,(I_r - C\,P)^{-1}\,C & P\,(I_r - C\,P)^{-1} \\ (I_r - C\,P)^{-1}\,C & (I_r - C\,P)^{-1} \end{pmatrix}.$$

Definition 1 ([27]). *A plant $P \in K^{q \times r}$ is* internally stabilizable *if there exists $C \in K^{r \times q}$ such that $H(P, C) \in A^{p \times p}$. Then, C is a* stabilizing controller *of P.*

If $H(P, C) \in A^{p \times p}$, then we can easily prove that all the entries of any transfer matrix between two signals appearing in Fig. 1 are A-stable (see, e.g., [27]).

A fundamental issue in control theory is to first test if a given plant P is internally stabilizable and if so, to explicitly compute a stabilizing controller of P, and by extension the family $\mathrm{Stab}(P)$ of all its stabilizing controllers.

To do that and to study other stabilization problems such as robust control, the fractional approach to systems was introduced in control theory [12, 27].

Definition 2. *A* fractional representation *of* $P \in K^{q\times p}$ *is defined by* $P = D^{-1}\,N = \widetilde{N}\,\widetilde{D}^{-1}$, *where* $R := (D \quad -N) \in A^{q\times p}$ *and* $\widetilde{R} = (\widetilde{N}^T \quad \widetilde{D}^T)^T \in A^{p\times r}$.

A plant $P \in K^{q\times r}$ always admits a fractional representation since we can always consider $D = d\,I_q$, $\widetilde{D} = d\,I_r$, where d is the product of all the denominators of the entries of P, which yields $N = D\,P \in A^{q\times r}$ and $\widetilde{N} = P\,\widetilde{D} \in A^{q\times r}$.

3 Testing Stability of Multidimensional Systems

3.1 Stability Tests for nD Systems

A fundamental issue in the fractional representation approach is to be able to solve the following *membership problem*: let $p \in K := Q(A)$, check whether or not $p \in A$. The answer to this problem depends on A.

In this paper, we shall focus on the case $A := \mathbb{Q}(z_1,\ldots,z_n)_S$ defined in the introduction, and mainly on 2D systems for the stabilization issue, i.e., on $A = \mathbb{Q}(z_1,z_2)_S$. Since we shall only consider exact computation methods based on computer algebra techniques, in what follows, we consider the ground field to be $\mathbb{Q}$ instead of $\mathbb{R}$. Tests for stability of multidimensional systems have largely been investigated in both the control theory and signal processing literatures. For more details, see the surveys [15,16] and the references therein.

Let $B := \mathbb{Q}[z_1,\ldots,z_n]$ be the commutative polynomial ring over $\mathbb{Q}$. We first note that $K := Q(A) = \mathbb{Q}(z_1,\ldots,z_n) = Q(B)$. Moreover, $A = \mathbb{Q}(z_1,\ldots,z_n)_S$ is the *localization* $B_S := S^{-1}\,B = \{b/s \mid b \in B,\ s \in S\}$ of the polynomial ring B with respect to the *(saturated) multiplicatively closed subset* of B defined by:

$$S := \{b \in B \mid V(\langle b\rangle) \cap \mathbb{U}^n = \emptyset\}.$$

Any element $p \in K$ can be written as $p = n/d$ with $0 \neq d,\ n \in B$. Moreover, we can always assume that the greatest common divisor $\gcd(d,n)$ is reduced to 1. Hence, given an element $p = n/d \in K$, $\gcd(d,n) = 1$, we get that $p \in A = B_S$ iff $d \in S$. The membership problem is reduced to checking whether or not $d \in S$. Let us study how this problem, i.e., the stability test, can be effectively checked.

Setting $z_k = u_k + i\,v_k$, where $u_k,\ v_k \in \mathbb{R}$, for $k = 1,\ldots,n$, and writing $b(z_1,\ldots,z_n) = c_1(u_1,\ldots,u_n,v_1,\ldots,v_n) + i\,c_2(u_1,\ldots,u_n,v_1,\ldots,v_n)$, where the c_i's are two real polynomials in the real variables u_k's and v_k's, $V(\langle b\rangle) \cap \mathbb{U}^n$ yields the following semi-algebraic set:

$$\begin{cases} c_1(u_1,\ldots,u_n,v_1,\ldots,v_n) = 0, \\ c_2(u_1,\ldots,u_n,v_1,\ldots,v_n) = 0, \\ u_k^2 + v_k^2 \leq 1,\ k = 1,\ldots,n. \end{cases} \tag{1}$$

Real algebraic methods such as CAD [1] can then be used to solve this problem for small n. But they quickly become impracticable in practice for nD systems, even for $n \geq 2$, since the number of unknowns has been doubled in (1). Hence, an algebraic formulation, more tractable for explicit computation, must be found.

The next theorem gives a mathematical characterization of $V(\langle b\rangle) \cap \mathbb{U}^n = \emptyset$.

Theorem 1 **([11]).** *Let* $b \in \mathbb{R}[z_1, \ldots, z_n]$ *and* $\mathbb{T}^n = \prod_{i=1}^n \{z_i \in \mathbb{C} \mid |z_i| = 1\}$. *Then, the following assertions are equivalent:*

1. $V(\langle b(z_1, \ldots, z_n)\rangle) \cap \mathbb{U}^n = \emptyset$.
2. $$\begin{cases} V(\langle b(z_1, 1, \ldots, 1)\rangle) \cap \mathbb{U} = \emptyset, \\ V(\langle b(1, z_2, 1, \ldots, 1)\rangle) \cap \mathbb{U} = \emptyset, \\ \qquad\vdots \\ V(\langle b(1, \ldots, 1, z_n)\rangle) \cap \mathbb{U} = \emptyset, \\ V(\langle b(z_1, \ldots, z_n)\rangle) \cap \mathbb{T}^n = \emptyset. \end{cases}$$

In [2], it is shown how 2 of Theorem 1 can be effectively tested by means of computer algebra methods. Let us shortly state again the main idea. The first n conditions of 2 of Theorem 1 can be efficiently checked by means of standard stability tests for 1D systems such as [26]. The only remaining difficulty is the last condition, which must be transformed into a more tractable algorithmic formulation. To do that, we can introduce *the Möbius transformation*

$$\begin{aligned} \varphi : \overline{\mathbb{R}}^n &\longrightarrow \mathbb{T}_\circ^n, \\ t := (t_1, \ldots, t_n) &\longmapsto z := (z_1, \ldots, z_n) = \left(\frac{t_1 - i}{t_1 + i}, \ldots, \frac{t_n - i}{t_n + i}\right), \end{aligned}$$

with the notations $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ and $\mathbb{T}_\circ = \mathbb{T} \setminus \{1\}$. Note that we have:

$$z_k = u_k + i\, v_k = \frac{t_k - i}{t_k + i} = \frac{t_k^2 - 1}{t_k^2 + 1} - i\, \frac{2\, t_k}{t_k^2 + 1}.$$

We can easily check that φ is bijective and that $t = \varphi^{-1}(z)$ is defined by:

$$t = \left(i\, \frac{1 + z_1}{1 - z_1}, \ldots, i\, \frac{1 + z_n}{1 - z_n}\right).$$

Now, substituting $z = \varphi(t)$ into $b(z)$, we get $b(\varphi(t)) = c_1(t) + i\, c_2(t)$, where the c_i's are two real rational functions of the real vector t. Writing $c_j = n_j/d_j$, where n_j, $d_j \in \mathbb{Q}[t]$ and $\gcd(d_j, n_j) = 1$, then $b(z) = 0$ is equivalent to $c_1(t) = 0$ and $c_2(t) = 0$, i.e., to $n_1(t) = 0$ and $n_2(t) = 0$. Hence, the problem of computing $V(\langle b\rangle) \cap \mathbb{T}_\circ^n$ is equivalent to the problem of computing $V(\langle n_1, n_2\rangle) \cap \mathbb{R}^n$. In particular, we get $V(\langle b\rangle) \cap \mathbb{T}_\circ^n = \emptyset$ iff $V(\langle n_1, n_2\rangle) \cap \mathbb{R}^n = \emptyset$. *Critical point methods* (see [1]) can be used to check the last condition. Indeed, they characterize a real point on every connected component of $V(\langle n_1, n_2\rangle) \cap \mathbb{R}^n$. Finally, while working over $\mathbb{T}_\circ^n$ and not over $\mathbb{T}^n$, we are missing to test the stability criterion on the particular set of points $\{(1, z_2, \ldots, z_n), \ldots, (z_1, \ldots, z_{n-1}, 1)\}$ of $\mathbb{T}^n \setminus \mathbb{T}_\circ^n$. Hence, we have to study the stability of the polynomials $b(1, z_2, \ldots, z_n), \ldots, b(z_1, \ldots, z_{n-1}, 1)$ separately based on the same method. This can be studied inductively on the dimension. The corresponding algorithm [2] is given in Algorithm 1 (see below).

Algorithm 1. IsStable

procedure `IsStable`$(b(z_1, \ldots, z_n))$ ▷ Return true if $V(\langle b \rangle) \cap \mathbb{U}^n = \emptyset$
  for $k = 0$ to $n - 1$ **do**
    Compute the set S_k of polynomials obtained by substituting k variables by 1 into $b(z_1, \ldots, z_n)$
    for each element b in S_k **do**
      $\{n_1, n_2\} =$ `Möbius_transform`(b)
      if $V(\langle n_1,\, n_2 \rangle) \cap \mathbb{R}^n \neq \emptyset$ **then**
        return False
      end if
    end for
  end for
  return True
end procedure

3.2 An Efficient Stability Test for 2D Systems

For $n = 2$, with the notations of (1), the last condition of 2 of Theorem 1, i.e., checking whether or not $V(\langle b \rangle) \cap \mathbb{T}^2$ is empty, amounts to search for the real solutions $(u_1,\, v_1,\, u_2,\, v_2)$ of the following *zero-dimensional polynomial system*

$$c_1(u_1, u_2, v_1, v_2) = 0,\; c_2(u_1, u_2, v_1, v_2) = 0,\; u_1^2 + v_1^2 = 1,\; u_2^2 + v_2^2 = 1,$$

i.e., for the real solutions of a polynomial system which only has a finite number of complex solutions. Hence, the problem is reduced to deciding if a polynomial system in two variables, which has finitely many complex solutions, admits real ones. Based on this idea, an efficient algorithm for testing stability of 2D systems is given in [3,4]. Let us explain it. According to the end of Sect. 3.1, using the Möbius transformation φ, $V(\langle b(z_1, z_2) \rangle) \cap \mathbb{T}^2 = \emptyset$ is equivalent to:

$$\begin{cases} V(\langle n_1,\, n_2 \rangle) \cap \mathbb{R}^2 = \emptyset, \\ b(1, z_2) \neq 0,\; |z_2| = 1, \\ b(z_1, 1) \neq 0,\; |z_1| = 1. \end{cases}$$

Since the last two conditions of the above system can be efficiently checked by means of, e.g., [26], let us concentrate on the first condition, i.e., on the problem of deciding when a zero-dimensional polynomial system $V(\langle n_1,\, n_2 \rangle)$ in two variables t_1 and t_2 has real solutions. The main idea is to reduce the problem of checking the existence of real solutions of such a polynomial system to the problem of checking the existence of real roots of a well-chosen univariate polynomial. A convenient method to do that is the so-called *univariate representation* of the solutions [24,25]. Let us recall this concept. Given a zero-dimensional polynomial ideal I in $\mathbb{Q}[t_1, t_2]$, a univariate representation consists in the datum of a linear form $s = a_1\, t_1 + a_2\, t_2$, with $a_1,\, a_2 \in \mathbb{Q}$, and three polynomials $h,\, h_{t_1},\, h_{t_2} \in \mathbb{Q}[s]$ such that the applications

$$\begin{array}{ll} \phi : V(I) \longrightarrow V(\langle h \rangle) = \{s \in \mathbb{C} \mid h(s) = 0\} & \psi : V(\langle h \rangle) \longrightarrow V(I) \\ t = (t_1,\, t_2) \longmapsto s = a_1\, t_1 + a_2\, t_2, & s \longmapsto t = (h_{t_1}(s),\, h_{t_2}(s)), \end{array}$$

provide a one-to-one correspondence between the zeros of I and the roots of h. In that case, the linear form $s = a_1\, t_1 + a_2\, t_2$ is said to be *separating*.

A key property of the 1–1 correspondence $\phi = \psi^{-1}$ is that it preserves the real zeros of $V(I)$ in the sense that any real zero of $V(I)$ corresponds a real root of h and conversely. As a consequence, deciding if $V(I)$ has real zeros is equivalent to deciding if the univariate polynomial h has real roots.

From a computational viewpoint, to conclude on the existence of real zeros of $V(I)$, it is sufficient to compute a separating linear form for $V(I)$ and the corresponding univariate polynomial h. In [3,4], an efficient algorithm based on *resultants* and *subresultant sequences* [1] is used to perform these computations and, then to conclude about the stability of 2D systems.

4 Testing Stabilizability of 2D Systems

4.1 Module-Theoretic Conditions for Stabilizability

As explained in Sect. 2, a fractional representation of a plant $P \in K^{q\times p}$ is defined by $P = D^{-1}\, N$, where $R = (D \quad -N) \in A^{q\times(q+r)}$. See Definition 2. Let us set $p = q + r$. Since A is an integral domain and R is a matrix, we are naturally in the realm of *module theory* [13,23], which is the extension of linear algebra for rings. Given $R \in A^{q\times p}$, we shall consider the following A-modules:

$$\begin{cases} \ker_A(.R) := \{\mu \in A^{1\times q} \mid \mu\, R = 0\}, \\ \operatorname{im}_A(.R) = A^{1\times q}\, R := \{\lambda \in A^{1\times p} \mid \exists\, \nu \in A^{1\times q} :\ \lambda = \nu\, R\}, \\ \operatorname{coker}_A(.R) = A^{1\times p}/(A^{1\times q}\, R). \end{cases}$$

We recall that factor A-module $M := \operatorname{coker}_A(.R) = A^{1\times p}/(A^{1\times q}\, R)$ is defined by the generators $\{y_j = \pi(f_j)\}_{j=1,\ldots,p}$, where $\{f_j\}_{j=1,\ldots,p}$ denotes the standard basis of the free A-module $A^{1\times p}$, namely, the basis formed by the standard basis vectors f_j's (i.e., the row vectors of length p with 1 at the j^{th} position and 0 elsewhere), and $\pi : A^{1\times p} \longrightarrow M$ is the A-homomorphism (i.e., A-linear map) which sends $\lambda \in A^{1\times p}$ onto its residue class $\pi(\lambda)$ in M (i.e., $\pi(\lambda') = \pi(\lambda)$ if there exists $\mu \in A^{1\times q}$ such that $\lambda' = \lambda + \mu\, R$). One can prove that the generators y_j's satisfy the A-linear relations $\sum_{j=1}^{p} R_{ij}\, y_j = 0$, $i = 1, \ldots, q$, where R_{ij} stands for the (i, j) entry of R. For more details, see, e.g., [8,20,22]. Hence, if we note by $y = (y_1, \ldots, y_p)^T$, then the above relation can formally be rewritten as $Ry = 0$, which explains why the A-module M is used to study the linear system $Ry = 0$. Using module theory, a characterization of stabilizability, namely, of the existence of a stabilizing controller C for a given plant P, was obtained.

Theorem 2 **([21]).** *Let $P \in K^{q\times r}$, $p = q + r$, $P = D^{-1}\, N$ be a fractional representation of P, where $R = (D \quad -N) \in A^{q\times p}$, and $M = A^{1\times p}/(A^{1\times q}\, R)$ the A-module finitely presented by R. Then, P is internally stabilizable iff the A-module $M/t(M)$ is projective, where $t(M) := \{m \in M \mid \exists\, a \in A\backslash\{0\} :\ a\, m = 0\}$ is the torsion submodule of M. In other words, P is internally stabilizable iff there exist an A-module L and $s \in \mathbb{Z}_{\geq 0}$ such that $L \oplus M/t(M) \cong A^{1\times s}$.*

According to Theorem 2, we have to study the following two problems:

1. Explicitly characterize $t(M)$, and thus $M/t(M)$.
2. Check whether or not $M/t(M)$ is a projective A-module.

Both problems can be solved by homological algebra methods [23]. In the rest of the paper, we shall suppose that A is a *coherent ring* [20,23]. For instance, we can consider the coherent but non-noetherian integral domain $H^\infty(\mathbb{C}_+)$ (see Introduction) or any *noetherian ring* such as $\mathbb{R}(z_1,\ldots,z_n)_S$ or RH_∞. The category of finitely presented modules over a coherent integral domain is a natural framework for mathematical systems theory [20].

Let us introduce the definition of an *extension module* [13,23]. Let us consider a finitely presented A-module $L := A^{1\times q_0}/(A^{1\times q_1}\, S_1)$, where $S_1 \in A^{q_1\times q_0}$. Since A is coherent, $\ker_A(.S_1)$ is a finitely generated A-module, and thus there exists a finite family of generators of $\ker_A(.S_1)$. Stacking these row vectors of $A^{1\times q_1}$, we obtain $S_2 \in A^{q_2\times q_1}$ such that $\ker_A(.S_1) = \operatorname{im}_A(.S_2) = A^{1\times q_2}\, S_2$. Repeating the same process with S_2, etc., we get the following *exact sequence* of A-modules

$$\ldots \xrightarrow{.S_3} A^{1\times q_2} \xrightarrow{.S_2} A^{1\times q_1} \xrightarrow{.S_1} A^{1\times q_0} \xrightarrow{\kappa} L \longrightarrow 0, \tag{2}$$

i.e., $\ker_A(.S_i) = \operatorname{im}_A(.S_{i+1})$ for $i \geq 1$, where κ is the epimorphism which maps $\eta \in A^{1\times q_0}$ onto its residue class $\kappa(\eta)$ in L. This exact sequence is called *free resolution* of L [13,23]. "Transposing" (2), i.e., applying the *contravariant left exact functor* $\mathrm{Rhom}_A(\,\cdot\,,A)$ to (2), we get the following *complex* of A-modules

$$\ldots \xleftarrow{.S_3^T} A^{1\times q_2} \xleftarrow{.S_2^T} A^{1\times q_1} \xleftarrow{.S_1^T} A^{1\times q_0} \longleftarrow 0,$$

i.e., $\operatorname{im}_A\left(.S_i^T\right) \subseteq \ker_A\left(.S_{i+1}^T\right)$ for $i \geq 1$. We introduce the following A-modules:

$$\begin{cases} \operatorname{ext}_A^0(L,A) \cong \ker_A\left(.S_1^T\right), \\ \operatorname{ext}_A^i(L,A) \cong \ker_A\left(.S_{i+1}^T\right)/\operatorname{im}_A\left(.S_i^T\right),\ i \geq 1. \end{cases}$$

It can be shown that, up to isomorphism, the A-modules $\operatorname{ext}_A^i(L,A)$'s depend only on L and not on the choice of the free resolution (2) of L, i.e., on the choice of the matrices S_i's. It explains the notation $\operatorname{ext}_A^i(L,A)$. See [13,23].

Theorem 3 **([20]).** *With the notations of Theorem 2, we have:*

1. *$t(M) \cong \operatorname{ext}_A^1(T(M),A)$, where $T(M) := A^{1\times q}/\left(A^{1\times p}\, R^T\right)$ is the so-called* Auslander transpose *of M (i.e., the A-module finitely presented by R^T).*
2. *Let us suppose that the* weak global dimension *of A is finite and is equal to n. Then, the obstruction for $M/t(M)$ to be projective is defined by*

$$I = \bigcap_{i=2}^{n} \operatorname{ann}_A\left(\operatorname{ext}_A^i(T(M/t(M)),A)\right),$$

where $\operatorname{ann}_A(L) := \{a \in A \mid a\, L = 0\}$ is the annihilator *of a A-module L. Hence, M is projective iff we have $I = A$.*

Let us now show how Theorem 3 can be used to check whether or not a nD system P is stabilizable. Let $A = \mathbb{Q}(z_1, \ldots, z_n)_S$, $B = \mathbb{Q}[z_1, \ldots, z_n]$, $K = Q(A) = \mathbb{Q}(z_1, \ldots, z_n)$, and $P \in K^{q \times r}$. Since $K = Q(B)$, we can write each entry P_{ij} of P as $P_{ij} = n_{ij}/d_{ij}$, where $0 \neq d_{ij}, n_{ij} \in B$, and $\gcd(d_{ij}, n_{ij}) = 1$. Let us denote by d the least common multiple of all the d_{ij}'s and set $D = d\, I_q \in B^{q \times q}$, $N = D\, P \in B^{q \times p}$, and $R = (D \quad -N) \in B^{q \times p}$. Let us also consider the finitely presented B-module $L := B^{1 \times p}/(B^{1 \times q}\, R)$. Since $A = S^{-1}\, B$ is a localization (see Sect. 3), A is a *flat* B-module [13,23], we then get that

$$A \otimes_B L \cong A^{1 \times p}/(A^{1 \times q}\, R) = M,$$

where $\otimes_B$ stands for the *tensor product* of B-modules [13,23]. Note that $A \otimes_B L$ can be understood as the A-module obtained by extending the coefficients of the B-module L to A. Using elimination theory over B (e.g., Gröbner bases, Janet bases), given the matrix R^T, we can compute $\ker_B\left(.R^T\right)$, i.e., the *second syzygy module* of $T(L) = B^{1 \times q}/\left(B^{1 \times p}\, R^T\right)$ [13,23]. For more details, see [8,13,22] and [9] for the OreModules package which handles such computations. Hence, we can compute the beginning of a free resolution of the B-module $T(L)$:

$$0 \longleftarrow T(L) \xleftarrow{\sigma} B^{1 \times q} \xleftarrow{.R^T} B^{1 \times p} \xleftarrow{.Q^T} B^{1 \times m}.$$

Applying the functor $\mathrm{Rhom}_B(\,\cdot\,, B)$ to it, we obtain the complex of B-modules:

$$0 \longrightarrow B^{1 \times q} \xrightarrow{.R} B^{1 \times p} \xrightarrow{.Q} B^{1 \times m}.$$

Therefore, we get $t(L) = \mathrm{ext}^1_B(T(L), B) = \ker_B(.Q)/\mathrm{im}_B(.R)$ and since we can also compute a matrix $R' \in B^{q' \times p}$ such that $\ker_B(.Q) = \mathrm{im}_B(.R')$, we obtain:

$$t(L) = (B^{1 \times q'}\, R')/(B^{1 \times q}\, R) \quad \Rightarrow \quad L/t(L) = B^{1 \times p}/(B^{1 \times q'}\, R').$$

The corresponding computations can be handled by the OreModules package [9]. Hence, based, e.g., on Gröbner basis techniques, we can find an explicit presentation $R' \in B^{q' \times p}$ of the B-module $L/t(L)$. Since $A = S^{-1}\, B$, we get [23]

$$A \otimes_B t(L) \cong A \otimes_B \mathrm{ext}^1_B(T(L), B) \cong \mathrm{ext}^1_A(A \otimes_B T(L), A) \cong \mathrm{ext}^1_A(T(M), A) \cong t(M),$$

which yields $A \otimes_B (L/t(L)) \cong (A \otimes_B L)/(A \otimes_B t(L)) \cong M/t(M)$. Thus, we have

$$M/t(M) \cong A \otimes_B \left(B^{1 \times p}/(B^{1 \times q'}\, R')\right) \cong A^{1 \times p}/(A^{1 \times q'}\, R'),$$

which shows that $R' \in B^{q' \times p}$ is a presentation matrix of the A-module $M/t(M)$, which explicitly solves the first point.

Let us now consider the second one, i.e., the problem of testing whether or not $M/t(M)$ is a projective A-module. Clearly, we have:

$$A \otimes_B T(L/t(L)) = A \otimes_B \left(B^{1 \times q'}/\left(B^{1 \times p}\, R'^T\right)\right) \cong A^{1 \times q'}/\left(A^{1 \times p}\, R'^T\right) = T(M/t(M)).$$

Moreover, since the localization $A = S^{-1}B$ commutes with the intersection of ideals and $S^{-1}\,\mathrm{ann}_B(P) = \mathrm{ann}_{S^{-1}B}(S^{-1}P)$ for a finitely generated B-module P (see, e.g., [13,23]), using the fact that the B-modules $\mathrm{ext}^i_B(T(L/t(L)), B)$'s are finitely generated, we then obtain:

$$A \otimes_B \left(\bigcap_{i=2}^{n} \mathrm{ann}_B(\mathrm{ext}^i_B(T(L/t(L)), B) \right) \cong \bigcap_{i=2}^{n} \mathrm{ann}_A \left(\mathrm{ext}^i_A(T(M/t(M)), A \right).$$

Hence, if we denote by $I = \bigcap_{i=2}^{n} \mathrm{ann}_A \left(\mathrm{ext}^i_A(T(M/t(M)), A \right)$ the obstruction for the A-module $M/t(M)$ to be projective (see 2 of Theorem 3) and similarly by $J = \bigcap_{i=2}^{n} \mathrm{ann}_B(\mathrm{ext}^i_B(T(L/t(L)), B)$ the obstruction for the B-module $L/t(L)$ to be projective, then $I \cong A \otimes_B J$. We note that the ideal J can be explicitly computed by means of elimination theory (e.g., Gröbner/Janet bases) and implemented in a computer algebra system (see the OreModules [9]). For more details, see [8]. Hence, $M/t(M)$ is a projective A-module iff $I = A$, i.e., iff $S^{-1}J = S^{-1}B$, i.e., iff $J \cap S \neq \emptyset$, i.e., iff there exists $b \in J$ such that $V(\langle b \rangle) \cap \mathbb{U}^n = \emptyset$.

4.2 Towards an Effective Version of the Polydisc Nullstellensatz

The condition $J \cap S \neq \emptyset$ yields the so-called *Polydisc Nullstellensatz*.

Theorem 4 (*Polydisc Nullstellensatz*, [7]). *Let J be a finitely generated ideal of $B = \mathbb{Q}[z_1, \ldots, z_n]$. Then, the two assertions are equivalent:*

1. *$V(J) \cap \mathbb{U}^n = \emptyset$.*
2. *There exists $b \in J$ such that $V(\langle b \rangle) \cap \mathbb{U}^n = \emptyset$.*

To our knowledge, there is no effective version of the Polydisc Nullstellensatz for a general ideal J. But, in [5], it is shown how the first condition of Theorem 4 can be effectively tested for a zero-dimensional polynomial system J, i.e., when B/J is a finite-dimensional $\mathbb{Q}$-vector space, or equivalently when $V(J)$ is defined by a finite number of complex points. Given a zero-dimensional polynomial ideal J in $\mathbb{Q}[z_1, \ldots, z_n]$, a first step consists in computing a univariate representation of $V(J)$. Such a representation characterizes the zeros $(z_1, \ldots, z_n)$ of $V(J)$ as:

$$\begin{cases} h(t) = 0, \\ z_1 = h_1(t), \\ \quad \vdots \\ z_n = h_n(t). \end{cases} \tag{3}$$

Using (3), we now have to check whether or not $|z_1| \leq 1, \ldots, |z_n| \leq 1$. To do, at the solutions $z_k = u_k + i\,v_k$ of (3), where $u_k,\, v_k \in \mathbb{R}$, we have to study the sign of the n polynomials $u_k^2 + v_k^2 - 1$ for $k = 1, \ldots, n$. From a computational viewpoint, a problem occurs when one of these polynomials vanishes. In that case, numerical computations are not sufficient to conclude. The algorithm, proposed in [5], proceeds by following the following three main steps:

1. Compute a set of hypercubes in $\mathbb{R}^{2n}$ isolating the zeros of $V(J)$. Each coordinate is represented by a box B in $\mathbb{R}^2$ obtained from the intervals containing its real and imaginary parts.
2. For each z_k, compute the number l_k of zeros of J satisfying $|z_k| = 1$. This can be obtained by using classical stability test for 1D systems applied to the *elimination polynomial* p_k of J with respect to z_k, i.e., $J \cap \mathbb{Q}[z_k] = \langle p_k \rangle$.
3. For each z_k, refine the isolating boxes of the solutions until exactly l_k intervals obtained from the evaluation of $u_k^2 + v_k^2 - 1$ at these boxes contains zero. The boxes that yield strictly positive evaluation are discarded.

At the end of these three steps, if all the isolating boxes were discarded, then $V(J) \cap \mathbb{U}^n = \emptyset$, and thus the plant P is stabilizable. Otherwise, the remaining boxes correspond to elements of $V(J) \cap \mathbb{U}^n$, which shows that $V(J) \cap \mathbb{U}^n \neq \emptyset$ and the plant P is not stabilizable. The algorithm testing 1 of Theorem 4 in the case of a zero-dimensional ideal J is given in Algorithm 2. It should be stressed that the symbol $\square f(B)$ used in this algorithm denotes the interval resulting from the evaluation of the polynomial f at the box B using interval arithmetic.

Based on Algorithm 2, we can effectively check whether or not a 2D system is stabilizable since when $n = 2$, $J = \mathrm{ann}_B(\mathrm{ext}_B^2(T(L/t(L)), B))$, and B/J is either B, which corresponds to $L/t(L)$ is a projective B-module, or zero-dimensional which corresponds to $L/t(L)$ is torsion-free but not projective. More generally, the method developed in [5] can be applied to a B-module $L/t(L)$ satisfying $\mathrm{ext}_B^i(T(L/t(L)), B) = 0$ for $i = 1, \ldots, n-1$. For instance, if $n = 3$, the last conditions mean that $L/t(L)$ is a *reflexive* B-module [8].

Algorithm 2. IsStabilizable

Input: A set of r polynomials $\{p_1, \ldots, p_r\} \subset \mathbb{Q}[z_1, \ldots, z_n]$ defining a zero-dimensional ideal $J = \langle p_1, \ldots, p_r \rangle$.
Output: `True` if $V(\langle p_1, \ldots, p_r \rangle) \cap \mathbb{U}^n = \emptyset$, and `False` otherwise.
Begin
◇ $\{h, h_{z_1}, \ldots, h_{z_n}\} :=$ `Univ_R`$(\{p_1, \ldots, p_r\})$;
◇ $\{B_1, \ldots, B_d\} :=$ `Isolate`(f);
◇ $L_B := \{B_1, \ldots, B_d\}$ and $\epsilon := \min_{i=1,\ldots,d}\{w(B_i)\}$;
For k from 1 to n **do**
  ◇ $l_k := \sharp\{z \in V(I) \mid |z_k| = 1\}$;
  While $\sharp\{i \mid 0 \in \square(\Re(h_{z_k})^2 + \Im(h_{z_k})^2 - 1)(B_i)\} > l_k$ **do**
    ◇ $\epsilon := \epsilon/2$;
    ◇ For $i = 1, \ldots, d$, set $B_i :=$`Isolate`(f, B_i, ϵ);
  End While
  ◇ $L_B := L_B \setminus \{B_i \mid \square(\Re(h_{z_k})^2 + \Im(h_{z_k})^2 - 1)(B_i) \subset \mathbb{R}_+\}$;
  ◇ If $L_B = \{\}$, then **Return** `True` **End If**;
End For
Return `False`.
End

5 Computing Stabilizing Controllers of 2D Systems

For a zero-dimensional ideal $J = \langle p_1, \ldots, p_r \rangle$, an algorithm is given in [5] for the computation $b \in J$ satisfying 2 of Theorem 4. In the following, we briefly outline the algorithm for $n = 2$. The method is an effective variant of the approach proposed in [14] which consists in considering the elimination polynomials r_{z_k} with respect to the variable z_k defined by $J \cap \mathbb{Q}[z_k] = \langle r_{z_k} \rangle$ and to factorize them into stable and unstable factors, i.e., $r_{z_k} = r_{z_k,s}\, r_{z_k,u}$, where the roots of $r_{z_k,s}$ (resp., $r_{z_k,u}$) are outside (resp., inside) the closed unit disc $\mathbb{U}$ for $k = 1, 2$. Then, we can define a stable polynomial $b = r_{z_1,s}(z_1)\, r_{z_2,s}(z_2)$ and Gröbner basis methods are finally used to get the cofactors u_i's defined by $b = \sum_{k=1}^{r} u_k\, p_k$.

Since the factorizations of the polynomials r_{z_1} and r_{z_2} are performed in $\mathbb{C}$, the resulting factors $r_{z_k,s}$ do not have usually their coefficients in $\mathbb{Q}$. This prevents the polynomial b (and the u_i's) from being computed exactly in $\mathbb{Q}[z_1, z_2]$. To overcome this issue, the method developed in [5] uses an univariate representation of the solutions of $V(J)$ to compute approximate factorizations of r_{z_i} over $\mathbb{Q}$ and construct a *Nullstelenstaz* relation on the corresponding approximated ideal. For a suitable approximation, the main result in [5] shows that the obtained cofactors u_i's for the approximated ideal hold for the initial ideal J, which yields a stable polynomial $b \in J$, i.e., $b \in J \cap S$.

More precisely, given an ideal $J \subset \mathbb{Q}[z_1, z_2]$, the method proceeds by first computing a univariate representation $\{h(t) = 0,\ z_1 = h_1(t),\ z_2 = h_2(t)\}$ of the zeros of J with respect to a separating form $t = a_1\, z_1 + a_2\, z_2$. Then, we consider the ideal $J_r = \langle h(t), z_1 - h_1(t), z_2 - h_2(t) \rangle \subset \mathbb{Q}[t, z_1, z_2]$ which is the intersection of the ideal J and $\langle t - a_1\, z_1 - a_2\, z_2 \rangle$. Using the univariate representation, approximations of the polynomials $r_{z_1,s}$, $r_{z_2,s}$, and b, respectively denoted by $\tilde{r}_{1,s}, \tilde{r}_{2,s}$ and $\tilde{b}$, are then computed as follows:

1. We approximate the complex roots $\gamma_1, \ldots, \gamma_n$ of $h(t)$ so that their real and imaginary parts are given by rational numbers. The resulting approximations are denoted by $\tilde{\gamma}_1, \ldots, \tilde{\gamma}_n$.
2. For each approximation $\tilde{\gamma}_k$, if $|h_{z_i}(\tilde{\gamma}_k)| > 1$, add the factor $z_i - h_{z_i}(\tilde{\gamma}_k)$ to the polynomial $\tilde{r}_{k,s}$.
3. Compute $\tilde{b} = \prod_{k=1}^{n} \tilde{r}_{k,s}$.

Finally, using the polynomial $\tilde{b}$, a Nullstellensatz relation is computed for the ideal $\tilde{J}_r = \langle \tilde{h}(t), z_1 - h_{z_1}(t), z_2 - g_{z_2}(t) \rangle$, where $\tilde{h}(t) = \prod_{i=1}^{d} (t - \tilde{\gamma}_i) \in \mathbb{Q}[t]$, which yields three cofactors u_1, u_2 and u_3 in $\mathbb{Q}[t, z_1, z_2]$. Moreover, in [8], it is shown that for close-enough approximations of the γ_k's, using these cofactors with the exact ideal J_r yields a stable polynomial b in J, i.e., $b \in J \cap S$.

The main algorithm, which can be generalized for zero-dimensional ideals J, is presented in Algorithm 3.

Now, given a 2D plant $P \in \mathbb{Q}(z_1, z_2)^{q \times p}$, we have shown that we can effectively test whether or not P is internally stabilizable. If so, an important issue is then to explicitly compute a stabilizing controller C (see Definition 1).

Algorithm 3. StabilizingPolynomial

Input: $J := \langle p_1, \ldots, p_r \rangle$ such that J is zero-dimensional and $V(J) \cap \mathbb{U}^n = \emptyset$.
Output: $b \in J$ such that $V(\langle b \rangle) \cap \mathbb{U}^n = \emptyset$.
Begin
◇ $\{h, h_{z_1}, \ldots, g_{z_2}\} :=$ `Univ_R`$(\{p_1, \ldots, p_r\})$;
◇ $\{B_1, \ldots, B_d\} :=$ `Isolate`(h);
◇ $L_B := \{B_1, \ldots, B_d\}$ and $\epsilon := \min_{i=1,\ldots,d}\{w(B_i)\}$;
Do
  ◇ $[r_1, \ldots, r_n] := [1, \ldots, 1]$ and $\tilde{f} := 1$;
  ◇ outside := `False`;
  For each B in L_B **do**
    While (outside=`False`) **do**
      For i from 1 to n **do**
        If $\square(\Re(h_{z_i})^2 + \Im(h_{z_i})^2 - 1)(B) \subset \mathbb{R}_+$ then
          ◇ $\gamma :=$ `midpoint`(B);
          ◇ $r_i := r_i\,(z_i - h_{z_i}(\gamma))$;
          ◇ outside := `True` and **Break For**;
        End If
      End For
      ◇ $\epsilon := \epsilon/2$;
      ◇ $B :=$`Isolate`(h, B, ϵ); (isolate the real roots of h inside B up to a precision ϵ)
    End While
    ◇ $\tilde{h} := \tilde{h}\,(t - \gamma)$;
    ◇ outside := `False`;
  End ForEach
  ◇ $\tilde{b} := \prod_{i=1}^{n} r_i$;
  ◇ $\tilde{b}_t := \tilde{b}$ evaluated at $z_i = h_{z_i}(t)$;
  ◇ $h_0 :=$ `quo`$(\tilde{b}_t, \tilde{h})$ in $\mathbb{Q}[t]$;
  ◇ $b := \tilde{b} - h_0\,(\tilde{h} - h)$ evaluated at $t = \sum_{k=1}^{n} a_k\, z_k$;
While (`IsStable`(s)=`False`)
◇ Return b.
End

Theorem 5 ([4]). *Let* $A = \mathbb{Q}(z_1, \ldots, z_n)_S$, $B = \mathbb{Q}[z_1, \ldots, z_n]$, $K = \mathbb{Q}(z_1, \ldots, z_n)$, *and* $P \in K^{q \times p}$ *be a stabilizable plant. Moreover, let* $P = D^{-1} N$ *a fractional representation of* P, *where* $R = (D \quad -N) \in B^{q \times p}$, $L = B^{1 \times p}/(B^{1 \times q}\, R)$ *the* B*-module finitely presented by* R, *and* $L/t(L) = B^{1 \times p}/(B^{1 \times q'}\, R')$. *Finally, let* $J = \bigcap_{i=2}^{n} \operatorname{ann}_B(\operatorname{ext}_B^i(T(L/t(L)), B)$ *and* $\pi \in J \cap S \neq \emptyset$.

Then, there exists a generalized inverse $S' \in B_\pi^{p \times q'}$ *of* R', *i.e.,* $R'\, S'\, R' = R'$, *where* $B_\pi = S_\pi^{-1}\, B$ *and* $S_\pi = \{1, \pi, \pi^2, \ldots\}$. *Writing* $R' = (D' \quad -N')$, *where* $D' \in B^{q' \times q'}$ *and* $N' \in B^{q' \times r}$, *and noting* $S = S'\, D'\, D^{-1} \in K^{p \times q}$, *then a stabilizing controller* C *of* P *is defined by* $C = Y\, X^{-1}$, *where:*

$$S = (X^T \quad Y^T)^T, \quad X \in A^{q \times q}, \quad Y \in A^{r \times q}.$$

According to Theorem 5, if $\pi \in J \cap S$ is known, then a stabilizing controller C of P can obtained by means of the computation of a generalized inverse S' of R' over B_π. Effective methods exist for solving this last point [8,13,22].

6 A Maple illustrating example

In this section, we demonstrate the results explained in the above sections on an explicit example first considered in [14]. To do that, we first load the `nDStab` package dedicated to the stability and stabilizability of multidimensional systems, as well as the OreModules package [9] dedicated to the study multidimensional linear systems theory based on algebraic analysis methods.

```
> with(LinearAlgebra):
> with(nDStab):
> with(OreModules):
```

We consider the plant $P \in \mathbb{Q}(z_1, z_2)^{2\times 2}$ defined by the transfer matrix:

```
> P := Matrix([[-(z[2]-3*z[1])/(2*z[1]-5), (2*z[1]-5)/(3*(2*z[1]-1))],
> [(2*z[1]-1)/(8*z[2]+6*z[1]-15),z[2]^2/(2*z[1]-1)]]);
```

$$P := \begin{bmatrix} -\frac{z_2-3\,z_1}{2\,z_1-5} & \frac{2\,z_1-5}{6\,z_1-3} \\ \frac{2\,z_1-1}{8\,z_2+6\,z_1-15} & \frac{z_2{}^2}{2\,z_1-1} \end{bmatrix}$$

We can check that some entries of P are unstable using the command `IsStable`:

```
> map(a->IsStable(denom(a)),P);
```

$$\begin{bmatrix} \mathit{true} & \mathit{false} \\ \mathit{true} & \mathit{false} \end{bmatrix}$$

Let us introduce the polynomial ring $B = \mathbb{Q}[z_1, z_2]$:

```
> B := DefineOreAlgebra(diff=[z[1],s[1]], diff=[z[2],s[2]],
> polynom=[s[1],s[2]]):
```

Now, we consider the fractional representation of P defined by $R = (d \quad -N)$, where $d \in B^{2\times 2}$ is the diagonal matrix defined by the polynomial `den` which is the least common multiple of all the denominators of the entries of P, i.e.:

```
> den := lcm(op(convert(map(denom,P),set)));
```

$$den := 3\,(2\,z_1-5)\,(2\,z_1-1)\,(8\,z_2+6\,z_1-15)$$

```
> d := DiagonalMatrix([den,den]);
```

$$d := \begin{bmatrix} 3\,(2\,z_1-5)\,(2\,z_1-1)\,(8\,z_2+6\,z_1-15) & 0 \\ 0 & 3\,(2\,z_1-5)\,(2\,z_1-1)\,(8\,z_2+6\,z_1-15) \end{bmatrix}$$

and the matrix $N = d\,P \in B^{2\times 2}$ is defined by:

```
> N := simplify(d.P);
```

$$N := \begin{bmatrix} 3\,(2\,z_1-1)\,(8\,z_2+6\,z_1-15)\,(-z_2+3\,z_1) & (8\,z_2+6\,z_1-15)\,(2\,z_1-5)^2 \\ 3\,(2\,z_1-5)\,(2\,z_1-1)^2 & 3\,(2\,z_1-5)\,(8\,z_2+6\,z_1-15)\,{z_2}^2 \end{bmatrix}$$

Since the notation D is prohibited by `Maple`, we use here d instead of D as it was done in the above sections. Then, we can define the matrix $R = (d \quad -N) \in B^{2\times 4}$

```
> R := Matrix([d, -N]):
```

and the finitely presented B-module $L = B^{1\times 4}/(B^{1\times 2}\,R)$. We first have to compute a presentation matrix for the B-module $L/t(L)$. Using the OreModules package, this can be done as follows:

```
> Ext1 := Exti(Involution(R,B),B,1):
```

The command `Exti` returns different matrices. Since the first matrix `Ext1`[1]

```
> Ext1[1];
```

$$\begin{bmatrix} 8\,z_2+6\,z_1-15 & 0 & 0 \\ 0 & 24\,{z_1}^3+32\,{z_1}^2 z_2-132\,{z_1}^2-96\,z_2 z_1+210\,z_1+40\,z_2-75 & 0 \\ 0 & 0 & 2\,z_1-5 \end{bmatrix}$$

is not reduced to an identity matrix, we deduce that the torsion submodule $t(L) = \{l \in L \mid \exists\, 0 \neq b \in B:\ b\,l = 0\}$ of L is not reduced to zero, i.e., $t(L) \neq 0$.

The second matrix `Ext1`[2] of `Ext1`, denoted by `Rp` in `Maple`,

```
> Rp := Ext1[2]:
```

is a presentation matrix of $L/t(L)$, i.e., $L/t(L) = B^{1\times 4}/(B^{1\times 3}\,R')$, where $R' =$ `Rp` $\in B^{3\times 4}$. For an easy display of `Rp`, we print it by means of its columns:

```
>  SubMatrix(Rp,1..3,1..2);
```

$$\begin{bmatrix} 12\,{z_1}^2-36\,z_1+15 & 0 \\ 36\,z_1{z_2}^3-54\,z_1{z_2}^2-90\,{z_2}^3+135\,{z_2}^2 & -24\,{z_1}^2z_2-32\,z_1{z_2}^2+36\,{z_1}^2+120\,z_2z_1+16\,{z_2}^2-180\,z_1-150\,z_2+225 \\ 0 & -12\,{z_1}^2-16\,z_2z_1+36\,z_1+8\,z_2-15 \end{bmatrix}$$

```
>  SubMatrix(Rp,1..3,3..3);
```

$$\begin{bmatrix} -18\,{z_1}^2+6\,z_2z_1+9\,z_1-3\,z_2 \\ -54\,z_1{z_2}^3+18\,{z_2}^4+8\,{z_1}^2z_2+81\,z_1{z_2}^2-27\,{z_2}^3-12\,{z_1}^2-8\,z_2z_1+36\,z_1+2\,z_2-15 \\ 4\,{z_1}^2-4\,z_1+1 \end{bmatrix}$$

```
>  SubMatrix(Rp,1..3,4..4);
```

$$\begin{bmatrix} -4\,{z_1}^2+20\,z_1-25 \\ 16\,{z_2}^4 \\ 6\,{z_2}^2z_1+8\,{z_2}^3-15\,{z_2}^2 \end{bmatrix}$$

If d_i denotes the i^{th} diagonal element of the matrix d, $R'_{i\bullet}$ the i^{th} row of R', and l_i the residue class of $R'_{i\bullet}$ in the B-module $L = B^{1\times 4}/(B^{1\times 3}\,R')$, then we have $d_i\,l_i = 0$. Moreover, $\{l_i\}_{i=1,2,3}$ is a generating set of the torsion B-submodule $t(L) = (B^{1\times 3}\,R')/(B^{1\times 2}\,R)$ of L.

By construction, the B-module $L/t(L) = B^{1\times 4}/(B^{1\times 3}\,R')$ is torsion-free. Since P is a 2D system, i.e., $n = 2$, the obstruction to projectivity for $L/t(L)$ is only defined by the B-module $\text{ext}^2_B(N', B)$, where $N' = B^{1\times 3}/\left(B^{1\times 4}\,{R'}^T\right)$ is the so-called *Auslander transpose* of $L/t(L)$. For more details, see [8, 22]. More precisely, one can prove that $L/t(L)$ is a projective B-module iff $\text{ext}^2_B(N', B) = 0$. If $\text{ext}^2_B(N', B) \neq 0$, then $\text{ext}^2_B(N', B)$ is *0-dimensional B-module*, i.e., it defines a finite-dimensional $\mathbb{Q}$-vector space. In particular, the following ideal J of B

$$J = \text{ann}_B(\text{ext}^2_B(N', B)) = \left\{b \in B \mid \forall\, e \in \text{ext}^2_B(N', B),\ b\,e = 0\right\}$$

is *zero-dimensional*, i.e., B/J is a finite-dimensional $\mathbb{Q}$-vector space. The ideal J can be directly computed by the OreModules command `PiPolynomial`:

```
>  pi := map(factor,PiPolynomial(Rp,B));
```

$$\pi := [4\,{z_2}^2-18\,z_1-30\,z_2+45, (2\,z_2-3)\,(2\,z_1-5)\,, (2\,z_1-5)\,(2\,z_1-1)]$$

Since $J \neq B$, we obtain that $L/t(L)$ is not a projective B-module.

According to our result on stabilizability, P is internally stabilizable iff the $A = \mathbb{Q}(z_1, z_2)_S = S^{-1}\,B$-module $A \otimes_B (L/t(L)) \cong A^{1\times 4}/(A^{1\times 3}\,R')$ is projective of rank 2, i.e., iff $S^{-1}\,J = A$, i.e., iff $J \cap S \neq \emptyset$, where

$$S = \{b \in B \mid V(\langle b\rangle) \cap \mathbb{U}^2 = \emptyset\}$$

is the multiplicatively closed subset of B formed by all the stable polynomials of B. Equivalently, by the Polydisk Nullstellensatz, P is internally stabilizable iff

$V(J) \cap \mathbb{U}^2 = \emptyset$. Since $V(J)$ is zero-dimensional, i.e., is formed by a finite number of complex points of $\mathbb{C}^2$, we can effectively test the Polydisk Nullstellensatz condition as follows:

```
> IsStabilizable(pi);
```

$$\mathit{true}$$

Hence, we obtain that P is internally stabilizable.

Let us now construct a stabilizing controller C of P. To do that, we must find an element $s \in J \cap S$. We can first try to test whether or not one of the generators of J belongs to S:

```
> map(IsStable,pi);
```

$$[\mathit{true}, \mathit{true}, \mathit{false}]$$

The first two generators of J belong to S. Let us denote them by π_1, resp. π_2.

Since the condition $J \cap S \neq \emptyset$ does not necessarily imply that at least one of the generators of J belongs to S, the algorithm which computes an element of J in S has to be used. This can be done by the command `StabilizingPolynomial`:

```
> factor(StabilizingPolynomial(pi));
```

$$(2\,z_2 - 3)\,(2\,z_2 - 15)\,(2\,z_1 - 5)$$

Since we have found elements in $J \cap S$, let us compute a stabilizing controller C of P. We note that R' has not full row rank since $\ker_B(.R')$ is defined by:

```
> Rp2 := SyzygyModule(Rp,B);
```

$$\mathrm{Rp2} := \begin{bmatrix} 6\,z_2{}^3 - 9\,z_2{}^2 & -2\,z_1 + 1 & 4\,z_2 z_1 - 6\,z_1 - 2\,z_2 + 15 \end{bmatrix}$$

Hence, $R'_2 = \texttt{Rp2}$ is such that $R'_2\, R' = 0$, which shows that the rows $\{R'_{i\bullet}\}_{i=1,\ldots,4}$ of R' satisfy $\sum_{i=1}^{4} \texttt{Rp2}[i]\, R'_{i\bullet} = 0$.

We consider $\pi = \pi_2$, i.e., $\pi = (2\,z_2 - 3)\,(2\,z_1 - 5)$. Similar results can be obtained by choosing $\pi = \pi_1$ instead of π_2 (but the outputs are larger to display).

We now have to find a generalized inverse $S' \in B_\pi^{4\times 3}$ of R', i.e., $R'\, S'\, R' = R'$, where $B_\pi = S_\pi^{-1}\, B$ is the localization of B with respect to the multiplicatively closed set $S_\pi = \{1, \pi, \pi^2, \ldots\}$. This can be done by first computing a right inverse of R'_2 over B_π. Using the OreModules package, we obtain that

```
> Sp2:= Transpose(LocalLeftInverse(Transpose(Rp2),[pi[2]],B));
```

$$Sp2 := \begin{bmatrix} 0 \\ -\frac{1}{12}\, \frac{\left(-4\,z_2{}^2 + 18\,z_1 + 30\,z_2 - 45\right)(2\,z_2 - 3)}{4\,z_2{}^2 - 18\,z_1 - 30\,z_2 + 45} \\ \frac{1}{4\,z_2{}^2 - 18\,z_1 - 30\,z_2 + 45}\left(\frac{1}{3}\,z_2{}^2 - \frac{3}{2}\,z_1 - \frac{5}{2}\,z_2 + \frac{15}{4}\right) \end{bmatrix}$$

is a right inverse of R'_2, i.e., $R'_2\, S'_2 = 1$. Then, defining $\Pi = I_3 - S'_2\, R'_2$, we get that $\Pi^2 = \Pi$, and thus there exists $S' \in B_\pi^{4\times 3}$ such that $\Pi = R'\, S'$. Using the OreModules package, this matrix can be obtained by factorization as follows:

```
> Proj := Transpose(simplify(1-Sp2.Rp2)):
> Sp := simplify(Transpose(Factorize(pi[2]*Proj,
> Transpose(Rp),B))/pi[2]);
```

$$Sp := \begin{bmatrix} -\frac{243\,z_1 z_2^2-81\,z_2^3+84\,z_1 z_2-16\,z_2^2+18\,z_1-186\,z_2+99}{(1152\,z_2-1728)(2\,z_1-5)} & -\frac{1}{16}\,\frac{-z_2+3\,z_1}{(2\,z_2-3)(2\,z_1-5)} & \frac{1}{16}\,\frac{-z_2+3\,z_1}{2\,z_1-5} \\ -\frac{2\,z_1-1}{576\,z_2-864} & -\frac{1}{48}\,\frac{1}{2\,z_2-3} & \frac{1}{48} \\ -\frac{81\,z_2^2+12\,z_1+16\,z_2-30}{1152\,z_2-1728} & -\frac{1}{16}\,\frac{1}{2\,z_2-3} & \frac{1}{16} \\ \frac{6\,z_1-8\,z_2+9}{128\,z_2-192} & 0 & 0 \end{bmatrix}$$

$S' =$ `Sp` satisfies $R'\,S'\,R' = R'$, i.e., S' is a generalized inverse of R' over B_π.

```
> simplify(Rp.Sp.Rp-Rp);
```

$$\begin{bmatrix} 0\,0\,0\,0 \\ 0\,0\,0\,0 \\ 0\,0\,0\,0 \end{bmatrix}$$

If we set $S := S'\,d'\,d^{-1}$, where $R' = (d' \quad -N')$, $d' \in B^{2\times 2}$ and $N' \in B^{2\times 2}$, and split S' as $S' = (X^T \quad Y^T)^T$, where $X \in K^{2\times 2}$, $Y \in K^{2\times 2}$, and $K = \mathbb{Q}(z_1, z_2)$

```
> dp := SubMatrix(Rp, 1..3,1..2):
> S := simplify(Sp.dp.MatrixInverse(d)):
> X := SubMatrix(Sp, 1..2,1..2):
> SubMatrix(X, 1..2,1..1):
```

$$\begin{bmatrix} -\frac{486\,z_1^2 z_2^2+486\,z_1 z_2^3-216\,z_2^4+168\,z_1^2 z_2-1247\,z_1 z_2^2+405\,z_2^3+36\,z_1^2-456\,z_1 z_2+16\,z_2^2+180\,z_1+186\,z_2-99}{(1152\,z_2-1728)(2\,z_1-5)(2\,z_1-1)(8\,z_2+6\,z_1-15)} \\ -\frac{36\,z_2^3+4\,z_1^2-54\,z_2^2-4\,z_1+1}{(2304\,z_2+1728\,z_1-4320)(2\,z_1-1)(2\,z_2-3)} \end{bmatrix}$$

```
> SubMatrix(X, 1..2,2..2):
```

$$\begin{bmatrix} \frac{1}{4}\,\frac{-z_2+3\,z_1}{(2\,z_1-1)(2\,z_1-5)^2(2\,z_2-3)} \\ \frac{1}{12}\,\frac{1}{(2\,z_1-1)(2\,z_2-3)(2\,z_1-5)} \end{bmatrix}$$

```
> Y := SubMatrix(S,3..4,1..2);
```

$$Y := \begin{bmatrix} -\frac{27\,z_2^2+4\,z_1-2}{(1152\,z_1-576)(2\,z_2-3)} & \frac{1}{4}\,\frac{1}{(2\,z_1-1)(2\,z_2-3)(2\,z_1-5)} \\ \frac{6\,z_1-8\,z_2+9}{(512\,z_2+384\,z_1-960)(2\,z_2-3)} & 0 \end{bmatrix}$$

then, the controller $C = Y\,X^{-1}$ internally stabilizes P. Hence, we obtain the following stabilizing controller of P

```
> C := map(factor,simplify(Y.MatrixInverse(X))):
```

$$C := \begin{bmatrix} -\frac{(81\,z_2^2+16\,z_2-24)(2\,z_1-5)}{-243\,z_1 z_2^2+81\,z_2^3+36\,z_1^2-96\,z_1 z_2+16\,z_2^2-36\,z_1+192\,z_2-99} & 9\,\frac{12\,z_1^2-16\,z_1 z_2-36\,z_1+72\,z_2-33}{-243\,z_1 z_2^2+81\,z_2^3+36\,z_1^2-96\,z_1 z_2+16\,z_2^2-36\,z_1+192\,z_2-99} \\ 9\,\frac{(6\,z_1-8\,z_2+9)(2\,z_1-5)}{-243\,z_1 z_2^2+81\,z_2^3+36\,z_1^2-96\,z_1 z_2+16\,z_2^2-36\,z_1+192\,z_2-99} & -27\,\frac{(6\,z_1-8\,z_2+9)(-z_2+3\,z_1)}{-243\,z_1 z_2^2+81\,z_2^3+36\,z_1^2-96\,z_1 z_2+16\,z_2^2-36\,z_1+192\,z_2-99} \end{bmatrix}$$

Finally, we check again that C stabilizes P. To do that, we can check again that all the entries of the matrix $H(P, C)$ belong to $A = \mathbb{Q}(z_1, z_2)_S$, i.e., are stable:

```
> H := MatrixInverse(Matrix([[DiagonalMatrix([1,1],2,2),-P],
> [-C,DiagonalMatrix([1,1],2,2)]])):
> denomH := convert(map(denom,H),set):
> map(IsStable,denomH);
```

$$\{true\}$$

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