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An Integro-differential Operator Approach to Linear Differential Systems

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Abstract: In this paper, we initiate a new algebraic analysis approach to linear differential systems based on rings of integro-differential operators. Within this algebraic analysis approach, we first interpret the method of variations of constants as an operator identity. Using this result, we show that the module associated with a state-space representation of a linear system is the same as the one associated with its standard convolution representation. This finitely presented module over the ring of integro-differential operators is proved to be stably free. Finally, we show how the reachability property can be expressed within this algebraic analysis approach.

Keywords: Linear systems, polynomial methods, algebraic analysis, rings of integro-differential operators, parametrization, reachability, behaviour theory

1. INTRODUCTION

In the 90's, a new approach to linear system theory (Kalman et al. (1969)) was developed based on ideas, methods, and results of *algebraic analysis*. Algebraic analysis is a mathematical theory, initiated in the 60's, that studies linear systems of differential equations (Kashiwara et al. (1971)). Within this mathematical approach, a *finitely presented left module* over *a ring of differential operators* is intrinsically associated with a linear differential system, and the properties of this linear system are studied using module theory, sheaf theory, homological algebra, etc. Control systems defined by linear differential equations have been intrinsically studied within an algebraic analysis approach using rings of differential operators. The deep connections between the algebraic analysis approach and the *behavioural theory* (Polderman et al. (1998)) have also been developed. See, e.g., Oberst (1990); Fliess (1990); Pommaret (2001); Quadrat (2010).

At the end of the 90's, the algebraic analysis approach to linear differential systems was extended to linear differential time-delay systems (Fliess and al. (1998)) using *rings of differential constant time-delay operators*. In Chyzak et al. (2005), an algebraic analysis approach to linear systems over *Ore algebras of functional operators* was developed. Different algorithmic aspects of this approach were studied using computer algebra methods for certain classes of noncommutative polynomial rings. Continuous, discrete, differential, constant delay, or mixed systems can be studied within this common effective mathematical approach. In Quadrat et al. (2016), an algebraic analysis approach to linear systems defined by differential time-dependent delay equations was initiated based on *Ore extensions of differential time-dependent delay operators*. *Rings of integro-differential-delay operators* were introduced in Quadrat (2015) to study the transformations between first-order linear systems with delayed inputs and purely differential linear systems (e.g., *Artstein's reduction*). Indeed, within the algebraic analysis approach to linear system theory, transformations and equivalences of linear functional systems can be studied by means of *homomorphisms* and *isomorphisms* between the finitely presented modules (Rotman (2009)) defined by the system matrices (Cluzeau et al. (2008)). This approach was further developed in Cluzeau et al. (2018) and computer algebra aspects of Quadrat (2015) were investigated.

In this paper, using rings of integro-differential operators, we initiate a new algebraic analysis approach to linear system theory (Kalman et al. (1969)). We first state again the general construction of rings of integro-differential operators (Quadrat (2015); Cluzeau et al. (2018)). Then, we show how the standard *method of variation of constants* can simply be rewritten as an identity of integro-differential operators and we explain its module-theoretic interpretation. Using this result, we can then prove that a state-space representation and its standard integral representation define the same module over a ring of integro-differential operators. Moreover, we prove that this module is *stably free* (Lam (1999); Rotman (2009)). Classical results of linear system theory (Kalman et al. (1969)) can be rewritten within this module-theoretic approach over rings of integro-differential operators. We finally shortly explain how the *reachability* of linear systems can be rewritten within this approach in a way that encapsulates both the behaviour and the parametrization philosophies.

2. ALGEBRAIC ANALYSIS APPROACH

Let us briefly state again the *algebraic analysis approach* to linear system theory. Let $\mathcal{D}$ be a ring of functional (e.g., differential, shift, time-delay) operators that is not necessarily supposed to be commutative. Let $R \in \mathcal{D}^{q\times p}$ be a $q \times p$ matrix with entries in $\mathcal{D}$ and $.R : \mathcal{D}^{1\times q} \longrightarrow \mathcal{D}^{1\times p}$ the *left $\mathcal{D}$-homomorphism* (i.e., the left $\mathcal{D}$-linear map) defined by $(.R)(\lambda) = \lambda\, R$ for all $\lambda \in \mathcal{D}^{1\times q}$. In what follows, we shall simply denote the image $\mathrm{im}_{\mathcal{D}}(.R)$ of $.R$ by $\mathcal{D}^{1\times q}\, R$. Then, we can define the *finitely presented left $\mathcal{D}$-module*:

$$\mathcal{M} = \mathrm{coker}_{\mathcal{D}}(.R) = \mathcal{D}^{1\times p}/(\mathcal{D}^{1\times q}\, R).$$

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See, e.g., Rotman (2009). Let $\pi : \mathcal{D}^{1\times p} \longrightarrow \mathcal{M}$ denote the canonical projection onto $\mathcal{M}$, i.e., the left $\mathcal{D}$-homomorphism which maps λ onto its *residue class* $\pi(\lambda)$ in $\mathcal{M}$ (i.e., $\pi(\lambda') = \pi(\lambda)$ for $\lambda' \in \mathcal{D}^{1\times p}$ if $\lambda' - \lambda \in \mathrm{im}_{\mathcal{D}}(.R)$, i.e., if there exists $\mu \in D^{1\times q}$ such that $\lambda' = \lambda + \mu\, R$). Let $\{f_j\}_{j=1,\ldots,p}$ denote the *standard basis* of $\mathcal{D}^{1\times p}$, namely, f_j is the row vector of size p defined by 1 at the j^{th} entry and 0 elsewhere. If we set $y_j = \pi(f_j)$ for $j = 1, \ldots, p$, then using the fact that every $m \in \mathcal{M}$ is of the form $\pi(\lambda)$ for a certain $\lambda = (\lambda_1 \ \ldots \ \lambda_p) \in \mathcal{D}^{1\times p}$, by the left $\mathcal{D}$-linearity of π, we obtain $m = \sum_{j=1}^{p} \lambda_j\, \pi(f_j) = \sum_{j=1}^{p} \lambda_j\, y_j$, i.e., m is a left $\mathcal{D}$-linear combination of the y_j's. Thus, the left $\mathcal{D}$-module $\mathcal{M}$ is *finitely generated* by $\{y_j\}_{j=1,\ldots,p}$.

Since the rows $R_{i\bullet} = (R_{i1} \ \ldots \ R_{ip})$ of R, $i = 1, \ldots, q$, belong to $\mathcal{D}^{1\times q}\, R$, their residue classes $\pi(R_{i\bullet})$ are reduced to 0, which, by the left $\mathcal{D}$-linearity of π, then yields:

$$\pi(R_{i\bullet}) = \sum_{j=1}^{p} R_{ij}\, \pi(f_j) = \sum_{j=1}^{p} R_{ij}\, y_j = 0, \quad i = 1, \ldots, q.$$

If we note $y = (y_1 \ \ldots \ y_p)^T$, then the above equations can be rewritten as $R\, y = 0$. Hence, the family of generators $\{y_j\}_{j=1,\ldots,p}$ satisfies the relations $R\, y = 0$ (and all their left $\mathcal{D}$-linear combinations). The left $\mathcal{D}$-module $\mathcal{M}$ is then said to be *finitely presented* (see Lam (1999); Rotman (2009)).

Let $\mathcal{F}$ be a left $\mathcal{D}$-module and $R \in \mathcal{D}^{q\times p}$. Then,

$$\ker_{\mathcal{F}}(R.) = \{\eta \in \mathcal{F}^{p\times 1} \mid R\, \eta = 0\}$$

is the *abelian group* (i.e., the $\mathbb{Z}$-module) formed by all the $\mathcal{F}$-solutions of the linear system $R\, \eta = 0$. Note that if $\mathcal{D}$ is a $\Bbbk$-algebra, where $\Bbbk$ is a field, then $\ker_{\mathcal{F}}(R.)$ inherits a $\Bbbk$-vector space structure. Within the *behaviour approach* (Polderman et al. (1998)), $\ker_{\mathcal{F}}(R.)$ is called a *behaviour*. A standard result in *homological algebra* asserts that

$$\ker_{\mathcal{F}}(R.) \cong \hom_{\mathcal{D}}(\mathcal{M}, \mathcal{F}),$$

where $\hom_{\mathcal{D}}(\mathcal{M}, \mathcal{F})$ denotes the abelian group (the $\Bbbk$-vector space) formed by all the left $\mathcal{D}$-homomorphisms from $\mathcal{M}$ to $\mathcal{F}$, and $\cong$ stands for an *isomorphism*, namely, a bijective homomorphism (Lam (1999); Rotman (2009)). Hence, the behaviour $\ker_{\mathcal{F}}(R.)$ can be intrinsically studied by the module-theoretic properties of $\hom_{\mathcal{D}}(\mathcal{M}, \mathcal{F})$, and thus, by the properties of the left $\mathcal{D}$-modules $\mathcal{M}$ and $\mathcal{F}$.

The main benefit of the algebraic analysis approach to linear system theory is that built-in system properties can be translated in terms of module properties, and those properties that can be effectively checked using homological algebra and computer algebra methods. For more details, see the references mentioned in Section 1.

Let $R' \in D^{q'\times p'}$ and $\mathcal{M}' = \mathrm{coker}_{\mathcal{D}}(.R')$ be the left $\mathcal{D}$-module finitely presented by R'. It can be shown that $f \in \hom_{\mathcal{D}}(\mathcal{M}, \mathcal{M}')$ is defined by a pair of matrices (P, Q), where $P \in \mathcal{D}^{p\times p'}$ and $Q \in \mathcal{D}^{q\times q'}$, satisfying the identity:

$$R\, P = Q\, R'. \tag{1}$$

More precisely, if $\pi' : D^{1\times p'} \longrightarrow \mathcal{M}'$ denotes the canonical projection onto $\mathcal{M}'$, then f is defined by:

$$\forall\, \lambda \in \mathcal{D}^{1\times p}, \quad f(\pi(\lambda)) = \pi'(\lambda\, P).$$

See, e.g., Cluzeau et al. (2008). Using (1), we have $R\,(P\, \eta') = Q\,(R'\, \eta') = 0$ for all $\eta' \in \ker_{\mathcal{F}}(R'.)$, i.e., f induces the following homomorphism of behaviours:

$$\begin{aligned} f^{\star} : \ker_{\mathcal{F}}(R'.) &\longrightarrow \ker_{\mathcal{F}}(R.) \\ \eta' &\longmapsto \eta = P\, \eta'. \end{aligned}$$

Thus, a natural way to study behaviour homomorphisms (e.g., equivalences) is through the study of the homomorphisms of the left modules finitely presented by the corresponding system matrices. See Cluzeau et al. (2008).

In the literature (see the references given in Section 1), $\mathcal{D}$ is usually the ring of ordinary or partial differential operators, the ring of differential (time-dependent) delay operators, the ring of multi-shift operators, etc. In the next section, we shall introduce the ring $\mathcal{D}$ of integro-differential operators and then use it to develop a new algebraic analysis approach to linear system theory.

3. RINGS OF INTEGRO-DIFFERENTIAL OPERATORS

Let $\mathcal{A}$ be a $\Bbbk$-algebra, where $\Bbbk$ is a field of characteristic 0 (e.g., $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$). We do not assume that $\mathcal{A}$ is a commutative ring. Let us now suppose that the ring $\mathrm{end}_{\Bbbk}(\mathcal{A})$ formed by the all the $\Bbbk$-endomorphisms of $\mathcal{A}$ contain a *derivation* ∂, namely, $\partial \in \mathrm{end}_{\Bbbk}(\mathcal{A})$ satisfies the standard Leibniz rule:

$$\forall\, a_1,\, a_2 \in \mathcal{A}, \quad \partial(a_1\, a_2) = \partial(a_1)\, a_2 + a_1\, \partial(a_2). \tag{2}$$

Then, $(\mathcal{A}, \partial)$ is called a *differential ring* and

$$\mathcal{C} = \{a \in \mathcal{A} \mid \partial(a) = 0\}$$

is the *subring of constants* of $\mathcal{A}$. Note that (2) yields

$$\partial(1) = \partial(1 \times 1) = \partial(1) + \partial(1) = 2\,\partial(1) \implies \partial(1) = 0,$$

and since ∂ is $\Bbbk$-linear, $\partial(c) = c\,\partial(1) = 0$ for all $c \in \Bbbk$, which shows that $\Bbbk \subseteq \mathcal{C}$. We also have $c_1\,(a\, c_2) = (c_1\, a)\, c_2$ for all c_1, $c_2 \in \mathcal{C}$ and for all $a \in \mathcal{A}$, which shows that $\mathcal{A}$ has a $\mathcal{C}$-*bimodule structure* (Rotman (2009)). Note that

$$\forall\, c_1,\, c_2 \in \mathcal{C},\ \forall\, a \in \mathcal{A} : \ \partial(c_1\, a\, c_2) = c_1\, \partial(a\, c_2) = \partial(c_1\, a)\, c_2,$$

i.e., ∂ is a $\mathcal{C}$-*bimodule endomomorphism*.

Example 1. The polynomial ring $\Bbbk[t]$ or the Laurent polynomial ring $\Bbbk[t, t^{-1}]$ in t with coefficients in a field $\Bbbk$, or the ring of $\Bbbk$-valued smooth (resp., analytic, meromorphic) functions in an open subset $\mathcal{U}$ of $\mathbb{R}$ (resp., of $\mathbb{C}$) with $\Bbbk = \mathbb{R}$ or $\mathbb{C}$ are standard examples of commutative differential rings with the derivation $\partial = d/dt$. If $(\mathcal{A}, \partial)$ is one of these differential ring and $n \in \mathbb{Z}_{>0}$, then $(\mathcal{A}^{n\times n}, \partial)$ is a noncommutative differential ring with $\mathcal{C} = \Bbbk^{n\times n}$.

Let $\mathbb{1}$ be the identity of $\mathrm{end}_{\Bbbk}(\mathcal{A})$: $\mathbb{1}(a) = a$ for all $a \in \mathcal{A}$.

Let us now consider a differential ring $(\mathcal{A}, \partial)$ for which there exists $I \in \mathrm{end}_{\Bbbk}(\mathcal{A})$ satisfying the conditions:

(1) I is a $\mathcal{C}$-bimodule endomorphism, i.e.:
$$\forall\, a \in \mathcal{A}, \quad \forall\, c_1,\, c_2 \in \mathcal{C}, \quad c_1\, I(a\, c_2) = I(c_1\, a)\, c_2.$$
(2) $\partial \circ I = \mathbb{1}$, i.e., $\partial(I(a)) = a$ for all $a \in \mathcal{A}$.
(3) The $\mathcal{C}$-bimodule endomorphism $e = \mathbb{1} - I \circ \partial$ of $\mathcal{A}$ is *multiplicative*, namely:
$$\forall\, a_1,\, a_2 \in \mathcal{A} : \ e(a_1\, a_2) = e(a_1)\, e(a_2).$$

Then, $(\mathcal{A}, \partial, I)$ is called an *integro-differential ring*.

Example 2. Let $\mathcal{U}$ be an open subset of $\mathbb{R}$, $t_0 \in \mathcal{U}$, and $(\mathcal{A} = C^{\infty}(\mathcal{U}), \partial = d/dt)$ be the differential ring of real-valued smooth functions on $\mathcal{U}$. Let $I(a)(t) = \int_{t_0}^{t} a(\tau)\, d\tau$ be the standard integral for all $a \in \mathcal{A}$. Using classical calculus identities, we then have

$$\frac{d}{dt}\int_{t_0}^{t} a(\tau)\,d\tau = a(t),\ e(a) = a(t) - \int_{t_0}^{t} \dot{a}(\tau)\,d\tau = a(t_0),$$

for all $a \in \mathcal{A}$. Thus, e is the *evaluation* at t_0, which is multiplicative. Moreover, $\mathcal{C} = \mathbb{R}$ and I is a $\mathbb{R}$-endomorphism of $\mathcal{A}$. Thus, (A, ∂, I) is an integro-differential ring. Finally, if $n \in \mathbb{Z}_{>0}$, then $(\mathcal{A} = C^\infty(\mathcal{U})^{n\times n}, \partial = d/dt, I = \int_{t_0}^{t} ds)$ is a noncommutative integro-differential ring with $\mathcal{C} = \Bbbk^{n\times n}$.

Now, note that every $a \in \mathcal{A}$ yields $\underline{a} \in \mathrm{end}_{\Bbbk}(\mathcal{A})$ defined by the multiplication by a, namely, $\underline{a} : b \in \mathcal{A} \longmapsto a\,b \in \mathcal{A}$.

We can now define the *ring of integro-differential operators* over $\mathcal{A}$ (Quadrat (2015); Cluzeau et al. (2018); Quadrat et al. (2020)) that plays an important role in what follows.

Definition 1. Let $(\mathcal{A}, \partial, I)$ be an integro-differential ring. Then, the *ring of integro-differential operators* is the $\Bbbk$-subalgebra of $\mathrm{end}_{\Bbbk}(\mathcal{A})$ generated by ∂, I, e, and by all the multiplications $\underline{a}$ for all $a \in \mathcal{A}$. Equivalently, $\mathcal{I}$ is the noncommutative polynomial ring $\mathcal{A}\langle\partial, I, e\rangle$ over $\mathcal{A}$ defined by ∂, I, e, and $\underline{a}$ for all $a \in \mathcal{A}$ which satisfy the relations:

$$\forall\, a \in \mathcal{A}, \quad \begin{cases} \partial \circ \underline{a} = \underline{a} \circ \partial + \underline{\partial(a)}, \\ \partial \circ I = \mathbb{1}, \\ I \circ \partial = \mathbb{1} - e, \\ e \circ \underline{a} = \underline{e(a)} \circ e. \end{cases} \tag{3}$$

To simplify the notations, in what follows, we shall remove the sign of composition $\circ$ in the expressions of the elements of $\mathcal{I}$. For instance, $\partial \circ I$ and $\mathbb{1} - I \circ \partial$ will be simply written $\partial\, I$ and $\mathbb{1} - I\,\partial$. Moreover, when the operator context is clear, $\underline{a}$ will be simply be denoted by a. For instance, the first and last identities of (3) are then rewritten as follows:

$$\forall\, a \in \mathcal{A}, \quad \partial\, a = a\,\partial + \partial(a), \quad e\,a = e(a)\,e. \tag{4}$$

With these conventions, we then have:

$$\forall\, a \in \mathcal{A},\ \partial(a),\ I(a),\ e(a) \in \mathcal{A},\ \partial\, a,\ I\, a,\ e\, a \in \mathrm{end}_{\Bbbk}(\mathcal{A}).$$

The only possible ambiguity is when $\partial(a)$, $I(a)$, and $e(a)$ are considered as multiplication operators as it is the case in (4). But the context will always be clear enough.

Remark 1. The first identity of (3) comes from:

$$\forall\, b \in \mathcal{A},\ (\partial\, a)(b) = \partial(a\, b) = \partial(a)\, b + a\, \partial(b) = (a\,\partial + \partial(a))(b).$$

The second and the third identities of (3) are by construction. Finally, the last identity of (3) is a direct consequence of the above third axiom of an integro-differential ring, i.e.:

$$\forall\, b \in \mathcal{A}, \quad (e\, a)(b) = e(a\, b) = e(a)\, e(b) = (e(a)\, e)(b).$$

Note that from (3), we can deduce the following identities:

$$\begin{gathered} e^2 = e, \quad e\, I = 0, \quad \partial\, e = 0, \\ \forall\, a \in \mathcal{A}, \quad I\, a\, \partial = -I\, \partial\, a + a - e(a)\, e, \\ I\, a\, I = [I(a), I] := I(a)\, I - I\, I(a). \end{gathered} \tag{5}$$

These identities can be proved using $\partial\, I = \mathbb{1}$ as follows:

$$\begin{gathered} e^2 = (\mathbb{1} - I\,\partial)\,(\mathbb{1} - I\,\partial) = \mathbb{1} - I\,\partial = e, \\ e\, I\, \partial = e\,(1 - e) = e - e^2 = 0 \ \Rightarrow\ e\, I = e\, I\, \partial\, I = 0, \\ \partial\, e = \partial\,(\mathbb{1} - I\,\partial) = \partial - \partial = 0. \end{gathered}$$

The last but one identity of (5) corresponds to the *integration by parts*: using (2) and $I\,\partial = \mathbb{1} - e$, we have:

$$I\, a\, \partial = I\,(\partial\, a - \partial(a)) = (\mathbb{1} - e)\, a - I\, \partial(a) = a - e(a)\, a - I\, \partial(a).$$

Finally, the last identity of (5) can be proved as follows. Using the first identity of (3) with $I(a)$ instead of a, we first get $\partial\, I(a) = I(a)\,\partial + \partial(I(a)) = I(a)\,\partial + a$, i.e.:

$$a = \partial\, I(a) - I(a)\,\partial = [\partial, I(a)].$$

Using $I\,\partial = \mathbb{1} - e$, $\partial\, I = \mathbb{1}$, and $e\, I = 0$, we finally obtain:

$$\begin{aligned} I\, a\, I &= I\,(\partial\, I(a) - I(a)\,\partial)\, I = (\mathbb{1} - e)\, I(a)\, I - I\, I(a) \\ &= I(a)\, I - I\, I(a) = [I(a), I]. \end{aligned}$$

Setting $a = 1$ in this identity and using $I(1) = t - t_0$, we have $I^2 = [t - t_0, I] = [t, I]$, which shows that I^2 can be expressed as a $\Bbbk[t]$-linear combination of I. More generally, this result holds for I^n for all $n \in \mathbb{Z}_{\geq 0}$.

Using (3), (5) and the last remark, every element of $\mathcal{I}$ is a finite sum of terms of the form $a\,\partial^j$, $b\, I\, c$, and $d\, e\,\partial^k$, where a, b, c, $d \in \mathcal{A}$ and $j, k \in \mathbb{Z}_{\geq 0}$ (Cluzeau et al. (2018)).

Example 3. We consider again the integro-differential ring $\mathcal{A}$ defined in Example 2. Then, $\Bbbk = \mathbb{R}$ and the ring of integro-differential operators $\mathcal{I}$ is the $\Bbbk$-subalgebra of $\mathrm{end}_{\Bbbk}(\mathcal{A})$ generated by the $\Bbbk$-endomorphisms of $\mathcal{A}$:

$$\begin{aligned} & a : b(\cdot) \longmapsto a(\cdot)\, b(\cdot), \\ & \partial : a(\cdot) \longmapsto \dot{a}(\cdot),\ \partial(a)(t) = \dot{a}(t) = \frac{da(t)}{dt}, \\ & I : a(\cdot) \longmapsto b(\cdot),\ I(a)(t) = b(t) = \int_{t_0}^{t} a(\tau)\, d\tau, \\ & e : a(\cdot) \longmapsto a(t_0). \end{aligned}$$

Hence, the element $a_1\, I\, a_2$ of $\mathcal{I}$ is the operator defined by:

$$a_1\, I\, a_2 : b(\cdot) \longmapsto c(\cdot),\ c(t) = a_1(t) \int_{t_0}^{t} a_2(\tau)\, b(\tau)\, d\tau.$$

For instance, the first identity of (3) comes from:

$$\begin{aligned} \forall\, b \in \mathcal{A},\ (\partial\, a)(b) = \frac{d}{dt}(a\, b) &= a \left(\frac{db}{dt}\right) + \left(\frac{da}{dt}\right) b \\ &= (a\,\partial + \partial(a))(b). \end{aligned}$$

Similarly, the other identities of (3) come from:

$$\forall\, b \in \mathcal{A}, \quad (\partial\, I)(b)(t) = \frac{d}{dt}\int_{t_0}^{t} b(\tau)\, d\tau = b(t),$$

$$(I\,\partial)(b)(t) = \int_{t_0}^{t} \dot{b}(\tau)\, d\tau = b(t) - b(t_0) = (\mathbb{1} - e)(b)(t),$$

$$\forall\, a,\ b \in \mathcal{A}, \quad (e\, a)(b) = e(a\, b) = a(t_0)\, b(t_0) = (e(a)\, e)(b).$$

Remark 2. The fact that ∂ is a left inverse but not a two-sided inverse of I implies that $\mathcal{I}$ is not a *Dedekind finite ring* (Lam (1999)), and thus, not a *noetherian ring*.

Finally, as shown in Quadrat et al. (2020), more evaluations can be added to $\mathcal{I}$. For instance, if we consider the ring of integro-differential operators $\mathcal{I}$ defined in Example 3, then we can add the evaluation e' defined by $e'(a)(t) = a(t_1)$ for all $a \in \mathcal{A}$, where $t_1 \in \mathcal{U}$ is fixed and different from t_0, to get the ring of integro-differential-evaluation operators $\mathcal{J} = \mathcal{A}\langle\partial, I, e, e'\rangle$, where e' satisfies:

$$\forall\, a \in \mathcal{A},\ e'\, a = e'(a)\, e',\ e\, e' = e',\ e'\, e = e,\ \partial\, e' = 0. \tag{6}$$

4. THE METHOD OF VARIATION OF CONSTANTS

In this section, we consider the ring of integro-differential operators $\mathcal{I} = \mathcal{A}\langle\partial, I, e\rangle$ defined in Example 3. Let us explain how the *variation of constants method* can be interpreted in terms of operators and within module theory.

Let Φ be the *transition matrix* of the first-order system:

$$\dot{x}(t) = A(t)\, x(t). \tag{7}$$

In what follows, we shall suppose that $\Phi : t \longmapsto \Phi(t, t_0)$ belongs to $\mathcal{A}^{n\times n}$. We recall that Φ is an invertible matrix, i.e., $\Phi^{-1} \in \mathcal{A}^{n\times n}$, satisfying (7) and $\Phi(t_0, t_0) = \mathbb{1}_n$.

Remark 3. If Φ is the transmission matrix of (7), then using (3), we get $I\,(\partial\,\mathbb{1}_n - A) = \mathbb{1}_n - e\,\mathbb{1}_n - I\,A$ and:

$$(\mathbb{1}_n - e\,\mathbb{1}_n - I\,A)\,\Phi = 0 \iff \Phi = \mathbb{1}_n + I(A\,\Phi).$$

The last equation can be used to define *Picard iteration*, and thus, *Peano–Baker series* for Φ, namely:

$$\Phi = \mathbb{1}_n + I(A) + (I\,A\,I)(A) + (I\,A\,I\,A\,I)(A) + \dots$$

Let us determine when $S = a_0\,I\,a_1 + a_2$, where a_0, a_1, and $a_2 \in \mathcal{A}^{n\times n}$, is a right inverse of R. Using (3), we have:

$$\begin{aligned} R\,S &= (\partial\,\mathbb{1}_n - A)\,(a_0\,I\,a_1 + a_2)\\ &= (\dot{a}_0 - A\,a_0)\,I\,a_1 + a_2\,\partial + \dot{a}_2 - A\,a_2 + a_0\,a_1. \end{aligned}$$

If $a_1 = 0$, then $R\,S = \mathbb{1}_n$ has no solution. If $a_1 \neq 0$, then $R\,S = \mathbb{1}_n$ if and only if:

$$\begin{cases} \dot{a}_0 - A\,a_0 = 0,\\ a_2 = 0,\\ \dot{a}_2 - A\,a_2 + a_0\,a_1 = \mathbb{1}_n, \end{cases} \iff \begin{cases} \dot{a}_0 - A\,a_0 = 0,\\ a_2 = 0,\\ a_0\,a_1 = \mathbb{1}_n, \end{cases}$$

$$\iff \begin{cases} a_0(t) = \Phi(t,t_0)\,c_0,\ c_0 \in \Bbbk^{n\times n},\\ a_2 = 0,\\ c_0\,a_1 = \Phi(t,t_0)^{-1} = \Phi(t_0,t). \end{cases}$$

The matrix R then has the following right inverse:

$$S = \Phi(t,t_0)\,c_0\,I\,a_1 = \Phi(t,t_0)\,I\,c_0\,a_1 = \Phi(t,t_0)\,I\,\Phi(t_0,t).$$

Using $R\,S = \mathbb{1}_n$, we get $R\,(S\,f) = f$ for all $f \in \mathcal{A}^{n\times 1}$, i.e.,

$$\begin{aligned} (S\,f)(t) &= \Phi(t,t_0)\,I\,\Phi(t_0,t)\,f(t) = \Phi(t,t_0)\int_{t_0}^{t}\Phi(t_0,\tau)\,f(\tau)\,d\tau\\ &= \int_{t_0}^{t}\Phi(t,t_0)\,\Phi(t_0,\tau)\,f(\tau)\,d\tau = \int_{t_0}^{t}\Phi(t,\tau)\,f(\tau)\,d\tau \end{aligned} \tag{8}$$

is a particular solution of the inhomogeneous system:

$$\dot{x}(t) - A(t)\,x(t) = f(t). \tag{9}$$

Note that $R\,S = \mathbb{1}_n$ also yields $\lambda = (\lambda\,R)\,S$, showing that $\ker_{\mathcal{I}}(.R) = 0$, i.e., the left $\mathcal{I}$-homomorphism $.R$ is injective. Then, we have the *short exact sequence* of $\mathcal{I}$-modules

$$0 \longrightarrow \mathcal{I}^{1\times n} \xrightarrow{.R} \mathcal{I}^{1\times n} \xrightarrow{\pi} \mathcal{M} \longrightarrow 0, \tag{10}$$

i.e., $.R$ is injective, $\ker\pi = \mathrm{im}_{\mathcal{I}}(.R)$, and π is surjective (Rotman (2009)). The identity $R\,S = \mathbb{1}_n$ then shows that $\Pi = S\,R \in \mathcal{I}^{n\times n}$ satisfies $\Pi^2 = \Pi$, i.e., Π is an idempotent of $\mathcal{I}^{n\times n}$, which yields the direct sum decomposition:

$$\mathcal{I}^{1\times n} = \ker_{\mathcal{I}}(.\Pi) \oplus \mathrm{im}_{\mathcal{I}}(.\Pi), \tag{11}$$

The fact that $.R$ is injective and $.S$ is surjective yield

$$\ker_{\mathcal{I}}(.\Pi) = \ker_{\mathcal{I}}(.S), \quad \mathrm{im}_{\mathcal{I}}(.\Pi) = \mathrm{im}_{\mathcal{I}}(.R) \cong \mathcal{I}^{1\times n}.$$

Using $\mathcal{I}^{1\times n} = \ker_{\mathcal{I}}(.S) \oplus \mathrm{im}_{\mathcal{I}}(.R)$, we first get

$$\mathcal{M} = \mathcal{I}^{1\times n}/\mathrm{im}_{\mathcal{I}}(.R) \cong \ker_{\mathcal{I}}(.S),$$

and $\mathcal{I}^{1\times n} = \ker_{\mathcal{I}}(.S) \oplus \mathrm{im}_{\mathcal{I}}(.R)$ and $\mathrm{im}_{\mathcal{I}}(.R) \cong \mathcal{I}^{1\times n}$ yield

$$\mathcal{M} \oplus \mathcal{I}^{1\times n} \cong \mathcal{I}^{1\times n}, \tag{12}$$

showing that $\mathcal{M}$ is a *stably free* left $\mathcal{I}$-module, and thus, a *projective* left $\mathcal{I}$-module (Lam (1999); Rotman (2009)). The fact that $\mathcal{M}$ is a stably free is also a direct consequence of the *spliting* of (10), which yields (12) (Rotman (2009)).

Let us now compute $\Pi = S\,R$. To do so, we first note that $\Psi = \Phi^{-1}$ satisfies $\dot{\Psi} + \Psi\,A = 0$ and, using $\Psi\,\partial = \partial\,\Psi - \dot{\Psi}$ (integration by parts; see the first identity of (3)), the third and fourth identities of (3), $e(\Psi) = \Phi(t_0,t_0) = \mathbb{1}_n$, yield:

$$\begin{aligned} \Pi = S\,R &= (\Phi\,I\,\Psi)\,(\partial\,\mathbb{1}_n - A) = \Phi\,I\,\partial\,\Psi - \Phi\,I\,(\dot{\Psi} + \Psi\,A)\\ &= \Phi\,(\mathbb{1} - e)\,\Psi = \mathbb{1}_n - \Phi\,e(\Psi)\,e = \mathbb{1}_n - \Phi\,e. \end{aligned}$$

Remark 4. Since $\Phi \neq 0$, the above computations show that $R\,S = \mathbb{1}_n$ but $S\,R \neq \mathbb{1}_n$, which proves that $\mathcal{I}$ is not *stably finite ring* (see, e.g., Lam (1999)), a fact which is consistent with the fact that (12) does not yield $\mathcal{M} = 0$.

The existence of a right inverse S of R implies that the short exact sequence (10) *splits*, i.e., the existence of a left $\mathcal{I}$-homomorphism $\rho : \mathcal{M} \longrightarrow \mathcal{I}^{1\times n}$ satisfying the identity $(.R)\circ(.S) + \rho\circ\pi = \mathrm{id}_{\mathcal{I}^{1\times n}}$. For more details, see, e.g., Rotman (2009). This split short exact sequence of left $\mathcal{I}$-modules can be displayed as follows:

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{I}^{1\times n} \arrow[r, shift left, ".R"] & \mathcal{I}^{1\times n} \arrow[l, shift left, ".S"] \arrow[r, shift left, "\pi"] & \mathcal{M} \arrow[l, shift left, "\rho"] \arrow[r] & 0. \end{tikzcd} \tag{13}$$

For $m = \pi(\lambda) \in \mathcal{M}$, where $\lambda \in \mathcal{I}^{1\times n}$, we have:

$$\rho(m) = \rho(\pi(\lambda)) = \lambda\,(\mathbb{1}_n - S\,R) = \lambda\,\Phi\,e. \tag{14}$$

Remark 5. Note that $\Phi\,e$ is an *idempotent* element of $\mathcal{I}^{n\times n}$ since, using $e^2 = e$ and $e(\Phi) = \mathbb{1}_n$, we then have:

$$(\Phi\,e)^2 = \Phi\,e(\Phi\,e) = \Phi\,e(\Phi)\,e^2 = \Phi\,e.$$

Similarly, $\mathbb{1}_n - \Phi\,e$ is an idempotent of $\mathcal{I}^{n\times n}$. Moreover, $\ker_{\mathcal{I}}(.(\mathbb{1}_n - \Phi\,e)) = \mathrm{im}_{\mathcal{I}}(.\Phi\,e)$ and $\mathrm{im}_{\mathcal{I}}(.(\mathbb{1}_n - \Phi\,e)) = \ker_{\mathcal{I}}(.\Phi\,e)$, which shows that (11) becomes:

$$\mathcal{I}^{1\times n} = \mathrm{im}_{\mathcal{I}}(.\Phi\,e) \oplus \ker_{\mathcal{I}}(.\Phi\,e).$$

Similarly, we have $\mathcal{I}^{n\times 1} = \mathrm{im}_{\mathcal{I}}(\Phi\,e.) \oplus \ker_{\mathcal{I}}(\Phi\,e.)$. The left (resp., right) $\mathcal{I}$-modules $\ker_{\mathcal{I}}(.\Phi\,e)$ and $\mathrm{im}_{\mathcal{I}}(.\Phi\,e)$ (resp., $\ker_{\mathcal{I}}(\Phi\,e.)$ and $\mathrm{im}_{\mathcal{I}}(\Phi\,e.)$) are thus direct summands of the direct sums of $\mathcal{I}$, i.e., they are *projective* left (resp., right) $\mathcal{I}$-modules. For more details, see, e.g., Rotman (2009).

Dualizing (13) with coefficients in $\mathcal{F}$, i.e., applying the *contravariant functor* $\hom_{\mathcal{I}}(\cdot,\mathcal{F})$ to split short exact sequence (13) and using the isomorphism $\ker_{\mathcal{F}}(R.) \cong \hom_{\mathcal{I}}(\mathcal{M},\mathcal{F})$ of $\Bbbk$-vector spaces, we obtain the following split short exact sequence of $\Bbbk$-vector spaces

$$\begin{tikzcd} 0 \arrow[r] & \ker_{\mathcal{F}}(R.) \arrow[r, shift left, "i"] & \mathcal{F}^{n\times 1} \arrow[l, shift left, "\Phi\,e."] \arrow[r, shift left, "R."] & \mathcal{F}^{n\times 1} \arrow[l, shift left, "S."] \arrow[r] & 0, \end{tikzcd} \tag{15}$$

where i is the canonical injection. See, e.g., Rotman (2009). Note that $\Phi\,e. : \mathcal{F}^{n\times 1} \longrightarrow \ker_{\mathcal{F}}(R.)$ is well-defined since:

$$\forall\,\eta \in \mathcal{F}^{n\times 1},\ (\partial\,\mathbb{1}_n - A)\,\Phi\,e\,\eta = (\Phi\,\partial + \dot{\Phi} - A\,\Phi)\,e\,\eta = 0.$$

In what follows, we shall simply denote $i \circ \Phi\,e.$ by $\Phi\,e$.

The splitting of (15) is equivalent to the following identity:

$$\Phi\,e + (\Phi\,I\,\Phi^{-1})\,(\partial\,\mathbb{1}_n - A) = \mathbb{1}_n. \tag{16}$$

Thus, for every $\xi \in \mathcal{A}^{n\times 1}$, if we set $\zeta(t) = \dot{\xi}(t) - A(t)\,\xi(t)$, then the operator identity (16) yields:

$$\begin{aligned} \xi(t) &= (\Phi\,e)\,\xi(t) + \Phi\,I\,\Phi^{-1}\,(\partial\,\mathbb{1}_n - A)\,\xi(t)\\ &= \Phi(t,t_0)\,\xi(t_0) + (\Phi\,I\,\Phi^{-1}\,\zeta(t)\\ &= \Phi(t,t_0)\,\xi(t_0) + \int_{t_0}^{t}\Phi(t,\tau)\,\zeta(\tau)\,d\tau. \end{aligned} \tag{17}$$

We find again *method of the variation of constants.*

If $f = B\,u$, where $B \in \mathcal{A}^{n\times m}$, then the general solution of

$$\dot{x}(t) = A(t)\,x(t) + B(t)\,u(t) \tag{18}$$

is then defined by:

$$x(t) = \Phi(t, t_0)\, x(t_0) + \int_{t_0}^{t} \Phi(t, \tau)\, B(\tau)\, u(\tau)\, d\tau. \qquad (19)$$

Remark 6. The natural triviality of the *Cauchy problem*

$$\begin{pmatrix} \partial\, \mathbb{1}_n - A \\ e\, \mathbb{1}_n \end{pmatrix} x(t) = 0 \;\Leftrightarrow\; \begin{cases} \dot{x}(t) = A(t)\, x(t), \\ x(t_0) = 0, \end{cases} \;\Leftrightarrow\; x = 0,$$

is a direct consequence of (16), which can be rewritten as

$$(S \quad \Phi)\, C = \mathbb{1}_n, \quad C := \begin{pmatrix} \partial\, \mathbb{1}_n - A \\ e\, \mathbb{1}_n \end{pmatrix}. \qquad (20)$$

Indeed, $C\, x(t) = 0$ then yields $(S \quad \Phi)\, C\, x(t) = x(t) = 0$. This result can be understood as a *Nullstellensatz theorem*. Note that the condition that all the entries of Φ belong to $\mathcal{A}$ plays a similar role as the *algebraically closed field* condition in algebraic geometry. Indeed, if this is not the case, then the Bézout identity (20) does not hold over $\mathcal{I}$.

Using (20), we have the following idempotent of $\mathcal{I}^{2n \times 2n}$:

$$\Theta = C\, (S \quad \Phi) = \begin{pmatrix} \mathbb{1}_n & \Phi\, \partial \\ 0 & e\, \mathbb{1}_n \end{pmatrix}.$$

Thus, we clearly have $\ker_{\mathcal{I}}(.\Theta) = \mathrm{im}_{\mathcal{I}}(.\Theta')$, where:

$$\Theta' = \mathbb{1}_{2n} - \Theta = \begin{pmatrix} 0 & -\Phi\, \partial \\ 0 & (1 - e)\, \mathbb{1}_n \end{pmatrix}.$$

Using $(1 - e)\, \mathbb{1}_n = I\, \partial\, \mathbb{1}_n$ and the fact that Φ is invertible, we have $\ker_{\mathcal{I}}(.\Theta) = \mathrm{im}_{\mathcal{I}}(.D)$, where $D = (0 \quad \partial\, \mathbb{1}_n) \in \mathcal{I}^{1 \times 2n}$. Note that $\ker_{\mathcal{I}}(.D) = 0$ since $\mu \in \ker_{\mathcal{I}}(.D)$ yields $\mu\, \partial\, \mathbb{1}_n = 0$, i.e., $\mu = \mu\, \partial\, \mathbb{1}_n\, I = 0$. Using (20), we have $\mathcal{I}^{1 \times 2n}\, C = \mathcal{I}^{1 \times n}$, i.e., $\mathcal{C} = \mathrm{coker}_{\mathcal{I}}(.C) = 0$, and the following split exact sequence of left $\mathcal{I}$-modules holds

$$0 \longrightarrow \mathcal{I}^{1 \times n} \underset{.F}{\overset{.D}{\rightleftarrows}} \mathcal{I}^{1 \times 2n} \underset{.E}{\overset{.C}{\rightleftarrows}} \mathcal{I}^{1 \times n} \longrightarrow 0, \qquad (21)$$

where $E = (S \quad \Phi)$ and $F = \left(-\Phi^T \quad (I\, \mathbb{1}_n)^T\right)^T$. Fix $\zeta, \omega \in \mathcal{F}^{n \times 1}$. Let us now study the solvability of the following inhomogeneous linear system

$$C\, \xi = \begin{pmatrix} \zeta \\ \omega \end{pmatrix}, \qquad (22)$$

where ξ is sought in $\mathcal{F}^{n \times 1}$. Applying the functor $\hom_{\mathcal{I}}(\,\cdot\,, \mathcal{F})$ to (21), we get the split exact sequence of $\Bbbk$-vector spaces

$$0 \longleftarrow \mathcal{F}^{n \times 1} \underset{D.}{\overset{F.}{\rightleftarrows}} \mathcal{F}^{2n \times 1} \underset{C.}{\overset{E.}{\rightleftarrows}} \mathcal{F}^{n \times 1} \longleftarrow 0,$$

which shows that a necessary and sufficient condition for the solvability of (22) is $D\, (\zeta^T \quad \omega^T)^T = 0$, i.e., $\partial\, \omega = 0$. Let $\mathcal{F} = \mathcal{A}$ and $\omega = x_{t_0} \in \Bbbk^{n \times 1}$. Then, (22) is a Cauchy problem, whose unique solution in $\mathcal{A}^{n \times 1}$ is defined by:

$$\xi(t) = E\, (\zeta(t)^T \quad x_{t_0}^T)^T = (S \quad \Phi)\, (\zeta(t)^T \quad x_{t_0}^T)^T.$$

Let us now give an equivalent representation of (7), i.e., another *presentation* of $\mathcal{M}$. We have previously shown that $\Pi = S\, R = \mathbb{1}_n - \Phi\, e$. Clearly, $\mathcal{I}^{1 \times n}\, \Pi \subseteq \mathcal{I}^{1 \times n}\, R$. Using the identity $R\, \Pi = (R\, S)\, R = R$, $\mathcal{I}^{1 \times n}\, R \subseteq \mathcal{I}^{1 \times n}\, \Pi$, which yields $\mathrm{im}_{\mathcal{I}}(.\Pi) = \mathrm{im}_{\mathcal{I}}(.R)$, and thus, $\mathcal{M} = \mathrm{coker}_{\mathcal{I}}(.\Pi)$, i.e., Π is another presentation matrix of $\mathcal{M}$. Hence, we have $\ker_{\mathcal{F}}(R.) = \ker_{\mathcal{F}}(\Pi.)$, i.e., (7) is equivalent to:

$$((\mathbb{1}_n - \Phi\, e)\, \eta(t) = \eta(t) - \Phi(t, t_0)\, \eta(t_0) = 0$$
$$\Leftrightarrow\; \eta(t) = \Phi(t, t_0)\, \eta(t_0).$$

Note that $\{e\, x_i\}_{i=1,\ldots,n}$ is another set of generators of $\mathcal{M}$. Indeed, the left $\mathcal{I}$-module generated by $\{e\, x_i\}_{i=1,\ldots,n}$ is a left $\mathcal{I}$-submodule of $\mathcal{M}$. Now, $\mathcal{M} = \mathrm{coker}_{\mathcal{I}}(.\Pi)$ yields $x = \Phi\, e\, x$, where $x = (x_1, \ldots, x_n)^T$. Thus, the x_i's are left $\mathcal{A}$-linear combinations of the $e\, x_j$'s, which yields $\mathcal{M} = \sum_{i=1}^{n} \mathcal{I}\, e\, x_i$. Equivalently, using (20), we have $\mathcal{I}^{1 \times 2\, n}\, C = \mathcal{I}^{1 \times n}$ and:

$$\mathcal{M} = \mathcal{I}^{1 \times n} / \left(\mathcal{I}^{1 \times n} R\right) = \left(\mathcal{I}^{1 \times 2\, n}\, C\right) / \left(\mathcal{I}^{1 \times n} R\right).$$

We characterize the relations among the $e\, x_i$'s. Lemma 3.1 of Cluzeau et al. (2008) and $\ker_{\mathcal{I}}(.C) = \mathrm{im}_{\mathcal{I}}(.D)$ yield:

$$\mathcal{M} \cong \mathrm{coker}_{\mathcal{I}}\left(.\begin{pmatrix} \mathbb{1}_n & 0 \\ 0 & \partial\, \mathbb{1}_n \end{pmatrix}\right) \cong \mathcal{M}' = \mathrm{coker}_{\mathcal{I}}(.\partial\, \mathbb{1}_n).$$

Using Lemma 3.1 of Cluzeau et al. (2008), the isomorphism $\psi : \mathcal{M}' \longrightarrow \mathcal{M}$ is defined $\psi(\pi'(\lambda)) = \pi(\lambda\, e\, \mathbb{1}_n)$ and $\psi^{-1} : \mathcal{M} \longrightarrow \mathcal{M}'$ is defined by $\psi^{-1}(\pi(\lambda)) = \pi'(\lambda\, \Phi)$ for all $\lambda \in \mathcal{I}^{1 \times n}$. Let us check again that $\mathcal{M}' \cong \mathcal{M}$. Using $\partial\, e = 0$, we obtain the following *commutative exact diagram*

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{I}^{1 \times n} \arrow[r, ".\partial\, \mathbb{1}_n"] \arrow[d, ".0"] & \mathcal{I}^{1 \times n} \arrow[r, "\pi'"] \arrow[d, ".e\, \mathbb{1}_n"] & \mathcal{M}' \arrow[r] \arrow[d, "\psi"] & 0 \\ 0 \arrow[r] & \mathcal{I}^{1 \times n} \arrow[r, ".R"] & \mathcal{I}^{1 \times n} \arrow[r, "\pi"] & \mathcal{M} \arrow[r] & 0, \end{tikzcd} \qquad (23)$$

where $\psi \in \hom_{\mathcal{I}}(\mathcal{M}', \mathcal{M})$ is given by $\psi(\pi'(\lambda)) = \pi(\lambda\, e\, \mathbb{1}_n)$ for all $\lambda \in \mathcal{I}^{1 \times n}$. Using (20), we get $\psi^{-1}(\pi(\lambda)) = \pi'(\lambda\, \Phi)$ for all $\lambda \in \mathcal{I}^{1 \times n}$, and the commutative exact diagram:

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{I}^{1 \times n} \arrow[r, ".R"] \arrow[d, ".\Phi"] & \mathcal{I}^{1 \times n} \arrow[r, "\pi"] \arrow[d, ".\Phi"] & \mathcal{M} \arrow[r] \arrow[d, "\psi^{-1}"] & 0 \\ 0 \arrow[r] & \mathcal{I}^{1 \times n} \arrow[r, ".\partial\, \mathbb{1}_n"] & \mathcal{I}^{1 \times n} \arrow[r, "\pi'"] & \mathcal{M}' \arrow[r] & 0. \end{tikzcd} \qquad (24)$$

Using (3), we can check again the commutativity of the above diagram: $R\, \Phi = (\partial\, \mathbb{1}_n - A)\, \Phi = \Phi\, \partial + \dot{\Phi} - A\, \Phi = \Phi\, \partial$.

As explained in Section 2, $\eta \in \ker_{\mathcal{F}}(R.)$ corresponds to $f \in \hom_{\mathcal{I}}(\mathcal{M}, \mathcal{F})$, where $\eta_i = f(x_i)$ for $i = 1, \ldots, n$. Using that $\{e\, x_i\}_{i=1,\ldots,n}$ also generates $\mathcal{M}$, η can be defined by $f(e\, x_i) = e\, f(x_i) = e\, \eta_i$ for $i = 1, \ldots, n$. For instance, if $\mathcal{F} = \mathcal{A}$, using $\ker_{\mathcal{A}}(\partial\, \mathbb{1}_n.) = \mathcal{C}(\mathcal{A})^{n \times 1}$, where $\mathcal{C}(\mathcal{A}) = \Bbbk$, then applying the functor $\hom_{\mathcal{I}}(\,\cdot\,, \mathcal{A})$ to (23) and (24), we obtain the following commutative exact diagram

$$\begin{tikzcd} \mathcal{A}^{n \times 1} & \mathcal{A}^{n \times 1} \arrow[l, "\partial\, \mathbb{1}_n."'] \arrow[d, shift left, "\Phi."] & \Bbbk^{n \times 1} \arrow[l] \arrow[d, shift left, "(\psi^{-1})^{\star}"] & 0 \arrow[l] \\ \mathcal{A}^{n \times 1} \arrow[u, shift left, "0."] & \mathcal{A}^{n \times 1} \arrow[l, "R."'] \arrow[u, shift left, "e\, \mathbb{1}_n."] & \ker_{\mathcal{A}}(R.) \arrow[l] \arrow[u, shift left, "\psi^{\star}"] & 0, \arrow[l] \end{tikzcd}$$

where $\psi^{\star}$ and $\left(\psi^{-1}\right)^{\star}$ are respectively defined by:

$$\forall\, \eta \in \ker_{\mathcal{A}}(R.),\;\; \psi^{\star}(\eta) = (e\, \mathbb{1}_n)(\eta) = (e(\eta_1), \ldots, e(\eta_n))^T,$$
$$\forall\, c \in \Bbbk^{n \times 1},\;\; \left(\psi^{-1}\right)^{\star}(c) = \Phi\, c \in \ker_{\mathcal{A}}(R.).$$

Theorem 7. Let $A \in \mathcal{A}^{n \times n}$, $R = \partial\, \mathbb{1}_n - A$, Φ be the transition matrix of $\dot{x}(t) = A(t)\, x(t)$, and $S = \Phi\, I\, \Phi^{-1}$. Then, the following results hold:

(1) $R\, S = \mathbb{1}_n$, $\Phi\, e + S\, R = \mathbb{1}_n$, and $\Pi := S\, R = \mathbb{1}_n - \Phi\, e$ satisfies $\Pi^2 = \Pi$, i.e., Π is an idempotent of $\mathcal{I}^{n \times n}$.

(2) If $\mathcal{F}$ is a left $\mathcal{I}$-module (e.g., $\mathcal{F} = \mathcal{A}$), then we have:

$$\forall\, \xi \in \mathcal{F}^{n \times 1}, \quad \xi = \Phi\, e\, \xi + S\, R\, \xi.$$

All the $\mathcal{F}$-solutions of $(\partial\, \mathbb{1}_n - A)\, \xi = \zeta$ are then of the form $\xi = \Phi\, e\, \xi + S\, \zeta$, i.e., of (17) if $\mathcal{F} = \mathcal{A}$.

(3) If $\mathcal{F}$ is a left $\mathcal{I}$-module (e.g., $\mathcal{F} = \mathcal{A}$) and $\xi_{t_0} \in \Bbbk^{n \times 1}$, then the following Cauchy problem

$$\begin{cases} (\partial\, \mathbb{1}_n - A)\, \xi = \zeta, \\ e\, \xi = \xi_{t_0} \end{cases}$$

has the unique solution $\xi = \Phi\, \xi_{t_0} + S\, \zeta \in \mathcal{F}^{n \times 1}$.

(4) $\mathrm{im}_{\mathcal{I}}(.R) = \mathrm{im}_{\mathcal{I}}(.\Pi)$, and thus, we have
$$\mathcal{M} = \mathrm{coker}_{\mathcal{I}}(.R) = \mathrm{coker}_{\mathcal{I}}(.\Pi),$$
which, for all left $\mathcal{F}$-modules (e.g., $\mathcal{F} = \mathcal{A}$), yields:
$$\ker_{\mathcal{F}}(R.) \cong \hom_{\mathcal{I}}(\mathcal{M}, \mathcal{F}) \cong \ker_{\mathcal{F}}(\Pi.).$$
(5) We have $\mathcal{M} \cong \mathcal{M}' = \mathrm{coker}_{\mathcal{I}}(.\partial \mathbb{1}_n)$, and thus:
$$\ker_{\mathcal{A}}(R.) \cong \ker_{\mathcal{A}}(\partial \mathbb{1}_n.) = \Bbbk^{n\times 1}.$$

5. REACHABILITY OF LINEAR SYSTEMS

Let us consider $P = (\partial \mathbb{1}_n - A \quad -B) \in \mathcal{I}^{n\times(n+m)}$, where $A \in \mathcal{A}^{n\times n}$ and $B \in \mathcal{A}^{n\times m}$. Then, the linear system (18) is defined by $P\,\eta = 0$, where $\eta = (x^T \quad u^T)^T$. As above, let Φ be the transition matrix of $\dot{x}(t) = A(t)\,x(t)$ and $S = \Phi\, I\, \Phi^{-1}$ a right inverse of $R = \partial \mathbb{1}_n - A$. Now, $P\,\eta = 0$ yields $(S\,P)\,\eta = 0$. Conversely, using $R\,S = \mathbb{1}_n$, $(S\,P)\,\eta = 0$ yields $R\,(S\,P)\,\eta = P\,\eta = 0$, which shows that $P\,\eta = 0$ and $S\,P\,\eta = 0$ are equivalent. Using (16), we have:
$$S\,P = (S\,R \quad -S\,B) = \left(\mathbb{1}_n - \Phi\, e \quad -\Phi\, I\, \Phi^{-1}\, B\right).$$
Hence, $(S\,P)\,\eta = 0$ is exactly (19). Within the algebraic analysis approach to linear system theory, we have just shown that $\mathcal{I}^{1\times n}\,P = \mathcal{I}^{1\times n}\,(S\,P)$, which then yields:
$$\mathcal{L} = \mathrm{coker}_{\mathcal{I}}(.P) = \mathrm{coker}_{\mathcal{I}}(.S\,P).$$
Thus, (18) and (19) are two equivalent representations of the same linear system, i.e., they define two different *presentations* of the same left $\mathcal{I}$-module $\mathcal{L}$ (Rotman (2009)).

Note that the identity $R\,S = \mathbb{1}_n$ shows that the matrix $T = (S^T \quad 0^T)^T \in \mathcal{I}^{(n+m)\times n}$ is a right inverse of P, and thus, the following exact sequence of $\mathcal{I}$-modules splits
$$0 \longrightarrow \mathcal{I}^{1\times n} \underset{.T}{\overset{.P}{\rightleftarrows}} \mathcal{I}^{1\times(n+m)} \underset{\sigma}{\overset{\kappa}{\rightleftarrows}} \mathcal{L} \longrightarrow 0,$$
which yields $\mathcal{L} \oplus \mathcal{I}^{1\times n} \cong \mathcal{I}^{1\times(n+m)}$, i.e., $\mathcal{L}$ is a *stably free* left $\mathcal{I}$-module (Lam (1999); Rotman (2009)). If $\mathcal{F}$ is a left $\mathcal{I}$-module, then applying the contravariant functor $\hom_{\mathcal{I}}(\,\cdot\,, \mathcal{F})$ to the above split exact sequence, we then obtain the following split exact sequence
$$0 \longrightarrow \ker_{\mathcal{F}}(P.) \underset{\varphi}{\overset{i}{\rightleftarrows}} \mathcal{F}^{(n+m)\times 1} \underset{T.}{\overset{P.}{\rightleftarrows}} \mathcal{F}^{n\times 1} \longrightarrow 0,$$
where $i \circ \varphi + T\,P. = \mathbb{1}_{n+m}$, i.e., $\varphi(\xi) = (\mathbb{1}_{n+m} - T\,P)\,\xi$ for all $\xi = (\xi_1^T \quad \xi_2^T)^T \in \mathcal{F}^{(n+m)\times 1}$. Now, using (16), we have:
$$\mathbb{1}_{n+m} - T\,P = \begin{pmatrix} \mathbb{1}_n - S\,R & S\,B \\ 0 & \mathbb{1}_m \end{pmatrix} = \begin{pmatrix} \Phi\, e & \Phi\, I\, \Phi^{-1}\, B \\ 0 & \mathbb{1}_m \end{pmatrix}.$$
Using $\ker_{\mathcal{F}}(P.) = \mathrm{im}\,\varphi = \varphi\left(\mathcal{F}^{(n+m)\times 1}\right)$, we get the following *parametrization* of the linear system $\ker_{\mathcal{F}}(P.)$
$$\begin{pmatrix} x(t) \\ u(t) \end{pmatrix} = \begin{pmatrix} \Phi\,\xi_1(t_0) + \Phi\, I\, \Phi^{-1}\, B\,\xi_2(t) \\ \xi_2(t) \end{pmatrix} \in \ker_{\mathcal{F}}(P.), \tag{25}$$
for all $\xi \in \mathcal{F}^{(n+m)\times 1}$. Therefore, we find again that $x(t) = \Phi\, e\,\xi_1(t) + \Phi\, I\, \Phi^{-1}\, B\, u(t)$ is a solution of (18) for all $\xi_1 \in \mathcal{F}^{n\times 1}$ and $u \in \mathcal{F}^{m\times 1}$, with $e(x) = \xi_1(t_0)$. Thus, (18) is parametrized by $x(t) = \Phi\, x(t_0) + \Phi\, I\, \Phi^{-1}\, B\, u(t)$.

Considering the ring $\mathcal{J}$ defined at the end of Section 3, we can then consider the value $e'(x) = x(t_1)$ of x at $t_1 > t_0$. Thus, we can study the *reachability property* of linear systems (Kalman et al. (1969)), namely, the fact that $(e'\,(\mathbb{1}_n - \Phi\, e))(z) = z(t_1) - \Phi(t_0, t_1)\, z(t_0)$ is of the form of $e'(\Phi\, I(\Phi^{-1}\, B\, u)) = \int_{t_0}^{t_1} \Phi(t_0, \tau)\, B(\tau)\, u(\tau)\, d\tau$, i.e., the fact that there is an input u such that (19) yields:
$$x(t_1) = \Phi(t_0, t_1)\, z(t_0) + \int_{t_0}^{t_1} \Phi(t_1, \tau)\, B(\tau)\, u(\tau)\, d\tau = z(t_1).$$
The reachability property corresponds to the inclusion:
$$\begin{aligned} &\mathrm{im}_{\mathcal{F}}(e'\,(\mathbb{1}_n - \Phi\, e).) = \mathrm{im}_{\mathcal{F}}((e'\,\mathbb{1}_n - e'(\Phi)\, e).) \\ &\subseteq \mathrm{im}_{\mathcal{F}}(e'\,\Phi\, I\, \Phi^{-1}\, B.) = \mathrm{im}_{\mathcal{F}}(e'(\Phi)\, e'\, I\, \Phi^{-1}\, B.). \end{aligned} \tag{26}$$
If $\mathcal{F}$ satisfies the condition that for every $z_{t_1} \in \mathbb{R}^{n\times 1}$, there exists $z \in \mathcal{F}^{n\times 1}$ such that $z(t_1) = z_{t_1}$ and $z(t_0) = 0$, then $z(t_1) - \Phi(t_0, t_1)\, z(t_0) = z_{t_1}$, i.e., $\mathrm{im}_{\mathcal{F}}(e'\,(\mathbb{1}_n - \Phi\, e).) = \mathbb{R}^{n\times 1}$. Hence, (26) becomes $\mathrm{im}_{\mathcal{F}}(e'\,\Phi\, I\, \Phi^{-1}\, B.) = \mathbb{R}^{n\times 1}$, i.e., $e'\,\Phi\, I\, \Phi^{-1}\, B. : \mathcal{F}^{m\times 1} \longrightarrow \mathbb{R}^{n\times 1}$ is surjective. It is well-known that $\mathrm{im}_{\mathcal{F}}(e'\,\Phi\, I\, \Phi^{-1}\, B.) = \mathrm{im}_{\mathbb{R}}(W_c.)$, where $W_c = e'\left(I\left(\Phi^{-1}\, B\, B^T\, \Phi^{-T}\right)\right) \in \mathbb{R}^{n\times n}$ is the *controllability gramian* (Kalman et al. (1969)). Thus, if the linear system (18) is reachable at t_1, i.e., if W_c is invertible, setting $L = B^T\, \Phi^{-T}\, W_c^{-1}\, e'(\Phi)^{-1}\,(e'\,\mathbb{1}_n - e'(\Phi)\, e)$, then we get the identity $e'\,\mathbb{1}_n - e'(\Phi)\, e = e'(\Phi)\, e'\, I\, \Phi^{-1}\, B\, L$. Setting $\xi_2 = L\,\xi_1$ for an arbitrary $\xi_1 \in \mathcal{F}^{n\times 1}$ into (25), i.e., $u = \xi_2 = L\,\xi_1$, then $x = \Phi\, e(\xi_1) + \Phi\, I(\Phi^{-1}\, B\, u)$ and:
$$\begin{aligned} x(t_0) &= e(x) = e(\Phi)\, e(\xi_1) + e(\Phi)\, e(I(\Phi^{-1}\, B\, u)) = \xi_1(t_0), \\ x(t_1) &= e'(x) = e'(\Phi)\, e'(e(\xi_1)) + e'(\Phi)\, e'(I\, \Phi^{-1}\, B\, u) \\ &= e'(\Phi)\, e(\xi_1) + e'(\xi_1) - e'(\Phi)\, e(\xi_1) = \xi_1(t_1). \end{aligned}$$
Since $\xi \in \mathcal{F}^{n\times 1}$ is arbitrary, u drives x from an arbitrary initial condition $\xi_1(t_0)$ to an arbitrary value $\xi_1(t_1)$ at t_1.

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