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Computation of Koszul homology and application to involutivity of partial differential systems

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Abstract: The formal integrability of systems of partial differential equations plays a fundamental role in different analysis and synthesis problems for both linear and nonlinear differential control systems. Following Spencer's theory, to test the formal integrability of a system of partial differential equations, we must study when the symbol of the system, namely, the top-order part of the linearization of the system, is 2-acyclic or involutive, *i.e.*, when certain Spencer cohomology groups vanish. Combining the fact that Spencer cohomology is dual to Koszul homology and symbolic computation methods, we show how to effectively compute the homology modules defined by the Koszul complex of a finitely presented module over a commutative polynomial ring. These results are implemented using the OreMorphisms package. We then use these results to effectively characterize 2-acyclicity and involutivity of the symbol of a linear system of partial differential equations. Finally, we show explicit computations on two standard examples.

Keywords: Systems of partial differential equations, multidimensional systems, behaviours, Cartan's involutivity, formal integrability, Koszul homology, Spencer cohomology

1. INTRODUCTION

It is well-known that the study of structural properties of both linear and nonlinear systems, such as controllability, observability, differential flatness, etc., boils down to the study of the *integrability* or the *involutivity* of systems of vector fields. Moreover, many standard problems in nonlinear control theory, such as linearization by static feedback, (input-state, input-output) linearizations, model matching, minimal realization, etc., can be reduced to the study of the existence of a local solution of (nonlinear) systems of partial differential equations (PDEs) in the unknowns of the problem. It is thus not surprising that *Frobenius theorem* on integrability of geometric distributions plays an important role in control theory. The study of the existence of formal/locally convergent power series solutions of systems of PDEs has a long mathematical history and is nowadays called the *formal integrability theory*. Thus, control theorists have to face the problem of studying the formal integrability of linear or nonlinear systems of PDEs.

According to Spencer's theory (Spencer (1969)), the formal integrability of linear PD systems can be reduced to the study of the *Spencer cohomology*, dual to the *Koszul homology* (Quillen (1964)), and integrability conditions.

In this paper, using computer algebra methods (*Gröbner bases*) and results on homomorphisms of *finitely presented modules* (Section 2) over the commutative polynomial ring $\mathcal{A} = \mathcal{B}[\chi_1, \ldots, \chi_n]$, where $\mathcal{B}$ is a commutative ring over a *computable* field $\mathcal{K}$ of characteristic 0, in Section 3, we first explain how to compute the *Koszul homology* $\mathcal{A}$-*modules* $H_i(\chi, \mathcal{M})$, $i = 0, \ldots, n$, of a finitely presented $\mathcal{A}$-module $\mathcal{M}$. To simplify the presentation, we shall only consider the case $n = 3$, *i.e.*, the most important situations in applications. But the general case exactly follows the same line. Koszul homology $\mathcal{A}$-modules can be computed by standard computer algebra systems, *e.g.*, using the OreMorphisms package (Cluzeau and Quadrat (2009)), developed in `Maple`, as demonstrated in Section 6 with two standard examples of the literature (Pommaret (1994)). Then, following Quillen (1964); Malgrange (2005), using the natural *total-order graduation* of rings of PD operators and the fact that the study of Spencer cohomology of the symbol of a linear PD system reduces to the study of the graded Koszul homology $\mathcal{A}$-modules $H_i(\chi, \mathrm{gr}(\mathcal{M}))$'s of the finitely presented graded $\mathcal{A}$-module $\mathrm{gr}(\mathcal{M})$, in Section 5, we give effective tests for i-acyclicity, $i = 0, \ldots, n$, and thus, for the involutivity of symbols of linear PD systems.

Notation. Let $\mathcal{K}$ be a field of *characteristic* 0 (*e.g.*, $\mathcal{K} = \mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$) and $\mathcal{K}[x_1, \ldots, x_n]$ the commutative ring formed by all the polynomials in the variables $x_1, \ldots, x_n$ with coefficients in $\mathcal{K}$. Let $\mathcal{D}$ be a noncommutative ring. Then, the set of all $q \times p$ matrices with entries in $\mathcal{D}$ is denoted by $\mathcal{D}^{q\times p}$. If $R \in \mathcal{D}^{q\times p}$, then we can consider the *left* $\mathcal{D}$-*homomorphism* (*i.e.*, left $\mathcal{D}$-linear map) defined by:

$$\begin{array}{rcl} .R : \mathcal{D}^{1\times q} & \longrightarrow & \mathcal{D}^{1\times p} \\ \lambda = (\lambda_1 \ \ldots \ \lambda_q) & \longmapsto & \lambda\, R. \end{array} \quad (1)$$

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We can also consider the following *left $\mathcal{D}$-modules*:

$$\ker_{\mathcal{D}}(.R) = \{\lambda \in \mathcal{D}^{1\times q} \mid \lambda R = 0\}, \ \operatorname{im}_{\mathcal{D}}(.R) = \mathcal{D}^{1\times q} R,$$
$$\operatorname{coker}_{\mathcal{D}}(.R) = \mathcal{D}^{1\times p}/(\mathcal{D}^{1\times q} R).$$

Note that $\mathcal{L} := \operatorname{coker}_{\mathcal{D}}(.R)$ is a *quotient* left $\mathcal{D}$-module, *i.e.*, $\mathcal{L}$ is the left $\mathcal{D}$-module defined by the *residue classes* $\pi(\mu) \in \mathcal{L}$ for all $\mu \in \mathcal{D}^{1\times p}$. By definition, $\pi(\mu') = \pi(\mu)$ if $\mu' - \mu \in \operatorname{im}_{\mathcal{D}}(.R)$, *i.e.*, if there exists $\lambda \in \mathcal{D}^{1\times q}$ such that $\mu' = \mu + \lambda R$. For all $d_1, d_2 \in \mathcal{D}$ and for all $\mu_1, \mu_2 \in \mathcal{D}^{1\times p}$, we then have $d_1 \pi(\mu_1) + d_2 \pi(\mu_2) := \pi(d_1 \mu_1 + d_2 \mu_2)$. Let $\pi : \mathcal{D}^{1\times p} \longrightarrow \mathcal{L}$ be the left $\mathcal{D}$-homomorphism defined by mapping $\mu \in \mathcal{D}^{1\times p}$ onto its residue class $\pi(\mu)$ in $\mathcal{L}$. If $\mathcal{L}'$ is a left $\mathcal{D}$-module, the $\mathbb{Z}$-module formed by all the homomorphisms from $\mathcal{L}$ to $\mathcal{L}'$ is denoted by $\hom_{\mathcal{D}}(\mathcal{L}, \mathcal{L}')$. Finally, if $\mathcal{D}$ is commutative, then $\hom_{\mathcal{D}}(\mathcal{L}, \mathcal{L}')$ inherits a natural $\mathcal{D}$-module structure.

2. ALGEBRAIC ANALYSIS APPROACH

2.1 Finitely presented left $\mathcal{D}$-modules & Behaviours

Let $R \in \mathcal{D}^{q\times p}$ and $\mathcal{L} = \operatorname{coker}_{\mathcal{D}}(.R)$. Moreover, let $\{e_i\}_{i=1,\ldots,p}$ be the standard basis of $\mathcal{D}^{1\times p}$ and $\{f_j\}_{j=1,\ldots,q}$ the standard basis of $\mathcal{D}^{1\times q}$. Finally, let $y_i = \pi(e_i)$ for $i = 1, \ldots, p$, and $y = (y_1 \ \ldots \ y_p)^T$. Using the fact that every $l \in \mathcal{L}$ is of the form $l = \pi(\mu)$ for a certain $\mu = (\mu_1 \ \ldots \ \mu_p) \in \mathcal{D}^{1\times p}$, we then have $l = \pi(\mu) = \pi\left(\sum_{i=1}^p \mu_i e_i\right) = \sum_{i=1}^p \mu_i y_i = \mu y$, which shows that $\{y_i\}_{i=1,\ldots,p}$ is a finite set of generators of $\mathcal{L}$. The left $\mathcal{D}$-module $\mathcal{M}$ is then said to be *finitely generated*. Moreover, we have $f_j R = (R_{j1} \ \ldots \ R_{jp}) \in \operatorname{im}_{\mathcal{D}}(.R)$, and thus, $\sum_{k=1}^p R_{jk} y_k = \pi(f_j R) = 0$ for $j = 1, \ldots, q$, which shows that the generators y_i's of $\mathcal{L}$ satisfy the finite generating set of left $\mathcal{D}$-linear relations $R y = 0$. The left $\mathcal{D}$-module $\mathcal{L}$ is thensaid to be *finitely presented*. Equivalently, $\mathcal{L}$ is defined by the following *exact sequence of left $\mathcal{D}$-modules*

$$\mathcal{D}^{1\times q} \xrightarrow{.R} \mathcal{D}^{1\times p} \xrightarrow{\pi} \mathcal{L} \longrightarrow 0,$$

i.e., π is surjective and $\ker \pi = \operatorname{im}_{\mathcal{D}}(.R)$.

Consider a left $\mathcal{D}$-module $\mathcal{F}$ and the left $\mathbb{Z}$-homomorphism:

$$\begin{aligned} R. : \mathcal{F}^{p\times 1} &\longrightarrow \mathcal{F}^{q\times 1} \\ \eta &\longmapsto R\,\eta. \end{aligned} \tag{2}$$

Then, $\ker_{\mathcal{F}}(R.) = \{\eta \in \mathcal{F}^{p\times 1} \mid R\,\eta = 0\}$ is the $\mathcal{F}$-solutions set of the linear system $R\,\eta = 0$, also called *behaviour*. A standard result in module theory asserts that:

$$\ker_{\mathcal{F}}(R.) \cong \hom_{\mathcal{D}}(\mathcal{L}, \mathcal{F}). \tag{3}$$

Indeed, for all $\phi \in \hom_{\mathcal{D}}(\mathcal{L}, \mathcal{F})$, we have $\phi(0) = 0$, *i.e.*, $\phi\left(\sum_{k=1}^p R_{jk} y_k\right) = \sum_{k=1}^p R_{jk}\, \phi(y_k) = 0$, which shows that $\psi(\phi) := (\phi(e_1) \ \ldots \ \phi(e_p))^T \in \ker_{\mathcal{F}}(R.)$. Conversely, if $\eta \in \ker_{\mathcal{F}}(R.)$, then $\phi_\eta(\pi(\lambda)) := \lambda\,\eta \in \mathcal{F}$ for all $\lambda \in \mathcal{D}^{1\times p}$ belongs to $\hom_{\mathcal{D}}(\mathcal{L}, \mathcal{F})$, $\psi(\phi_\eta) = \eta$, and $\phi_{\psi(\phi)} = \phi$.

Within the *algebraic analysis approach* to linear PD systems, $\mathcal{D}$ is usually a (noncommutative) *ring of PD operators*. Let us state again this definition. Let $\mathcal{B}$ be a commutative ring and $\partial_1, \ldots, \partial_n$ *n-commuting derivations* of $\mathcal{B}$, namely, $\partial_i : \mathcal{B} \longrightarrow \mathcal{B}$ is additive and:

$$\forall\, b_1, b_2 \in \mathcal{B}, \ \partial_i(b_1 b_2) = \partial_i(b_1)\, b_2 + b_1\, \partial_i(b_2), \ i = 1, \ldots, n,$$
$$\partial_i \circ \partial_j = \partial_j \circ \partial_i, \ 1 \leq i < j \leq n.$$

If $\operatorname{end}_{\mathbb{Z}}(\mathcal{B})$ denotes the ring of all the $\mathbb{Z}$-linear maps from $\mathcal{B}$ to $\mathcal{B}$, and if $b \in \mathcal{B}$, then we can define $\overline{b} \in \operatorname{end}_{\mathbb{Z}}(\mathcal{B})$ by $\overline{b}(a) = b\,a$ for all $a \in \mathcal{B}$. Then, $\mathcal{D} = \mathcal{B}\langle \partial_1, \ldots, \partial_n \rangle$ is the subring of $\operatorname{end}_{\mathbb{Z}}(\mathcal{B})$ generated by the ∂_i's and the $\overline{b}$'s for all $b \in \mathcal{B}$. For all $a \in \mathcal{B}$, we have

$$(\partial_i \circ \overline{b})(a) = \partial_i(b\,a) = \partial_i(b)\,a + b\,\partial_i(a) = (\overline{b} \circ \partial_i + \overline{\partial_i(b)})(a),$$

i.e., $\partial_i \circ \overline{b} = \overline{b} \circ \partial_i + \overline{\partial_i(b)}$ holds in $\mathcal{D}$ for all $b \in \mathcal{B}$ and $i = 1, \ldots, n$, as well as $\partial_i \circ \partial_j = \partial_j \circ \partial_i$ for $1 \leq i < j \leq n$. If we simply note $\overline{b}$ by b and write multiplicatively the compositions in $\mathcal{D}$, then $\mathcal{D}$ is the noncommutative polynomial ring in $\partial_1, \ldots, \partial_n$ with coefficients in $\mathcal{B}$ satisfying:

$$\begin{aligned} &\partial_i\, b = b\,\partial_i + \partial_i(b), \quad b \in \mathcal{B}, \quad i = 1, \ldots, n, \\ &\partial_i\, \partial_j = \partial_j\, \partial_i, \quad 1 \leq i < j \leq n. \end{aligned} \tag{4}$$

Any $P \in \mathcal{D}$ can be uniquely written as $P = \sum_{0 \leq |\nu| \leq r} b_\nu\, \partial^\nu$, where $\nu = (\nu_1, \ldots, \nu_n) \in \mathbb{Z}^n_{\geq 0}$ is a multi-index of *length* $|\nu| = \nu_1 + \cdots + \nu_n$, $\partial^\nu = \partial_1^{\nu_1} \ldots \partial_n^{\nu_n}$, and $b_\nu \in \mathcal{B}$.

Examples of rings of PD operators are the commutative polynomial ring $\mathcal{D} = \mathcal{B}[\partial_1, \ldots, \partial_n]$, when $\partial_i(b) = 0$ for all $b \in \mathcal{B}$ and $i = 1, \ldots, n$, or $\mathcal{D} = \mathcal{B}\langle \partial_1, \ldots, \partial_n \rangle$, where $\mathcal{B} = \mathcal{K}[x_1, \ldots, x_n]$, $\mathcal{K}$ is a field, and $\partial_i(b) = \partial b / \partial x_i$ for all $b \in \mathcal{B}$ and $i = 1, \ldots, n$, the so-called the *Weyl algebra* of PD operators with polynomial coefficients.

2.2 Homomorphisms of finitely presented left $\mathcal{D}$-modules

In this section, we briefly state again useful results on homomorphisms of finitely presented left $\mathcal{D}$-modules.

Lemma 1. (Cluzeau and Quadrat (2008)). Let us consider two finitely presented left $\mathcal{D}$-modules $\mathcal{L} = \mathcal{D}^{1\times p}/(\mathcal{D}^{1\times q} R)$ and $\mathcal{L}' = \mathcal{D}^{1\times p'}/(\mathcal{D}^{1\times q'} R')$. The existence of a homomorphism $f \in \hom_{\mathcal{D}}(\mathcal{L}, \mathcal{L}')$ is equivalent to the existence of matrices $P \in \mathcal{D}^{p\times p'}$ and $Q \in \mathcal{D}^{q\times q'}$ satisfying the relation:

$$R\,P = Q\,R'. \tag{5}$$

Then, the *commutative exact diagram*

$$\begin{tikzcd} \mathcal{D}^{1\times q} \arrow[r, ".R"] \arrow[d, ".Q"] & \mathcal{D}^{1\times p} \arrow[r, "\pi"] \arrow[d, ".P"] & \mathcal{L} \arrow[r] \arrow[d, "f"] & 0 \\ \mathcal{D}^{1\times q'} \arrow[r, ".R'"] & \mathcal{D}^{1\times p'} \arrow[r, "\pi'"] & \mathcal{L}' \arrow[r] & 0 \end{tikzcd}$$

holds, where $f \in \hom_{\mathcal{D}}(\mathcal{L}, \mathcal{L}')$ is defined by:

$$\forall\, \mu \in \mathcal{D}^{1\times p}, \quad f(\pi(\lambda)) \ = \ \pi'(\mu\,P).$$

Note that the identity (5) yields the *homomorphism of behaviours* $f^\star : \ker_{\mathcal{F}}(R'.) \longrightarrow \ker_{\mathcal{F}}(R.)$ defined by:

$$\forall\, \eta' \in \ker_{\mathcal{F}}(R'.), \quad f^\star(\eta') = P\,\eta \in \ker_{\mathcal{F}}(R.).$$

For the computation of the *Koszul homology* (see Section 3), we shall need to compute kernels and images of homomorphisms. Let us explicitly characterize the kernel and the image of $f \in \hom_{\mathcal{D}}(\mathcal{L}, \mathcal{L}')$, where $\mathcal{L}$ and $\mathcal{L}'$ are two presented left $\mathcal{D}$-modules.

Lemma 2. (Cluzeau and Quadrat (2008)). Let us consider two finitely presented left $\mathcal{D}$-modules $\mathcal{L} = \mathcal{D}^{1\times p}/(\mathcal{D}^{1\times q} R)$ and $\mathcal{L}' = \mathcal{D}^{1\times p'}/(\mathcal{D}^{1\times q'} R')$. Let $f \in \hom_{\mathcal{D}}(\mathcal{L}, \mathcal{L}')$ be a homomorphism defined by $P \in \mathcal{D}^{p\times p'}$ and $Q \in \mathcal{D}^{q\times q'}$ satisfying (5). Let $L = (P^T \quad R'^T)^T \in \mathcal{D}^{(p+q')\times p'}$. Then:

(1) $\ker(f) = (\mathcal{D}^{1\times r} S')/(\mathcal{D}^{1\times q} R)$, where $S' \in \mathcal{D}^{r\times p}$ is a matrix defined by:

$$\ker_{\mathcal{D}}(.L) = \mathcal{D}^{1\times r}\,(S' \quad -S''), \quad S'' \in \mathcal{D}^{r\times q'}.$$

(2) $\mathrm{im}(f) = (\mathcal{D}^{1\times(p+q')}\, L)/(\mathcal{D}^{1\times q'}\, R')$.
(3) $\mathrm{coker}(f) = \mathcal{D}^{1\times p'}/(\mathcal{D}^{1\times(p+q')}\, L)$.

3. THE KOSZUL COMPLEX

In this section, we shall briefly introduce the notions of *Koszul complex* and *Koszul homology* (Serre (1989)). In Section 4, we shall explain how to characterize the Koszul homology. In Section 5, we shall show how to use these results to effectively check the i-acyclicity and the *involutivity* of linear PD systems.

Let $\mathcal{A}$ be a commutative ring and T a free $\mathcal{A}$-module of rank n, *i.e.*, $T \cong \mathcal{A}^n$. Let $u_1, \ldots, u_n$ be a basis of T, *i.e.*, $T = \mathcal{A}\, u_1 \oplus \cdots \oplus \mathcal{A}\, u_n$. We can then consider the *exterior algebra* of T defined by $\Lambda\, T = \bigoplus_{r=0}^n \Lambda^r\, T$, where $\Lambda^0\, T = \mathcal{A}$ and $\Lambda^r\, T$ is the *homogeneous component of degree* r of $\Lambda\, T$. As an $\mathcal{A}$-algebra, $(\Lambda\, T, +, \wedge)$ is generated by $u_1, \ldots, u_n$ satisfying the following relations:

$$u_i \wedge u_i = 0, \quad u_i \wedge u_j = -u_j \wedge u_i, \quad i \neq j.$$

Then, $\omega \in \Lambda^r\, T$ has the form $\omega = \sum_{i \in I} a_i\, u_{i_1} \wedge \cdots \wedge u_{i_r}$, where $1 \leq i_1 < i_2 < \cdots < i_r \leq n$, $a_i \in \mathcal{A}$, I is a finite set,

$$\forall\, \omega \in \Lambda^r\, T, \quad \forall\, \sigma \in \Lambda^s\, T, \quad \omega \wedge \sigma = (-1)^{r\, s}\, \sigma \wedge \omega$$

and thus, $\omega \wedge \omega = 0$ for r is odd. Hence, the $\Lambda^r\, T$'s are the free $\mathcal{A}$-modules generated by $u_{i_1} \wedge \cdots \wedge u_{i_r}$, where $1 \leq i_1 < i_2 < \cdots < i_r \leq n$, *i.e.*, we have $\Lambda^r\, T \cong \mathcal{A}^{\binom{n}{r}}$.

Let $\mathcal{M}$ be a finitely generated $\mathcal{A}$-module, $\chi_1, \ldots, \chi_n \in \mathcal{A}$. For $r = 1, \ldots, n$, let $d_r : \Lambda^r\, T \otimes_{\mathcal{A}} \mathcal{M} \longrightarrow \Lambda^{r-1}\, T \otimes_{\mathcal{A}} \mathcal{M}$ be the $\mathcal{A}$-homomorphism defined by

$$\begin{aligned} & d_r\, (u_{i_1} \wedge \cdots \wedge u_{i_r} \otimes m) \\ & = \sum_{j=1}^{r} (-1)^{j+1} u_{i_1} \wedge \cdots \wedge \hat{u}_{i_j} \wedge \cdots \wedge u_{i_r} \otimes\, \chi_{i_j}\, m, \end{aligned} \tag{6}$$

for all $u_{i_1} \wedge \cdots \wedge u_{i_r} \otimes m \in \Lambda^r\, T \otimes_{\mathcal{A}} \mathcal{M}$, where $\hat{u}_{i_j}$ means that this term is omitted. Set $\chi = (\chi_1, \ldots, \chi_n)$. The d_r's induce the following *complex* $K(\chi, \mathcal{M})$ of $\mathcal{A}$-modules

$$0 \xrightarrow{d_{n+1}} \Lambda^n\, T \otimes_{\mathcal{A}} \mathcal{M} \xrightarrow{d_n} \Lambda^{n-1}\, T \otimes_{\mathcal{A}} \mathcal{M} \xrightarrow{d_{n-1}} \ldots$$

$$\ldots \xrightarrow{d_3} \Lambda^2\, T \otimes_{\mathcal{A}} \mathcal{M} \xrightarrow{d_2} T \otimes_{\mathcal{A}} \mathcal{M} \xrightarrow{d_1} \mathcal{M} \xrightarrow{d_0} 0,$$

i.e., $d_r \circ d_{r+1} = 0$ for $r = 0, \ldots, n$, called the *Koszul complex* of $\mathcal{M}$, and we can define the *homology $\mathcal{A}$-modules*:

$$H_r(\chi, \mathcal{M}) = \ker(d_r)/\mathrm{im}(d_{r+1}), \quad r = 0, \ldots, n.$$

If $\mathcal{A}$ is a *noetherian ring* and $\mathcal{M}$ a finitely generated $\mathcal{A}$-module, then the $H_r(\chi, \mathcal{M})$'s are finitely generated $\mathcal{A}$-modules. They only depend on T and not on the choice of the basis $u_1, \ldots, u_n$ of T. Moreover, we can check that:

$$H_0(\chi, \mathcal{M}) = \mathcal{M}/\mathrm{im}(d_1) \cong \mathcal{M}/\left(\sum_{i=1}^n \chi_i\, \mathcal{M}\right),$$

$$H_n(\chi, \mathcal{M}) = \ker(d_n) = \{m \in \mathcal{M} \mid \chi_i\, m = 0,\ i = 1, \ldots, n\}.$$

Lemma 3. (Serre (1989)). The Koszul homology $\mathcal{A}$-modules $H_r(\chi, \mathcal{M})$'s satisfy that $\chi_i\, H_r(\chi, \mathcal{M}) = 0$, for $i = 1, \ldots, n$.

If $\mathcal{A} = \mathcal{B}[\chi_1, \ldots, \chi_n]$, where $\mathcal{B}$ is a commutative ring, then Lemma 3 shows that any element $t \in \mathcal{A}$ which has no constant term acts trivially over $H_r(\chi, \mathcal{M})$, *i.e.*, for every homology class $\alpha \in H_r(\chi, \mathcal{M})$, the homology class $t\, \alpha$ vanishes. Thus, the $H_r(\chi, \mathcal{M})$'s are finitely generated $\mathcal{B}$-modules and if $\mathcal{B} = \mathcal{K}$ is a field, then the $H_r(\chi, \mathcal{M})$'s are finite-dimensional $\mathcal{K}$-vector spaces (Malgrange (2005)).

4. CHARACTERIZATION OF KOSZUL HOMOLOGY

Let $\mathcal{A}$ be a commutative noetherian ring, $R \in \mathcal{A}^{q\times p}$, and $\mathcal{M} = \mathrm{coker}_{\mathcal{A}}(.R) = \mathcal{A}^{1\times p}/(\mathcal{A}^{1\times q}\, R)$. In this section, we explain how to characterize the homology $\mathcal{A}$-modules $H_r(\chi, \mathcal{M})$'s as $H_r(\chi, \mathcal{M}) = (\mathcal{A}^{1\times s'_r}\, S'_r)/(\mathcal{A}^{1\times s_r}\, S_r)$, where S'_r and S_r are matrices having $p\binom{n}{r}$ columns. The rows of S'_r and S_r represent elements of $\Lambda^r\, T \otimes_{\mathcal{A}} \mathcal{A}^{1\times p}$ whose classes modulo $\mathcal{A}^{1\times q}\, R$ are generators as $\mathcal{A}$-modules of $\ker(d_r)$ and $\mathrm{im}(d_{r+1})$, respectively. We shall only consider the case $n = 3$ that cover most of the examples studied in applications. The general case can be handled similarly.

The free $\mathcal{A}$-modules $\Lambda^3\, T$ and $\Lambda^2\, T$ have respectively rank 1 and 3 with the following chosen bases:

- $\{e_1 \wedge e_2 \wedge e_3\}$ for $\Lambda^3\, T$,
- $\{e_2 \wedge e_3\,, e_3 \wedge e_1,\, e_1 \wedge e_2\}$ for $\Lambda^2\, T$.

The Koszul differential d_3 acts on $\Lambda^3\, T \otimes_{\mathcal{A}} \mathcal{M}$ as:

$$\begin{aligned} d_3(e_1 \wedge e_2 \wedge e_3 \otimes m) = e_2 \wedge e_3 \otimes \chi_1\, m + e_3 \wedge e_1 \otimes \chi_2\, m \\ + e_1 \wedge e_2 \otimes \chi_3\, m, \end{aligned}$$

for all $m \in \mathcal{M}$, and d_2 acts on $\Lambda^2\, T \otimes_{\mathcal{A}} \mathcal{M}$ as:

$$\forall\, m \in \mathcal{M}, \begin{cases} d_2(e_2 \wedge e_3 \otimes m) &= -e_2 \otimes \chi_3\, m + e_3 \otimes \chi_2\, m, \\ d_2(e_3 \wedge e_1 \otimes m) &= e_1 \otimes \chi_3\, m - e_3 \otimes \chi_1\, m, \\ d_2(e_1 \wedge e_2 \otimes m) &= -e_1 \otimes \chi_2\, m + e_2 \otimes \chi_1\, m, \end{cases}$$

and $d_1(e_i \otimes m) = \chi_i\, m$ for all $m \in \mathcal{M}$ and $i = 1, 2, 3$. Let:

$$P_1 = \begin{pmatrix} \chi_1 \\ \chi_2 \\ \chi_3 \end{pmatrix} \in \mathcal{A}^{3\times 1}, \quad P_2 = \begin{pmatrix} 0 & -\chi_3 & \chi_2 \\ \chi_3 & 0 & -\chi_1 \\ -\chi_2 & \chi_1 & 0 \end{pmatrix} \in \mathcal{A}^{3\times 3},$$

$$P_3 = (\chi_1 \quad \chi_2 \quad \chi_3) \in \mathcal{A}^{1\times 3}.$$

Using $T \otimes_{\mathcal{A}} \mathcal{A}^{1\times r} \simeq \mathcal{A}^{1\times 3\, r}$, $\Lambda^2\, T \otimes_{\mathcal{A}} \mathcal{A}^{1\times r} \simeq \mathcal{A}^{1\times 3\, r}$, and $\Lambda^3\, T \otimes_{\mathcal{A}} \mathcal{A}^{1\times r} \simeq \mathcal{A}^{1\times r}$, for $r = q, p$, we then obtain the following commutative diagram of $\mathcal{A}$-modules with exact rows and exact first two columns

$$\begin{tikzcd} & & 0 \arrow[d] & \\ \mathcal{A}^{1\times q} \arrow[r, ".R"] \arrow[d, ".P_3\otimes I_q"] & \mathcal{A}^{1\times p} \arrow[r] \arrow[d, ".P_3\otimes I_p"] & \Lambda^3\, T \otimes_{\mathcal{A}} \mathcal{M} \arrow[r] \arrow[d, "d_3"] & 0 \\ \mathcal{A}^{1\times 3q} \arrow[r, ".I_3\otimes R"] \arrow[d, ".P_2\otimes I_q"] & \mathcal{A}^{1\times 3p} \arrow[r] \arrow[d, ".P_2\otimes I_p"] & \Lambda^2\, T \otimes_{\mathcal{A}} \mathcal{M} \arrow[r] \arrow[d, "d_2"] & 0 \\ \mathcal{A}^{1\otimes 3q} \arrow[r, ".I_3\otimes R"] \arrow[d, ".P_1\otimes I_q"] & \mathcal{A}^{1\times 3p} \arrow[r] \arrow[d, ".P_1\otimes I_p"] & T \otimes_{\mathcal{A}} \mathcal{M} \arrow[r] \arrow[d, "d_1"] & 0 \\ \mathcal{A}^{1\times q} \arrow[r, ".R"] & \mathcal{A}^{1\times p} \arrow[r] & \mathcal{M} \arrow[r] \arrow[d] & 0, \\ & & 0 & \end{tikzcd}$$

where $P_i \otimes I_p$ denotes the *Kronecker product* of the matrices P_i and I_p (and similarly for $P_i \otimes I_q$) for $i = 1, 2, 3$. Then, the Kozsul homology $\mathcal{A}$-modules are defined by:

$$H_0(\chi, \mathcal{M}) = \mathrm{coker}(d_1),\ H_1(\chi, \mathcal{M}) = \ker(d_1)/\mathrm{im}(d_2),$$
$$H_2(\chi, \mathcal{M}) = \ker(d_2)/\mathrm{im}(d_1),\ H_3(\chi, \mathcal{M}) = \ker(d_3).$$

Using Lemma 2, the third isomorphism theorem yields:

$$H_0(\chi, \mathcal{M}) = \mathcal{A}^{1\times p}/\left(\mathcal{A}^{1\times(3p+q)} \begin{pmatrix} P_1 \otimes I_p \\ R \end{pmatrix}\right),$$

$$H_1(\chi, \mathcal{M}) = (\mathcal{A}^{1\times s_1}\, S'_1)/\left(\mathcal{A}^{1\times 3(p+q)} \begin{pmatrix} P_2 \otimes I_p \\ I_3 \otimes R \end{pmatrix}\right),$$

$$H_2(\chi,\mathcal{M}) = (\mathcal{A}^{1\times s_2}\, S_2')/\left(\mathcal{A}^{1\times (p+3q)}\begin{pmatrix} P_3\otimes I_p \\ I_3\otimes R\end{pmatrix}\right),$$
$$H_3(\chi,\mathcal{M}) = (\mathcal{A}^{1\times s_3}\, S_3')/(\mathcal{A}^{1\times q}\, R),$$

where, for $i=1,\,2,\,3$, $S_i'\in\mathcal{A}^{s_i\times\binom{3}{i}\,p}$ are defined by:

$$\ker_{\mathcal{A}}\left(.\begin{pmatrix} P_i\otimes I_p \\ I_{\binom{3}{i-1}}\otimes R\end{pmatrix}\right) = \mathcal{A}^{1\times s_i}\,(S_i' \quad -S_i''),$$

with $S_i''\in\mathcal{A}^{s_1\times\binom{3}{i}\,q}$.

If $\mathcal{A}$ is a commutative polynomial ring over a computable field $\mathcal{K}$ (*e.g.*, $\mathcal{K}=\mathbb{Q}$), then, using, *e.g.*, Gröbner basis methods, we can effectively compute the $\mathcal{A}$-modules $H_r(\chi,\mathcal{M})$, $r=0,\ldots,n$. See Cluzeau and Quadrat (2009).

5. ACYCLICITY AND INVOLUTIVITY OF LINEAR PD SYSTEMS

In this section, we first define the concepts of *formal integrability*, *i-acyclicity*, and *involutivity* of a linear PD system, and then explain how to effectively test them.

5.1 Formal integrability of linear PD systems

With the notations of Section 2, the *order* of $P\in\mathcal{D}$ is defined by $\mathrm{ord}(P)=\sup\{|\nu| \mid b_\nu\neq 0\}$. By extension, the *order* of a matrix $R\in\mathcal{D}^{q\times p}$ is given by $\mathrm{ord}(R)=\max\{\mathrm{ord}(R_{ij})\mid i=1,\ldots,q,\ j=1,\ldots,p\}$. Let $\mathcal{D}_s=\{P\in\mathcal{D}\mid 0\le\mathrm{ord}(P)\le s\}$ for $s\ge 0$ and set $\mathcal{D}_s=0$ for $s<0$. Then, we have $\mathcal{D}=\bigcup_{s\in\mathbb{Z}}\mathcal{D}_s$ and $\mathcal{D}_s\,\mathcal{D}_t\subseteq\mathcal{D}_{s+t}$ for all $s,\,t\in\mathbb{Z}$, *i.e.*, $\mathcal{D}$ is a *filtered ring*. We note that $\{\partial^\nu \mid 0\le|\nu|\le s\}$ is a basis of $\mathcal{D}_s$ as a left $\mathcal{B}=\mathcal{D}_0$-module, *i.e.*, $\mathcal{D}_s$ is a finitely generated free $\mathcal{B}$-module.

Definition 4. A linear PD system defined by $R\in\mathcal{D}^{q\times p}$ with $r=\mathrm{ord}(R)$ is said to be *formally integrable* if:

$$\forall\, s\in\mathbb{Z},\quad (\mathcal{D}^{1\times q}\,R)\cap\mathcal{D}_{r+s}^{1\times p}=\mathcal{D}_s^{1\times q}\,R. \qquad (7)$$

Conditions (7) are called *integrability conditions*.

The integrability conditions (7) mean that for all $s\in\mathbb{Z}$, the system equations of order $r+s$, *i.e.*, $(\mathcal{D}^{1\times q}\,R)\cap\mathcal{D}_{r+s}^{1\times p}$, are exactly the s^{th}*-prolongation* $\rho_s(R):=\mathcal{D}_s^{1\times q}\,R$ of the system equations. Note that we always have:

$$\forall\, s\in\mathbb{Z},\quad \rho_s(R)\subseteq(\mathcal{D}^{1\times q}\,R)\cap\mathcal{D}_{r+s}^{1\times p}.$$

Example 1. Let $\mathcal{D}=\mathcal{B}\langle\partial_1,\partial_2\rangle$ be the Weyl algebra, *i.e.*, $\mathcal{B}=\mathcal{K}[x_1,x_2]$, $R=(\partial_1\quad \partial_2+x_1)^T$ so that $r=1$. We have $-(\partial_2+x_1)\,\partial_1+\partial_1\,(\partial_2+x_1)=1$, which yields $d_1=(-d_1\,(\partial_2+x_1))\,\partial_1+(d_1\,\partial_1)\,(\partial_2+x_1)$ for all $d_1\in\mathcal{D}_1$. This implies that $(\mathcal{D}^{1\times 2}\,R)\cap\mathcal{D}_1=\mathcal{D}_1$. The linear PD system defined by R is then not formally integrable since $\rho_0(R)=\mathcal{B}\,\partial_1+\mathcal{B}\,(\partial_2+x_1)\subsetneq(\mathcal{D}^{1\times 2}\,R)\cap\mathcal{D}_1=\mathcal{D}_1$.

Remark 5. From Section 2, we have $\mathcal{L}=\sum_{i=1}^p\mathcal{D}\,y_i$. Then, $\mathcal{L}_t=\sum_{i=1}^p\mathcal{D}_t\,y_i$ is the *quotient filtration* of $\mathcal{L}$ induced by the filtration $\{\mathcal{D}_t\}_{t\in\mathbb{Z}}$ of $\mathcal{D}$, namely, $\mathcal{L}=\bigcup_{t\in\mathbb{Z}}\mathcal{L}_t$, where we set $\mathcal{L}_t=0$ for $t<0$, and $\mathcal{D}_s\,\mathcal{L}_t\subseteq\mathcal{L}_{s+t}$ for all $s,\,t\in\mathbb{Z}$. In other words, we have:

$$\forall\, s\in\mathbb{Z},\quad \mathcal{L}_{r+s}=(\mathcal{D}_{r+s}^{1\times p}+\mathcal{D}^{1\times q}\,R)/(\mathcal{D}^{1\times q}\,R) = \mathcal{D}_{r+s}^{1\times p}/\left((\mathcal{D}^{1\times q}\,R)\cap\mathcal{D}_{r+s}^{1\times p}\right).$$

Alternatively, we can consider the restriction of the left $\mathcal{D}$-homomorphism $.R$ defined by (1) to $\mathcal{D}_s^{1\times q}\subset\mathcal{D}^{1\times q}$ to get the left $\mathcal{B}$-homomorphism $.R:\mathcal{D}_s^{1\times q}\longrightarrow\mathcal{D}_{r+s}^{1\times p}$, and then its cokernel left $\mathcal{B}$-module, *i.e.*, $\mathcal{L}'_{r+s}=\mathcal{D}_{r+s}^{1\times p}/(\mathcal{D}_s^{1\times q}\,R)$ for all $s\in\mathbb{Z}$. Hence, we have $\mathcal{L}'_{r+s}=\mathcal{L}_{r+s}$ for all $s\in\mathbb{Z}$ if and only if (7) holds. If so, $.R$ is said to be a *strict morphism*.

Note that (7) corresponds to an infinite number of integrability conditions. Fundamental results due to Quillen and Goldschmidt (Quillen (1964); Spencer (1969)) show that they can be reduced to a single integrability condition $(\mathcal{D}^{1\times q}\,R)\cap\mathcal{D}_{r+1}^{1\times p}=\mathcal{D}_1^{1\times q}\,R$, if the so-called *symbol* of the linear PD system is *2-acyclic* (see Section 5.2). More generally, a linear PD system is said to be *involutive* if it is formally integrable and if its symbol is *n-acyclic*, namely, *involutive*. In the next section, we shall introduce these concepts and show that they can be effectively tested using Koszul homology computations.

5.2 Acyclicity and involutivity of the symbol module

With the notations of Section 5.1, for all $r\in\mathbb{Z}$, let $\mathrm{gr}(\mathcal{D})_r=\mathcal{D}_r/\mathcal{D}_{r-1}$ and $\sigma_r:\mathcal{D}_r\longrightarrow\mathrm{gr}(\mathcal{D})_r$ be the canonical projection. Then, the *graded ring* of $\mathcal{D}$ is defined by $\mathrm{gr}(\mathcal{D})=\bigoplus_{r\in\mathbb{Z}}\mathrm{gr}(\mathcal{D})_r$. If we set $\chi_i=\sigma_1(\partial_i)$ for $i=1,\ldots,n$, then the identity $\partial_i\,b=b\,\partial_i+\partial_i(b)$ in $\mathcal{D}_1$ yields $\chi_i\,b=b\,\chi_i$ in $\mathrm{gr}(\mathcal{D})_1$ for all $b\in\mathcal{B}$ and $i=1,\ldots,n$. Moreover, we have $\chi_i\,\chi_j=\chi_j\,\chi_i$ in $\mathrm{gr}(\mathcal{D})_2$. Thus, we have:

$$\forall\, P=\sum_{0\le|\nu|\le r}b_\nu\,\partial^\nu\in\mathcal{D}_r,\quad \sigma_r(P)=\sum_{|\nu|=r}b_\nu\,\chi^\nu.$$

Let us note $\mathcal{A}=\mathrm{gr}(\mathcal{D})$ and $\mathcal{A}_r=\mathrm{gr}(\mathcal{D})_r$ for all $r\in\mathbb{Z}$. Then, $\mathcal{A}_r=\bigoplus_{|\nu|=r}\mathcal{B}\,\chi^\nu$ is a finitely generated free $\mathcal{B}$-module and $\mathcal{A}=\mathcal{B}[\chi_1,\ldots,\chi_n]$, *i.e.*, $\mathcal{A}$ is the commutative polynomial ring in $\chi_1,\ldots,\chi_n$ with coefficients in $\mathcal{B}$. Elements of $\mathcal{A}_r$ are homogeneous polynomials of degree r. Finally, let $\mathcal{A}(s)$ be the *shifted graded ring* defined by $\mathcal{A}(s)_r=\mathcal{A}_{r+s}$ for all $r,\,s\in\mathbb{Z}$.

Set $T=\mathcal{A}_1$. Then, $T=\bigoplus_{i=1}^n\mathcal{B}\,\chi_i$ is a free $\mathcal{B}$-module of rank n. Note that the *graded symmetric algebra* of T, *i.e.*, $S(T)=\bigoplus_{t\ge 0}S^t\,T$ – where $S^0\,T=\mathcal{B}$ and $S^t\,T\cong\bigoplus_{|\nu|=t}\mathcal{B}\,\chi^\nu=\mathcal{A}_t$ is the set of *t-symmetric tensors* – is isomorphic to $\mathcal{A}=\mathcal{B}[\chi_1,\ldots,\chi_n]$. If $\mathcal{N}$ is a $\mathcal{A}$-module, it is then a $\mathcal{B}$-module. As explained in Section 3, we can define the $\mathcal{B}$-modules $\Lambda^r\,T\otimes_{\mathcal{B}}\mathcal{N}$ for $r=0,\ldots,n$, which naturally inherit $\mathcal{A}$-modules structures from the $\mathcal{A}$-module structure of $\mathcal{N}$. Moreover, $d_r:\Lambda^r\,T\otimes_{\mathcal{B}}\mathcal{N}\longrightarrow\Lambda^{r-1}\,T\otimes_{\mathcal{B}}\mathcal{N}$ defined by (6) is a well-defined $\mathcal{A}$-homomorphism. Thus, we can construct the Koszul complex $K(\chi,\mathcal{N})$ and define its Koszul homology $\mathcal{A}$-modules $H_r(\chi,\mathcal{N})$ for $r=0,\ldots,n$.

Let r be the *order* of R and $\sigma_r(R)\in\mathcal{A}^{q\times p}$ be the *principal symbol* of R, namely, the matrix obtained by the application of the $\mathcal{B}$-homomorphism σ_r to the entries of R. Let $\mathcal{M}=\mathrm{coker}_{\mathcal{A}}(.\sigma_r(R))$ be the $\mathcal{A}$-module finitely presented by $\sigma_r(R)$. All the entries of the rows $\sigma_r(R)_{i\bullet}$'s of $\sigma_r(R)$ are homogeneous polynomials of degree r. Thus, $.\sigma_r(R)\in\hom_{\mathcal{A}}(\mathcal{A}^{1\times q},\mathcal{A}^{1\times p})$ yields the graded homomorphism $\mathcal{A}(-r)^{1\times q}\longrightarrow\mathcal{A}(0)^{1\times p}$ of degree 0. Its image $\mathcal{P}:=\sum_{i=1}^q\mathcal{A}(-r)\,\sigma_r(R)_{i\bullet}$ is then a graded $\mathcal{A}$-submodule of $\mathcal{A}(0)^{1\times p}$ and the graded $\mathcal{A}$-module $\mathrm{gr}(\mathcal{M})$ is defined by:

$$\mathrm{gr}(\mathcal{M})=\mathcal{A}(0)^{1\times p}/\left(\sum_{i=1}^q\mathcal{A}(-r)\,\sigma_r(R)_{i\bullet}\right)=\bigoplus_{s\in\mathbb{Z}}\mathrm{gr}(\mathcal{M})_s, \qquad (8)$$

where $\mathrm{gr}(\mathcal{M})_s = \mathcal{A}_s^{1\times p} / (\sum_{i=1}^{q} \mathcal{A}_{s-r}\, \sigma_r(R)_{i\bullet})$ for all $s \in \mathbb{Z}$ and $\sigma_r(P)\, \pi_s(m) = \pi_{r+s}(P\, m)$ for all $P \in \mathcal{D}$ and for all $m \in \mathrm{gr}(\mathcal{M})$, where $\pi_s : \mathcal{A}_s^{1\times p} \longrightarrow \mathrm{gr}(\mathcal{M})_s$ denotes the canonical projection for all $s \in \mathbb{Z}$. The graded $\mathcal{A}$-module $\mathrm{gr}(\mathcal{M})$ is called the *symbol module*. Note that:

$$\forall\, s \in \mathbb{Z},\ l \in \mathbb{Z}_{\geq 0}, \quad \mathcal{A}_s^{1\times l} \cong \mathcal{B}^{1\times l} \otimes_{\mathcal{B}} S^s\, T \cong \mathcal{B}^{1\times l\,\binom{n+s-1}{s}}.$$

The $\mathcal{B}$-module $\mathrm{gr}(\mathcal{M})_s$ is generated by $y_{i,\nu} = \pi_s(\chi^\nu\, e_i)$ for $|\nu| = s$ and $i = 1,\ldots,p$. If $\sigma_r(R)_{ij} = \sum_{|\alpha|=r} b_{ij,\alpha}\, \chi^\alpha$, where $b_{ij,\alpha} \in \mathcal{B}$, then the $y_{i,\nu}$'s satisfy the $\mathcal{B}$-linear relations $\sum_{j=1}^{p} \sum_{|\alpha|=r} b_{ij,\alpha}\, y_{j,\alpha+\beta} = 0$ for $i = 1,\ldots,q$ and $|\beta| = s - r$. If $p_s = p\binom{n+s-1}{s}$, $q_s = q\binom{n+s-r-1}{s-r}$, and $\Sigma_s \in \mathcal{B}^{q_s\times p_s}$ is the matrix of coefficients of the latter linear system, then we have $\mathrm{gr}(\mathcal{M})_s = \mathrm{coker}_{\mathcal{B}}(.\Sigma_s)$.

As explained above, we can now define the Koszul complex $K(\chi, \mathrm{gr}(\mathcal{M}))$ of the finitely generated $\mathcal{A}$-module $\mathrm{gr}(\mathcal{M})$. This Koszul complex has a natural grading defined by:

$$\Lambda^r\, T \otimes_{\mathcal{B}} \mathrm{gr}(\mathcal{M}) = \bigoplus_{s\in\mathbb{Z}} \Lambda^r\, T \otimes_{\mathcal{B}} \mathrm{gr}(\mathcal{M})_s,\ r = 0,\ldots,n,$$

and using $\chi_i \in \mathcal{A}_1$, the map d_r defined by (6) yields the following homogeneous $\mathcal{B}$-homomorphism of degree 1:

$$d_r : \Lambda^r\, T \otimes_{\mathcal{B}} \mathrm{gr}(\mathcal{M})_s \longrightarrow \Lambda^{r-1}\, T \otimes_{\mathcal{B}} \mathrm{gr}(\mathcal{M})_{s+1},$$
$$d_r(\chi_{i_1} \wedge \ldots \wedge \chi_{i_r} \otimes m_s)$$
$$= \sum_{i=1}^{r} (-1)^{i+1}\, \chi_{i_1} \wedge \ldots \wedge \hat{\chi}_{i_j} \wedge \ldots \wedge \chi_{i_r} \otimes \chi_{i_j}\, m_s,$$

for all $m_s \in \mathrm{gr}(\mathcal{M})_s$. Then, $K(\chi, \mathrm{gr}(\mathcal{M}))$ is the direct sum of the following complexes of $\mathcal{B}$-modules:

$$0 \longrightarrow \Lambda^n\, T \otimes_{\mathcal{B}} \mathrm{gr}(\mathcal{M})_s \xrightarrow{d_n} \Lambda^{n-1}\, T \otimes_{\mathcal{B}} \mathrm{gr}(\mathcal{M})_{s+1} \xrightarrow{d_{n-1}} \cdots$$
$$\cdots \xrightarrow{d_3} \Lambda^2\, T \otimes_{\mathcal{B}} \mathrm{gr}(\mathcal{M})_{s+n-2} \xrightarrow{d_2} T \otimes_{\mathcal{B}} \mathrm{gr}(\mathcal{M})_{s+n-1} \xrightarrow{d_1}$$
$$\mathrm{gr}(\mathcal{M})_{s+n} \xrightarrow{d_0} 0, \quad s \in \mathbb{Z}. \tag{9}$$

Thus, we have

$$H_r(\chi, \mathrm{gr}(\mathcal{M})) = \bigoplus_{k\in\mathbb{Z}} H_r(\chi, \mathrm{gr}(\mathcal{M}))_k, \quad r = 0,\ldots,n, \tag{10}$$

where $H_r(\chi, \mathrm{gr}(\mathcal{M}))_k$ is the homology at $\Lambda^r\, T \otimes_{\mathcal{B}} \mathrm{gr}(\mathcal{M})_k$.

If $\mathcal{B}$ is a noetherian ring (*e.g.*, $\mathcal{B} = \mathcal{K}$ or $\mathcal{B} = \mathcal{K}[x_1,\ldots,x_n]$), then $\mathcal{A}$ is a noetherian ring and the $H_r(\chi, \mathrm{gr}(\mathcal{M}))$'s are finitely generated $\mathcal{A}$-modules, and thus, finitely generated $\mathcal{B}$-modules by Lemma 3. Then, (10) implies that only finitely many $H_r(\chi, \mathrm{gr}(\mathcal{M}))_k$'s are non-zero for $k \in \mathbb{Z}$. This result holds for $r = 0,\ldots,n$, and thus, we have:

$$\exists\, K \geq 0,\ \forall\, r \in \{0,\ldots,n\},\ \forall\, k \geq K,\ H_r(\chi, \mathrm{gr}(\mathcal{M}))_k = 0.$$

Let $T^\star := \hom_{\mathcal{B}}(T, \mathcal{B})$, $g_s := \hom_{\mathcal{B}}(\mathrm{gr}(\mathcal{M})_s, \mathcal{B}) \cong \ker_{\mathcal{B}}(\Sigma_s.)$, and recall that $\hom_{\mathcal{B}}(\Lambda^r\, T, \mathcal{B}) \cong \Lambda^r\, T^\star$. The application of the *contravariant functor* $\hom_{\mathcal{B}}(\,\cdot\,, \mathcal{B})$ to (9) yields the dual of the Koszul complex (9), *i.e.*,

$$0 \to g_{s+n} \xrightarrow{d_1^\star} T^\star \otimes_{\mathcal{B}} g_{s+n-1} \xrightarrow{d_2^\star} \Lambda^2\, T^\star \otimes_{\mathcal{B}} g_{s+n-2} \xrightarrow{d_3^\star} \cdots$$
$$\cdots \xrightarrow{d_{n-1}^\star} \Lambda^{n-1}\, T^\star \otimes_{\mathcal{B}} g_{s+1} \xrightarrow{d_n^\star} \Lambda^n\, T^\star \otimes_{\mathcal{B}} g_s \longrightarrow 0, \tag{11}$$

i.e., the so-called *Spencer sequence*. The *Spencer cohomology* of (11) at $\Lambda^j\, T^\star \otimes_{\mathcal{B}} g_k$ is denoted by $H^{k,j}(g_r)$. If $\mathcal{B} = \mathcal{K}$, then the $H_j(\chi, \mathrm{gr}(\mathcal{M}))_k$'s are finite dimensional $\mathcal{K}$-vector spaces. If $\mathcal{K} = \mathbb{R}$, Quillen (1964) proves that the $\mathcal{K}$-vector space $\mathrm{gr}(\mathcal{M})_s$ is locally the dual of the *symbol* g_s of the *sheafification version* of the *differential operator* $R. : \mathcal{F}^{p\times 1} \longrightarrow \mathcal{F}^{q\times 1}$, and $H_j(\chi, \mathrm{gr}(\mathcal{M}))_k$ is the dual $\mathcal{K}$-vector space of the *Spencer cohomology* $H^{k,j}(g_r)$.

Definition 6. Given integers $\ell \geq 0$ and $i \in \{0,\ldots,n\}$, the symbol $\mathcal{A}$-module $\mathrm{gr}(\mathcal{M})$ is said to be *i-acyclic* at degree ℓ if $H_k(\mathrm{gr}(\mathcal{M}))_d = 0$ for all $d \geq \ell$ and $k = 0,\ldots,i$. If $\mathrm{gr}(\mathcal{M})$ is n-acyclic at degree ℓ, then we simply say that $\mathrm{gr}(\mathcal{M})$ is *involutive* at degree ℓ.

One can prove that $H_0(\chi, \mathrm{gr}(\mathcal{M}))_{k+1} = 0$ for $k \geq r$ and $\mathrm{gr}(\mathcal{M})$ is 1-acyclic at degree r (Malgrange (2005)).

Using Section 4, matrices S_2' and S_2 with homogeneous rows exist such that we have the following representation:

$$H_2(\chi, \mathcal{M}) = (\mathcal{A}^{1\times s_2'}\, S_2')/(\mathcal{A}^{1\times s_2}\, S_2). \tag{12}$$

Theorem 7. Let $\mathcal{M} = \mathcal{A}^{1\times p}/(\mathcal{A}^{1\times q}\, \sigma_r(R))$ be a finitely presented $\mathcal{A}$-module, where $\sigma_r(R)$ has homogeneous rows with degree r. Let $S_2' \in \mathcal{A}^{s_2'\times p\binom{n}{2}}$ and $S_2 \in \mathcal{A}^{s_2\times p\binom{n}{2}}$ be two matrices with homogeneous rows such that $H_2(\chi, \mathcal{M})$ is defined by (12). Let $S_{2,r}' \in \mathcal{A}^{s_{2,r}'\times p\binom{n}{2}}$ be the matrix obtained by deleting all the rows of S_2' of degree less than r. Then, $\mathrm{gr}(\mathcal{M})$ is 2-acyclic at degree r if and only if there exists a matrix $T_{2,r} \in \mathcal{A}^{s_{2,r}'\times s_2}$ such that $S_{2,r}' = T_{2,r}\, S_2$.

Proof. Let us write $S_2' = ({S_{2,r}'}^T \quad {S_{2,<r}'}^T)^T$, where the rows of $S_{2,<r}'$ have degree strictly less than r. According to Lemma 3, the residue classes of the rows of $\chi_i\, S_{2,<r}'$ vanish in $H_2(\chi, \mathcal{M})$. Hence, $H_2(\chi, \mathrm{gr}(\mathcal{M}))_r$ is trivial if and only if the rows of S_2' with degree not less than r belong to the $\mathcal{A}$-submodule generated by rows of S_2. The latter statement means that the matrix $S_{2,r}'$ admits a factorisation $T_{2,r}\, S_2$.

Remark 8. If $\mathcal{B}$ is a commutative polynomial ring with coefficients in a computable field $\mathcal{K}$, then, using Gröbner bases methods, the characterizations of the $H_r(\chi, \mathcal{M})$'s obtained in Section 4 can be explicitly computed. See, *e.g.*, the `Maple` package OreMorphisms (Cluzeau and Quadrat (2009)). Then, given $S_{2,r}'$ and S_2, testing whether or not there exists a factorisation $S_{2,r}' = T_{2,r}\, S_2$ can also be done using Gröbner bases methods. For instance, the `Factorize` command of the OreModules package outputs a matrix $T_{2,r}$ if it exists.

From Theorem 7, we can effectively test the 2-acyclicity.

(1) Use Section 4 with $\mathcal{M} = \mathrm{coker}_{\mathcal{A}}(.\sigma_r(R)))$, where $r = \mathrm{ord}(R)$, to compute S_2' and S_2 such that (12).
(2) Let $S_{2,r}'$ be the matrix obtained by deleting rows of S' with degrees strictly less than r.
(3) Test if a factorisation of the form $S_{2,r}' = T_{2,r}\, S_2$ exists, where $T_{2,r}$ is a matrix with entries in $\mathcal{A}$.

Hence, $\mathrm{gr}(\mathcal{M})$ is 2-acyclic at degree r if and only if a factorization such as in Step (3) exists.

Similarly, we can test whether or not $H_i(\chi, \mathrm{gr}(\mathcal{M}))$ is trivial at a degree r for $i = 3,\ldots,n$, and thus, test the involutivity of the symbol $\mathcal{A}$-module $\mathrm{gr}(\mathcal{M})$ at degree r. As explained above, $H_i(\chi, \mathrm{gr}(\mathcal{M}))_k = 0$ for a large enough k, and thus, the above approach can be used with the prolongation $\mathcal{M}_{r+s} = \mathrm{coker}_{\mathcal{A}}(.\rho_s(R))$ for a large s.

6. EXAMPLES

6.1 Example 2 (Pommaret, 1994, III.D.Example 4).

Let $\mathcal{B} = \mathbb{Q}[\alpha]$, $\mathcal{D} = \mathcal{B}[\partial_1, \partial_2, \partial_3]$, $\mathcal{A} = \mathcal{B}[\chi_1, \chi_2, \chi_3]$, $\mathcal{L} = \mathcal{D}/(\mathcal{D}^{1\times 5} R)$, $r = 2$, $\mathcal{M} = \mathcal{A}/(\mathcal{A}^{1\times 5}\,\sigma_2(R))$, where $R \in \mathcal{D}^{1\times 5}$ and $\sigma_2(R) \in \mathcal{A}^{1\times 5}$ are defined by

$$R = \begin{pmatrix} \partial_3^2 \\ \partial_2\,\partial_3 - \alpha\,\partial_1^2 \\ \partial_2^2 \\ \partial_1\,\partial_3 \\ \partial_1\,\partial_2 \end{pmatrix}, \quad \sigma_2(R) = \begin{pmatrix} \chi_3^2 \\ \chi_2\,\chi_3 - \alpha\,\chi_1^2 \\ \chi_2^2 \\ \chi_1\,\chi_3 \\ \chi_1\,\chi_2 \end{pmatrix}$$

and $\mathrm{gr}(\mathcal{M}) = \mathcal{A}(0)/(\mathcal{A}(-2)^{1\times 5}\,\sigma_2(R))$. We study the 2-acyclicity of $\mathrm{gr}(\mathcal{M})$ at degree 2. Let $p = \chi_2\,\chi_3 - \alpha\,\chi_1^2$ and:

$$S_2' = \begin{pmatrix} \chi_2 & 0 & \alpha\,\chi_1 \\ \chi_3 & \alpha\,\chi_1 & 0 \\ \chi_1 & 0 & 0 \\ \chi_3^2 & 0 & 0 \\ \chi_2\,\chi_3 & 0 & 0 \\ \chi_2^2 & 0 & 0 \\ 0 & \chi_3 & 0 \\ 0 & \chi_2 & \chi_3 \\ 0 & 0 & \chi_2 \\ 0 & 0 & \chi_3^2 \\ 0 & 0 & \chi_1\,\chi_3 \end{pmatrix}, \quad S_2 = \begin{pmatrix} \chi_1 & \chi_2 & \chi_3 \\ \chi_3^2 & 0 & 0 \\ p & 0 & 0 \\ \chi_2^2 & 0 & 0 \\ \chi_1\,\chi_3 & 0 & 0 \\ \chi_1\,\chi_2 & 0 & 0 \\ 0 & \chi_3^2 & 0 \\ 0 & p & 0 \\ 0 & \chi_2^2 & 0 \\ 0 & \chi_1\,\chi_3 & 0 \\ 0 & \chi_1\,\chi_2 & 0 \\ 0 & 0 & \chi_3^2 \\ 0 & 0 & p \\ 0 & 0 & \chi_2^2 \\ 0 & 0 & \chi_1\,\chi_3 \\ 0 & 0 & \chi_1\,\chi_2 \end{pmatrix}.$$

We have $H_2(\chi, \mathcal{M}) = (\mathcal{A}^{1\times 11}\,S_2')/(\mathcal{A}^{1\times 16}\,S_2)$. Then, we can check that the matrix $S_{2,2}'$ formed by the last four rows of S_2 satisfies $S_{2,2}' = T_{2,2}\,S_2$ for a certain matrix $T_{2,2}$. Thus, $\mathrm{gr}(\mathcal{M})$ is 2-acyclic at degree 2 for all α.

To test the involutivity at degree 2, we have to consider $H_3(\chi, \mathcal{M})$. Computing a representation of $H_3(\chi, \mathcal{M})$, we obtain $H_3(\chi, \mathcal{M}) = (\mathcal{A}^{1\times 6}\,S_3')/(\mathcal{A}^{1\times 5}\,S_3)$, where:

$$S_3' = (\chi_3^2 \quad \chi_2\,\chi_3 \quad \chi_1\,\chi_3 \quad \chi_2^2 \quad \chi_1\,\chi_2 \quad \alpha\,\chi_1^2)^T$$
$$S_3 = \sigma_2(R) = (\chi_3^2 \quad \chi_2\,\chi_3 - \alpha\,\chi_1^2 \quad \chi_2^2 \quad \chi_1\,\chi_3 \quad \chi_1\,\chi_2)^T.$$

We then have $S_{3,2}' = S_3'$ and we can check that there is no factorization of the form $S_{3,2}' = T_{3,2}\,S_3$, which yields $H_3(\chi, \mathrm{gr}(\mathcal{M}))$ does not vanish at degree 2 for almost all α, *i.e.*, $\mathrm{gr}(\mathcal{M})$ is not generically involutive at degree 2.

Considering the last entry of S_3', the only particular value to check is $\alpha = 0$. We substitute $\alpha = 0$ into R. In the same manner, we can compute S_3' and S_3 such that $H_3(\chi, \mathcal{M}) = (\mathcal{A}^{1\times 2}\,S_3')/(\mathcal{A}^{1\times 5}\,S_3)$, where $S_3' = (\chi_3 \quad \chi_2)^T$ and $S_3 = \sigma_2(R)$. Since the degree of S_3' is strictly less than $r = 2$, $\mathrm{gr}(\mathcal{M})$ is 3-acyclic, *i.e.*, involutive at degree 2. We find again the results obtained in Pommaret (1994).

6.2 Janet's Example (Pommaret, 1994, III.D.Example 1).

Let $\mathcal{B} = \mathbb{Q}[x_1, x_2, x_3]$, $\mathcal{D} = \mathcal{B}\langle\partial_1, \partial_2, \partial_3\rangle$ be the Weyl algebra, and $\mathcal{A} = \mathrm{gr}(\mathcal{D}) = \mathcal{B}[\chi_1, \chi_2, \chi_3]$. We consider the finitely presented left $\mathcal{D}$-module $\mathcal{L} = \mathrm{coker}_{\mathcal{D}}(.R)$ and the finitely presented $\mathcal{A}$-module $\mathcal{M} = \mathrm{coker}_{\mathcal{A}}(.\sigma_2(R))$, where:

$$R = \begin{pmatrix} \partial_3^2 - x_2\,\partial_1^2 \\ \partial_2^2 \end{pmatrix} \in \mathcal{D}^{2\times 1}, \ \sigma_2(R) = \begin{pmatrix} \chi_3^2 - x_2\,\chi_1^2 \\ \chi_2^2 \end{pmatrix} \in \mathcal{A}^{2\times 1}.$$

Let us consider the graded $\mathcal{A}$-module $\mathrm{gr}(\mathcal{M})$ defined by $\mathrm{gr}(\mathcal{M}) = \mathcal{A}(0)/\left(\mathcal{A}(-2)\,(\chi_3^2 - x_2\,\chi_1^2) + \mathcal{A}(-2)\,\chi_2^2\right)$. We have $\mathrm{gr}(\mathcal{M})_0 = \mathcal{A}_0 = \mathcal{B}$, $\mathrm{gr}(\mathcal{M})_1 = \mathcal{A}_1 = \bigoplus_{i=1}^3 \mathcal{B}\,\chi_i$, $\mathrm{gr}(\mathcal{M})_2 = \mathcal{A}_2/\left(\mathcal{A}_0\,(\chi_3^2 - x_2\,\chi_1^2) + \mathcal{A}_0\,\chi_2^2\right)$, *i.e.*, $\mathrm{gr}(\mathcal{M})_2$ is the $\mathcal{B}$-module defined by the 6 generators y_ν, where y_ν denotes the residue class of χ^ν in $\mathrm{gr}(\mathcal{M})_2$ and $|\nu| = 2$, modulo $\mathcal{B}$-linear combinations of the two relations defined by the entries of $\sigma_2(R)$, *i.e.*, $y_{(0,0,2)} - x_2\,y_{(2,0,0)} = 0$ and $y_{(0,2,0)} = 0$. $\mathrm{gr}(\mathcal{M})_3 = \mathcal{A}_3/\left(\mathcal{A}_1\,(\chi_3^2 - x_2\,\chi_1^2) + \mathcal{A}_1\,\chi_2^2\right)$, *i.e.*, $\mathrm{gr}(\mathcal{M})_3$ is the $\mathcal{B}$-module defined by the 10 generators y_ν, with $|\nu| = 3$, modulo the $\mathcal{A}_1$-linear combinations of the entries of $\sigma_2(R)$, *i.e.*, modulo the following 6 relations:

$$y_{(0,0,2)+1_i} - x_2\,y_{(2,0,0)+1_i} = 0, \ y_{(0,2,0)+1_i} = 0, \ i = 1, 2, 3.$$

To study the formal integrability of the linear PD system defined by R, we study when $\mathrm{gr}(\mathcal{M})$ is 2-acyclic or involutive. We have $H_2(\chi, \mathcal{M}) = (\mathcal{A}^{1\times 8}\,S_2')/(\mathcal{A}^{1\times 7}\,S_2)$, where, with the notation $p = \chi_3^2 - x_2\,\chi_1^2$:

$$S_2' = \begin{pmatrix} \chi_1 & \chi_2 & \chi_3 \\ \chi_2^2 & 0 & 0 \\ \chi_2\,\chi_3 & 0 & \chi_1\,\chi_2 \\ \chi_3^2 & x_2\,\chi_1\,\chi_2 & x_2\,\chi_1\,\chi_3 \\ 0 & \chi_2^2 & 0 \\ 0 & -p & 0 \\ 0 & 0 & \chi_2^2 \\ 0 & 0 & -p \end{pmatrix}, \quad S_2 = \begin{pmatrix} \chi_1 & \chi_2 & \chi_3 \\ p & 0 & 0 \\ \chi_2^2 & 0 & 0 \\ 0 & p & 0 \\ 0 & \chi_2^2 & 0 \\ 0 & 0 & p \\ 0 & 0 & \chi_2^2 \end{pmatrix}.$$

Then, we can check that the matrix $S_{2,2}'$ obtained by removing the rows of S_2' of order strictly less than 2, cannot be factorized as $T_{2,2}\,S_2$, which shows that $\mathrm{gr}(\mathcal{M})$ is not 2-acyclic at degree 2 (see Pommaret (1994)).

Let us now study if $\mathrm{gr}(\mathcal{M})$ is 2-acyclic/involutive at degree 3. We consider the first prolongation $\rho_1(R)$ of R, its principal symbol $\sigma_3(\rho_1(R))$, *i.e.*,

$$\sigma_3\left(\begin{pmatrix} \partial_1\,\partial_3^2 - x_2\,\partial_1^3 \\ \partial_2\,\partial_3^2 - x_2\,\partial_1^2\,\partial_2 - \partial_1^2 \\ \partial_3^3 - x_2\,\partial_1^2\,\partial_3 \\ \partial_1\,\partial_2^2 \\ \partial_2^3 \\ \partial_2^2\,\partial_3 \end{pmatrix}\right) = \begin{pmatrix} \chi_1\,p \\ \chi_2\,p \\ \chi_3\,p \\ \chi_1\,\chi_2^2 \\ \chi_2^3 \\ \chi_2^2\,\chi_3 \end{pmatrix},$$

which corresponds to $\mathrm{gr}(\mathcal{M})_3$, and the finitely presented $\mathcal{A}$-module $\mathcal{M}_3 = \mathrm{coker}_{\mathcal{A}}(.\rho_1(R))$. Then, we get $H_2(\chi, \mathcal{M}_3) = (\mathcal{A}^{1\times 8}\,U_2')/(\mathcal{A}^{1\times 19}\,U_2)$ and $H_3(\chi, \mathcal{M}_3) = (\mathcal{A}^{1\times 2}\,V_3')/(\mathcal{A}^{1\times 6}\,V_3)$, where the entries of U_2' and V_3' have degree strictly less than 3, which yields $H_i(\chi, \mathcal{M}_3) = 0$, $i = 2, 3$, *i.e.*, $\mathrm{gr}(\mathcal{M})$ is involutive at degree 3 (Pommaret (1994)). For the explicit computations (and more examples), see the webpage of Cluzeau and Quadrat (2008).

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