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This paper provides a step forward to developing an algorithmic study of linear systems of polynomial ordinary integro-differential equations over a field $\mathbb{k}$ of characteristic zero. Such a study can be achieved by first obtaining a constructive proof of the coherence property of the ring $\mathbb{I}_1(\mathbb{k})$ of linear ordinary integro-differential operators with coefficients in $\mathbb{k}[t]$. To do that, the finiteness of the intersection of two finitely generated ideals has to be algorithmically studied. Three cases must be considered: first when evaluation operators generate the two ideals; second, when only one ideal is generated by evaluation operators; and third, when none is generated by evaluation operators. In this paper, we first explicitly characterize the intersection of two finitely generated ideals defined by evaluation operators. As for the second case, a key result is that the ideals generated by evaluations are semisimple $\mathbb{I}_1$-modules. We develop an algorithmic proof of this result. In particular, we show how a finite set of generators, defined by "simple" evaluations, can be obtained, that characterizes the class of finitely generated evaluation ideals of $\mathbb{I}_1$ as finitely generated $\mathbb{k}[t]$-modules. Due to lack of space, the second and third cases will be developed in other publications.

CCS Concepts: • **Computing methodologies** → **Symbolic and algebraic manipulation**; • **Symbolic and algebraic algorithms** → *Algebraic algorithms*.

Additional Key Words and Phrases: Linear systems of integro-differential equations, rings of integro-differential operators, noncommutative polynomial rings, elimination theory, coherent rings, semisimple modules

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1 INTRODUCTION

Calculus is fundamental in mathematics, mathematical physics, and engineering sciences. The idea of mechanizing the computation of integro-differential expressions is rather old in computer algebra. The problem of simplifying integro-differential expressions using a consistent system of rewriting rules – which corresponds to the standard calculus relations between the differential, integral, function multiplication, and evaluation operators – has recently regained interest in computer algebra, control theory, physics, etc.

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The fine algebraic study of the ring $\mathbb{I}_1(\mathbb{k})$ of polynomial ordinary integro-differential operators, where $\mathbb{k}$ is a field of characteristic 0, was initiated in [1]. See also [11]. It can easily be shown that $\mathbb{I}_1(\mathbb{k})$ is not a left nor a right noetherian ring. An important result of Bavula (see [1, Theorem 4.4]) proves that $\mathbb{I}_1(\mathbb{k})$ is a *coherent ring*, namely, the left/right $\mathbb{I}_1(\mathbb{k})$-module of relations between the generators of any finitely generated left/right ideal of $\mathbb{I}_1(\mathbb{k})$ is finitely generated.

A linear system of polynomial ordinary integro-differential equations naturally defines a *finitely presented* left $\mathbb{I}_1(\mathbb{k})$-module: its representation as a linear system $R\,y = 0$ yields a matrix of integro-differential operators $R \in \mathbb{I}_1(\mathbb{k})^{q\times p}$, and thus, the finitely presented left $\mathbb{I}_1$-module $\mathcal{M} = \operatorname{coker}_{\mathbb{I}_1(\mathbb{k})}(.R) = \mathbb{I}_1(\mathbb{k})^{1\times p}/(\mathbb{I}_1(\mathbb{k})^{1\times q}\,R)$. Within the *algebraic analysis approach* [2, 8, 9], the solution space of the corresponding integro-differential linear system can then be interpreted as the $\mathbb{k}$-vector space $\hom_{\mathbb{I}_1(\mathbb{k})}(\mathcal{M}, \mathcal{F})$ of all the left $\mathbb{I}_1(\mathbb{k})$-homomorphisms from $\mathcal{M}$ to a left $\mathbb{I}_1$-module $\mathcal{F}$ where the solutions are sought. Hence, the study of linear systems of polynomial ordinary integro-differential naturally lies in *the category of finitely presented left* $\mathbb{I}_1(\mathbb{k})$*-modules.* In homological algebra, it is well-known that finitely presented left modules over a coherent ring are stable by all the standard algebraic operations, exactly as the category of finitely generated left modules over a left noetherian ring. Bavula's result on the coherence of $\mathbb{I}_1(\mathbb{k})$ thus opens the possibility to develop an algorithmic study of polynomial ordinary integro-differential systems within the algebraic analysis approach. Such an approach would be the extension of the effective algebraic analysis approach to linear systems of polynomial ordinary differential equations defined over the *Weyl algebra* $\mathbb{A}_1(\mathbb{k})$ (see, e.g., [3, 9]).

To develop such a research program, we first have to obtain an algorithmic version of Bavula's proof, i.e., an *algorithmic elimination theory* for polynomial ordinary integro-differential linear systems.

The goal of this paper is to further develop the algorithmic proof of the coherence of $\mathbb{I}_1(\mathbb{k})$ initiated in [5].

The coherence property relies on a characterization that includes two conditions. The first one states that the left/right annihilator of an element of $\mathbb{I}_1(\mathbb{k})$ is finitely generated. An algorithmic proof of this condition was developed in [10] for the elements of $\mathbb{I}_1(\mathbb{k})$ that do not belong to the only two-sided ideal $\langle e\rangle$ of $\mathbb{I}_1(\mathbb{k})$ defined by *evaluation operators*, namely, the elements of $\langle e\rangle = \mathbb{k}[t]\,e\,\mathbb{k}[\partial]$, where $\partial = d/dt$ denotes the differential operator with respect to t, $I = \int_0^t \cdot\, d\tau$ the indefinite integral operator, and $e = 1 - I\,\partial$ the evaluation at $t = 0$. The case of the annihilator of the elements of $\langle e\rangle$ was algorithmically developed in [5].

In this paper, we study the second condition of the coherence property that states that the intersection of two finitely generated left/right ideals of $\mathbb{I}_1(\mathbb{k})$ is finitely generated. We first show how the method proposed in [5] can be extended to effectively characterize the intersection of two left ideals $\mathcal{I}$ and $\mathcal{J}$ finitely generated by evaluation operators, i.e., $\mathcal{I},\ \mathcal{J} \subseteq \langle e\rangle$. To study the case where only one ideal is included in $\langle e\rangle$, a key ingredient of the proof of [1, Theorem 4.4] relies on the fact that ideals defined by evaluations are *semisimple* $\mathbb{I}_1(\mathbb{k})$-modules. We then develop an algorithmic proof of this result by showing how a finite set of generators $\{g_i\}_{i=1,\ldots,s}$ of $\mathcal{I}$, formed by "simple" evaluations, namely, elements of $e\,\mathbb{k}[\partial]$, can be computed. This result exhibits the semisimple property of $\mathcal{I}$ since we then have $\mathcal{I} = \sum_{i=1}^{s} \mathbb{I}_1(\mathbb{k})\, g_i = \sum_{i=1}^{s} \mathbb{k}[t]\, g_i \cong \mathbb{k}[t]^s$, where $\mathbb{k}[t]$ is a *simple* left $\mathbb{I}_1(\mathbb{k})$-module (see [1]). The fact that $\mathcal{I} \cap \mathcal{J}$ is finitely generated in the case of two finitely generated ideals $\mathcal{I} \subseteq \langle e\rangle$ and $\mathcal{J} \not\subset \langle e\rangle$ or $\mathcal{I} \not\subset \langle e\rangle$ and $\mathcal{J} \not\subset \langle e\rangle$ will be constructively studied in other publications.

2 GENERALITIES ON THE RING OF INTEGRO-DIFFERENTIAL OPERATORS

In the rest of the paper, $\mathbb{k}$ will denote a field of characteristic zero ($\mathbb{Q}$ while studying the computational aspects) and $\mathbb{k}[t]$ the ring of polynomials with coefficients in $\mathbb{k}$.

2.1 The ring of integro-differential operators

Let $\operatorname{end}_{\mathbb{k}}(\mathbb{k}[t])$ be the endomorphism ring of $\mathbb{k}[t]$, namely, the ring of all the $\mathbb{k}$-linear maps from $\mathbb{k}[t]$ to itself. This paper studies some properties of the ring $\mathbb{I}_1$ of *ordinary integro-differential operators with polynomial coefficients* which is defined as follows.

Definition 2.1. The ring $\mathbb{I}_1(\mathbb{k})$ of *ordinary integro-differential operators with polynomial coefficients* is the $\mathbb{k}$-subalgebra of $\operatorname{end}_{\mathbb{k}}(\mathbb{k}[t])$ generated by the following three linear operators:

$$t : \mathbb{k}[t] \longrightarrow \mathbb{k}[t], \quad p \longmapsto t\,p,$$

$$\partial : \mathbb{k}[t] \longrightarrow \mathbb{k}[t], \quad p \longmapsto \frac{dp(t)}{dt},$$

$$I : \mathbb{k}[t] \longrightarrow \mathbb{k}[t], \quad p \longmapsto \int_0^t p(\tau)\, d\tau.$$

In what follows, $\mathbb{I}_1(\mathbb{k})$ will simply be denoted by $\mathbb{I}_1$.

Notice that $\mathbb{I}_1$ contains the *Weyl algebra* $\mathbb{A}_1$ of ordinary linear differential operators with polynomial coefficients defined as the $\mathbb{k}$-subalgebra of $\operatorname{end}_{\mathbb{k}}(\mathbb{k}[t])$ generated by the operators t and ∂ above.

Among the elements of the ring $\mathbb{I}_1$, let 1 denote the identity of $\operatorname{end}_{\mathbb{k}}(\mathbb{k}[t])$ and let us consider $e := 1 - I\,\partial$. It satisfies:

$$\forall\, p \in \mathbb{k}[t], \quad e(p) = (1 - I\,\partial)(p) = p(0).$$

Hence, e corresponds to the *evaluation operator* at 0. Note that e is *multiplicative*, namely, $e(p_1\, p_2) = e(p_1)\, e(p_2)$ for all $p_1,\ p_2 \in \mathbb{k}[t]$.

In the ring $\mathbb{I}_1$, certain identities between the operators t, ∂, I, and e hold [1, 5, 10, 11]. For instance, we have $\partial\, I = 1$ and $I\,\partial = 1 - e$, which correspond respectively to the first and second fundamental theorems of calculus. It can be proved that every element $f \in \mathbb{I}_1$ can be written uniquely as

$$f = \sum_{i=0}^{n} a_i\, \partial^i + \sum_{j=0}^{m} b_j\, I\, c_j + \sum_{k=0}^{r} d_k\, e\, \partial^k,$$

where $a_i, b_j, c_j, d_k \in \mathbb{k}[t]$ and $n, m, r \in \mathbb{N}$. For more details, see [1, 5, 10, 11]. The above writing is called the *normal form* of f. In what follows, an element of $\mathbb{I}_1$ with a normal form of the form $\sum_{k=0}^{r} d_k\, e\, \partial^k$, where $d_k \in \mathbb{k}[t]$, will be called an *evaluation operator*. We shall see that ideals of $\mathbb{I}_1$ generated by such evaluation operators have interesting properties. We recall that the set $\langle e\rangle := \mathbb{I}_1\, e\, \mathbb{I}_1$ is the only two-sided ideal of $\mathbb{I}_1$ (see [1]). We further have:

$$\langle e\rangle = \mathbb{k}[t]\, e\, \mathbb{k}[\partial] = \left\{ d \in \mathbb{I}_1 \mid d = \sum_{j=0}^{q} d_k\, e\, \partial^k,\ d_k \in \mathbb{k}[t],\ q \in \mathbb{N} \right\}.$$

Thus, $\langle e\rangle$ is the ideal formed by all the evaluation operators of $\mathbb{I}_1$. A left/right ideal $I \subseteq \langle e\rangle$ will be called an *evaluation ideal*.

An interesting family of evaluation operators is formed by the *Taylor operators* T_n defined by:

$$\forall\, n \in \mathbb{N}, \quad T_n = \sum_{k=0}^{n} \frac{t^k}{k!}\, e\, \partial^k. \tag{1}$$

One can first prove the following identity

$$\forall\, n \in \mathbb{N}, \quad T_n + I^{n+1}\, \partial^{n+1} = 1, \tag{2}$$

which is the operator-theoretic interpretation of Taylor's theorem with integral remainder. See [5, Lemma 2.4]. In particular, (2) shows that the sequence $(T_n)_{n\in\mathbb{N}}$ plays the role of an *approximate identity* in $\langle e\rangle$. It explains why (2) plays a central role in [5] and in this paper.

The operators T_n's satisfy $T_m\, T_n = T_m$ for $0 \le m \le n$. They provide the strictly ascending chain $(\mathbb{I}_1\, T_n)_{n\in\mathbb{N}}$ of left ideals of $\mathbb{I}_1$ ([10]), which proves that $\mathbb{I}_1$ is not left noetherian. Using the involution θ of $\mathbb{I}_1$ defined by $\theta(\partial) = I$, $\theta(I) = \partial$, and $\theta(t) = \partial\, t\, p = (t\,\partial{+}1)\, p$ (see [1]), we can deduce that $\mathbb{I}_1$ is also not a right noetherian ring.

Therefore, contrary to its noetherian subring $\mathbb{A}_1$ (see, e.g., [2]), $\mathbb{I}_1$ is not noetherian. This negative result seems to be an important obstruction for the development of an algorithmic study of the linear systems of integro-differential operators. In the next section, we shall explain why this is fortunately not the end of the story.

2.2 The ring of integro-differential operators is coherent

As explained in [5], the fact that one can develop an algorithmic study of linear systems over $\mathbb{I}_1$ relies on the so-called *coherence* property of $\mathbb{I}_1$. Let us recall it hereafter.

Let $\mathcal{R}$ be a noncommutative ring. A left $\mathcal{R}$-module $\mathcal{M}$ is said to be *finitely generated* if it admits a finite set of generators, i.e., there is a family $\{g_i\}_{i=1,\ldots,p}$, where $g_i \in \mathcal{M}$ and $p \in \mathbb{N}$, such that every element $m \in \mathcal{M}$ can be written as $m = \sum_{i=1}^{p} r_i\, g_i$ for some $r_i \in \mathcal{R}$.

A left $\mathcal{R}$-module $\mathcal{M}$, finitely generated by $\{g_i\}_{i=1,\ldots,p}$, is said to be *finitely presented* if the left $\mathcal{R}$-module of relations between the generators $\{g_i\}_{i=1,\ldots,p}$, i.e.,

$$\operatorname{Syz}(\mathcal{M}) = \left\{ (\lambda_1, \ldots, \lambda_p) \in \mathcal{R}^{1\times p} \;\middle|\; \sum_{i=1}^{p} \lambda_i\, g_i = 0 \right\}$$

is finitely generated.

Let $\{e_i\}_{i=1,\ldots,p}$ be the standard basis of $\mathcal{R}^{1\times p}$ (i.e., the i^{th} entry of e_i is 1 and the other entries are 0) and $\pi : \mathcal{R}^{1\times p} \longrightarrow \mathcal{M}$ the left $\mathcal{R}$-epimorphism defined by sending e_i onto g_i for $i = 1, \ldots, p$, i.e., $\pi(\lambda) = \sum_{i=1}^{p} \lambda_i\, g_i$ for all $\lambda \in \mathcal{R}^{1\times p}$. Then, we have $\ker \pi = \operatorname{Syz}(\mathcal{M})$.

Let us suppose that $\operatorname{Syz}(\mathcal{M})$ is a finitely generated left $\mathcal{R}$-module and $\{R_{i\bullet}\}_{i=1,\ldots,q}$ is a set of generators of $\operatorname{Syz}(\mathcal{M})$, where $R_{i\bullet} \in \mathcal{R}^{1\times p}$

and $R = (R_{1\bullet}^T \ \dots \ R_{q\bullet}^T)^T \in \mathcal{R}^{q\times p}$. If $.R : \mathcal{R}^{1\times q} \longrightarrow \mathcal{R}^{1\times p}$ is the left $\mathcal{R}$-homomorphism defined by $(.R)(\mu) := \mu\, R$ for all $\mu \in \mathcal{R}^{1\times q}$, then $\mathcal{S} = \mathrm{im}_{\mathcal{R}}(.R) := \mathcal{R}^{1\times q}\, R$ and we have the following *exact sequence*

$$\mathcal{R}^{1\times q} \xrightarrow{.R} \mathcal{R}^{1\times p} \xrightarrow{\pi} \mathcal{M} \longrightarrow 0,$$

namely, $\ker \pi = \mathrm{im}_{\mathcal{R}}(.R)$ and π is surjective, which then yields $\mathcal{M} \cong \mathrm{coker}(.R) = \mathcal{R}^{1\times p} / (\mathcal{R}^{1\times q}\, R)$. The left $\mathcal{R}$-module $\mathcal{M}$ is said to be *finitely presented* by $R \in \mathcal{R}^{q\times p}$. For more details, see [2, 12].

We can now give the definition of a coherent ring.

Definition 2.2. Let $\mathcal{R}$ be a noncommutative ring. A left $\mathcal{R}$-module $\mathcal{M}$ is said to be *left coherent* if $\mathcal{M}$ is a finitely generated left $\mathcal{R}$-module and if every finitely generated left $\mathcal{R}$-submodule of $\mathcal{M}$ is finitely presented. The ring $\mathcal{R}$ is said to be *left coherent* if $\mathcal{R}$ is a left coherent $\mathcal{R}$-module, i.e., if every finitely generated left ideal of $\mathcal{R}$ is finitely presented. Symetric definitions hold for right $\mathcal{R}$-modules and a ring is said to be *coherent* if it is both left and right coherent.

In [1], Bavula proved the following important result for $\mathbb{I}_1$.

Theorem 2.3 ([1], Theorem 4.4). *$\mathbb{I}_1$ is a coherent ring.*

To prove the latter theorem, Bavula used the following standard characterization of coherent rings.

Proposition 2.4 ([13], Proposition 13.3). *A ring $\mathcal{R}$ is left coherent if and only if the following two conditions hold:*

1. *For every $a \in \mathcal{R}$, $\mathrm{ann}_{\mathcal{R}}(.a) := \{r \in \mathcal{R} \mid r\, a = 0\}$ is a finitely generated left ideal.*
2. *For all finitely generated left ideals $\mathcal{I}$ and $\mathcal{J}$, the left ideal $\mathcal{I} \cap \mathcal{J}$ is finitely generated.*

A similar result holds for a right coherent ring ($\mathrm{ann}_{\mathcal{R}}(.a)$ is then replaced by $\mathrm{ann}_{\mathcal{R}}(a.)$ and left ideals by right ideals).

The proof of the coherence of $\mathbb{I}_1$ given in [1] is not algorithmic. In order to develop an effective algebraic analysis approach to linear systems over $\mathbb{I}_1$, we aim at providing such a constructive proof. To do that, we rely on the characterization of Proposition 2.4. In [10], an algorithmic characterization of $\mathrm{ann}_{\mathcal{R}}(.a)$ for $a \in \mathbb{I}_1 \setminus \langle e\rangle$ was obtained. In [5], we provided an explicit characterization of $\mathrm{ann}_{\mathcal{R}}(.a)$ for $a \in \langle e\rangle$ and gave an algorithm for computing a finite set of generators for $\mathrm{ann}_{\mathcal{R}}(.a)$. Therefore, Condition (1) on the annihilators of elements of $\mathbb{I}_1$ was made algorithmic. It thus remains to obtain a constructive version of Condition (2). It is the goal of the present paper. More precisely, to do that, we distinguish three cases depending on the fact that both, one, or none of the ideals is an evaluation ideal, i.e., is included in $\langle e\rangle$. In this paper, we shall handle the first case. Due to lack of space, the second one will be developed in another publication (based on the results of Section 4 in the present paper). The last one will be investigated in the future (see Section 5).

3 THE INTERSECTION OF EVALUATION IDEALS

In this section, we shall provide an algorithmic version of Condition (2) of Proposition 2.4 in the case where both $\mathcal{I}$ and $\mathcal{J}$ are finitely generated left evaluation ideals.

For instance, if we consider the principal ideals generated by the Taylor operators defined by (2), namely, $\mathcal{I} = \mathbb{I}_1\, T_r$ and $\mathcal{J} = \mathbb{I}_1\, T_s$ for two integers $r,\, s \in \mathbb{N}$ such that $r \leq s$, then, the relation $T_r\, T_s = T_r$ implies that $\mathbb{I}_1\, T_r \subset \mathbb{I}_1 T_s$ so that we have $\mathcal{I} \cap \mathcal{J} = \mathbb{I}_1 T_r = \mathcal{I}$.

To develop a method for all evaluations ideals, we shall first generalize the algorithm developed in [5], which computes generators of the annihilator of a scalar evaluation operator, to the matrix case.

3.1 Annihilator of a matrix evaluation operator

Let $R = \sum_{k=0}^n R_k(t)\, e\, \partial^k \in \langle e\rangle^{q\times p}$, where $R_k \in \mathbb{k}[t]^{q\times p}$, be a matrix whose entries are all evaluation operators. Let us show how to determine a finite set of generators for $\ker_{\mathbb{I}_1}(.R) = \left\{u \in \mathbb{I}_1^{1\times q} \mid u\, R = 0\right\}$.

To do that, we shall need the following definition.

Definition 3.1. If A is a matrix with entries in $\mathbb{k}[t]$, then the *degree* of A, denoted by $\deg(A)$, is the maximal degree of all its entries.

Note that the proofs of Lemma 4.5, Proposition 4.7, and Theorem 4.9 of [5] can easily be generalized to the matrix case by adapting the dimensions of the matrices. Hence, we have the following result.

Theorem 3.2. *Let $R \in \langle e\rangle^{q\times p}$ and write $R = \sum_{k=0}^n R_k(t)\, e\, \partial^k$, where $R_k \in \mathbb{k}[t]^{q\times p}$. Let $m = \max_{k\in [\![0,n]\!]} \deg(R_k)$, and*

$$C = \begin{pmatrix} R_0 & \dots & R_n \\ \vdots & & \vdots \\ R_0^{(m+1)} & \dots & R_n^{(m+1)} \end{pmatrix} \in \mathbb{k}^{q\,(m+2)\times p\,(n+1)}, \quad J_{m+1} = \begin{pmatrix} I_q \\ I_q\, \partial \\ \vdots \\ I_q\, \partial^{m+1} \end{pmatrix}.$$

Finally, let $D \in \mathbb{k}[t]^{r\times q\,(m+2)}$ and $E \in \mathbb{k}^{s\times q\,(m+2)}$ be two full row rank matrices satisfying

$$\ker_{\mathbb{k}[t]}(.C) = \mathrm{im}_{\mathbb{k}[t]}(.D), \quad \ker_{\mathbb{k}}(.e(C)) = \mathrm{im}_{\mathbb{k}}(.E),$$

and let us define the following matrices

$$\begin{pmatrix} u_1 \\ \vdots \\ u_r \end{pmatrix} = D\, J_{m+1}, \quad \begin{pmatrix} v_1 \\ \vdots \\ v_s \end{pmatrix} = E\, e\, J_{m+1},$$

where $u_1, \dots, u_r$ and $v_1, \dots, v_s$ belong to $\mathbb{I}_1^{1\times q}$. Then, we have:

$$\ker_{\mathbb{I}_1}(.R) = \sum_{i=1}^r \mathbb{I}_1\, u_i + \sum_{j=1}^s \mathbb{I}_1\, v_j = \mathrm{im}_{\mathbb{I}_1}\left(.(u_1^T \ \dots \ u_r^T \ v_1^T \ \dots \ v_s^T)^T\right).$$

Thus, $\ker_{\mathbb{I}_1}(.R)$ is a finitely generated left $\mathbb{I}_1$-module and a set of generators $\{u_1, \dots, u_r,\, v_1, \dots, v_s\}$ of $\ker_{\mathbb{I}_1}(.R)$ can effectively be computed.

In [5, Corollary 4.7], for $a \in \mathbb{I}_1$, it is proved that $\ker_{\mathbb{I}_1}(.a)$ can be generated by the sole u_i's. This result was obtained by proving that the $\mathbb{k}[t]$-module finitely presented by C, i.e., $\mathcal{N} := \mathrm{coker}_{\mathbb{k}[t]}(.C)$, is reduced to 0, a fact that implies that we can take $E = e(D)$, so that $s = r$ and $v_i = e\, u_i$ for $i = 1, \dots, r$. For more details, we refer to [5].

In the matrix case, we can extend [5, Corollary 4.7] by showing that the $\mathbb{k}[t]$-module $\mathcal{N}$ is *free*, i.e., $\mathcal{N}$ is *isomorphic* to $\mathbb{k}[t]^d$ for a certain $d \in \mathbb{N}$, which is denoted by $\mathcal{N} \cong \mathbb{k}[t]^d$. Note that d is called the *rank* of $\mathcal{N}$ and the $\mathbb{k}[t]$-module 0 is free of rank 0. For more details, see [12, Ch. 2, p. 56–60]. As a consequence, $\ker_{\mathbb{I}_1}(.R)$ can be generated by the u_i's. To prove this result, we shall need the following lemma.

Lemma 3.3. *Let* $A \in \Bbbk[t]^{q\times p}$, $m = \deg(A)$, *and*

$$D_m(A) = J_m(A) = \begin{pmatrix} A \\ \vdots \\ A^{(m)} \end{pmatrix} \in \Bbbk[t]^{q(m+1)\times p},$$

$$U = \begin{pmatrix} I_q & t\,I_q & \dots & \frac{t^m}{m!}\,I_q \\ 0 & I_q & \dots & \frac{t^{m-1}}{(m-1)!}\,I_q \\ \vdots & & \ddots & \vdots \\ 0 & \dots & & I_q \end{pmatrix} \in \Bbbk[t]^{q(m+1)\times q(m+1)}.$$

Then, we have $U^{-1} \in \Bbbk[t]^{q(m+1)\times q(m+1)}$ *and*

$$U^{-1}\,D_m(A) = D_m(A)(0) = \begin{pmatrix} A(0) \\ \vdots \\ A^{(m)}(0) \end{pmatrix}. \tag{3}$$

Proof. Clearly the matrix U is invertible and

$$U^{-1} = \begin{pmatrix} I_q & -t\,I_q & \dots & \frac{(-t)^m}{m!}\,I_q \\ 0 & I_q & \dots & \frac{(-t)^{m-1}}{(m-1)!}\,I_q \\ \vdots & & I_q & \vdots \\ 0 & \dots & & I_q \end{pmatrix}.$$

Then, we have

$$U^{-1}\,D_m(A) = \begin{pmatrix} A - t\,A^{(1)} + \frac{t^2}{2!}\,A^{(2)} + \dots + \frac{(-t)^m}{m!}\,A^{(m)} \\ \vdots \\ A^{(i)} - t\,A^{(i+1)} + \dots + \frac{(-t)^{m-i}}{(m-i)!}\,A^{(m)} \\ \vdots \\ A^{(m)} \end{pmatrix}.$$

Let $T_i = A^{(i)} - t\,A^{(i+1)} + \dots + \frac{(-t)^{m-i}}{(m-i)!}\,A^{(m)}$ for $i = 0, \dots, m$. Differentiating T_i with respect to t produces the telescoping sum

$$\frac{dT_i}{dt} = A^{(i+1)} - A^{(i+1)} - t\,A^{(i+2)} + \cdots$$
$$\cdots + \frac{(-t)^{m-i-1}}{(m-i-1)!}\,A^{(m)} + \frac{(-t)^{m-i-1}}{(m-i-1)!}\,A^{(m+1)} = 0,$$

which finally shows that $T_i = T_i(0) = A^{(i)}(0)$ for $i = 0, \dots, m$. □

With the notations of Theorem 3.2, if we consider the matrix $A = (R_0 \ \dots \ R_n) \in \Bbbk[t]^{q\times p(n+1)}$, then using Lemma 3.3, we obtain

$$C(0) = \begin{pmatrix} D_m(A)(0) \\ 0 \end{pmatrix} = \begin{pmatrix} U^{-1}\,D_m(A) \\ 0 \end{pmatrix} = \begin{pmatrix} U^{-1} & 0 \\ 0 & I_q \end{pmatrix} \begin{pmatrix} D_m(A) \\ 0 \end{pmatrix}$$
$$= \begin{pmatrix} U^{-1} & 0 \\ 0 & I_q \end{pmatrix} C. \tag{4}$$

Proposition 3.4. *With the notations of Theorem 3.2, the* $\Bbbk[t]$*-module* $\mathcal{N} = \operatorname{coker}_{\Bbbk[t]}(.C)$ *is such that* $\mathcal{N} = \operatorname{coker}_{\Bbbk[t]}(.C(0))$*, and thus, is a* free $\Bbbk[t]$*-module, namely,* $\mathcal{N} \cong \Bbbk[t]^d$*, where*

$$d = p(n+1) - \operatorname{rank}_{\Bbbk}(C(0)).$$

Proof. Let us consider $A = (R_0 \ \dots \ R_n) \in \Bbbk[t]^{q\times p(n+1)}$. Using (4) and the fact that U is *unimodular*, i.e., $U^{-1} \in \Bbbk[t]^{q(m+1)\times q(m+1)}$, $\mathcal{R}^{1\times q(m+2)}\,C = \mathcal{R}^{1\times q(m+2)}\,C(0)$, which yields

$$\mathcal{N} = \operatorname{coker}_{\Bbbk[t]}(.C) = \operatorname{coker}_{\Bbbk[t]}(.C(0)) = \operatorname{coker}_{\Bbbk[t]}(.D_m(A)(0)),$$

where

$$D_m(A)(0) = \begin{pmatrix} R_0 & \dots & R_n \\ \vdots & \dots & \vdots \\ R_0^{(m)}(0) & \dots & R_n^{(m)}(0) \end{pmatrix} \in \Bbbk^{q(m+1)\times p(n+1)}.$$

Finally, using the fact that $D_m(A)(0)$ is a matrix with entries in $\Bbbk$, $\mathcal{N} \cong \Bbbk[t] \otimes_{\Bbbk} \operatorname{coker}_{\Bbbk}(.D_m(A)(0))$, where $\operatorname{coker}_{\Bbbk}(.D_m(A)(0))$ is a $\Bbbk$-vector space of dimension

$$d = p(n+1) - \operatorname{rank}_{\Bbbk}(D_m(A)(0)) = p(n+1) - \operatorname{rank}_{\Bbbk}(C(0)),$$

and thus, $\mathcal{N} \cong \Bbbk[t]^d$, i.e., $\mathcal{N}$ is a free $\Bbbk[t]$-module of rank d. □

Example 3.5. Let us consider $R = t\,e + t\,e\,\partial \in \langle e \rangle$. Thus, we have $p = q = 1$, $R = R_0\,e + R_1\,e\,p$, where $R_0 = R_1 = t$, $m = n = 1$, and

$$C = \begin{pmatrix} t & t \\ 1 & 1 \\ 0 & 0 \end{pmatrix},\ C(0) = \begin{pmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 0 \end{pmatrix},\ D = \begin{pmatrix} 1 & -t & 0 \\ 0 & 0 & 1 \end{pmatrix},\ E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 1 - t\,\partial \\ \partial^2 \end{pmatrix}, \quad v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} e \\ e\,\partial^2 \end{pmatrix}.$$

By Theorem 3.2, we then have

$$\ker_{\mathbb{I}_1}(.R) = \operatorname{ann}_{\mathbb{I}_1}(.R) = \mathbb{I}_1\,(1 - t\,\partial) + \mathbb{I}_1\,\partial^2 + \mathbb{I}_1\,e + \mathbb{I}_1\,e\,\partial^2.$$

Moreover, we can check that (4) holds

$$U = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \quad U^{-1} = \begin{pmatrix} 1 & -t \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} U^{-1} & 0 \\ 0 & 1 \end{pmatrix} C = C(0).$$

We have $\operatorname{rank}_{\Bbbk}(C(0)) = 1$, which, by Proposition 3.4, shows that $\mathcal{N} = \Bbbk[t]^{1\times 2}/(\Bbbk[t]^{1\times 3}\,C) = \Bbbk[t]^{1\times 2}/(\Bbbk[t]^{1\times 3}\,C(0))$ is a free $\Bbbk[t]$-module of rank 1. This last result can easily be checked again [3, 7].

Note that Proposition 3.4 generalizes and simplifies the proof of Proposition 4.16 of [5] as follows.

Corollary 3.6. *With the above notations,* $\mathcal{N} = 0$ *if and only if* $C(0)$ *is a full column rank matrix.*

If $p = q = 1$ *and if the polynomials* R_k*'s are supposed to be* $\Bbbk$*-linearly independent, then* $\mathcal{N} = 0$.

Proof. The first result is a direct consequence of Proposition 3.4. Let us now write $R_k = \sum_{l=0}^m R_{kl}\,t^l$, where $R_{kl} \in \Bbbk$. Using Proposition 3.4, $\mathcal{N} = \operatorname{coker}_{\Bbbk[t]}(.C)$ is then a free $\Bbbk[t]$-module of rank $d = n + 1 - \operatorname{rank}_{\Bbbk}(C(0))$, where

$$C(0) = \begin{pmatrix} R_{00} & \dots & R_{n0} \\ \vdots & \dots & \vdots \\ m!\,R_{0m} & \dots & m!\,R_{nm} \\ 0 & \dots & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & m! & 0 \\ 0 & \dots & 0 & 1 \end{pmatrix} \begin{pmatrix} R_{00} & \dots & R_{n0} \\ \vdots & \vdots & \vdots \\ R_{0m} & \dots & R_{nm} \\ 0 & \dots & 0 \end{pmatrix}.$$

Over the infinite field $\Bbbk$, the fact that the R_k's are $\Bbbk$-linearly independent as formal polynomials implies that the right matrix factor of $C(0)$ has column rank $n+1$. Thus $\operatorname{rank}_{\Bbbk}(C(0)) = n+1$, i.e., $d = 0$, and thus, $\mathcal{N} = 0$. □

Theorem 3.7. *Let $R = \sum_{k=0}^{n} R_k(t)\, e\, \partial^k \in \langle e \rangle^{q\times p}$ with matrices $R_k \in \Bbbk[t]^{q\times p}$. With the notations of Theorem 3.2, we have*

$$\ker_{\mathbb{I}_1}(.R) = \sum_{i=1}^{r} \mathbb{I}_1\, u_i = \operatorname{im}_{\mathbb{I}_1}(.U), \quad U = \left(u_1^T \ldots u_r^T\right)^T,$$

where $r = q(m+2) - \operatorname{rank}_{\Bbbk}(C(0))$ and $m = \max_{k\in[\![0,n]\!]} \deg(R_k)$.

Proof. With the notations of Theorem 3.2, we have the following long exact sequence of $\Bbbk[t]$-modules

$$0 \longrightarrow \Bbbk[t]^{1\times r} \xrightarrow{.D} \Bbbk[t]^{1\times q(m+2)} \xrightarrow{.C} \Bbbk[t]^{1\times p(n+1)} \xrightarrow{\sigma} \mathcal{N} \longrightarrow 0,$$

where σ denotes the canonical projection onto $\mathcal{N}$. By Proposition 3.4, $\mathcal{N}$ is a free $\Bbbk[t]$-module of rank $d = p(n+1) - \operatorname{rank}_{\Bbbk}(C(0))$. Since the alternative sum of ranks in an exact sequence is zero (see [12, Exercise 3.16(i), p. 129]), we have

$$r - q(m+2) + p(n+1) - d = 0 \;\Rightarrow\; r = q(m+2) - \operatorname{rank}_{\Bbbk}(C(0)).$$

The freeness of $\mathcal{N}$ also implies that the latter exact sequence *splits* (in other words, we have a *contractible complex* [12, Ch. 6, p. 337]), i.e., that there exist $X \in \Bbbk[t]^{q(m+2)\times r}$ and $Y \in \Bbbk[t]^{p(n+1)\times q(m+2)}$ satisfying the identities $D\,X = I_r$ and $X\,D + C\,Y = I_{q(m+2)}$. Using $D\,C = 0$, we have $D(0)\,C(0) = 0$, i.e., $\operatorname{im}_{\Bbbk}(.D(0)) \subseteq \ker_{\Bbbk}(.C(0))$. Now, if $v \in \ker_{\Bbbk}(.C(0))$, using $X(0)\,D(0) + C(0)\,Y(0) = I_{q(m+2)}$, we obtain $v = (v\,X(0))\,D(0)$, which yields $v \in \operatorname{im}_{\Bbbk}(.D(0))$, and thus, $\ker_{\Bbbk}(.C(0)) = \operatorname{im}_{\Bbbk}(.D(0))$. Thus, we can take $E = e(D)$ in Theorem 3.2. With the notations of Theorem 3.7, using the identity $e\,a = e(a)\,e$ for all $a \in \Bbbk[t]$, $v = E\,e\,J_{m+1} = e(D)\,e\,J_{m+1} = e\,D\,J_{m+1} = e\,u$, i.e., the v_j's are evaluations of the u_i's, which ends the proof. □

Example 3.8. Continuing Example 3.5, $e\,u_1 = e\,(1 - t\,\partial) = e = v_1$ and $e\,u_2 = e\,\partial^2 = v_2$ yield $\ker_{\mathbb{I}_1}(.R) = \mathbb{I}_1\,(1 - t\,\partial) + \mathbb{I}_1\,\partial^2$.

Remark. Let us explain how a full row matrix $D \in \Bbbk[t]^{r\times q(m+2)}$ satisfying $\ker_{\Bbbk[t]}(.C) = \operatorname{im}_{\Bbbk[t]}(.D)$ can directly be computed. Let $E \in \Bbbk^{s\times q(m+2)}$ be a full row rank satisfying $\ker_{\Bbbk}(.C(0)) = \operatorname{im}_{\Bbbk}(.E)$. Such a matrix can easily be computed using linear algebra methods. If we denote by $V \in \Bbbk[t]^{q(m+2)\times q(m+2)}$ the first matrix in the right-hand side of (4), using (4), we then get $\ker_{\Bbbk[t]}(.C) = \operatorname{im}_{\Bbbk[t]}(.(E\,V))$.

3.2 Generators of the intersection of evaluation ideals

We shall now use the results of Section 3.1 to show how to compute a finite set of generators of the intersection of two finitely generated evaluation ideals, i.e., finitely generated ideals of $\mathbb{I}_1$ included in $\langle e \rangle$.

Theorem 3.9. *Let $\mathcal{I} = \sum_{i=1}^{n_1} \mathbb{I}_1\, p_i$ and $\mathcal{J} = \sum_{j=1}^{n_2} \mathbb{I}_1\, q_j$ be two finitely generated evaluation ideals. Moreover, let*

$$p = \left(p_1 \cdots p_{n_1}\right)^T, \quad q = \left(q_1 \cdots q_{n_2}\right)^T,$$

and

$$R = \left(p^T \quad q^T\right)^T = \left(p_1 \cdots p_{n_1}\; q_1 \cdots q_{n_2}\right)^T \in \langle e \rangle^{(n_1+n_2)\times 1}.$$

If $u_1, \ldots, u_r$ are the generators of $\ker_{\mathbb{I}_1}(.R)$ given by Theorem 3.7, i.e., $\ker_{\mathbb{I}_1}(.R) = \sum_{i=1}^{r} \mathbb{I}_1\, u_i$, where $u_i = (u_{i,1} \quad u_{i,2}) \in \mathbb{I}_1^{1\times(n_1+n_2)}$, $u_{i,1} \in \mathbb{I}_1^{1\times n_1}$, and $u_{i,2} \in \mathbb{I}_1^{1\times n_2}$, then, we have

$$\mathcal{I} \cap \mathcal{J} = \sum_{i=1}^{r} \mathbb{I}_1\,(u_{i,1}\, p) = \sum_{i=1}^{r} \mathbb{I}_1\,(u_{i,2}\, q).$$

In particular, $\mathcal{I} \cap \mathcal{J}$ is finitely generated and a set of generators can explicitly be computed.

Proof. With the notations of Theorem 3.7, $\ker_{\mathbb{I}_1}(.R) = \sum_{i=1}^{r} \mathbb{I}_1\, u_i$, where $u_i = (u_{i,1} \quad u_{i,2}) \in \mathbb{I}_1^{1\times(n_1+n_2)}$, $u_{i,1} \in \mathbb{I}_1^{1\times n_1}$, and $u_{i,2} \in \mathbb{I}_1^{1\times n_2}$. Therefore, we have $u_{i,1}\, p = -u_{i,2}\, q$ for $i = 1, \ldots, r$, which shows that $\sum_{i=1}^{r} \mathbb{I}_1\,(u_{i,1}\, p) = \sum_{i=1}^{r} \mathbb{I}_1\,(u_{i,2}\, q) \subseteq \mathcal{I} \cap \mathcal{J}$.

Let us now consider $x \in \mathcal{I} \cap \mathcal{J}$. Thus, $x \in \mathcal{I}$ so that there exist $a_1, \ldots, a_{n_1} \in \mathbb{I}_1$ such that $x = \sum_{i=1}^{n_1} a_i\, p_i = a\, p$ with $a := \left(a_1 \cdots a_{n_1}\right)$. Similarly, $x \in \mathcal{J}$ so that there exist $b_1, \ldots, b_{n_2} \in \mathbb{I}_1$ satisfying $x = \sum_{i=1}^{n_2} b_i\, q_i = b\, q$ with $b := \left(b_1 \cdots b_{n_2}\right)$. Therefore, we have the relation $x = a\,p = b\,q$, which implies $(a \quad -b)\,R = 0$ so that $(a \quad -b) \in \ker_{\mathbb{I}_1}(.R)$. We thus have $(a \quad -b) = \sum_{i=1}^{r} f_i\, u_i$ for some $f_i \in \mathbb{I}_1$, and thus, $a = \sum_{i=1}^{r} f_i\, u_{i,1}$ and $b = -\sum_{i=1}^{r} f_i\, u_{i,2}$. Then, we have $x = a\,p = \sum_{i=1}^{r} f_i\, u_{i,1}\, p = b\,q = -\sum_{i=1}^{r} f_i\, u_{i,2}\, q$, which proves the reverse inclusion and the result. □

We obtain Algorithm 1 displayed at the end of the paper.

Let us illustrate our algorithm for computing generators of the intersection of two finitely generated evaluation ideals.

Example 3.10. Let $\mathcal{I} = \mathbb{I}_1\,(t^2+1)\,e$ and $\mathcal{J} = \mathbb{I}_1(t\,e + t^2\,e\,\partial)$. To compute a finite set of generators of $\mathcal{I} \cap \mathcal{J}$, we first define

$$R = \begin{pmatrix} (t^2+1)\,e \\ t\,e + t^2\,e\,\partial \end{pmatrix} = \underbrace{\begin{pmatrix} t^2+1 \\ t \end{pmatrix}}_{R_0} e + \underbrace{\begin{pmatrix} 0 \\ t^2 \end{pmatrix}}_{R_1} e\,\partial,$$

and then we can consider the following matrix

$$C = \begin{pmatrix} R_0 & R_1 \\ \vdots & \vdots \\ R_0^{(3)} & R_1^{(3)} \end{pmatrix} = \begin{pmatrix} t^2+1 & 0 \\ t & t^2 \\ 2\,t & 0 \\ 1 & 2\,t \\ 2 & 0 \\ 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \in \Bbbk[t]^{8\times 2}.$$

Using, e.g., the OreModules package ([4]), we can compute a full row rank matrix D satisfying $\ker_{\Bbbk[t]}(.C) = \operatorname{im}_{\Bbbk[t]}(.D)$. We get

$$D = \begin{pmatrix} -2 & 0 & t & 0 & 1 & 0 & 0 & 0 \\ 0 & -4 & 1 & 2\,t & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & t & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 1 & 2\,t & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

so that $\ker_{\mathbb{I}_1}(.R) = \operatorname{im}_{\mathbb{I}_1}(.U)$, where

$$U = D \begin{pmatrix} I_2 \\ I_2\,\partial \\ I_2\,\partial^2 \\ I_2\,\partial^3 \end{pmatrix} = \begin{pmatrix} \partial^2 + t\,\partial - 2 & 0 \\ \partial & 2\,t\,\partial - 4 \\ t\,\partial^2 - \partial & 0 \\ \partial^2 & 2\,t\,\partial^2 - 2\,\partial \\ \partial^3 & 0 \\ 0 & \partial^3 \end{pmatrix}.$$

Using Theorem 3.7, we have $p = 1$, $q = 2$, $m = 2$, $\operatorname{rank}_{\mathbb{k}}(C(0)) = 2$, and thus, $r = 8 - 2 = 6$. Partitioning $U = (u_1\ u_2)$, where u_1 (resp., u_2) is the first (resp., second) column of u, we finally have

$$u_1\,(t^2+1)\,e = u_2\,(t\,e + t^2\,e\,\partial) = \begin{pmatrix} 0 & 2\,t\,e & 0 & 2\,e & 0 & 0 \end{pmatrix}^T,$$

so that $\mathcal{I} \cap \mathcal{J} = \mathbb{I}_1\,t\,e + \mathbb{I}_1\,e = \mathbb{I}_1\,e$ because $e\,\partial\,t\,e = e\,(t\,\partial + 1)\,e = e$.

Finally, the above computations show that the syzygy module $\operatorname{Syz}(\mathcal{K}) = \{(\alpha \quad \beta) \in \mathbb{I}_1^{1\times 2} \mid \alpha\,(t^2+1)\,e + \beta\,(t\,e + t^2\,e\,\partial) = 0\}$ of the ideal $\mathcal{K} = \mathbb{I}_1\,(t^2+1)\,e + \mathbb{I}_1(t\,e + t^2\,e\,\partial)$ is generated by the rows of U.

In Example 3.10, we have considered two principal ideals. First note that the algorithm applies similarly for ideals which are not principal. Moreover, in Proposition 4.7, we shall constructively prove that any finitely generated ideal of $\langle e \rangle$ is principal.

4 STRUCTURE OF FINITELY GENERATED EVALUATION IDEALS AS SEMISIMPLE MODULES

In this section, we study the left ideal structure of finitely generated evaluation ideals. Using the involution θ of $\mathbb{I}_1$ given in Section 2.1, which satisfies $\theta(\langle e \rangle) \subseteq \langle e \rangle$, we can similarly handle finitely generated right evaluation ideals of $\mathbb{I}_1$. In [1], these ideals are proved to be finitely generated *semisimple* left $\mathbb{I}_1$-modules (see the definition below). More precisely, they are finitely generated $\mathbb{k}[t]$-modules. Below, we show how to constructively prove these results. In particular, we explain how to effectively compute a finite set of generators of these ideals as $\mathbb{k}[t]$-modules. In the next section, this last result will be used to algorithmically characterize the intersection of two finitely generated ideals in the case when at least one is in $\langle e \rangle$.

Using the fact that $\mathbb{k}[t]$ is a subring of $\mathbb{I}_1$, a left ideal $\mathcal{I}$ of $\mathbb{I}_1$ inherits a $\mathbb{k}[t]$-module structure. Hence, the generators of an evaluation ideal $\mathcal{I}$ of $\mathbb{I}_1$ as a $\mathbb{k}[t]$-module also generate $\mathcal{I}$ as a $\mathbb{I}_1$-module.

Definition 4.1 ([12], Ch. 4, p. 154). Let $\mathcal{M}$ be a left module over a ring $\mathcal{R}$.

- $\mathcal{M}$ is said to be *simple* if it is non-zero and has no non-zero proper left $\mathcal{R}$-submodules.
- $\mathcal{M}$ is said to be *semisimple* if it is a direct sum of simple left $\mathcal{R}$-modules.

We give examples that will play important roles in what follows.

Example 4.2. In [1], it is shown that $\mathbb{k}[t]$ is a simple left $\mathbb{I}_1$-module. It is finitely presented as left $\mathbb{I}_1$-module because (2) and $e^2 = e$ yield

$$\mathbb{I}_1/(\mathbb{I}_1\,\partial) = \mathbb{I}_1/(\mathbb{I}_1\,(1-e)) \cong \mathbb{I}_1\,e = \mathbb{k}[t]\,e \cong \mathbb{k}[t].$$

As a consequence, $\mathbb{k}[t]^n$ is a finitely generated semisimple left $\mathbb{I}_1$-module for all $n \in \mathbb{N}$.

Before stating the main result of this section, we need to introduce another concept.

Definition 4.3. An element a of $\langle e \rangle$ is called a *simple evaluation* if it is of the form $a = e\,q(\partial)$, where $q \in \mathbb{k}[\partial]$, i.e., $a \in e\,\mathbb{k}[\partial]$.

Using [5, Lemma 2.4], if $P \in \mathbb{I}_1$ and $a = \sum_{k=0}^r d_k\,e\,\partial^k \in \langle e \rangle$, where $d_k \in \mathbb{k}[t]$ for $k = 0, \ldots, r$, then we have

$$P\,a = \sum_{k=0}^{r} P(d_k)\,e\,\partial^k, \tag{5}$$

where $P(d_k)$ is an element of $\mathbb{k}[t]$ obtained by applying the operator $P \in \mathbb{I}_1$ to the polynomial d_k.

If $a = e\,q(\partial)$ is a simple evaluation, then, using (5), for any $b \in \mathbb{I}_1$, we have $b\,a = b(1)\,e\,q(\partial) = b(1)\,a \in \mathbb{k}[t]\,e\,q(\partial)$, which shows that $\mathbb{I}_1\,a = \mathbb{k}[t]\,a$. For instance, we have $\mathbb{I}_1\,e = \mathbb{k}[t]\,e$. More generally, we have $b\,t^k\,a = b\,t^k\,e\,q(\partial) = b(t^k)\,e\,q(\partial) = b(t^k)\,a$ for $n \in \mathbb{N}$.

The main result of this section is Theorem 4.5, stated below, which shows that, as a $\mathbb{k}[t]$-module, every finitely generated evaluation ideal of $\mathbb{I}_1$ can be generated by simple evaluations.

Let us consider an evaluation ideal $\mathcal{I} \subseteq \langle e \rangle$ finitely generated by evaluation operators $a_1, \ldots, a_q$. We can then form the column vector $A = (a_1\ \ldots\ a_q)^T$ of $\langle e \rangle^{q\times 1}$ and write its normal form $A = \sum_{k=0}^n A_k(t)\,e\,\partial^k$, where $A_k \in \mathbb{k}[t]^{q\times 1}$ for $k = 0, \ldots, n$. Using the notations of Theorems 3.2 and 3.7, let $m = \max_{k\in [\![0,r]\!]} \deg(A_k)$,

$$C = \begin{pmatrix} A_0 & \ldots & A_n \\ \vdots & & \vdots \\ A_0^{(m+1)} & \ldots & A_n^{(m+1)} \end{pmatrix}, \quad J_{m+1} = \begin{pmatrix} I_q \\ I_q\,\partial \\ \vdots \\ I_q\,\partial^{m+1} \end{pmatrix},$$

$D \in \mathbb{k}[t]^{r\times q(m+2)}$ be a full row rank matrix generating $\ker_{\mathbb{k}[t]}(.C)$, i.e., satisfying $\ker_{\mathbb{k}[t]}(.C) = \operatorname{im}_{\mathbb{k}[t]}(.D) \cong \mathbb{k}[t]^r$ (see Remark 3.1). If we note $D = (D_0\ \ldots\ D_{m+1})$, where $D_i \in \mathbb{k}[t]^{r\times q}$ for $i = 0, \ldots, m+1$, and $B = D\,J_{m+1} = D_0 + D_1\,\partial + \cdots + D_{m+1}\,\partial^{m+1}$, then we have $\ker_{\mathbb{I}_1}(.A) = \operatorname{im}_{\mathbb{I}_1}(.B)$.

Notice that, by definition of m, we have $C = (C'^T \quad 0^T)^T$, where

$$C' = \begin{pmatrix} A_0 & \ldots & A_n \\ \vdots & & \vdots \\ A_0^{(m)} & \ldots & A_n^{(m)} \end{pmatrix} \in \mathbb{k}[t]^{q(m+1)\times(n+1)}, \tag{6}$$

so that we can choose D with a block-partition $D = \begin{pmatrix} D' & 0 \\ 0 & I_q \end{pmatrix}$, where $D' \in \mathbb{k}[t]^{(r-q)\times q(m+1)}$ is a full row rank matrix satisfying $\ker_{\mathbb{k}[t]}(.C') = \operatorname{im}_{\mathbb{k}[t]}(.D')$. As a consequence, the matrix $B = D\,J_{m+1}$ can be written as $B = (B'^T \quad \partial^{m+1}\,I_q)^T$, where

$$B' = \sum_{i=0}^{m} D'_i\,\partial^i \in \mathbb{I}_1^{(r-q)\times q}, \tag{7}$$

and $D' = (D'_0 \quad \ldots \quad D'_m)$, where $D'_i \in \mathbb{k}[t]^{(r-q)\times q}$ for $i = 0, \ldots, m$.

By the proof of Theorem 3.7, D has a right inverse $X \in \mathbb{k}[t]^{q(m+2)\times r}$, i.e., $D\,X = I_r$. Using the above structure of D, we then get

$$\begin{pmatrix} D' & 0 \\ 0 & I_q \end{pmatrix} \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} = \begin{pmatrix} I_{r-q} & 0 \\ 0 & I_q \end{pmatrix},$$

which yields $D'\,X_{11} = I_{r-q}$, i.e., D' has a right inverse.

We first have the following lemma.

Lemma 4.4. *With the above notations, if $\mathcal{M} = \operatorname{coker}_{\mathbb{I}_1}(.B)$ is the left $\mathbb{I}_1$-module finitely presented by the matrix B and π is the canonical projection onto $\mathcal{M}$, then the following map*

$$\begin{array}{rcll} \psi: \mathcal{M} & \longrightarrow & \mathcal{I} = \operatorname{im}_{\mathbb{I}_1}(.A) = \sum_{k=0}^{q} \mathbb{I}_1 \, a_k, & \\ \pi(\lambda) & \longmapsto & \lambda \, A, \quad \forall \, \lambda \in \mathbb{I}_1^{1\times q}, & \end{array} \tag{8}$$

is well-defined and is an isomorphism of left $\mathbb{I}_1$-modules, i.e., $\mathcal{M} \cong \mathcal{I}$.

Proof. Let us check that ψ is an isomorphism whose inverse is

$$\begin{array}{rcll} \varphi: \mathcal{I} & \longrightarrow & \mathcal{M} & \\ \lambda \, A & \longmapsto & \pi(\lambda), & \forall \, \lambda \in \mathbb{I}_1^{1\times q}. \end{array}$$

Let us first show that the two maps ψ and φ are well-defined. First, if $\pi(\lambda) = \pi(\mu)$, where $\lambda, \, \mu \in \mathbb{I}_1^{1\times q}$, then we have $\pi(\lambda - \mu) = 0$ which implies $\lambda - \mu = \nu \, B$ for some $\nu \in \mathbb{I}_1^{1\times r}$. Thus, $\lambda \, A - \mu \, A = \nu \, B \, A = 0$ so that $\lambda \, A = \mu \, A$. Regarding φ, if $x \in \mathcal{I}$ and $x = \lambda \, A = \mu \, A$ for $\lambda, \, \mu \in \mathbb{I}_1^{1\times q}$, then $(\lambda - \mu) \, A = 0$, so that $\lambda - \mu \in \ker_{\mathbb{I}_1}(.A) = \operatorname{im}_{\mathbb{I}_1}(.B)$. Thus, there exists $\nu \in \mathbb{I}_1^{1\times r}$ such that $\lambda - \mu = \nu \, B$. Then, we have $\pi(\lambda - \mu) = \pi(\nu \, B) = 0$, so that $\pi(\lambda) = \pi(\mu)$. Finally, we have $(\varphi \circ \psi)(\pi(\lambda)) = \pi(\lambda)$ and $(\psi \circ \varphi)(\lambda \, A) = \lambda \, A$ for all $\lambda \in \mathbb{I}_1^{1\times q}$, which proves that ψ and φ are isomorphisms, $\mathcal{M} \cong \mathcal{I}$, and $\varphi = \psi^{-1}$. □

Theorem 4.5. *Let $\mathcal{I} \subseteq \langle e \rangle$ be an evaluation ideal finitely generated by elements $a_1, \ldots a_q$ and let $A = (a_1 \ \ldots \ a_q)^T$. Then, $\mathcal{I}$ is a semisimple $\mathbb{k}[t]$-module that can be generated by a finite set of simple evaluations.*

More precisely, if we consider

- *C' the matrix defined by (6),*
- *$s = \operatorname{rank}_{\mathbb{k}}(C(0))$, $\quad r = q(m+2) - s$,*
- *U the unimodular matrix defined in Lemma 3.3,*
- *$\{e_k\}_{k=1,\ldots,q}$ the standard basis of $\mathbb{I}_1^{1\times q}$,*
- *$y = (\pi(e_1) \ \ldots \ \pi(e_q))^T$, where $\pi : \mathbb{I}_1^{1\times q} \longrightarrow \mathcal{M} = \operatorname{coker}_{\mathbb{I}_1}(.B)$ is the canonical projection onto $\mathcal{M}$ and the matrix $B \in \mathbb{I}_1^{r\times q}$ satisfies $\ker_{\mathbb{I}_1}(.A) = \operatorname{im}_{\mathbb{I}_1}(.B)$ as explained in Theorem 3.7,*
- *$z = (e\, y^T \quad e \, \partial \, y^T \quad \ldots \quad e \, \partial^m \, y^T)^T$,*
- *$Q' \in \mathbb{k}^{q(m+1)\times s}$ a full column rank matrix whose columns define a basis of $\operatorname{im}_{\mathbb{k}}(C'(0).)$,*
- *$T \in \mathbb{k}^{s\times q(m+1)}$ a left inverse of Q',*
- *$w = T \, z \in \mathcal{M}^s$,*

then the entries w_i of the vector w are simple evaluations and the finite family $\{g_i = \psi(w_i) = \pi^{-1}(w_i) \, A\}_{i=1,\ldots,s}$ generates the left ideal $\mathcal{I}$ as a $\mathbb{k}[t]$-module, i.e., $\mathcal{I} = \sum_{i=1}^{s} \mathbb{k}[t] \, g_i$.

Proof. Let us consider the following exact sequence

$$\mathbb{I}_1^{1\times r} \xrightarrow{.B} \mathbb{I}_1^{1\times q} \xrightarrow{\pi} \mathcal{M} = \operatorname{coker}_{\mathbb{I}_1}(.B) \longrightarrow 0,$$

where π is the canonical projection onto $\mathcal{M}$. If $e_1, \ldots, e_q$ denotes the canonical basis of $\mathbb{I}_1^{1\times q}$, then $\{y_i = \pi(e_i)\}_{i=1,\ldots,q}$ is a set of generators of $\mathcal{M}$ and these generators of $\mathcal{M}$ satisfy the left $\mathbb{I}_1$-linear relations $B \, y = 0$, where $y = (y_1 \ \ldots \ y_q)^T$. For more details, see, e.g., [3, 9].

Now, using $B = (B'^T \quad \partial^{m+1} \, I_q)^T$, where $B' \in \mathbb{I}_1^{(r-q)\times q}$, in the left $\mathbb{I}_1$-relation $B \, y = 0$, we then have $\partial^{m+1} \, y = 0$.

Let $T_m = \sum_{k=0}^{m} \frac{t^k}{k!} \, e \, \partial^k$ be the m^{th} Taylor operator. Using (2), $\partial^{m+1} \, y = 0$ yields $I^{m+1} \, \partial^{m+1} \, y = 0$, and thus, $(1 - T_m) \, y = 0$, i.e., $y = T_m \, y$. Conversely, using (5), $y = T_m \, y$ yields

$$\partial^{m+1} \, y = \partial^{m+1} \, T_m \, y = \sum_{k=0}^{m} \partial^{m+1} \, \frac{t^k}{k!} \, e \, \partial^k = \sum_{k=0}^{m} \partial^{m+1} \left(\frac{t^k}{k!}\right) e \, \partial^k = 0.$$

Therefore, $\partial^{m+1} \, y = 0$ is equivalent to $y = T_m \, y$. Hence, if we set $z_k = e \, \partial^k y \in \mathcal{M}^q$ for $k = 0, \ldots, m$ and denote by $z_{k,j}$ the j^{th} entry of z_k for $j = 1, \ldots, q$, we then obtain

$$y = T_m \, y = \sum_{k=0}^{m} \frac{t^k}{k!} \, e \, \partial^k \, y = \sum_{k=0}^{m} \frac{t^k}{k!} \, z_k, \tag{9}$$

which shows that $\{z_{k,j}\}_{k=0,\ldots,m, j=1,\ldots,q}$ is another set of generators of $\mathcal{M}$ defined by means of simple evaluations, which yields

$$\mathcal{M} = \sum_{k=0}^{m} \sum_{j=1}^{q} \mathbb{I}_1 \, z_{k,j} = \sum_{k=0}^{m} \sum_{j=1}^{q} \mathbb{k}[t] \, z_{k,j}.$$

Thus, $\{z_{k,j}\}_{k=0,\ldots,m, j=1,\ldots,q}$ also generates $\mathcal{M}$ as a $\mathbb{k}[t]$-module.

Using (5), for any $P \in \mathbb{I}_1$, we have $P \, y = \sum_{k=0}^{m} P\left(\frac{t^k}{k!}\right) z_k$, where $P\left(\frac{t^k}{k!}\right) \in \mathbb{k}[t]$ for $k = 0, \ldots, m$. If $P \in \mathbb{k}[t]$, then we state again that P acts in $\mathbb{I}_1$ as the multiplication by P, and thus, $P \, y = P \, \sum_{k=0}^{m} \frac{t^k}{k!} \, z_k$.

Let us now find the relations satisfied by the generators $z_{k,j}$'s. We first have that $e \, z_k = z_k$ for all $k = 0 \ldots m$. Using (9), we obtain:

$$0 = B' \, y = B' \sum_{k=0}^{m} \frac{t^k}{k!} \, z_k = \sum_{k=0}^{m} B' \left(\frac{t^k}{k!}\right) z_k.$$

If we note $z = (z_0^T \ \ldots \ z_m^T)^T$ and

$$P = \left(B'(1) \quad B'(t) \quad \ldots \quad B'\left(\tfrac{t^m}{m!}\right)\right) \in \mathbb{k}[t]^{(r-q)\times q(m+1)},$$

then the relations satisfied by the generators $z_{k,j}$'s are $e \, z = z$ and $P \, z = 0$, so that we have

$$\mathcal{M} \cong \mathcal{M}' = \operatorname{coker}_{\mathbb{I}_1}\left(.\begin{pmatrix} P \\ (1-e) \, I_{q(m+1)} \end{pmatrix}\right).$$

Let us now compute $\ker_{\mathbb{k}[t]}(P.) = \{\eta \in \mathbb{k}[t]^{q(m+1)\times 1} \mid P \, \eta = 0\}$. Using (7), we first have

$$\begin{aligned} P &= \left(D'_0 \quad t \, D'_0 + D'_1 \quad \ldots \quad \tfrac{t^m}{m!} \, D'_0 + \cdots + D'_m\right), \\ &= \left(D'_0 \quad D'_1 \quad \ldots \quad D'_m\right) \, U = D' \, U, \end{aligned}$$

where U is the unimodular matrix introduced in Lemma 3.3. Now, if $\eta \in \ker_{\mathbb{k}[t]}(P.)$, then we have $D' \, U \, \eta = 0$ which is equivalent to $D' \, \zeta = 0$ and $\zeta = U \, \eta$. Now, $\ker_{\mathbb{k}[t]}(D'.) = \operatorname{im}_{\mathbb{k}[t]}(C'.)$ implies $\zeta = D' \, \xi$ for a certain $\xi \in \mathbb{k}[t]^{q(m+1)\times 1}$, and thus, $\eta = U^{-1} \, C' \, \xi$. This shows $\ker_{\mathbb{k}[t]}(P.) = \operatorname{im}_{\mathbb{k}[t]}((U^{-1} \, C').)$. If we note $Q = U^{-1} \, C'$, then Lemma 3.3 shows that $Q = C'(0) \in \mathbb{k}^{(n+1)\times q(m+1)}$, and thus, $\ker_{\mathbb{k}[t]}(P.) = \operatorname{im}_{\mathbb{k}[t]}(C'(0).)$. Let us consider a basis of the finite-dimensional $\mathbb{k}$-vector space $\operatorname{im}_{\mathbb{k}}(C'(0).)$ and stack the corresponding column vectors into a matrix $Q' \in \mathbb{k}^{q(m+1)\times s}$, where $s = \operatorname{rank}_{\mathbb{k}}(C'(0))$. Then, we have $\ker_{\mathbb{k}[t]}(P.) = \operatorname{im}_{\mathbb{k}[t]}(Q'.)$

We state again that D' has a right inverse $X_{11} \in \mathbb{k}[t]^{q(m+1)\times(r-q)}$, i.e., $D' \, X_{11} = I_{r-q}$, which shows that $S = U^{-1} \, X_{11} \in \mathbb{k}[t]^{q(m+1)\times(r-q)}$

is a right inverse of $P = D' U$ because U is unimodular. Therefore, we have the following split exact sequence of $\mathbb{k}[t]$-modules

$$0 \longrightarrow \mathbb{k}[t]^{s\times 1} \xrightarrow{Q'.} \mathbb{k}[t]^{q(m+1)\times 1} \xrightarrow{P.} \mathbb{k}[t]^{(r-q)\times 1} \longrightarrow 0,$$

which implies $s = q(m+1) - (r-q) = q(m+2) - r$ (see [12, Exercise 3.16(i), p. 129]). From Theorem 3.7, we then have $r = q(m+2) - \mathrm{rank}_{\mathbb{k}}(C'(0))$, which shows again that $s = \mathrm{rank}_{\mathbb{k}}(C'(0))$.

Using $P\,S = I_{r-q}$, we have $P\,(I_{q(m+1)} - S\,P) = 0$, which shows that $\mathrm{im}_{\mathbb{k}[t]}((I_{q(m+1)} - S\,P).) \subseteq \ker_{\mathbb{k}[t]}(P.) = \mathrm{im}_{\mathbb{k}[t]}(Q'.)$ and proves the existence of a matrix $T \in \mathbb{k}[t]^{s\times q(m+1)}$ satisfying the identity $I_{q(m+1)} - S\,P = Q'\,T$, i.e., $S\,P + Q'\,T = I_{q(m+1)}$. Hence, using $P\,Q' = 0$, we have $S\,P\,Q' + Q'\,T\,Q' = Q'$, i.e., $Q'\,T\,Q' = Q'$ or, equivalently, $Q'\,(T\,Q' - I_s) = 0$, and thus, $T\,Q' = I_s$ because $\ker_{\mathbb{k}[t]}(Q'.) = 0$.

Now, we have $\ker_{\mathbb{I}_1}(.P) = 0$ because $\mu\,P = 0$ and $P\,S = I_{r-q}$ yield $\mu = \mu\,P\,S = 0$. Therefore, we have the following split exact sequence

$$0 \longrightarrow \mathbb{k}[t]^{1\times(r-q)} \underset{.S}{\overset{.P}{\rightleftarrows}} \mathbb{k}[t]^{1\times q\,(m+1)} \xrightarrow{\gamma} \mathrm{coker}_{\mathbb{k}[t]}(.P) \longrightarrow 0,$$

which shows that $\mathrm{coker}_{\mathbb{k}[t]}(.P)$ is a *stably free* $\mathbb{k}[t]$-module [12, Ch. 4, p. 204], and thus, a free $\mathbb{k}[t]$-module of rank $q(m+2) - r = \mathrm{rank}_{\mathbb{k}}(C'(0))$ because $\mathbb{k}[t]$ is a principal ideal domain (see, e.g., [9]).

Using $P\,Q' = 0$, i.e., $\mathrm{im}_{\mathbb{k}[t]}(.P) \subseteq \ker_{\mathbb{k}[t]}(.Q')$, we can consider the following complex

$$0 \longrightarrow \mathbb{k}[t]^{1\times(r-q)} \xrightarrow{.P} \mathbb{k}[t]^{1\times q(m+1)} \xrightarrow{.Q'} \mathbb{k}[t]^{s\times 1} \longrightarrow 0.$$

The identiy $P\,S = I_{r-q}$ (resp., $T\,Q' = I_s$) shows that $.P$ is injective (resp., $.Q'$ is surjective). Now, let $\lambda \in \ker_{\mathbb{k}[t]}(.Q')$. Using the identities $S\,P + Q'\,T = I_{q(m+1)}$, we then have $\lambda = (\lambda\,S)\,P \in \mathrm{im}_{\mathbb{k}[t]}(.P)$, which shows that $\ker_{\mathbb{k}[t]}(.Q') = \mathrm{im}_{\mathbb{k}[t]}(.P)$ and proves that the above complex is a split short exact sequence (see [12, Ch. 2, p. 52]).

We have the following commutative exact diagram of $\mathbb{k}[t]$-modules

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{k}[t]^{1\times(r-q)} & \underset{.S}{\overset{.P}{\rightleftarrows}} & \mathbb{k}[t]^{1\times q(m+1)} & \underset{.T}{\overset{.Q'}{\rightleftarrows}} & \mathbb{k}[t]^{s\times 1} & \longrightarrow & 0 \\
 & & \| & & \| & & \phi \uparrow\downarrow \phi^{-1} & & \\
0 & \longrightarrow & \mathbb{k}[t]^{1\times(r-q)} & \underset{.S}{\overset{.P}{\rightleftarrows}} & \mathbb{k}[t]^{1\times q(m+1)} & \xrightarrow{\gamma} & \mathrm{coker}_{\mathbb{k}[t]}(.P) & \longrightarrow & 0,
\end{array}$$

where γ is the canonical projection and ϕ the isomorphism of $\mathbb{k}[t]$-modules defined by: for $\lambda \in \mathbb{k}[t]^{1\times q\,(m+1)}$, $\phi(\gamma(\lambda)) = \lambda\,Q'$. This application is well-defined: if $\gamma(\lambda) = \gamma(\mu)$, where $\lambda,\, \mu \in \mathbb{k}[t]^{1\times q\,(m+1)}$, then there is $\nu \in \mathbb{k}[t]^{1\times(r-q)}$ such that $\lambda - \mu = \nu\,P$. Then, we have $\lambda\,Q' - \mu\,Q' = \nu\,P\,Q' = 0$, which yields $\lambda\,Q' = \mu\,Q'$, i.e., $\phi(\gamma(\lambda)) = \phi(\gamma(\mu))$. Clearly, ϕ is injective and surjective. Finally, ϕ^{-1} is defined by $\phi^{-1}(\mu\,Q') = \gamma(\mu)$ for all $\mu \in \mathbb{k}[t]^{1\times q\,(m+1)}$.

Let us set $w = T\,z \in \mathcal{M}^{s\times 1}$. Since the $z_k = e\,\partial^k\,y$'s are formed by simple evaluations and the entries of T belong to $\mathbb{k}$, the w_k's are formed by simple evaluations. Now, $S\,P + Q'\,T = I_{q\,(m+1)}$ yields $z = Q'\,w$ because $P\,z = 0$. The entries w_i of w are then generators of $\mathcal{M}$ as a $\mathbb{I}_1$-module, and thus, as a $\mathbb{k}[t]$-module. Finally, if $\psi : \mathcal{M} \longrightarrow \mathcal{I}$ is the isomorphism of Lemma 4.4, $\mathcal{I}$ is then finitely generated by $\{g_i\}_{i=1,\ldots,s}$, where $g_i = \psi(w_i) = \pi^{-1}(w_i)\,A$ for $i = 1, \ldots, s$. □

We obtain Algorithm 2 displayed at the end of the paper.

Let $\Gamma \in \mathbb{k}[t]^{q\times q(m+1)}$ be the matrix satisfying $y = \Gamma\,z$ (see (9)). Using $z = Q'\,w$, we have $y = (\Gamma\,Q')\,w$, which yields $A = \Gamma\,Q'\,G$, where A and G are defined in Algorithm 2, and expresses the original generators a_i's of $\mathcal{I}$ in terms of the second set of generators g_j's.

Example 4.6. Let $\mathcal{I} = \mathbb{I}_1\,(t^2+1)\,e$ and $\mathcal{J} = \mathbb{I}_1(t\,e + t^2\,e\partial)$. Let us explicitly show that $\mathcal{I}$ and $\mathcal{J}$ are two semisimple $\mathbb{k}[t]$-modules, that they can be generated by simple evaluations, and finally find again the result of Example 3.10.

For $\mathcal{I}$, we have $A = ((t^2+1)\,e)$, $q = 1$, $n = 0$, $m = 2$, $r = 3$, and

$$C'_{\mathcal{I}} = \begin{pmatrix} t^2+1 \\ 2\,t \\ 2 \end{pmatrix},\ U_{\mathcal{I}}^{-1} = \begin{pmatrix} 1 & -t & \frac{t^2}{2} \\ 0 & 1 & -t \\ 0 & 0 & 1 \end{pmatrix},\ Q_{\mathcal{I}} = U_{\mathcal{I}}^{-1}\,C'_{\mathcal{I}} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}.$$

Then, we have $s = \mathrm{rank}_{\mathbb{k}}(C(0)) = 1$ and $T_{\mathcal{I}} = \begin{pmatrix} 0 & 0 & \frac{1}{2} \end{pmatrix}$ is a left inverse of $Q_{\mathcal{I}}$, and thus,

$$w_{\mathcal{I}} = T_{\mathcal{I}}\,z = T_{\mathcal{I}} \begin{pmatrix} e \\ e\,\partial \\ e\,\partial^2 \end{pmatrix} = \frac{1}{2}\,e\,\partial^2 \ \Rightarrow\ w_{\mathcal{I}}\,A = \frac{1}{2}\,e\,\partial^2\,(t^2+1)\,e = \frac{1}{2}\,e,$$

which yields $\mathcal{I} = \mathbb{k}[t]\,e$. Note that if we consider, e.g., the left inverses $T'_{\mathcal{I}} = (1 \quad 0 \quad 0)$ or $T''_{\mathcal{I}} = (3 \quad 0 \quad -1)$ of $Q_{\mathcal{I}}$, we obtain

$$w'_{\mathcal{I}} = T'_{\mathcal{I}}\,z = e \ \Rightarrow\ w'_{\mathcal{I}}\,A = e\,(t^2+1)\,e = e,$$
$$w''_{\mathcal{I}} = T''_{\mathcal{I}}\,z = 3\,e - 2\,e\,\partial^2 \ \Rightarrow\ w''_{\mathcal{I}}\,A = (3\,e - 2\,e\,\partial^2)\,(t^2+1)\,e = e,$$

which also yields $\mathcal{I} = \mathbb{k}[t]\,e$.

For $\mathcal{J}$, we have $A = (t\,e + t^2\,e\,\partial)$, $q = 1$, $n = 1$, $m = 2$, $r = 2$, and

$$C'_{\mathcal{J}} = \begin{pmatrix} t & t^2 \\ 1 & 2\,t \\ 0 & 2 \end{pmatrix},\ U_{\mathcal{J}}^{-1} = \begin{pmatrix} 1 & -t & \frac{t^2}{2} \\ 0 & 1 & -t \\ 0 & 0 & 1 \end{pmatrix},\ Q_{\mathcal{J}} = U_{\mathcal{J}}^{-1}\,C'_{\mathcal{J}} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 2 \end{pmatrix}.$$

Then, we have $s = 2$,

$$T_{\mathcal{J}} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix},\ w_{\mathcal{J}} = T_{\mathcal{J}}\,z = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} e \\ e\,\partial \\ e\,\partial^2 \end{pmatrix} = \begin{pmatrix} e\,\partial \\ \frac{1}{2}\,e\,\partial^2 \end{pmatrix}$$

$$\Rightarrow\ w_{\mathcal{J}}\,A = \begin{pmatrix} e\,\partial \\ \frac{1}{2}\,e\,\partial^2 \end{pmatrix} (t\,e + t^2\,e\,\partial) = \begin{pmatrix} e\,\partial\,(t\,e + t^2\,e\,\partial) \\ \frac{1}{2}\,e\,\partial^2\,(t\,e + t^2\,e\,\partial) \end{pmatrix} = \begin{pmatrix} e \\ e\,\partial \end{pmatrix}$$

and thus, we have $\mathcal{J} = \mathbb{k}[t]\,e + \mathbb{k}[t]\,e\,\partial$. Finally, we can then check that $\mathcal{I} \cap \mathcal{J} = \mathbb{k}[t]\,e = \mathbb{I}_1\,e$ (see Example 3.10).

The next proposition shows that every finitely generated evaluation ideal is *principal*, i.e., can be generated by a single element. This result first appears in [1, Theorem 4.5]. We give here an explicit proof.

Proposition 4.7. *Let $\mathcal{I} \subseteq \langle e\rangle$ be a finitely generated evaluation ideal. Then, $\mathcal{I}$ is principal, i.e., can be generated by a single element.*

Proof. Let $\mathcal{I} \subseteq \mathbb{I}_1$ be a finitely generated evaluation ideal. From Theorem 4.5, there is a finite number of simple evaluation $\{h_i\}_{i=1,\ldots,n}$ that generate $\mathcal{I}$ as a $\mathbb{k}[t]$-module, i.e., $\mathcal{I} = \sum_{i=1}^n \mathbb{k}[t]\,h_i$. Now, set $h = \sum_{k=1}^n \frac{t^k}{k!}\,h_k \in \mathcal{I}$. Then, using the results in the paragraph after (5), we have $e\,\partial^l\,h = \sum_{k=1}^n e\,\partial^l \left(\frac{t^k}{k!}\right) h_k = h_l$ for all $l = 1, \ldots, n$. Thus, h_l belongs to the ideal generated by h for all $l = 1, \ldots, n$, so that $\mathcal{I} = \mathbb{I}_1\,h$, which ends the proof. □

5 PERSPECTIVES

In this paper, we have first generalized a result obtained in [5] – on the explicit characterization of the annihilator of an *evaluation operator* – to the matrix case. Using this result, we have shown how to compute a finite set of generators for the intersection of two finitely generated ideals included in $\langle e \rangle$. Finally, we have effectively characterized the fact that a finitely generated ideal in $\langle e \rangle$ is semisimple. More precisely, we have explained how a finite set generators, defined by simple evaluations, can be obtained for such an ideal. It gives an explicit description of this ideal as a finitely generated $\mathbb{k}[t]$-module.

This characterization of finitely generated ideals in $\langle e \rangle$ as semisimple modules can be used to compute a finite set of generators of the intersection of two finitely generated ideals in the case where one is in $\langle e \rangle$. The main idea is to transform this problem into a simple $\mathbb{k}[t]$ problem. For lack of space, this result will be explained elsewhere.

Finally, the last case to be considered for an effective proof of the coherence of $\mathbb{I}_1$ is the case where both finitely generated ideals $\mathcal{I}$ and $\mathcal{J}$ are not in $\langle e \rangle$. The main idea of the proof given in [1] is first to determine an element $0 \neq a \in \mathcal{I} \cap J$ which is not in $\langle e \rangle$ and then use the fact that the *length* of the left $\mathbb{I}_1$-module $\mathbb{I}_1/(\mathbb{I}_1\, a)$ is finite. Such an element a can be obtained as follows. We can consider $h \in \mathcal{I}$ and $g \in \mathcal{J}$ such that neither h nor g belongs to $\langle e \rangle$. Using [5], there are $N,\, M \in \mathbb{N}$ such that $\partial^N h,\, \partial^M g \in \mathbb{A}_1 \setminus \{0\}$. Finally, using the *left Ore property* of $\mathbb{A}_1$, there are $u,\, v \in \mathbb{A}_1 \setminus \{0\}$ such that $a = u\, \partial^N h = v\, \partial^M g \in \mathcal{I} \cap \mathcal{J} \setminus \langle e \rangle$. But the use of the finite length condition of $\mathbb{I}_1/(\mathbb{I}_1\, a)$ still has to be made algorithmic.

Algorithm 1 Compute generators of $\mathcal{I} \cap \mathcal{J}$ where $\mathcal{I}, \mathcal{J} \subseteq \langle e \rangle$

Require: $p_1, \ldots, p_{n_1}$ generators of $\mathcal{I}$, $q_1, \ldots, q_{n_2}$ generators of $\mathcal{J}$
- Set $R = (p_1 \; \ldots \; p_{n_1} \; q_1 \; \ldots \; q_{n_2})^T$.
- Compute the matrix C corresponding to R.
- Compute D such that $\ker_{\mathbb{k}[t]}(.C) = \mathrm{im}_{\mathbb{k}[t]}(.D)$.
- Compute $u = (u_1,\; \ldots,\; u_r)^T = D\, J_{m+1}$, where $u_i = (u_{i,1} \;\; u_{i,2})$.

return $\{u_{1,1}\, p,\; \ldots,\; u_{n_1,1}\, p\}$

Algorithm 2 Compute simple evaluation generators of a finitely generated evaluation ideal $\mathcal{I}$ as a $\mathbb{k}[t]$-module

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Require: $a_1, \ldots, a_q$ generators of $\mathcal{I}$
- Set $A = (a_1 \; \ldots \; a_q)^T$ and compute the matrix C' defined by (6).
- Compute a full column rank matrix Q' whose columns define a basis of $\mathrm{im}_{\mathbb{k}}(C'(0).)$.
- Compute a left inverse T of Q'.
- Compute $G = (g_1 \; \ldots \; g_s)^T = T\, \left(e\, I_q \quad e\, \partial\, I_q \; \ldots \; e\, \partial^m\, I_q\right)^T A$

return $\{g_1, \ldots, g_s\}$.

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