

New perspectives in algebraic systems theory

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The purpose of this talk is to develop the following 3 ideas:

- 1 A large class of multidimensional systems can be studied by means of a **non-commutative polynomial approach** over skew polynomial rings and Ore algebras of functional operators.
Non-commutative Gröbner bases \Rightarrow **constructive approach**.
- 2 **Algebraic analysis** is a natural mathematical framework for the intrinsic study of linear systems theory (**module theory**).
- 3 **Constructive homological algebra** allows us to develop algorithms and symbolic packages dedicated to the study of the structural properties of multidimensional linear systems.

Matrices of differential operators

- **Newton:** Fluxion calculus (1666) (“dot-age”)

$$\begin{cases} \ddot{x}_1(t) + \alpha x_1(t) - \alpha u(t) = 0, \\ \ddot{x}_2(t) + \alpha x_2(t) - \alpha u(t) = 0, \end{cases} \quad \alpha = g/l.$$

- **Leibniz:** Infinitesimal calculus (1676) (“d-ism”)

$$\begin{cases} \frac{d^2 x_1(t)}{dt^2} + \alpha x_1(t) - \alpha u(t) = 0, \\ \frac{d^2 x_2(t)}{dt^2} + \alpha x_2(t) - \alpha u(t) = 0. \end{cases}$$

- **Boole:** Operational calculus (1859-60)

$$\begin{pmatrix} \frac{d^2}{dt^2} + \alpha & 0 & -\alpha \\ 0 & \frac{d^2}{dt^2} + \alpha & -\alpha \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ u(t) \end{pmatrix} = 0.$$

⇒ Ring of differential operators $D = \mathbb{Q}(\alpha) \left[\frac{d}{dt} \right]$:

$$\sum_{i=0}^n a_i \left(\frac{d}{dt} \right)^i \in D, \quad a_i \in \mathbb{Q}(g, l), \quad \left(\frac{d}{dt} \right)^i = \frac{d}{dt} \circ \dots \circ \frac{d}{dt} = \frac{d^i}{dt^i}.$$

Functional operators

- Differential operators: $\left(\sum_{j=0}^m b_j(t) \frac{d^j}{dt^j}\right) \left(\sum_{i=0}^n a_i(t) \frac{d^i}{dt^i}\right)$

$$\frac{d}{dt}(a y) = a \frac{d}{dt} y + \left(\frac{da}{dt}\right) y \Rightarrow \frac{d}{dt} a \cdot = a \frac{d}{dt} \cdot + \frac{da}{dt} \cdot$$

- Shift operators: $\delta a(t) = a(t-h)$, $\sigma a_n = a_{n+1}$.

$$\delta(a(t)y(t)) = a(t-h)y(t-h) = \delta a \delta y \Rightarrow \delta a \cdot = (\delta a) \delta \cdot$$

$$\sigma(a_n y_n) = a_{n+1} y_{n+1} = \sigma a \sigma y \Rightarrow \sigma a \cdot = (\sigma a) \sigma \cdot$$

- Difference operators: $\Delta a(x) = a(x+1) - a(x)$.

$$\Delta a(x) \cdot = a(x+1) \Delta \cdot + (\Delta a) \cdot$$

- Divided difference operators: $d_{x_0} a(x) = \frac{a(x) - a(x_0)}{x - x_0}$.

$$d_{x_0} a(x) \cdot = a(x_0) d_{x_0} \cdot + (d_{x_0} a) \cdot$$

- q -difference, q -shift, q -dilation, Frobenius, Euler operators.

Skew polynomial rings (Ore, 1933)

- **Definition:** A **skew polynomial ring** $A[\partial; \alpha, \beta]$ is a non-commutative polynomial ring in ∂ with coefficients in A satisfying

$$\forall a \in A, \quad \partial a = \alpha(a) \partial + \beta(a)$$

where $\alpha : A \rightarrow A$ and $\beta : A \rightarrow A$ are such that:

$$\begin{cases} \alpha(1) = 1, \\ \alpha(a + b) = \alpha(a) + \alpha(b), \\ \alpha(ab) = \alpha(a)\alpha(b), \end{cases} \quad \begin{cases} \beta(a + b) = \beta(a) + \beta(b), \\ \beta(ab) = \alpha(a)\beta(b) + \beta(a)b. \end{cases}$$

- $P \in A[\partial; \alpha, \beta]$ has a unique form $P = \sum_{i=0}^n a_i \partial^i$, $a_i \in A$.
 - Ring of differential operators: $A[\partial; \text{id}, \frac{d}{dt}]$.
 - Ring of shift operators: $A[\partial; \delta, 0]$, $A[\partial; \sigma, 0]$.
 - Ring of difference operators: $A[\partial; \tau, \tau - \text{id}]$, $\tau a(x) = a(x + 1)$.

Ore algebras (Chyzak-Salvy, 1996)

- We can iterate skew polynomial rings to get **Ore extensions**:

$$A[\partial_1; \alpha_1, \beta_1] \dots [\partial_n; \alpha_n, \beta_n]$$

- **Definition**: An Ore extension $A[\partial_1; \alpha_1, \beta_1] \dots [\partial_n; \alpha_n, \beta_n]$ is called an **Ore algebras** if the ∂_i 's commute, i.e., if we have

$$1 \leq j < i \leq m, \quad \alpha_i(\partial_j) = \partial_j, \quad \beta_i(\partial_j) = 0,$$

and the $\alpha_{i|_A}$'s and $\beta_{j|_A}$'s commute for $i \neq j$.

- Ring of differential operators: $A \left[\partial_1; \text{id}, \frac{\partial}{\partial x_1} \right] \dots \left[\partial_n; \text{id}, \frac{\partial}{\partial x_n} \right]$.
- Ring of differential delay operators: $A \left[\partial_1; \text{id}, \frac{d}{dt} \right] [\partial_2; \delta, 0]$.
- Ring of shift operators: $A[\partial_1; \sigma_1, 0] \dots [\partial_n; \sigma_n, 0]$.

Matrix of functional operators

- The **wind tunnel model** (Manitius, IEEE TAC 84):

$$\begin{cases} \dot{x}_1(t) + a x_1(t) - k a x_2(t - h) = 0, \\ \dot{x}_2(t) - x_3(t) = 0, \\ \dot{x}_3(t) + \omega^2 x_2(t) + 2 \zeta \omega x_3(t) - \omega^2 u(t) = 0. \end{cases} \quad (*)$$

- We introduce the **commutative Ore algebra**:

$$D = \mathbb{Q}(a, k, \omega, \zeta) \left[\partial_1; \text{id}, \frac{d}{dt} \right] [\partial_2; \delta, 0].$$

- The system (*) can be rewritten as:

$$\begin{pmatrix} \partial_1 + a & -k a \partial_2 & 0 & 0 \\ 0 & \partial_1 & -1 & 0 \\ 0 & \omega^2 & \partial_1 + 2 \zeta \omega & -\omega^2 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ u(t) \end{pmatrix} = 0.$$

Matrix of functional operators

- Linearization of the Navier-Stokes \sim a parabolic Poiseuille profile

$$\begin{cases} \partial_t u_1 + 4y(1-y)\partial_x u_1 - 4(2y-1)u_2 - \frac{1}{Re}(\partial_x^2 + \partial_y^2)u_1 + \partial_x p = 0, \\ \partial_t u_2 + 4y(1-y)\partial_x u_2 - \frac{1}{Re}(\partial_x^2 + \partial_y^2)u_2 + \partial_y p = 0, \\ \partial_x u_1 + \partial_y u_2 = 0. \end{cases} \quad (*)$$

(e.g., Vazquez-Krstic, IEEE 07)

- Let us introduce the so-called Weyl algebra $(\partial_x x = x \partial_x + 1)$:

$$D = \mathbb{Q}(Re)[t, x, y] \left[\partial_t; \text{id}, \frac{\partial}{\partial t} \right] \left[\partial_x; \text{id}, \frac{\partial}{\partial x} \right] \left[\partial_y; \text{id}, \frac{\partial}{\partial y} \right].$$

- The system (*) is defined by the matrix of PD operators:

$$\begin{pmatrix} \partial_t + 4y(1-y)\partial_x - \frac{1}{Re}(\partial_x^2 + \partial_y^2) & -4(2y-1) & \partial_x \\ 0 & \partial_t + 4y(1-y)\partial_x - \frac{1}{Re}(\partial_x^2 + \partial_y^2) & \partial_y \\ \partial_x & \partial_y & 0 \end{pmatrix}.$$

Non-commutative Gröbner bases

- Let $D = A[\partial_1; \alpha_1, \beta_1] \dots [\partial_m; \alpha_m, \beta_m]$ be an Ore algebra.
- **Theorem:** (Kredel, 93) Let $A = k[x_1, \dots, x_n]$ a commutative polynomial ring ($k = \mathbb{Q}, \mathbb{F}_p$) and D an Ore algebra satisfying

$$\alpha_i(x_j) = a_{ij} x_j + b_{ij}, \quad \beta_i(x_j) = c_{ij},$$

for certain $0 \neq a_{ij} \in k$, $b_{ij} \in k$, $c_{ij} \in A$ and $\deg(c_{ij}) \leq 1$. Then, a non-commutative version of **Buchberger's algorithm** terminates for any term order and its result is a **Gröbner basis**.

- **Implementation** in the Maple package **Ore_algebra** (Chyzak) (Singular, Macaulay 2, NCAAlgebra (Helton et al), JanetOre...).
- Gröbner bases can be used to **effectively compute over D** .

Algebraic analysis

- Let D be an Ore algebra and $R \in D^{q \times p}$.
- Let us consider the **left D -morphism** (i.e., D -linear application):

$$\begin{array}{ccc} D^{1 \times q} & \xrightarrow{\cdot R} & D^{1 \times p} \\ \lambda = (\lambda_1 \dots \lambda_q) & \longmapsto & \lambda R. \end{array}$$

- We introduce the **finitely presented left D -module**:

$$M = D^{1 \times p} / \text{im}_D(\cdot R) = D^{1 \times p} / (D^{1 \times q} R).$$

- M is formed by the equivalence classes $\pi(\mu)$ of $\mu \in D^{1 \times p}$ for the **equivalence relation** \sim on $D^{1 \times p}$:

$$\mu_1 \sim \mu_2 \Leftrightarrow \exists \lambda \in D^{1 \times q} : \mu_1 = \mu_2 + \lambda R \Leftrightarrow \mu_1 - \mu_2 \in D^{1 \times q} R.$$

- ① **Number theory:** $\mathbb{C} = \mathbb{R}[x]/(x^2 + 1)$, $\mathbb{Z}[i\sqrt{5}] = \mathbb{Z}[x]/(x^2 + 5)$.
- ② **Algebraic geometry:** $\mathbb{C}[x, y]/(x^2 + y^2 - 1, x - y)$.

Linear systems of equations

- $M = D^{1 \times p} / (D^{1 \times q} R)$ can be defined by **generators and relations**:
- Let $\{f_k\}_{k=1, \dots, p}$ the **standard basis** of $D^{1 \times p}$ ($f_k = (0 \dots 1 \dots 0)$).
- Let $\pi : D^{1 \times p} \longrightarrow M$ be the **D -morphism** sending μ to $\pi(\mu)$.

$$\forall m \in M, \exists \mu = (\mu_1 \dots \mu_p) \in D^{1 \times p} : m = \pi(\mu) = \sum_{k=1}^p \mu_k \pi(f_k),$$

$\Rightarrow \{y_k = \pi(f_k)\}_{k=1, \dots, p}$ is a **family of generators** of M .

$$\pi((R_{l1} \dots R_{lp})) = \pi \left(\sum_{k=1}^p R_{lk} f_k \right) = \sum_{k=1}^p R_{lk} y_k = 0, \quad l = 1, \dots, q,$$

$\Rightarrow y = (y_1 \dots y_p)^T$ satisfies the **relation $Ry = 0$** .

Duality modules — systems

- Let \mathcal{F} be a left D -module

$$\forall f_1, f_2 \in \mathcal{F}, \quad \forall d_1, d_2 \in D : d_1 f_1 + d_2 f_2 \in \mathcal{F},$$

and $\text{hom}_D(M, \mathcal{F})$ the abelian group:

$$\text{hom}_D(M, \mathcal{F}) = \{f : M \rightarrow \mathcal{F} \mid f(d_1 m_1 + d_2 m_2) = d_1 f(m_1) + d_2 f(m_2)\}.$$

- Theorem (Malgrange):

$$\text{hom}_D(M, \mathcal{F}) \cong \ker_{\mathcal{F}}(R.) = \{\eta \in \mathcal{F}^p \mid R\eta = 0\}$$

- $\text{hom}_D(M, \mathcal{F})$ intrinsically characterizes the system $\ker_{\mathcal{F}}(R.)$ as it does not depend on the embedding of $\ker_{\mathcal{F}}(R.)$ into \mathcal{F}^p .
- We assume that \mathcal{F} is an injective cogenerator left D -module
 \Rightarrow study of the system $\text{hom}_D(M, \mathcal{F})$ by means of properties of M .

- **Definition:** 1. M is **free** if $\exists r \in \mathbb{Z}_+$ such that $M \cong D^r$.
- 2. M is **projective** if $\exists r \in \mathbb{Z}_+$ and a D -module P such that:

$$M \oplus P \cong D^r.$$

- 3. M is **reflexive** if $\varepsilon : M \longrightarrow \text{hom}_D(\text{hom}_D(M, D), D)$ is an isomorphism, where:

$$\varepsilon(m)(f) = f(m), \quad \forall m \in M, \quad f \in \text{hom}_D(M, D).$$

- 4. M is **torsion-free** if:

$$t(M) = \{m \in M \mid \exists 0 \neq P \in D : P m = 0\} = 0.$$

- 5. M is **torsion** if $t(M) = M$.

Classification of modules

- **Theorem:** 1. We have the following implications:

free \Rightarrow projective \Rightarrow reflexive \Rightarrow torsion-free.

2. If D is a principal domain (e.g., $K[\partial; \text{id}, \frac{d}{dt}]$), then:

torsion-free = free.

3. If D is a hereditary ring (e.g., $\mathbb{Q}[t][\partial; \text{id}, \frac{d}{dt}]$), then:

torsion-free = projective.

4. If $D = k[x_1, \dots, x_n]$ and k a field, then:

projective = free (Quillen-Suslin theorem).

4. If $D = A_n(k)$ or $B_n(k)$, k is a field of characteristic 0, then

projective = free (Stafford theorem),

for modules of rank at least 2.

Dictionary systems – modules

Module M	Structural properties $\ker_{\mathcal{F}}(R.)$	Stabilization problems Optimal control
Torsion	Autonomous system Poles/zeros classifications	
With torsion	Existence of autonomous elements	
Torsion-free	No autonomous elements, Controllability, Parametrizability, π -freeness	Variational problem without constraints (Euler-Lagrange equations)
Projective	Bézout identities, Internal stabilizability	Computation of Lagrange parameters without integration Existence of a parametrization all stabilizing controllers
Free	Flatness, Poles placement, Doubly coprime factorization	Youla-Kučera parametrization Optimal controller

Involutions, adjoints and dual systems

- **Definition:** A linear map $\theta : D \longrightarrow D$ is an **involution** of D if:

$$\forall P, Q \in D : \theta(PQ) = \theta(Q)\theta(P), \quad \theta^2 = \text{id}.$$

- **Example:** 1. If D is a commutative ring, then $\theta = \text{id}$.
- 2. An involution of $D = A \left[\partial_1; \text{id}, \frac{\partial}{\partial x_1} \right] \dots \left[\partial_n; \text{id}, \frac{\partial}{\partial x_n} \right]$ is:

$$\forall a \in A, \quad \theta(a(x)) = a(x), \quad \theta(\partial_i) = -\partial_i, \quad i = 1, \dots, n.$$

- 3. An involution of $D = A \left[\partial_1; \text{id}, \frac{d}{dt} \right] \left[\partial_2; \delta, 0 \right]$ is defined by:

$$\forall a \in A, \quad \theta(a(t)) = a(-t), \quad \theta(\partial_i) = \partial_i, \quad i = 1, 2.$$

- The **adjoint** of $R \in D^{q \times p}$ is defined by $\theta(R) = (\theta(R_{ij}))^T \in D^{p \times q}$.
- $N = D^{1 \times q} / (D^{1 \times p} \theta(R))$ is called the **transposed** of M .

Projectiveness, observability and controllability

- **Theorem:** If $R \in D^{q \times p}$ has full row rank, then the left D -module $M = D^{1 \times p} / (D^{1 \times q} R)$ is **projective** iff:

$$N = D^{1 \times q} / (D^{1 \times p} \theta(R)) = 0 \Leftrightarrow \exists S \in D^{p \times q} : RS = I_q.$$

- Let $D = \mathcal{A}(I) \left[\partial; \text{id}, \frac{d}{dt} \right]$ and $R = (\partial I_n - A \quad -B) \in D^{n \times (n+m)}$.
 $M = D^{1 \times (n+m)} / (D^{1 \times n} R)$ is **projective** iff $\theta(R) \lambda = 0 \Leftrightarrow \lambda = 0$:

$$\begin{cases} -\partial \lambda - A^T \lambda = 0, \\ -B^T \lambda = 0, \end{cases} \Rightarrow \begin{cases} \partial \lambda = -A^T \lambda, \\ B^T \lambda = 0, \\ B^T \partial \lambda + \dot{B}^T \lambda = (-B^T A^T + \dot{B}^T) \lambda = 0. \end{cases}$$

Hence, M is **projective** iff, for all $t_0 \in I$, we have:

$$\text{rank}_{\mathbb{R}}(B \mid AB - \dot{B} \mid A^2 B + \dots \mid A^{n-1} B + \dots \mid \dots)(t_0) = n.$$

- $D^{1 \times p} / (D^{1 \times q} (P(\partial) - Q(\partial)))$ **proj.** iff $P(\partial) X(\partial) - Q(\partial) Y(\partial) = I_q$.

Module M	Homological algebra	\mathcal{F} injective cogenerator
with torsion	$t(M) \cong \text{ext}_D^1(N, D)$	\emptyset
torsion-free	$\text{ext}_D^1(N, D) = 0$	$\ker_{\mathcal{F}}(R.) = Q \mathcal{F}^1$
reflexive	$\text{ext}_D^i(N, D) = 0$ $i = 1, 2$	$\ker_{\mathcal{F}}(R.) = Q_1 \mathcal{F}^1$ $\ker_{\mathcal{F}}(Q_1.) = Q_2 \mathcal{F}^2$
projective	$\text{ext}_D^i(N, D) = 0$ $1 \leq i \leq n$	$\ker_{\mathcal{F}}(R.) = Q_1 \mathcal{F}^1$ $\ker_{\mathcal{F}}(Q_1.) = Q_2 \mathcal{F}^2$ \dots $\ker_{\mathcal{F}}(Q_{n-1}.) = Q_n \mathcal{F}^n$
free	Quillen-Suslin theorem Stafford's theorem	$\ker_{\mathcal{F}}(R.) = Q \mathcal{F}^l$ $\exists T \in D^{l \times p} : TQ = I_l$

Extension functor $\text{ext}_D^1(\cdot, D)$

- **Parametrizability:** $Ry = 0 \stackrel{?}{\iff} \exists Q \in D^{p \times m} : y = Qz.$

$$4. \quad \theta(P)z = y \implies Ry = 0 \quad 1.$$

$$\begin{array}{ccc} \uparrow & & \downarrow \\ \text{involution } \theta & & \text{involution } \theta \\ \uparrow & & \downarrow \end{array}$$

$$3. \quad 0 = P\mu \stackrel{\text{Gb}}{\iff} \theta(R)\lambda = \mu \quad 2.$$

$$\begin{aligned} P \circ \theta(R) = 0 &\implies \theta(P \circ \theta(R)) = \theta^2(R) \circ \theta(P) \\ &= R \circ \theta(P) = 0. \end{aligned}$$

$$5. \quad \theta(P)z = y \stackrel{\text{Gb}}{\iff} R'y = 0, \quad R' \in D^{q' \times p}.$$

$$\text{ext}_D^1(N, D) \cong (D^{1 \times q'} R') / (D^{1 \times q} R)$$

- Using **Gb**, we can test whether or not $\text{ext}_D^1(N, D) = 0$

Wind tunnel model (Manitius, IEEE TAC 84)

1. The w.t.m. is defined by the **under-determined system**:

$$\begin{pmatrix} \partial_1 + a & -k a \partial_2 & 0 & 0 \\ 0 & \partial_1 & -1 & 0 \\ 0 & \omega^2 & \partial_1 + 2\zeta\omega & -\omega^2 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ u(t) \end{pmatrix} = 0.$$

2. We compute $\theta(R) = R^T$ and define $\theta(R) \lambda = \mu$:

$$\begin{cases} (\partial_1 + a) \lambda_1 = \mu_1, \\ -k a \partial_2 \lambda_1 + \partial_1 \lambda_2 + \omega^2 \lambda_3 = \mu_2, \\ -\lambda_2 + (\partial_1 + 2\zeta\omega) \lambda_3 = \mu_3, \\ -\omega^2 \lambda_3 = \mu_4. \end{cases} \quad (2)$$

- (2) is **over-determined** $\xrightarrow{\text{Gb}}$ **compatibility conditions** $P \mu = 0$.

3. We obtain the **compatibility condition** $P \mu = 0$:

$$\begin{aligned} \omega^2 k a \partial_2 \mu_1 + \omega^2 (\partial_1 - a) \mu_2 + \omega^2 (\partial_1^2 + a \partial_1) \mu_3 \\ + (\partial_1^3 + 2 \zeta \omega \partial_1^2 + a \partial_1^2 + \omega^2 \partial_1 + 2 a \zeta \omega \partial_1 + a \omega^2) \mu_4 = 0. \end{aligned}$$

4. We consider the **over-determined system** $P^T z = y$.

$$\begin{cases} \omega^2 k a \partial_2 z = x_1, \\ \omega^2 (\partial_1 - a) z = x_2, \\ \omega^2 (\partial_1^2 + a \partial_1) z = x_3, \\ (\partial_1^3 + (2 \zeta \omega + a) \partial_1^2 + (\omega^2 + 2 a \omega \zeta) \partial_1 + a \omega) z = u. \end{cases} \quad (4)$$

5. The **compatibility conditions** of $P^T z = y$ are **exactly generated** by $R y = 0$ and (4) is a **parametrization** of the w.t.m.

1. The **model of a moving tank** is defined by:

$$\begin{pmatrix} \partial_1 & -\partial_1 \partial_2^2 & a \partial_1^2 \partial_2 \\ \partial_1 \partial_2^2 & -\partial_1 & a \partial_1^2 \partial_2 \end{pmatrix} \begin{pmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{pmatrix} = 0.$$

2. We compute $\theta(R) = R^T$ and define $\theta(R) \lambda = \mu$:

$$\begin{cases} \partial_1 \lambda_1 + \partial_1 \partial_2^2 \lambda_2 = \mu_1, \\ -\partial_1 \partial_2^2 \lambda_1 - \partial_1 \lambda_2 = \mu_2, \\ a \partial_1^2 \partial_2 \lambda_1 + a \partial_1^2 \partial_2 \lambda_2 = \mu_3. \end{cases} \quad (2)$$

(2) is **over-determined** $\xrightarrow{\text{Gb}}$ **compatibility conditions** $P \mu = 0$.

3. We obtain the **compatibility condition** $P \mu = 0$:

$$a \partial_1 \partial_2 \mu_1 - a \partial_1 \partial_2 \mu_2 - (1 + \partial_2^2) \mu_3 = 0.$$

4. We consider the **over-determined system** $P^T z = y$.

$$\begin{cases} a \partial_1 \partial_2 z = y_1, \\ -a \partial_1 \partial_2 z = y_2, \\ -(1 + \partial_2^2) z = y_3. \end{cases} \quad (4)$$

5. The **compatibility conditions** of $P^T z = y$ are $R' y = 0$:

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & (1 + \partial_2^2) & -a \partial_1 \partial_2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = 0.$$

$$t(M) \cong \text{ext}_D^1(N, D) \cong \left(D^{1 \times 2} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 + \partial_2^2 & -a \partial_1 \partial_2 \end{pmatrix} \right) / \left(D^{1 \times 2} \begin{pmatrix} \partial_1 & -\partial_1 \partial_2^2 & a \partial_1^2 \partial_2 \\ \partial_1 \partial_2^2 & -\partial_1 & a \partial_1^2 \partial_2 \end{pmatrix} \right)$$

$$\begin{cases} y_1 + y_2 = z_1, \\ \partial_1 y_1 - \partial_1 \partial_2^2 y_2 + a \partial_1^2 \partial_2 y_3 = 0, \\ \partial_1 \partial_2^2 y_1 - \partial_1 y_2 + a \partial_1^2 \partial_2 y_3 = 0, \end{cases} \xrightarrow{\text{Gb}} \begin{cases} \partial_1 (\partial_2^2 - 1) z_1 = 0. \end{cases}$$

$$\begin{cases} (1 + \partial_2^2) y_2 - a \partial_1 \partial_2 y_3 = z_2, \\ \partial_1 y_1 - \partial_1 \partial_2^2 y_2 + a \partial_1^2 \partial_2 y_3 = 0, \\ \partial_1 \partial_2^2 y_1 - \partial_1 y_2 + a \partial_1^2 \partial_2 y_3 = 0, \end{cases} \xrightarrow{\text{Gb}} \begin{cases} \partial_1 (\partial_2^2 - 1) z_2 = 0. \end{cases}$$

$\Rightarrow z_1(t)$ and $z_2(t)$ are autonomous elements.

Examples

- 2D Stokes equations:

$$\begin{pmatrix} -\nu(\partial_x^2 + \partial_y^2) & 0 & \partial_x \\ 0 & -\nu(\partial_x^2 + \partial_y^2) & \partial_y \\ \partial_x & \partial_y & 0 \end{pmatrix} \begin{pmatrix} u \\ v \\ p \end{pmatrix} = 0$$

$$\Rightarrow \begin{cases} (\partial_x^2 + \partial_y^2)^2 u = 0, \\ (\partial_x^2 + \partial_y^2)^2 v = 0, \\ (\partial_x^2 + \partial_y^2) p = 0. \end{cases} \quad \text{torsion module}$$

- Moving tank (Petit, Rouchon, IEEE TAC 02):

$$\begin{cases} \dot{y}_1(t) - \dot{y}_2(t - 2h) + \alpha \ddot{y}_3(t - h) = 0, \\ \dot{y}_1(t - 2h) - \dot{y}_2(t) + \alpha \ddot{y}_3(t - h) = 0, \end{cases}$$

$$\Rightarrow \begin{cases} z_1(t) = y_1(t) + y_2(t), \\ z_2(t) = y_2(t) + y_2(t - 2h) - a \dot{y}_3(t - h), \\ \frac{d}{dt} (1 - \delta^2) z_i(t) = 0, \quad i = 1, 2. \end{cases} \quad \text{module with torsion}$$

Examples: torsion-free modules

- Wind tunnel model (Manitius, IEEE TAC 84):

$$\begin{cases} \dot{x}_1(t) + a x_1(t) - k a x_2(t - h) = 0, \\ \dot{x}_2(t) - x_3(t) = 0, \\ \dot{x}_3(t) + \omega^2 x_2(t) + 2 \zeta \omega x_3(t) - \omega^2 u(t) = 0, \end{cases}$$

$$\Leftrightarrow \begin{cases} x_1(t) = \omega^2 k a z(t - h), \\ x_2(t) = \omega^2 \dot{z}(t) - a \omega^2 z(t), \\ x_3(t) = \omega^2 \ddot{z}(t) + \omega^2 a \dot{z}(t), \\ u(t) = z(t)^{(3)} + (2 \zeta \omega + a) \ddot{z}(t) + (\omega^2 + 2 a \omega \zeta) \dot{z}(t) + a \omega z(t). \end{cases}$$

\Rightarrow motion planning and tracking (Fliess et al).

- 2D stress tensor (elasticity theory):

$$\begin{cases} \partial_x \sigma^{11} + \partial_y \sigma^{12} = 0, \\ \partial_x \sigma^{12} + \partial_y \sigma^{22} = 0, \end{cases} \Leftrightarrow \begin{cases} \sigma^{11} = \partial_y^2 \lambda, \\ \sigma^{12} = -\partial_x \partial_y \lambda, \\ \sigma^{22} = \partial_x^2 \lambda, \end{cases} \text{ Airy function } \lambda.$$

Examples: reflexive modules

- div-curl-grad: $\vec{\nabla} \cdot \vec{B} = 0 \Leftrightarrow \vec{B} = \vec{\nabla} \wedge \vec{A}, \vec{\nabla} \wedge \vec{A} = \vec{0} \Leftrightarrow \vec{A} = \vec{\nabla} f.$
- First group of Maxwell equations:

$$\left\{ \begin{array}{l} \frac{\partial \vec{B}}{\partial t} + \vec{\nabla} \wedge \vec{E} = \vec{0}, \\ \vec{\nabla} \cdot \vec{B} = 0, \end{array} \right. \Leftrightarrow \left\{ \begin{array}{l} \vec{E} = -\frac{\partial \vec{A}}{\partial t} - \vec{\nabla} V, \\ \vec{B} = \vec{\nabla} \wedge \vec{A}. \end{array} \right.$$

$$\left\{ \begin{array}{l} -\frac{\partial \vec{A}}{\partial t} - \vec{\nabla} V = \vec{0}, \\ \vec{\nabla} \wedge \vec{A} = \vec{0}, \end{array} \right. \Leftrightarrow \left\{ \begin{array}{l} \vec{A} = \vec{\nabla} \xi, \\ V = -\frac{\partial \xi}{\partial t}. \end{array} \right.$$

- 3D stress tensor: Maxwell, Morera parametrizations. . .
- Linearized Einstein equations (system of PDEs 10×10)?

\Rightarrow **OREMODULES** (Chyzak, Robertz, Q.)

Variational problems

- Let us extremize **the electromagnetic action**

$$\int \left(\frac{1}{2\mu_0} \|\vec{B}\|^2 - \frac{\epsilon_0}{2} \|\vec{E}\|^2 \right) dx_1 dx_2 dx_3 dt, \quad (1)$$

where \vec{B} and \vec{E} satisfy:

$$\begin{cases} \frac{\partial \vec{B}}{\partial t} + \vec{\nabla} \wedge \vec{E} = \vec{0}, \\ \vec{\nabla} \cdot \vec{B} = 0, \end{cases} \Leftrightarrow \begin{cases} \vec{E} = -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t}, \\ \vec{B} = \vec{\nabla} \wedge \vec{A}. \end{cases} \quad (3)$$

- Substituting (3) in (1) and using **Lorentz gauge**

$$\vec{\nabla} \cdot \vec{A} + \frac{1}{c^2} \frac{\partial V}{\partial t} = 0, \quad c^2 = 1/(\epsilon_0 \mu_0),$$

$$\Rightarrow \begin{cases} \frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2} - \Delta \vec{A} = 0, \\ \frac{1}{c^2} \frac{\partial^2 V}{\partial t^2} - \Delta V = 0. \end{cases} \quad (\text{electromagnetic waves}).$$

Constructive version of the Quillen-Suslin theorem

- **Constructive proofs** of the Quillen-Suslin theorem exist and one was **implemented** by Fabiańska in the package **QUILLENUSULIN**.

⇒ Computation of **bases of free $k[x_1, \dots, x_n]$ -modules**.

⇒ Computation of **flat outputs of flat systems** (Fliess et al).

$$(\text{Logemann, SCL 87}) \begin{cases} \dot{x}_1(t) + x_1(t) - u(t) = 0, \\ \dot{x}_2(t) - \dot{x}_2(t-h) - x_1(t) + a x_2(t) = 0, \end{cases}$$

$$\Leftrightarrow \begin{cases} x_1(t) = \dot{z}(t) - z(t-h) + a z(t), \\ x_2(t) = z(t), \\ u(t) = \ddot{z}(t) + \dot{z}(t) - \dot{z}(t-h) - z(t-h) + a \dot{z}(t) + a z(t). \end{cases}$$

- **Completion problem:** $\begin{pmatrix} R \\ T \end{pmatrix} (S \quad Q) = \begin{pmatrix} I_q & 0 \\ 0 & I_{p-q} \end{pmatrix} = I_p$.

A flat time-delay system is equivalent to the system without delay!

Constructive version of Stafford's theorem

- The time-varying linear control system (Sontag)

$$\begin{cases} \dot{x}_1(t) - t u_1(t) = 0, \\ \dot{x}_2(t) - u_2(t) = 0, \end{cases}$$

is **injectively parametrized** by (STAFFORD, Robertz, Q.)

$$\begin{cases} x_1(t) = t^2 z_1(t) - t \dot{z}_2(t) + z_2(t), \\ x_2(t) = t(t+1) z_1(t) - (t+1) \dot{z}_2(t) + z_2(t), \\ u_1(t) = t \dot{z}_1(t) + 2 z_1(t) - \ddot{z}_2(t), \\ u_2(t) = t(t+1) \dot{z}_1(t) + (2t+1) z_1(t) - (t+1) \ddot{z}_2(t), \end{cases}$$

and $\{z_1, z_2\}$ is a **basis** of the **free** left $A_1(\mathbb{Q})$ -module M as:

$$\begin{cases} z_1(t) = (t+1) u_1(t) - u_2(t), \\ z_2(t) = (t+1) x_1(t) - t x_2(t). \end{cases}$$

- Idem for $\partial_1 y_1 + \partial_2 y_2 + \partial_3 y_3 + x_3 y_1 = 0$.

Morphisms and transformations

- We consider $R' \in D^{q' \times p'}$, $\ker_{\mathcal{F}}(R'.)$, $M' = D^{1 \times p'} / (D^{1 \times q'} R')$.

How can we send elements of $\ker_{\mathcal{F}}(R'.)$ to elements of $\ker_{\mathcal{F}}(R.)$?

- **Theorem:** Any element $f \in \text{hom}_D(M, M')$ is defined by two matrices $P \in D^{p \times p'}$ and $Q \in D^{q \times q'}$ satisfying that:

$$R P = Q R'$$

- If $f \in \text{hom}_D(M, M')$, then we can define ($R P \eta' = Q R' \eta' = 0$):

$$\begin{aligned} f^* : \ker_{\mathcal{F}}(R'.) &\longrightarrow \ker_{\mathcal{F}}(R.), \\ \eta' &\longmapsto \eta = P \eta'. \end{aligned}$$

- $\text{hom}_D(M, M')$ can be **totally** (resp., **partially**) computed if D is a **commutative** (resp., **non-commutative**) polynomial ring.
- $\text{end}_D(M) = \text{hom}_D(M, M)$ defines the **internal symmetries** of M .

Factorization, reduction and decomposition problems

- The knowledge of the ring $\text{end}_D(M) = \text{hom}_D(M, M)$ and of its idempotents allows us to **constructively study** the problems:

① $\exists R_1 \in D^{r \times p}, R_2 \in D^{q \times r} : R = R_2 R_1 ?$

② $\exists W \in \text{GL}_p(D), V \in \text{GL}_q(D) : V R W = \begin{pmatrix} S_{11} & S_{12} \\ 0 & S_{22} \end{pmatrix} ?$

③ $\exists W \in \text{GL}_p(D), V \in \text{GL}_q(D) : V R W = \begin{pmatrix} S_{11} & 0 \\ 0 & S_{22} \end{pmatrix} ?$

- Basis computations are used for Problems 2 and 3

\Rightarrow OREMODULES, JACOBSON, QUILLENUSLIN, STAFFORD.

- Algorithms are implemented in ORE MORPHISMS (Cluzeau, Q.).

$$R = \begin{pmatrix} \partial_1 & -\partial_1 \partial_2^2 & \alpha \partial_1^2 \partial_2 \\ \partial_1 \partial_2^2 & -\partial_1 & \alpha \partial_1^2 \partial_2 \end{pmatrix}.$$

$$U = \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \text{GL}_3(D), \quad V = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \in \text{GL}_2(D),$$

$$\bar{R} = V R U^{-1} = \begin{pmatrix} \partial_1 (1 - \partial_2^2) & 0 & 0 \\ 0 & \partial_1 (1 + \partial_2^2) & 2 \alpha \partial_1^2 \partial_2 \end{pmatrix}.$$

Classical systems of PDEs

$$U \begin{pmatrix} \partial_t - k \partial_x^2 - a_1 & -b_1 \\ -a_2 & \partial_t - k \partial_x^2 - b_2 \end{pmatrix} U^{-1} \\ = \begin{pmatrix} \partial_t - k \partial_x^2 - \frac{(a_1 + b_2)}{2} + \frac{1}{2\alpha} & 0 \\ 0 & \partial_t - k \partial_x^2 - \frac{(a_1 + b_2)}{2} - \frac{1}{2\alpha} \end{pmatrix}.$$

$$U = V = \begin{pmatrix} 2 a_2 \alpha & (b_2 - a_1) \alpha - 1 \\ 2 a_2 \alpha & (b_2 - a_1) \alpha + 1 \end{pmatrix},$$

$$((a_1 - b_2)^2 + 4 a_2 b_1) \alpha^2 - 1 = 0.$$

- Wave/Cauchy-Riemann/Dirac/Beltrami eqs, electrical line...

Wind tunnel model (Manitius, IEEE TAC 84)

$$V \begin{pmatrix} \partial_1 + a & -k a \partial_2 & 0 & 0 \\ 0 & \partial_1 & -1 & 0 \\ 0 & \omega^2 & \partial_1 + 2\zeta\omega & -\omega^2 \end{pmatrix} U^{-1} \\ = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & \partial_1 + a & -\omega^2 k a \partial_2 \end{pmatrix},$$

$$U = \begin{pmatrix} 0 & \omega^2 & \partial_1 + 2\zeta\omega & -\omega^2 \\ 0 & \partial_1 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\omega^2} & 0 & 0 \end{pmatrix}, \quad V = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

- Computations obtained by **OREMORPHISMS** or **SERRE** (Q.).

String with an interior mass (Fliess et al, COCV 98)

$$(\star) \begin{cases} \phi_1(t) + \psi_1(t) - \phi_2(t) - \psi_2(t) = 0, \\ \dot{\phi}_1(t) + \dot{\psi}_1(t) + \eta_1 \phi_1(t) - \eta_1 \psi_1(t) - \eta_2 \phi_2(t) + \eta_2 \psi_2(t) = 0, \\ \phi_1(t - 2h_1) + \psi_1(t) - u(t - h_1) = 0, \\ \phi_2(t) + \psi_2(t - 2h_2) - v(t - h_2) = 0. \end{cases}$$

$$V \begin{pmatrix} 1 & 1 & -1 & -1 & 0 & 0 \\ \partial_1 + \eta_1 & \partial_1 - \eta_1 & -\eta_2 & \eta_2 & 0 & 0 \\ \partial_2^2 & 1 & 0 & 0 & -\partial_2 & 0 \\ 0 & 0 & 1 & \partial_3^2 & 0 & -\partial_3 \end{pmatrix} U^{-1}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \partial_1 + \eta_1 + \eta_2 & 2\eta_1 \partial_2 & 2\eta_2 \partial_3 \end{pmatrix},$$

$$(\star) \Leftrightarrow \dot{z}_1(t) + (\eta_1 + \eta_2) z_1(t) + 2\eta_1 z_2(t - h_1) + 2\eta_2 z_3(t - h_2) = 0.$$

String with an interior mass (Fliess et al, COCV 98)

The **unimodular matrices** U and V are defined by:

$$U = \begin{pmatrix} 1 & 0 & 0 & 1 & \partial_2 & 0 \\ 0 & -1 & 0 & 0 & -\partial_2 & 0 \\ 0 & 0 & 1 & 0 & 0 & -\partial_3 \\ 0 & -1 & -1 & 1 & 0 & \partial_3 \\ 0 & 0 & 0 & \partial_2 & \partial_2^2 - 1 & 0 \\ 0 & -\partial_3 & -\partial_3 & \partial_3 & 0 & \partial_3^2 - 1 \end{pmatrix},$$

$$V = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \partial_2^2 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ \partial_2^2 (\partial_1 - \eta_1 + \eta_2) - \partial_1 - \eta_1 & 1 & -\partial_1 + \eta_1 - \eta_2 & 2\eta_2 \end{pmatrix}.$$

- The computations were obtained by **OREMORPHISMS** or **SERRE**.

Conclusion

- Based on algebraic analysis, constructive homological algebra and Ore algebras, we have developed a general **non-commutative polynomial approach to multidimensional linear systems**.
- The different results have been implemented in packages:

OREMODULES, JANETORE, ORE MORPHISMS, SERRE,
STAFFORD, QUILLEN SUSLIN, HOMALG, CONTROL.LIB.

This new approach allowed us to:

- ① Develop an intrinsic approach (independent of the form).
- ② Develop generic algorithms and generic implementations.
- ③ Constructively study certain classes of flat systems.
- ④ Extend the concepts of primeness, solve conjectures. . .

Conclusion

- An **algebraic analysis approach** was developed for **stabilization problems of ∞ -dimensional systems** within the **fractional representation approach** developed by Desoer, Vidyasagar, Callier...
- This new approach is based on **module theory** over **Banach algebras** such as $H^\infty(\mathbb{C}_+)$, $\hat{\mathcal{A}}$, $A(\mathbb{D})$, W_+ ...

- 1 Solve Vidyasagar-Schneider-Francis' question.
- 2 Extend the Youla-Kučera parametrization of all stabilizing controllers for stabilizing plants.
- 3 Solve Feintuch's question and Lin's conjecture.
- 4 Develop a **duality** between the **module-theoretic** and the **operator-theoretic approaches (domains, graphs)**:

A forced marriage of Willems and Curtain? :)