Further results on the computation of the annihilators of integro-differential operators

Thomas Cluzeau
Univ. Limoges, CNRS, XLIM, UMR 7252, F-87000 Limoges
France
thomas.cluzeau@unilim.fr

Camille Pinto
Sorbonne University – Inria Paris
Paris, France
camille.pinto@inria.fr

Alban Quadrat
Sorbonne University – Inria Paris
Paris, France
alban.quadrat@inria.fr

ABSTRACT

This paper exposes some effective aspects of the algebra of linear ordinary integro-differential operators with polynomial coefficients. More precisely, we prove that the annihilator of an evaluation operator is a finitely generated ideal which can be explicitly characterized and computed. This is an advance towards the development of an effective elimination theory for ordinary integro-differential operators and an effective study of linear systems of integro-differential equations with polynomial coefficients.

CCS CONCEPTS

• **Computing methodologies** → **Symbolic and algebraic manipulation**; • **Symbolic and algebraic algorithms** → *Algebraic algorithms*.

KEYWORDS

Non-commutative operators, integro-differential operators, annihilator, coherence

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1 INTRODUCTION AND MOTIVATION

Algebraic analysis, more specifically, *algebraic D-module theory*, where D stands for "differential", is a mathematical field which studies linear systems of ordinary or partial differential equations using algebraic theories such as ring theory of differential operators, module theory, homological algebra [2, 6, 9]. The main idea of this theory is to use a correspondence between linear systems of differential equations and *finitely presented left modules over a ring of differential operators* (e.g., the *Weyl algebra* of differential operators with polynomial coefficients). This theory is nowadays well-known in fundamental mathematics. In the last decades, the development of an effective approach to algebraic D-module theory was studied by the computer algebra community. It relies on an effective differential elimination theory using, e.g., Gröbner or Janet basis methods. Different softwares can nowadays handle effective aspects of algebraic analysis: `Macaulay2`, `Maple` (`OreModules`), `Singular`, `HomAlg`, etc.

Based on our experience in effective algebraic D-modules theory, we aim at extending the algebraic analysis approach to handle linear systems of ordinary integro-differential equations with polynomial coefficients. In other words, we would like to replace the Weyl algebra by the ring of ordinary integro-differential operators with polynomial coefficients, denoted by $\mathbb{I}_1$ in what follows. Contrary to the Weyl algebra case, Bavula proved in [1] that $\mathbb{I}_1$ is not a noetherian ring, a fact which seems to compromise the possibility to develop an effective integro-differential elimination theory, and thus, an effective algebraic analysis approach to ordinary integro-differential linear systems. However, he also proved that $\mathbb{I}_1$ is *coherent* [1], namely, that every finitely generated left/right ideal of $\mathbb{I}_1$ is *finitely presented* [2, 18, 20]. As explained below, this result is at the core of the future development of an effective integro-differential theory. Yet, Bavula's proof of the coherence of $\mathbb{I}_1$ remains not algorithmic.

This paper aims at effectively characterizing the *annihilator* of an integro-differential operator with polynomial coefficients, i.e., of an element of $\mathbb{I}_1$. In [17], such an effective characterization was obtained and implemented for an element of $\mathbb{I}_1$ which is not a so-called *evaluation operator*. In this paper, we handle the second case, namely, the case of an element of $\mathbb{I}_1$ which is an evaluation operator. This result completes the algorithmic characterization of the first of the two standard conditions characterizing the coherence property of $\mathbb{I}_1$. The second one asserts that the intersection of two finitely generated left/right ideals is also finitely generated. This problem will be studied in a future publication.

To further motivate this work, let us explain why the development of an effective version of the coherence property of $\mathbb{I}_1$ plays a central role towards an effective study of linear systems of integro-differential equations with polynomial coefficients and towards the development of dedicated implementations built upon modern computer algebra systems. Within the algebraic analysis approach, a linear system of integro-differential equations with polynomial coefficients is defined by a matrix $R \in \mathbb{I}_1^{q\times p}$, i.e., by $R\,\eta = 0$, where $\eta \in \mathcal{F}^{p\times 1}$ and $\mathcal{F}$ is a left $\mathbb{I}_1$-module (e.g., $\Bbbk[t]$, $C^\infty(\mathbb{R})$). It can be proved that the linear integro-differential system $R\,\eta = 0$ is associated with the finitely presented left $\mathbb{I}_1$-module $\mathcal{M} = \operatorname{coker}_{\mathbb{I}_1}(.R) = \mathbb{I}_1^{1\times p}/\left(\mathbb{I}_1^{1\times q}\,R\right)$. Hence, the theory of linear integro-differential systems deals with the category of finitely presented left $\mathbb{I}_1$-modules [2, 9]. Now, a standard result in module

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theory [2, 18, 20] asserts that if $\mathcal{R}$ is a left coherent ring, then a left $\mathcal{R}$-module $\mathcal{M}$ is *coherent* (namely, $\mathcal{M}$ is a finitely generated left $\mathcal{R}$-module and all of its finitely generated left $\mathcal{R}$-submodules are finitely presented) if and only if $\mathcal{M}$ is finitely presented. Combining this result with the fact that $\mathbb{I}_1$ is coherent, we obtain that the finitely presented left $\mathbb{I}_1$-module $\mathcal{M} = \operatorname{coker}_{\mathbb{I}_1}(.R)$ – associated with the linear system $R\,\eta = 0$ – is coherent. In other words, due to the coherence property of $\mathbb{I}_1$, the linear system theory over $\mathbb{I}_1$ deals with the study of the category of left coherent $\mathbb{I}_1$-modules. Now, standard theorems on finitely generated modules over noetherian rings can be extended to finitely presented modules over coherent rings. Moreover, the coherence property is compatible with all the standard algebraic operations (e.g., direct sum, intersection, quotient, tensor product, homomorphism, kernel, image, cokernel). For more details, see [2, 18, 20]. Therefore, if the coherence property of $\mathbb{I}_1$ is made algorithmic and implemented in computer algebra systems, then the algebraic side of linear system theory over $\mathbb{I}_1$ can also be made effective. Note that, to our knowledge, $\mathbb{I}_1$ would be the first example of a coherent but not noetherian ring implemented in a computer algebra system, that has important applications (e.g., calculus).

2 THE RING OF INTEGRO-DIFFERENTIAL OPERATORS

In what follows, let $\mathbb{k}$ be an algebraically closed field of characteristic 0 (e.g., $\mathbb{k} = \overline{\mathbb{Q}}$ or $\mathbb{C}$), t_0 a base point of $\mathbb{k}$. Let us consider the noncommutative $\mathbb{k}$-endomorphism ring $\mathcal{E} = \operatorname{end}_{\mathbb{k}}(\mathbb{k}[t])$ of $\mathbb{k}[t]$ and the $\mathbb{k}$-linear endomorphisms defined on the basis $(t^n)_{n\in\mathbb{N}}$ of $\mathbb{k}[t]$ as follows:

$$\begin{aligned} t &: \mathbb{k}[t] \longrightarrow \mathbb{k}[t], \quad t^n \longmapsto t^{n+1},\\ \partial &: \mathbb{k}[t] \longrightarrow \mathbb{k}[t], \quad t^n \longmapsto n\,t^{n-1},\\ I &: \mathbb{k}[t] \longrightarrow \mathbb{k}[t], \quad t^n \longmapsto \frac{t^{n+1}}{n+1} - \frac{t_0^{n+1}}{n+1}. \end{aligned} \tag{1}$$

These $\mathbb{k}$-endomorphisms respectively define the following linear operators acting on $\mathbb{k}[t]$:

$$\begin{aligned} t &: \mathbb{k}[t] \longrightarrow \mathbb{k}[t], \quad p \longmapsto t\,p,\\ \partial &: \mathbb{k}[t] \longrightarrow \mathbb{k}[t], \quad p \longmapsto p',\\ I &: \mathbb{k}[t] \longrightarrow \mathbb{k}[t], \quad p \longmapsto \int_{t_0}^{t} p(\tau)\,d\tau. \end{aligned}$$

Definition 2.1 ([1]). $\mathbb{A}_1$ is the $\mathbb{k}$-subalgebra of $\mathcal{E}$ generated by t and ∂, and $\mathbb{I}_1$ is the $\mathbb{k}$-subalgebra of $\mathcal{E}$ generated by t, ∂, and I.

Then, $\mathbb{A}_1(\mathbb{k})$, or simply $\mathbb{A}_1$, is called the *Weyl algebra* defining the ordinary differential operators with polynomial coefficients. Similarly, $\mathbb{I}_1(\mathbb{k})$, or simply $\mathbb{I}_1$, is the *ring of ordinary integro-differential operators* in the variable t with polynomial coefficients in $\mathbb{k}[t]$.

In particular, we have the inclusion $\mathbb{A}_1 \subset \mathbb{I}_1$. *The first fundamental theorem of calculus* can be rewritten as $\partial I = 1$, where 1 stands for the identity of $\mathcal{E}$. Moreover, we can see that for every $p \in \mathbb{k}[t]$, $(1 - I\,\partial)(p) = p(t_0)$, which shows that the operator

$$e : \mathbb{k}[t] \longrightarrow \mathbb{k}[t], \quad p \longmapsto p(t_0),$$

belongs to $\mathbb{I}_1$. We shall refer to it as the *evaluation* operator. Note that e is multiplicative, i.e., $e(p_1\,p_2) = e(p_1)\,e(p_2)$, for all $p_1,\ p_2 \in \mathbb{k}[t]$. *The second fundamental theorem of calculus* then rewrites $I\,\partial = 1 - e$. In what follows, we shall simply set t_0 to 0.

Contrary to its subring $\mathbb{A}_1$, the ring $\mathbb{I}_1$ has nontrivial zero divisors since $e\,t = 0$ and $e\,I = 0$.

For $p \in \mathbb{k}[t]$, we have the following fundamental identities in $\mathbb{I}_1$:

$$\partial\,p = p\,\partial + p', \quad \partial\,I = 1, \quad I\,\partial = 1 - e, \quad e\,p = e(p)\,e = p(0)\,e.$$

See, e.g., [15]. We can deduce the following extra identities:

$$\begin{gathered} e^2 = e, \quad \partial\,e = 0, \quad I\,p\,\partial = p - e(p)\,e - I\,p',\\ I\,p\,I = I(p)\,I - I\,I(p), \quad I\,p\,e = I(p)\,e. \end{gathered}$$

Note that the first of the aforementioned identities corresponds to the *Leibniz rule* and the last but one to *the integration by parts*.

We state again that an element P of $\mathbb{I}_1$ can be written uniquely as

$$P = \sum_{i=0}^{n} a_i\,\partial^i + \sum_{j=0}^{m} b_j\,I\,c_j + \sum_{k=0}^{q} f_k\,e\,\partial^k, \tag{2}$$

where $a_i,\ b_j,\ c_j,\ f_k \in \mathbb{k}[t]$ and $n,\ m,\ q \in \mathbb{N}$. For more details, see [1, 10, 17]. The identity (2) is called the *normal form* of P.

For instance, let us give the explicit normal form of I^n. First, setting $p = 1$ in the identity $I = I(p)\,I - I\,I(p)$ and using $I(1) = t$, we obtain the identity $I^2 = t\,I - I\,t$, which corresponds to *the double integration*. More generally, we have the following explicit result on *multiple integrations* (that does not seem to appear in the literature).

Proposition 2.2. *The operator I^n can be written as a polynomial of degree* 1 *in I. More precisely, we have*

$$\forall\, n \geq 1, \quad I^n = \sum_{k=0}^{n-1} \frac{t^k}{k!}\,I\,\frac{(-t)^{n-1-k}}{(n-1-k)!}. \tag{3}$$

Proof. The operators I^n and $\sum_{k=0}^{n-1} \frac{t^k}{k!}\,I\,\frac{(-t)^{n-1-k}}{(n-1-k)!}$ are two $\mathbb{k}$-endomorphisms of $\mathbb{k}[t]$, i.e., two linear operators uniquely determined by their value on the basis $(t^n)_{n\in\mathbb{N}}$ of $\mathbb{k}[t]$. Let m be a nonnegative integer. On the one hand, we have

$$\begin{aligned} \sum_{k=0}^{n-1} \frac{t^k}{k!}\,I\,\frac{(-t)^{n-1-k}}{(n-1-k)!}\left(t^m\right) &= \sum_{k=0}^{n-1} \frac{t^k}{k!}\,\frac{(-1)^{n-1-k}}{(n-1-k)!}\,I\left(t^{n-1-k+m}\right)\\ &= \sum_{k=0}^{n-1} \frac{t^k}{k!}\,\frac{(-1)^{n-1-k}}{(n-1-k)!}\,\frac{t^{n-k+m}}{(n-k+m)}\\ &= \sum_{k=0}^{n-1} \frac{t^{n+m}}{k!\,(n-k+m)}\,\frac{(-1)^{n-1-k}}{(n-1-k)!}. \end{aligned}$$

On the other hand, we have

$$I^n\left(t^m\right) = I^{n-1}\left(\frac{t^{m+1}}{m+1}\right) = \cdots = \frac{t^{n+m}}{(m+1)(m+2)\cdots(m+n)}. \tag{4}$$

To conclude, we thus have to prove the following identity:

$$\sum_{k=0}^{n-1} \frac{(-1)^{n-1-k}}{k!\,(n-1-k)!\,(n-k+m)} = \frac{1}{(m+1)(m+2)\cdots(m+n)}. \tag{5}$$

To do so, let us note $c = \frac{1}{(m+1)\cdots(m+n)}$ and compute its partial fraction expansion, i.e., write it as $c = \frac{\alpha_1}{m+1} + \cdots + \frac{\alpha_n}{m+n}$, where $\alpha_1, \ldots, \alpha_n \in \mathbb{k}$. The k^{th} coefficient α_k of this decomposition is given by $\alpha_k = [(m+k)\,c]|_{m=-k} = \frac{1}{(-k+1)(-k+2)\cdots(-1)(n-k)!}$ so that we get $\alpha_k = \frac{1}{(-1)^{k-1}(k-1)!(n-k)!}$. Hence, we have $\alpha_{n-k} = \frac{(-1)^{n-k-1}}{k!\,(n-k-1)!}$, which proves (5) and thus (3). □

Note that Formula (3) holds for any $t_0 \in \mathbb{k}$ in the definition of I (see (1)) and not only for $t_0 = 0$.

Proposition 2.3 ([1]). *The set* $\langle e \rangle = \mathbb{I}_1\, e\, \mathbb{I}_1$ *is the only two-sided ideal of* $\mathbb{I}_1$*. Moreover, we have:*

$$\langle e \rangle = \mathbb{k}[t]\, e\, \mathbb{k}[\partial] = \left\{ \sum_{k=0}^{q} f_k\, e\, \partial^k \mid f_k \in \mathbb{k}[t],\ q \in \mathbb{N} \right\}.$$

Using (2), every $P \in \mathbb{I}_1$ can be decomposed as $P = P_1 + P_2 + P_3$, where $P_1 \in \mathbb{A}_1$, $P_2 \in \mathcal{I} = \left\{ \sum_{j=0}^{m} b_j\, I\, c_j \mid b_j,\, c_j \in \mathbb{k}[t],\ m \in \mathbb{N} \right\}$, and $P_3 \in \langle e \rangle$. Moreover, using the identity $I\, p\, I = I(p)\, I - I\, I(p)$ for all $p \in \mathbb{k}[t]$, we can see that $\mathcal{I}$ is a nonunital ring.

If $p \in \mathbb{k}[t]$, then we note

$$\mathcal{K}_{\mathbb{I}_1}(p) = \{P \in \mathbb{I}_1 \mid P(p) = 0\},$$

where $P(p)$ stands for the application of (1) to p. Note that $\mathcal{K}_{\mathbb{I}_1}(p)$ is a left ideal of $\mathbb{I}_1$. Similarly, we note $\mathcal{K}_{\mathbb{A}_1}(p) = \{P \in \mathbb{A}_1 \mid P(p) = 0\}$.

Let us state a result that will be useful in what follows.

Lemma 2.4. *Let* $P \in \mathbb{I}_1$ *and* $a = \sum_{k=0}^{l} \alpha_k\, e\, \partial^k \in \langle e \rangle$*. We have*

$$P\, a = \sum_{k=0}^{l} P(\alpha_k)\, e\, \partial^k,$$

where $P(\alpha_k) \in \mathbb{k}[t]$ *is obtained by applying* $P \in \mathcal{E}$ *to* α_k *using (1).*

Proof. Let us consider $P \in \mathbb{I}_1$. Then, there are $P_1 \in \mathbb{A}_1$, $P_2 \in \mathcal{I}$, and $P_3 \in \langle e \rangle$ such that $P = P_1 + P_2 + P_3$. By linearity and associativity properties, we only have to prove that $P\, p\, e = P(p)\, e$, for all $p \in \mathbb{k}[t]$. If $P_1 = \sum_{i=0}^{n} a_i\, \partial^i$, then, using the identity $\partial\, e = 0$, we get

$$\begin{aligned} P_1\, p\, e &= \sum_{i=0}^{n} a_i\, \partial^i\, p\, e = \sum_{i=0}^{n} a_i \sum_{j=0}^{i} \binom{i}{j} p^{(j)}\, \partial^{i-j}\, e \\ &= \sum_{i=0}^{n} a_i\, p^{(i)}\, e = P_1(p)\, e. \end{aligned}$$

If $P_2 = \sum_{j=0}^{m} b_j\, I\, c_j$, then, using the identity $I\, p\, e = I(p)\, e$, for all $p \in \mathbb{k}[t]$, we have

$$P_2\, p\, e = \sum_{j=0}^{m} b_j\, I\, c_j\, p\, e = \sum_{j=0}^{m} b_j I(c_j\, p)\, e = P_2(p)\, e.$$

If $P_3 = \sum_{k=0}^{q} f_k\, e\, \partial^k$, then, using $e\, p = e(p)\, e$, for all $p \in \mathbb{k}[t]$, and $e^2 = e$, we obtain

$$P_3\, p\, e = \sum_{k=0}^{q} f_k\, e\, \partial^k\, p\, e = \sum_{k=0}^{q} f_k\, e(\partial^k(p))\, e^2 = P_3(p)\, e.$$

Thus, we have $P\, p\, e = P(p)\, e$, for all $p \in \mathbb{k}[t]$. This ends the proof. □

We state again that the *annihilator* of $a \in \mathbb{I}_1$ is defined by

$$\mathrm{ann}_{\mathbb{I}_1}(.a) = \{P \in \mathbb{I}_1 \mid P\, a = 0\}.$$

By Lemma 2.4, the annihilator of $a = \sum_{k=0}^{q} f_k\, e\, \partial^k \in \langle e \rangle$ can be described by $\mathrm{ann}_{\mathbb{I}_1}(.a) = \{P \in \mathbb{I}_1 \mid\ P(f_k) = 0,\ k = 0, \ldots, q\}$.

It is well-known that $\mathbb{A}_1$ is a *noetherian ring*, i.e., is a left and a right noetherian ring (see, e.g., [2, 6]). As for $\mathbb{I}_1$, the situation is different. Indeed, we have seen that the identities $\partial\, I = 1$ and $I\, \partial = 1 - e$ hold in $\mathbb{I}_1$. Now, a theorem due to Jacobson [8] asserts that the existence of a left/right inverse, which is not a two-sided inverse, of an element in a noncommutative ring $\mathcal{R}$ implies that $\mathcal{R}$ is not left/right noetherian. We thus have the following result.

Proposition 2.5 ([1]). *The ring* $\mathbb{I}_1$ *is neither a left nor a right noetherian ring.*

A more explicit proof of Proposition 2.5 consists in considering the chain of left ideals generated by the *Taylor operators* defined by

$$\forall\, n \in \mathbb{N}, \quad T_n = \sum_{k=0}^{n} \frac{t^k}{k!}\, e\, \partial^k. \tag{6}$$

One can check that $T_n\, T_{n+1} = T_n$ and $\mathbb{I}_1\, T_n \neq \mathbb{I}_1\, T_{n+1}$, for all $n \in \mathbb{N}$, so that $\mathbb{I}_1\, T_0 \subsetneq \mathbb{I}_1\, T_1 \subsetneq \mathbb{I}_1\, T_2 \subsetneq \cdots$ is a strictly ascending chain of left ideals of $\mathbb{I}_1$. This proves that $\mathbb{I}_1$ is not a left noetherian ring.

Recall that an *involution* θ of a $\mathbb{k}$-algebra $\mathcal{R}$ is a $\mathbb{k}$-linear endomorphism of $\mathcal{R}$ satisfying $\theta(d_1\, d_2) = \theta(d_2)\, \theta(d_1)$, for all $d_1,\, d_2 \in \mathcal{R}$, and $\theta^2 = 1$.

Proposition 2.6 ([1]). $\mathbb{I}_1$ *admits the involution* θ *defined by*

$$\theta(t) = (t\, \partial + 1)\, \partial, \quad \theta(\partial) = I, \quad \theta(I) = \partial. \tag{7}$$

An important consequence of (7) is that $\theta(e) = e$ and $\theta(\langle e \rangle) = \langle e \rangle$. Moreover, the involution θ defined in Proposition 2.6 can be used to turn the strictly ascending chain of left ideals exhibited above into a strictly ascending chain of right ideals of $\mathbb{I}_1$. This proves that $\mathbb{I}_1$ is also not a right noetherian ring.

At first sight, the fact that $\mathbb{I}_1$ is not a noetherian ring seems to be a strong obstruction to a pure algebraic, and thus to an effective, study of $\mathbb{I}_1$. The next section explains why an effective study of linear systems of integro-differential equations with polynomial coefficients remains feasible.

3 THE COHERENCE PROPERTY OF $\mathbb{I}_1$

In this section, let $\mathcal{R}$ be a ring and $\mathcal{M}$ a left $\mathcal{R}$-module.

Definition 3.1. A left $\mathcal{R}$-module $\mathcal{M}$ is said to be *finitely generated* if there is a finite family $(g_i)_{i \in [\![1,p]\!]}$ of elements of $\mathcal{M}$ satisfying

$$\forall\, m \in \mathcal{M}, \quad \exists\, r_1, \ldots, r_p \in \mathcal{R}, \quad m = \sum_{i=1}^{p} r_i\, g_i. \tag{8}$$

$(g_i)_{i \in [\![1,p]\!]}$ is then a finite *set of generators* of the left $\mathcal{R}$-module $\mathcal{M}$.

A left $\mathcal{R}$-module $\mathcal{M}$ is finitely generated if there exist some surjective $\mathcal{R}$-homomorphism $\pi : \mathcal{R}^{1\times p} \longrightarrow \mathcal{M}$, i.e., a $\mathcal{R}$-epimorphism. Let $e_i = (0, \ldots, 1, \ldots, 0)$ be the i^{th} element of the *standard basis* of $\mathcal{R}^{1\times p}$, namely, the row vector of length p with 1 at the i^{th} position and 0 elsewhere. If $(g_i)_{i \in [\![1,p]\!]}$ is a set of generators of $\mathcal{M}$, then we can consider the following $\mathcal{R}$-epimorphism:

$$\pi : \mathcal{R}^{1\times p} \longrightarrow \mathcal{M}, \quad e_i \longmapsto g_i,\ i = 1, \ldots, p.$$

Definition 3.2. Let $\mathcal{M}$ be a finitely generated left $\mathcal{R}$-module and $(g_i)_{i \in [\![1,p]\!]}$ a finite set of generators of $\mathcal{M}$. Then, $\mathcal{M}$ is said to be *finitely presented* if the left $\mathcal{R}$-module of all the relations among the g_i's, namely,

$$\ker \pi = \left\{ (\lambda_1, \ldots, \lambda_p) \in \mathcal{R}^{1\times p} \mid \sum_{i=1}^{p} \lambda_i \, g_i = 0 \right\}$$

is finitely generated.

By definition, a finitely presented module is finitely generated.

The fact that $\ker \pi$ is finitely generated is equivalent to the existence of a finite set of generators of $\ker \pi$, i.e., a finite set of elements $R_{1\bullet}, \ldots, R_{q\bullet} \in \mathcal{R}^{1\times p}$ satisfying that, for all $\lambda \in \ker \pi$, there are $\mu_1, \ldots, \mu_q \in \mathcal{R}$ such that

$$\lambda = \sum_{i=1}^{q} \mu_i \, R_{i\bullet} = \underbrace{\begin{pmatrix} \mu_1 & \ldots & \mu_q \end{pmatrix}}_{\mu} \underbrace{\begin{pmatrix} \text{———} R_{1\bullet} \text{———} \\ \vdots \\ \text{———} R_{q\bullet} \text{———} \end{pmatrix}}_{R} = \mu \, R.$$

Thus, we can write $\ker \pi = \operatorname{im}_{\mathcal{R}}(\cdot R) = \{\mu \, R \mid \mu \in \mathcal{R}^{1\times q}\}$, where $R \in \mathcal{R}^{q\times p}$ is the matrix having the $R_{i\bullet}$'s as rows, which is equivalent to the following *exact sequence* of left $\mathcal{R}$-modules:

$$\mathcal{R}^{1\times q} \xrightarrow{\cdot R} \mathcal{R}^{1\times p} \xrightarrow{\pi} \mathcal{M} \longrightarrow 0.$$

Equivalently, we have $\mathcal{M} \cong \operatorname{coker}_{\mathcal{R}}(.R) = \mathcal{R}^{1\times p}/(\mathcal{R}^{1\times q} R)$, where $\cong$ means "isomorphic to". For more details, see [2, 18, 20].

Definition 3.3. Let $\mathcal{R}$ be a noncommutative ring.

- A left $\mathcal{R}$-module $\mathcal{M}$ is said to be *left coherent* if $\mathcal{M}$ is a finitely generated left $\mathcal{R}$-module and if every finitely generated left $\mathcal{R}$-submodule of $\mathcal{M}$ is finitely presented.
- The ring $\mathcal{R}$ is said to be *left coherent* if $\mathcal{R}$ is a left coherent $\mathcal{R}$-module, i.e., if every finitely generated left ideal of $\mathcal{R}$ is finitely presented.

Similar definitions hold for right $\mathcal{R}$-modules and a ring is said to be *coherent* if it is both left and right coherent.

According to Definitions 3.3 and 3.2, a ring $\mathcal{R}$ is left coherent if for every finitely generated ideal $\mathcal{J}$ of $\mathcal{R}$, the left $\mathcal{R}$-module of the relations among a finite set of generators of $\mathcal{J}$ is finitely generated.

Example 3.4. Left (resp., right) noetherian rings are left (resp., right) coherent rings. Two examples of coherent rings which are not noetherian are the ring $\Bbbk[x_i \mid i \in \mathbb{N}]$ of polynomials in an infinite number of variables $\{x_i\}_{i\in\mathbb{N}}$ with coefficients in a field $\Bbbk$, and the ring of the entire functions on $\mathbb{C}$. For more details, see [18].

Let us state a useful characterization of a coherent ring.

Proposition 3.5 ([18, 20]). *Let $\mathcal{R}$ be a ring. The following two conditions are equivalent:*

(1) *$\mathcal{R}$ is a left coherent ring.*
(2)(a) *For every $a \in \mathcal{R}$, $\operatorname{ann}_{\mathcal{R}}(.a)$ is a finitely generated left ideal.*
(b) *For all finitely generated left ideals $\mathcal{J}_1$ and $\mathcal{J}_2$, the left ideal $\mathcal{J}_1 \cap \mathcal{J}_2$ is finitely generated.*

A similar result holds for a right coherent ring ($\operatorname{ann}_{\mathcal{R}}(.a)$ is then replaced by $\operatorname{ann}_{\mathcal{R}}(a.)$ and left ideals by right ideals).

We can now state a result that is at the core of this paper.

Theorem 3.6 ([1]). *$\mathbb{I}_1$ is a coherent ring.*

Using the involution θ of $\mathbb{I}_1$ (see Proposition 2.6), the left coherence property yields the right coherence property, and vice versa.

We point out that the proof of Theorem 3.6 given in [1] is not constructive. The main goal of the present paper is to contribute to the development of an effective version of the coherence property of $\mathbb{I}_1$ and its implementation in the computer algebra software `Maple`. Here, we shall focus on Condition (2)(a) of Proposition 3.5. The second condition will be studied in a future work.

Note that, to our knowledge, $\mathbb{I}_1$ would be the first example of a coherent but not noetherian ring implemented in a computer algebra system, which has important applications (e.g., calculus).

4 ANNIHILATOR OF AN EVALUATION OPERATOR

4.1 Preliminary remarks and results

In Section 3, we have recalled that the ring $\mathbb{I}_1$ was coherent. Yet, the proof of the coherence property given in [1] remains not algorithmic. To make it so, we shall rely on Proposition 3.5 which shows that the coherence property is equivalent to two conditions: one on the annihilator of elements of $\mathbb{I}_1$ and one on the intersection of finitely generated ideals. In this paper, we shall only focus on the first one, letting the second one for a future work.

Let us then consider Condition (2) (a) of Proposition 3.5. In the case $a \in \mathbb{I}_1 \setminus \langle e \rangle$, the characterization of a finite set of generators for $\operatorname{ann}_{\mathbb{I}_1}(.a)$ was obtained in [17] and implemented in the `IntDiffOp` package [10]. Hence, it remains to study the case $a \in \langle e \rangle$.

Note that $\operatorname{ann}_{\mathbb{I}_1}(.a)$ is the left ideal of $\mathbb{I}_1$ defining all the *compatibility conditions* of the inhomogeneous linear equation $a\,h = g$, where g is fixed in a left $\mathbb{I}_1$-module $\mathcal{F}$ and h is sought in $\mathcal{F}$. Indeed, if $P \in \operatorname{ann}_{\mathbb{I}_1}(.a)$, then by definition, $P\,a = 0$, i.e., $P\,g = 0$. This last equation is a necessary condition for the above system to have a solution, i.e., $P\,g = 0$ is a compatibility condition.

We now state two lemmas that will be used in what follows.

Lemma 4.1 ([4]). *Let $P \in \mathbb{I}_1$. Then, there is $N \in \mathbb{N}$ such that the operator $\partial^N P$ belongs to $\mathbb{A}_1$. In particular, using the normal form* (2) *of P, we can take $N = \max_{(j,k)\in[\![0,m]\!]\times[\![0,q]\!]}\{\deg_t(b_j), \deg_t(f_k)\}+1$.*

Lemma 4.2. *For all $n \in \mathbb{N}$, we have $I^n \partial^n + T_{n-1} = 1$, where T_n is defined by (6). For every $P \in \mathbb{I}_1$, there is $N \in \mathbb{N}$ such that P can be written as $P = I^N \partial^N P + T_{N-1} P$, where $\partial^N P \in \mathbb{A}_1$ and $T_{N-1} P \in \langle e \rangle$.*

Proof. Using (1), let us compare $(I^n \partial^n)\left(t^l\right)$ and $(1 - T_{n-1})\left(t^l\right)$, for all $l \geq 0$. If $l \leq n-1$, we have

$$(1 - T_{n-1})\left(t^l\right) = 0, \quad (I^n \partial^n)\left(t^l\right) = I^n(0) = 0.$$

Now, for $l > n-1$, we have

$$(1 - T_{n-1})\left(t^l\right) = t^l, \quad (I^n \partial^n)\left(t^l\right) = l \, \ldots \, (l-n+1) \, I^n\left(t^{l-n}\right).$$

Using (4), we obtain

$$I^n\left(t^{l-n}\right) = \frac{t^l}{(l-n+1)\,(l-n+2)\ldots l}.$$

Then, we have

$$(I^n \partial^n)\left(t^l\right) = l \, \ldots \, (l-n+1) \, I^n\left(t^{l-n}\right) = t^l.$$

The second assertion is a direct consequence of the first one, Lemma 4.1, and Proposition 2.3. □

Using Proposition 2.2, for all $n \geq 1$ and $p \in \mathbb{k}[t]$, we have

$$I^n(p) = \sum_{k=0}^{n-1} \frac{t^k}{k!} \int_0^t \frac{(-\tau)^{n-1-k}}{(n-1-k)!}\, p(\tau)\, d\tau = \int_0^t \frac{(t-\tau)^{n-1}}{(n-1)!}\, p(\tau)\, d\tau.$$

Hence, the identity $I^n \partial^n + T_{n-1} = 1$ of Lemma 4.2 yields

$$\begin{aligned} \forall\, p \in \mathbb{k}[t], \quad p &= T_{n-1}(p) + \left(I^n \partial^n\right)(p) \\ &= \sum_{k=0}^{n-1} \frac{t^k}{k!}\, p^{(k)}(0) + \int_0^t \frac{(t-\tau)^{n-1}}{(n-1)!}\, p^{(n)}(\tau)\, d\tau, \end{aligned}$$

for all $n \geq 1$. This shows that Lemma 4.2 encapsulates the *Taylor's theorem with an integral form of the remainder* into a simple identity.

4.2 Annihilator of a simple evaluation operator

Let $p \in \mathbb{k}[t]$ and let us exhibit a generating set of the left $\mathbb{I}_1$-ideal $\mathrm{ann}_{\mathbb{I}_1}(.p\, e)$. From Lemma 2.4, we have

$$\mathrm{ann}_{\mathbb{I}_1}(.p\, e) = \{P \in \mathbb{I}_1 \mid P\, p\, e = P(p)\, e = 0\} = \{P \in \mathbb{I}_1 \mid P(p) = 0\}.$$

According to [5, Proposition 3.2], $\mathcal{K}_{\mathbb{A}_1}(p) := \{Q \in \mathbb{A}_1 \mid Q(p) = 0\}$ is the left ideal of $\mathbb{A}_1$ generated by $Q_1 := \partial^{m+1}$ and $Q_2 := p\, \partial^m - p^{(m)}$. In other words, we have $\mathcal{K}_{\mathbb{A}_1}(p) = \mathbb{A}_1\, Q_1 + \mathbb{A}_1\, Q_2$.

Lemma 4.3. *Let $p \in \mathbb{k}[t]$ be a polynomial of degree m, $Q_1 = \partial^{m+1}$, and $Q_2 = p\, \partial^m - p^{(m)}$. Then, we have*

$$\mathrm{ann}_{\mathbb{I}_1}(.p\, e) \cap \langle e \rangle \subseteq \mathbb{I}_1\, Q_1 + \mathbb{I}_1\, Q_2.$$

Proof. If $P \in \mathrm{ann}_{\mathbb{I}_1}(.p\, e) \cap \langle e \rangle$, then, from Lemma 2.4, we have $P = \sum_{k=0}^{q} f_k\, e\, \partial^k$, where $f_k \in \mathbb{k}[t]$, $k = 0, \ldots, q$, and P satisfies $P(p) = 0$. If $q > m$, then we can write

$$P = \sum_{k=0}^{m} f_k\, e\, \partial^k + \left(\sum_{k=m+1}^{q} f_k\, e\, \partial^{k-m-1} \right) Q_1.$$

Thus, $P \in \mathbb{I}_1\, Q_1 + \mathbb{I}_1\, Q_2$ if and only if $\overline{P} = \sum_{k=0}^{m} f_k\, e\, \partial^k \in \mathbb{I}_1\, Q_1 + \mathbb{I}_1\, Q_2$. Since $P(p) = 0$, then $\overline{P}$ satisfies $\overline{P}(p) = 0$. Then, we have:

$$\overline{P}(p) = 0 \;\Leftrightarrow\; \sum_{k=0}^{m} f_k\, e\, \partial^k(p) = 0 \;\Leftrightarrow\; \sum_{k=0}^{m} f_k\, p^{(k)}(0) = 0.$$

Since $\deg_t(p)$ is exactly m, we get $p^{(m)} = p^{(m)}(0) \neq 0$. Therefore, we obtain the identity $f_m = -\frac{1}{p^{(m)}} \sum_{k=0}^{m-1} f_k\, p^{(k)}(0)$. Substituting this expression into $\overline{P}$ and using $p^{(k)}(0)\, e = e(p^{(k)})\, e = e\, p^{(k)}$, we obtain

$$\begin{aligned} \overline{P} &= \sum_{k=0}^{m-1} \left(f_k\, e\, \partial^k - \frac{1}{p^{(m)}}\, f_k\, p^{(k)}(0)\, e\, \partial^m \right), \\ &= \sum_{k=0}^{m-1} \left(f_k\, e\, \partial^k - \frac{1}{p^{(m)}}\, f_k\, e\, p^{(k)} \partial^m \right), \\ &= \sum_{k=0}^{m-1} f_k\, e\, \frac{1}{p^{(m)}} \left(p^{(m)}\, \partial^k - p^{(k)}\, \partial^m \right). \end{aligned}$$

Therefore, $\overline{P}(p) = 0$ yields $p^{(m)}\, \partial^k - p^{(k)}\, \partial^m \in \mathcal{K}_{\mathbb{A}_1}(p)$, for $k \in [\![0, m-1]\!]$. From [5, Proposition 3.2], we get $\mathcal{K}_{\mathbb{A}_1}(p) = \mathbb{A}_1\, Q_1 + \mathbb{A}_1\, Q_2$, which shows that the term $f_k\, e\, \frac{1}{p^{(m)}} \left(p^{(m)}\, \partial^k - p^{(k)}\, \partial^m \right)$ belongs to $\mathbb{I}_1\, Q_1 + \mathbb{I}_1\, Q_2$, for $k = 0, \ldots, m-1$. This proves $\overline{P} \in \mathbb{I}_1\, Q_1 + \mathbb{I}_1\, Q_2$. □

Proposition 4.4. *Let $p \in \mathbb{k}[t]$ be of degree m, $Q_1 = \partial^{m+1}$, and $Q_2 = p\, \partial^m - p^{(m)}$. Then, we have*

$$\mathrm{ann}_{\mathbb{I}_1}(.p\, e) = \mathbb{I}_1\, Q_1 + \mathbb{I}_1\, Q_2.$$

Proof. Let $P \in \mathrm{ann}_{\mathbb{I}_1}(.p\, e)$. From Lemma 2.4, we have $P(p) = 0$. According to Lemma 4.1, there is $N \in \mathbb{N}$ such that $\partial^N P \in \mathbb{A}_1$. Thus, we have $\partial^N P \in \mathcal{K}_{\mathbb{A}_1}(p)$. Since $\mathcal{K}_{\mathbb{A}_1}(p) = \mathbb{A}_1\, Q_1 + \mathbb{A}_1\, Q_2$, there is $(\alpha, \beta) \in \mathbb{A}_1^2$ such that $\partial^N P = \alpha\, Q_1 + \beta\, Q_2$, which yields $I^N\, \partial^N P = (I^N\, \alpha)\, Q_1 + (I^N\, \beta)\, Q_2$. Using the identity of Lemma 4.2, i.e., $I^N \partial^N + T_{N-1} = 1$, we get $T_{N-1}\, P = P - (I^N\, \alpha)\, Q_1 - (I^N\, \beta)\, Q_2 \in \mathrm{ann}_{\mathbb{I}_1}(.p\, e) \cap \langle e \rangle$ because P, Q_1, $Q_2 \in \mathrm{ann}_{\mathbb{I}_1}(.p\, e)$ and $T_{N-1}\, P \in \langle e \rangle$ since $T_{N-1} \in \langle e \rangle$ and $\langle e \rangle$ is a right ideal of $\mathbb{I}_1$ (by Proposition 2.3). Using Lemma 4.3, $T_{N-1}\, P \in \mathbb{I}_1\, Q_1 + \mathbb{I}_1\, Q_2$ and so is $P = T_{N-1}\, P + (I^N\, \alpha)\, Q_1 + (I^N\, \beta)\, Q_2$, which finally proves the result since we clearly have $\mathbb{I}_1\, Q_1 + \mathbb{I}_1\, Q_2 \subseteq \mathrm{ann}_{\mathbb{I}_1}(.p\, e)$. □

4.3 Main results

Let us now extend Proposition 4.4 to a general element $a \in \langle e \rangle$. In what follows, for $R \in \mathbb{k}[t]^{q \times p}$, we use the following $\mathbb{k}[t]$-modules:

$$\begin{aligned} \ker_{\mathbb{k}[t]}(.R) &= \left\{ \lambda \in \mathbb{k}[t]^{1 \times q} \mid \lambda\, R = 0 \right\}, \\ \mathrm{im}_{\mathbb{k}[t]}(.R) &= \left\{ \lambda\, R \mid \lambda \in \mathbb{k}[t]^{1 \times q} \right\}. \end{aligned}$$

We shall need the next lemma which characterizes the differential operators which annihilate a finite family of polynomials $\{a_i\}_{i=0,\ldots,r}$.

Lemma 4.5. *Let $a_i \in \mathbb{k}[t]$, $i = 0, \ldots, r$, $m = \max_{i \in [\![0, r]\!]}\{\deg_t(a_i)\}$,*

$$C = \begin{pmatrix} a_0 & \ldots & a_r \\ \vdots & & \vdots \\ a_0^{(m+1)} & \ldots & a_r^{(m+1)} \end{pmatrix} \in \mathbb{k}[t]^{(m+2) \times (r+1)}, \quad J_{m+1} = \begin{pmatrix} 1 \\ \partial \\ \vdots \\ \partial^{m+1} \end{pmatrix}. \tag{9}$$

Let $D \in \mathbb{k}[t]^{u \times (m+2)}$ be a full row rank matrix, where $u \in [\![1, m+2]\!]$ is the rank of D such that

$$\ker_{\mathbb{k}[t]}(.C) = \mathrm{im}_{\mathbb{k}[t]}(.D),$$

and the vector

$$\begin{pmatrix} f_1 & \ldots & f_u \end{pmatrix}^T = D\, J_{m+1} \in \mathbb{A}_1^{u \times 1}.$$

Then, we have

$$\bigcap_{i=0}^{r} \mathcal{K}_{\mathbb{A}_1}(a_i) = \sum_{j=1}^{u} \mathbb{A}_1\, f_j.$$

Proof. Since $\partial^{m+1}(a_i) = 0$, for $i = 0, \ldots, r$, $\partial^{m+1} \in \bigcap_{i=0}^{r} \mathcal{K}_{\mathbb{A}_1}(a_i)$. Hence, if $P = \sum_{j=0}^{n} c_j\, \partial^j \in \bigcap_{i=0}^{r} \mathcal{K}_{\mathbb{A}_1}(a_i)$, where $n > m+1$ and $c_j \in \mathbb{k}[t]$, for $j = 0, \ldots, n$, then we can write

$$P = \sum_{j=0}^{m} c_j\, \partial^j + \left(\sum_{j=m+1}^{n} c_j\, \partial^{j-m-1} \right) \partial^{m+1},$$

and we get $\overline{P} = \sum_{j=0}^{m} c_j\, \partial^j \in \bigcap_{j=0}^{r} \mathcal{K}_{\mathbb{A}_1}(a_i)$. To simplify the exposition below, we keep the term of order $m+1$ in $\overline{P}$. We then have:

$$\forall i \in [\![0, r]\!],\ \overline{P}(a_i) = \sum_{j=0}^{m+1} c_j\, a_i^{(j)} = 0 \Leftrightarrow (c_0\ \dots\ c_{m+1})\, C = (0\ \dots\ 0),$$

where the matrix $C \in \mathbb{k}[t]^{(m+2)\times(r+1)}$ is defined by (9). This shows that $(c_0\ \dots\ c_{m+1}) \in \ker_{\mathbb{k}[t]}(.C)$. Now, since $\mathbb{k}[t]$ is a noetherian ring, the $\mathbb{k}[t]$-module $\ker_{\mathbb{k}[t]}(.C)$ is finitely generated (see, e.g., [7]), and thus, there is a finite set of generators for $\ker_{\mathbb{k}[t]}(.C)$. Stacking the corresponding polynomial rows into a matrix, we obtain a matrix $D \in \mathbb{k}[t]^{u\times(m+2)}$ such that $\ker_{\mathbb{k}[t]}(.C) = \operatorname{im}_{\mathbb{k}[t]}(.D)$. Note that $\ker_{\mathbb{k}[t]}(.C)$ is a $\mathbb{k}[t]$-submodule of the free $\mathbb{k}[t]$-module $\mathbb{k}[t]^{1\times(m+2)}$, and thus, $\ker_{\mathbb{k}[t]}(.C)$ is a free $\mathbb{k}[t]$-module because $\mathbb{k}[t]$ is a principal ideal domain (see, e.g., [18]). Hence, the rows of the matrix D can be chosen so that they are $\mathbb{k}[t]$-linearly independent, i.e., such that the matrix D has *full row rank* (i.e., $\nu\, D = 0$ yields $\nu = 0$). Therefore, we have the following exact sequence of $\mathbb{k}[t]$-modules:

$$0 \longrightarrow \mathbb{k}[t]^{1\times u} \xrightarrow{.D} \mathbb{k}[t]^{1\times(m+2)} \xrightarrow{.C} \mathbb{k}[t]^{1\times(r+1)}. \quad (10)$$

Thus, $(c_0\ \dots\ c_{m+1}) = \alpha\, D$ for a certain $\alpha \in \mathbb{k}[t]^{1\times u}$. Let us consider $(f_1\ \dots\ f_u)^T := D\, J_{m+1}$. Using $\overline{P} = (c_0\ \dots\ c_{m+1})\, J_{m+1}$, we then get $\overline{P} = \alpha\, D\, J_{m+1} = \sum_{j=1}^{u} \alpha_j\, f_j$, i.e., $\overline{P} \in \sum_{j=1}^{u} \mathbb{A}_1\, f_j$. Now $(0\ \dots\ 0\ 1) \in \operatorname{im}_{\mathbb{k}[t]}(.D)$ since the last row of C has zero entries. This yields $(0\ \dots\ 0\ 1)\, J_{m+1} = \partial^{m+1} \in \sum_{j=1}^{u} \mathbb{k}[t]\, f_j$. Hence, we have $P \in \overline{P} + \mathbb{A}_1\, \partial^{m+1} \subset \sum_{j=1}^{u} \mathbb{A}_1\, f_j$, which proves the first inclusion.

For the converse, we note that each entry of the column vector $D\, J_{m+1}$ is an element of $\mathbb{A}_1$ which annihilates all the a_i's because the rows of D belong to $\ker_{\mathbb{k}[t]}(.C)$. Hence, we obtain that $(f_1\ \dots\ f_u)^T = D\, J_{m+1}$ is a generating set of $\bigcap_{i=0}^{r} \mathcal{K}_{\mathbb{A}_1}(a_i)$. □

Note that the generators f_j's of Lemma 4.5 can be explicitly obtained by the computation of a *Hermite normal form* (or a *Smith normal form*) of the matrix C. For more details, see, e.g., [11]. Such a computation is implemented in computer algebra softwares.

Example 4.6. Let us consider $a_0 = t$, $a_1 = t^2 + 1$, and $a_2 = t^3 - 2\,t$. Then, we have $m = \max_{i \in [\![0,2]\!]}\{\deg_t(a_i)\} = 3$ and

$$C = \begin{pmatrix} t & 1 & 0 & 0 & 0 \\ t^2+1 & 2\,t & 2 & 0 & 0 \\ t^3 - 2\,t & 3\,t^2 - 2 & 6\,t & 6 & 0 \end{pmatrix}^T \in \mathbb{k}[t]^{5\times 3}.$$

Using, e.g., `Maple`, we can compute a matrix D whose rows generate the $\mathbb{k}[t]$-module $\ker_{\mathbb{k}[t]}(.C)$. We find

$$D = \begin{pmatrix} -6 & 6\,t & -3\,t^2+3 & t^3 - 3\,t & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Thus, two generators f_1 and f_2 of $\bigcap_{i=0}^{2} \mathcal{K}_{\mathbb{A}_1}(a_i)$ are defined by

$$\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = D\, J_4 = \begin{pmatrix} -6 + 6\,t\,\partial + (-3\,t^2+3)\,\partial^2 + (t^3 - 3\,t)\,\partial^3 \\ \partial^4 \end{pmatrix},$$

i.e., we have $\bigcap_{i=0}^{2} \mathcal{K}_{\mathbb{A}_1}(a_i) = \mathbb{A}_1\, f_1 + \mathbb{A}_1\, f_2$.

Using Lemma 4.5, the next proposition characterizes a finite set of generators of the left ideal $\operatorname{ann}_{\mathbb{I}_1}(.a) \bigcap \langle e\rangle$ of $\mathbb{I}_1$, where $a \in \langle e\rangle$.

Proposition 4.7. *Let $a = \sum_{i=0}^{r} a_i\, e\, \partial^i \in \langle e\rangle$, $a_i \in \mathbb{k}[t]$, for $k = 0, \dots, r$, $m = \max_{i\in[\![0,r]\!]}\{\deg_t(a_i)\}$, and $C \in \mathbb{k}[t]^{(m+2)\times(r+1)}$ and J_{m+1} be the matrices defined by (9). If $E \in \mathbb{k}^{v\times(m+2)}$ is a full row rank matrix such that $\ker_{\mathbb{k}}(.e(C)) = \operatorname{im}_{\mathbb{k}}(.E)$ (where $e(C)$ is the evalutation of C at $t = 0$), then we have*

$$\operatorname{ann}_{\mathbb{I}_1}(.a) \bigcap \langle e\rangle = \sum_{k=1}^{v} \mathbb{I}_1\, g_k,$$

where the g_k's are defined by

$$\begin{pmatrix} g_1 & \dots & g_v \end{pmatrix}^T = E\, e\, J_{m+1} \in \langle e\rangle^{v\times 1}. \quad (11)$$

Proof. Let $P \in \operatorname{ann}_{\mathbb{I}_1}(.a) \bigcap \langle e\rangle$. We have $P = \sum_{j=0}^{l} \alpha_j\, e\, \partial^j$ and $P\,a = 0$. Using Lemma 2.4, $P\,a = \sum_{i=0}^{r} P(a_i)\, e\, \partial^i = 0$ if and only if $P(a_i) = 0$, for $i = 0, \dots, r$. As a consequence, $\operatorname{ann}_{\mathbb{I}_1}(.a) \bigcap \langle e\rangle = \left(\bigcap_{i=0}^{r} \mathcal{K}_{\mathbb{I}_1}(a_i)\right) \bigcap \langle e\rangle$. Now, for $i = 0, \dots, r$, if we note $m_i = \deg_t(a_i)$, then we have $\partial^{m_i+1} \in \mathcal{K}_{\mathbb{I}_1}(a_i)$. Hence, we get $\partial^{m+1} \in \bigcap_{i=0}^{r} \mathcal{K}_{\mathbb{I}_1}(a_i)$. Consequently, $e\,\partial^{m+1} \in \operatorname{ann}_{\mathbb{I}_1}(.a) \bigcap \langle e\rangle$. Therefore, we can, without loss of generality, assume $l = m+1$, i.e., $P = \sum_{j=0}^{m+1} \alpha_j\, e\, \partial^j \in \operatorname{ann}_{\mathbb{I}_1}(.a)$. We then have

$$\begin{aligned} P\,a = 0 &\Longleftrightarrow P(a_i) = 0, \quad i = 0, \dots, r, \\ &\Longleftrightarrow P(a_i) = \sum_{j=0}^{m+1} \alpha_j\, e(\partial^j a_i) = 0, \quad i = 0, \dots, r, \\ &\Longleftrightarrow (\alpha_0\ \dots\ \alpha_{m+1})\, e(C) = (0\ \dots\ 0), \\ &\Longleftrightarrow (\alpha_0\ \dots\ \alpha_{m+1}) \in \ker_{\mathbb{k}[t]}(.e(C)), \end{aligned}$$

where the matrix C is defined by (9). Let $E \in \mathbb{k}^{v\times(m+2)}$ be a full row rank matrix whose rows define a basis of $\ker_{\mathbb{k}}(.e(C))$, where $v = \dim_{\mathbb{k}} \ker_{\mathbb{k}}(.e(C))$. Then, we have the following exact sequence of $\mathbb{k}$-vector spaces:

$$0 \longrightarrow \mathbb{k}^{1\times v} \xrightarrow{.E} \mathbb{k}^{1\times(m+2)} \xrightarrow{.e(C)} \mathbb{k}^{1\times(r+1)}.$$

The above exact sequence of $\mathbb{k}$-vector spaces *splits* (see, e.g., [18]). Thus, taking the tensor product of this exact sequence by the free $\mathbb{k}$-module $\mathbb{k}[t]$ yields the following exact sequence of $\mathbb{k}[t]$-modules (see, e.g., [18]):

$$0 \longrightarrow \mathbb{k}[t]^{1\times v} \xrightarrow{.E} \mathbb{k}[t]^{1\times(m+2)} \xrightarrow{.e(C)} \mathbb{k}[t]^{1\times(r+1)}, \quad (12)$$

i.e., $\ker_{\mathbb{k}[t]}(.e(C)) = \operatorname{im}_{\mathbb{k}[t]}(.E)$. From the exactness of (12), we can deduce that $P\,a = 0$ if and only if $(\alpha_0\ \dots\ \alpha_{m+1}) \in \operatorname{im}_{\mathbb{k}[t]}(.E)$, i.e., if and only if there is $\nu \in \mathbb{k}[t]^{1\times v}$ such that $(\alpha_0\ \dots\ \alpha_{m+1}) = \nu\, E$. Hence, P can be written $P = \nu\, E\, e\, J_{m+1}$. Therefore, $P\,a = 0$ if and only if $P \in \mathbb{I}_1^{1\times v}\, E\, e\, J_{m+1}$. Now, using the standard basis of $\mathbb{k}[t]^{1\times v}$, we obtain that the g_k's defined by (11) generate the left ideal $\mathbb{I}_1^{1\times v}\, E\, e\, J_{m+1}$ of $\mathbb{I}_1$. The last row of the matrix $e(C)$ is the zero row. Thus, we have $(0\ \dots\ 0\ 1) \in \ker_{\mathbb{k}}(.e(C))$, which implies that $(0\ \dots\ 0\ 1)\, e\, J_{m+1} = e\,\partial^{m+1} \in \sum_{k=1}^{v} \mathbb{I}_1\, g_k$. This proves that (11) is a set of generators of $\operatorname{ann}_{\mathbb{I}_1}(.a) \bigcap \langle e\rangle$, i.e., $\operatorname{ann}_{\mathbb{I}_1}(.a) \bigcap \langle e\rangle = \sum_{k=1}^{v} \mathbb{I}_1\, g_k$. □

Example 4.8. We continue Example 4.6. The evaluation of the matrix C defined in Example 4.6 at $t = 0$ is

$$e(C) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 & 0 \\ 0 & -2 & 0 & 6 & 0 \end{pmatrix}^T,$$

and $\ker_{\mathbb{k}}(.e(C))$ is generated by the rows of the matrix

$$E = \begin{pmatrix} 2 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Hence, a set of generators of $\mathrm{ann}_{\mathbb{I}_1}(.a) \bigcap \langle e \rangle$ is defined by

$$\begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = E\, e\, J_4 = \begin{pmatrix} 2\,e - e\,\partial^2 \\ e\,\partial^4 \end{pmatrix},$$

i.e., $\mathrm{ann}_{\mathbb{I}_1}(.a) \bigcap \langle e \rangle = \left\{ \alpha_1\,(2\,e - e\,\partial^2) + \alpha_2\, e\,\partial^4 \mid \alpha_1,\, \alpha_2 \in \mathbb{I}_1 \right\}$.

We can now state the main result of the present article.

Theorem 4.9. *Let $a = \sum_{i=0}^{r} a_i\, e\, \partial^i \in \langle e \rangle$, where $a_i \in \mathbb{k}[t]$ for $i = 0, \ldots, r$. Then, we have*

$$\mathrm{ann}_{\mathbb{I}_1}(.a) = \sum_{j=1}^{u} \mathbb{I}_1\, f_j + \sum_{k=1}^{v} \mathbb{I}_1\, g_k,$$

where the f_j's (resp., g_k's) are the differential (resp., evaluation) operators defined in Lemma 4.5 (resp., Proposition 4.7). In particular, the left ideal $\mathrm{ann}_{\mathbb{I}_1}(.a)$ of $\mathbb{I}_1$ is finitely generated.

Proof. Let us consider $a = \sum_{i=0}^{r} a_i\, e\, \partial^i$ and $P \in \mathrm{ann}_{\mathbb{I}_1}(.a)$. From Lemma 4.1, there is $N \in \mathbb{N}$ such that $\partial^N P \in \mathbb{A}_1$. Then, $(\partial^N P)\, a = \partial^N (P\, a) = 0$, i.e., $\sum_{i=0}^{r} (\partial^N P)(a_i)\, e\, \partial^i = 0$, and thus, according to Lemma 2.4, $(\partial^N P)(a_i) = 0$, for $i = 0, \ldots, r$. Therefore $\partial^N P \in \bigcap_{i=0}^{r} \mathcal{K}_{\mathbb{A}_1}(a_i) = \sum_{j=1}^{u} \mathbb{A}_1\, f_j$ (see Lemma 4.5). Using Lemma 4.2, P can be written as $P = T_{N-1}\, P + S$, where $S = I^N\, \partial^N\, P \in \sum_{j=1}^{u} \mathbb{I}_1\, f_j$. Now, note that $T_{N-1}\, P \in \mathrm{ann}_{\mathbb{I}_1}(.a) \bigcap \langle e \rangle$. Thus, by Proposition 4.7, $T_{N-1}\, P \in \sum_{k=1}^{v} \mathbb{I}_1\, g_k$, which shows that $P \in \sum_{j=1}^{u} \mathbb{I}_1\, f_j + \sum_{k=1}^{v} \mathbb{I}_1\, g_k$, and $\mathrm{ann}_{\mathbb{I}_1}(.a) \subseteq \sum_{j=1}^{u} \mathbb{I}_1\, f_j + \sum_{k=1}^{v} \mathbb{I}_1\, g_k$. Finally, by construction, we have $\sum_{j=1}^{u} \mathbb{I}_1\, f_j + \sum_{k=1}^{v} \mathbb{I}_1\, g_k \subseteq \mathrm{ann}_{\mathbb{I}_1}(.a)$, which ends the proof. □

Example 4.10. Applying Theorem 4.9 to $a \in \langle e \rangle$ defined in Example 4.8 and using the results obtained in Examples 4.6 and 4.8, we obtain $\mathrm{ann}_{\mathbb{I}_1}(.a) = \mathbb{I}_1\, f_1 + \mathbb{I}_1\, f_2 + \mathbb{I}_1\, g_1 + \mathbb{I}_1\, g_2$, where f_1, f_2, g_1, and g_2 are defined in Examples 4.6 and 4.8.

In the case $r = 0$, Proposition 4.4 shows that $\mathrm{ann}_{\mathbb{I}_1}(.a)$ can be generated by two differential operators. Thus, we can wonder if the same result holds for the general case, i.e., for all $r \in \mathbb{N}$. Comparing the exact sequences (10) and (12), we note that $\ker_{\mathbb{k}[t]}(.C) = \mathrm{im}_{\mathbb{k}[t]}(.D)$ and $\ker_{\mathbb{k}[t]}(.e(C)) = \mathrm{im}_{\mathbb{k}[t]}(.E)$, where $D \in \mathbb{k}[t]^{u \times (m+2)}$ and $E \in \mathbb{k}^{v \times (m+2)}$ are full row rank matrices. Therefore, we can wonder if we can take $E = e(D)$ in Proposition 4.7.

Example 4.11. Considering again Example 4.10, we can check that $g_1 = -e\, f_1/3$ and $g_2 = e\, f_2$, which shows that $\mathrm{ann}_{\mathbb{I}_1}(.a) = \mathbb{I}_1\, f_1 + \mathbb{I}_1\, f_2$.

Proposition 4.12. *Let $a = \sum_{i=0}^{r} a_i\, e\, \partial^i \in \langle e \rangle$, $a_i \in \mathbb{k}[t]$, for $k = 0, \ldots, r$, $m = \max_{i \in [\![0,\, r]\!]}\{\deg_t(a_i)\}$, and C and J_{m+1} be the matrices defined by (9). Let $\mathcal{N} = \mathrm{coker}_{\mathbb{k}[t]}(.C)$ denote the $\mathbb{k}[t]$-module finitely presented by C. If $\mathcal{N} = 0$, then, in Proposition 4.7, we can always choose E to be $e(D)$, a fact which implies*

$$v = u, \quad \begin{pmatrix} g_1 & \ldots & g_u \end{pmatrix}^T = e\, D\, J_{m+1} = e\, \begin{pmatrix} f_1 & \ldots & f_u \end{pmatrix}^T,$$

and thus

$$\mathrm{ann}_{\mathbb{I}_1}(.a) \bigcap \langle e \rangle = \sum_{j=1}^{u} \mathbb{I}_1\, e\, f_j, \quad \mathrm{ann}_{\mathbb{I}_1}(.a) = \sum_{j=1}^{u} \mathbb{I}_1\, f_j.$$

Proof. According to the proof of Lemma 4.5 (see (10)) and by definition of $\mathcal{N} = \mathrm{coker}_{\mathbb{k}[t]}(.C)$, we have the following exact sequence

$$0 \longrightarrow \mathbb{k}[t]^{1 \times u} \xrightarrow{.D} \mathbb{k}[t]^{1 \times (m+2)} \xrightarrow{.C} \mathbb{k}[t]^{1 \times (r+1)} \xrightarrow{\pi} \mathcal{N} \longrightarrow 0,$$

where π denotes the standard projection of $\mathbb{k}[t]^{1 \times (r+1)}$ onto $\mathcal{N}$. Now, if $\mathcal{N} = 0$, then we deduce the short exact sequence

$$0 \longrightarrow \mathbb{k}[t]^{1 \times u} \xrightarrow{.D} \mathbb{k}[t]^{1 \times (m+2)} \xrightarrow{.C} \mathbb{k}[t]^{1 \times (r+1)} \longrightarrow 0,$$

which ends with the free $\mathbb{k}[t]$-module $\mathbb{k}[t]^{1 \times (r+1)}$. Hence, the above exact sequence splits (see, e.g., [18]). Thus, the application of the *functor* $\mathbb{k}[t]/\langle t \rangle \otimes_{\mathbb{k}[t]} \cdot$ to this split exact sequence yields the following exact sequence of $\mathbb{k}$-vector spaces (see, e.g., [18]):

$$0 \longrightarrow \mathbb{k}^{1 \times u} \xrightarrow{.e(D)} \mathbb{k}^{1 \times (m+2)} \xrightarrow{.e(C)} \mathbb{k}^{1 \times (r+1)} \longrightarrow 0, \qquad (13)$$

which yields $\ker_{\mathbb{k}}(.e(C)) = \mathrm{im}_{\mathbb{k}}(.e(D))$, i.e., we can take $E = e(D)$. □

Proposition 4.12 shows that, under the hypothesis $\mathcal{N} = 0$, the left ideal $\mathrm{ann}_{\mathbb{I}_1}(.a)$ of $\mathbb{I}_1$ can be generated by differential operators. In the following, we shall prove (see Proposition 4.16 below) that the condition $\mathcal{N} = 0$ can always be assumed to be fulfilled. To do so, let us first introduce the concept of the *Fitting ideals* of a module.

Definition 4.13 ([7], Section 20.2). Let $\mathcal{R}$ be a commutative ring, $\mathcal{M}$ a finitely presented left $\mathcal{R}$-module, and $C \in \mathcal{R}^{q \times p}$ a presentation matrix of $\mathcal{M}$, i.e., $\mathcal{M} \cong \mathrm{coker}_{\mathbb{k}[t]}(.C)$. The i^{th} *Fitting ideal* of $\mathcal{M}$ is

$$\mathrm{Fitt}_i(\mathcal{M}) := \begin{cases} \mathcal{R} & \text{if } \ p - i \leq 0, \\ \mathcal{F}_{p-i} & \text{if } \ 0 < p - i \leq \min(p, q), \\ \langle 0 \rangle & \text{if } \ p - i > \min(p, q), \end{cases}$$

where $\mathcal{F}_j$ stands for the ideal of $\mathcal{R}$ generated by all the minors (determinants of submatrices) of size j of the matrix C.

We recall that a finitely generated $\mathcal{R}$-module $\mathcal{M}$ is said to be *projective* if there are a $\mathcal{R}$-module $\mathcal{P}$ and $s \in \mathbb{N}$ such that we have $\mathcal{M} \oplus \mathcal{P} \cong \mathcal{R}^s$. If $\mathcal{R}$ is an integral domain, the *rank* of a $\mathcal{R}$-module $\mathcal{M}$ is the dimension of the $Q(\mathcal{R})$-vector space $Q(\mathcal{R}) \otimes_{\mathcal{R}} \mathcal{M}$ obtained by extending the coefficients of $\mathcal{M}$ from $\mathcal{R}$ to its *quotient field* $Q(\mathcal{R}) = \{r_1/r_2 \mid 0 \neq r_2,\, r_1 \in \mathcal{R}\}$. See [7, 18]. Let us now state a standard result on Fitting ideals.

Theorem 4.14 ([7], Prop. 20.8, p. 495). *Let $\mathcal{R}$ be a commutative ring, $\mathcal{M}$ a finitely presented left $\mathcal{R}$-module, and $C \in \mathcal{R}^{q \times p}$ a presentation matrix of $\mathcal{M}$. The following assertions are equivalent:*

(1) *$\mathcal{M}$ is a projective module of rank r.*

(2) $\mathrm{Fitt}_i(\mathcal{M}) = \langle 0\rangle$, *for* $i = 0, \ldots, r-1$, *and* $\mathrm{Fitt}_r(\mathcal{M}) = \mathcal{R}$.

Note that if $\mathcal{M}$ is a projective $\mathcal{R}$-module of rank 0, then $\mathcal{M} = 0$. Also, while considering the annihilator $\mathrm{ann}_{\mathbb{I}_1}(.a)$ of an element $a = \sum_{i=0}^r a_i\, e\, \partial^i \in \langle e\rangle$, we can always suppose that the a_i's are $\mathbb{k}$-linearly independent. Indeed, if we have $\sum_{i=0}^r c_i\, a_i = 0$, where $c_i \in \mathbb{k}$, for $i = 0, \ldots, r$ and $c_k \neq 0$, then $a_k = -\sum_{0\le i\neq k\le r}(c_i/c_k)\, a_i$, which yields $\bigcap_{i=0}^r \mathcal{K}_{\mathbb{A}_1}(a_i) = \bigcap_{0\le i\neq k\le r}\mathcal{K}_{\mathbb{A}_1}(a_i)$. Consequently, for our purpose, the matrix C defined by (9) can always be assumed to have $\mathbb{k}$-linearly independent columns.

Lemma 4.15. *Let $a_0, \ldots, a_r \in \mathbb{k}[t]$ be $\mathbb{k}$-linearly independent polynomials and $m = \max_{i\in[\![0,r]\!]}\{\deg_t(a_i)\}$. If $C \in \mathbb{k}[t]^{(m+2)\times(r+1)}$ is the matrix defined by (9), then we have $r+1 \le m+2$ and, for all $h \in \mathbb{k}$, $\mathrm{rank}_{\mathbb{k}}\, C(h) = r+1$, i.e., the columns of $C(h)$ are $\mathbb{k}$-linearly independent.*

Proof. Let us fix $h \in \mathbb{k}$. Considering the Taylor expansion of the polynomial a_i at h, we have

$$a_i(t) = \sum_{j=0}^{m+1} a_{i,j}\, t^j = \sum_{j=0}^{m+1} \frac{a_i^{(j)}(h)}{j!}\,(t-h)^j.$$

Expanding the latter expression yields

$$\begin{aligned}\sum_{j=0}^{m+1} \frac{a_i^{(j)}(h)}{j!}\,(t-h)^j &= \sum_{j=0}^{m+1}\sum_{k=0}^{j}\binom{j}{k}\frac{a_i^{(j)}(h)}{j!}\,(-h)^{j-k}\, t^k,\\ &= \sum_{k=0}^{m+1}\left(\sum_{j=k}^{m+1}\binom{j}{k}\frac{(-h)^{j-k}}{j!}\, a_i^{(j)}(h)\right) t^k.\end{aligned}$$

Hence, we obtain the linear system

$$\sum_{j=k}^{m+1}\frac{(-h)^{j-k}}{k!\,(j-k)!}\, a_i^{(j)}(h) = a_{i,k},\ \ i = 0, \ldots, r,\ \ k = 0, \ldots, m+1,$$

which can be written as

$$\underbrace{\begin{pmatrix} 1 & -h & \frac{h^2}{2} & \ldots & \ldots & \frac{(-h)^{m+1}}{0!(m+1)!}\\ 0 & 1 & -h & \frac{h^2}{2} & \ldots & \frac{(-h)^m}{1!m!}\\ 0 & 0 & \frac{1}{2} & -\frac{h}{2} & \ldots & \frac{(-h)^{m-1}}{2!(m-1)!}\\ \vdots & \vdots & & \ddots & \vdots & \\ \vdots & \vdots & & & \ddots & \vdots\\ 0 & 0 & \ldots & \ldots & 0 & \frac{1}{(m+1)!0!}\end{pmatrix}}_{G} C(h) = \underbrace{\begin{pmatrix} a_{0,0} & \ldots & a_{r,0}\\ a_{0,1} & \ldots & a_{r,1}\\ \vdots & & \vdots\\ a_{0,m+1} & \ldots & a_{r,m+1}\end{pmatrix}}_{H},$$

where $C(h)$ denotes the value of the matrix C defined by (9) at $h \in \mathbb{k}$. Now, the fact that $a_0, \ldots, a_r$ are $\mathbb{k}$-linearly independent means that $\sum_{i=0}^r a_i\, c_i = 0$, i.e., $\sum_{i=0}^r a_{i,l}\, c_i = 0$, for $l = 0, \ldots, m+1$, implies $c_i = 0$, for $i = 0, \ldots, r$. Consequently, considering the linear map $H. : \mathbb{k}^{(r+1)\times 1} \longrightarrow \mathbb{k}^{(m+2)\times 1}$, we have $\ker_{\mathbb{k}}(H.) = 0$, which shows that $\mathrm{rank}_{\mathbb{k}}(H) = \dim_{\mathbb{k}}\mathrm{im}_{\mathbb{k}}(H.) = r+1 \le m+2$. The fact that $\det(G) \neq 0$ finally yields $\mathrm{rank}_{\mathbb{k}}(C(h)) = r+1$. □

Proposition 4.16. *Let $a_0, \ldots, a_r \in \mathbb{k}[t]$ be $\mathbb{k}$-linearly independent polynomials, $m = \max_{i\in[\![0,r]\!]}\{\deg_t(a_i)\}$, and $C \in \mathbb{k}[t]^{(m+2)\times(r+1)}$ the matrix defined by (9). Then, we have $\mathcal{N} = \mathrm{coker}_{\mathbb{k}[t]}(.C) = 0$.*

Proof. First, from Lemma 4.15, we have $r+1 \le m+2$. Then, $\mathrm{Fitt}_0(\mathcal{N})$ is the ideal of $\mathbb{k}[t]$ formed by all the $r+1$ minors of the matrix C. Moreover, Lemma 4.15 also implies that the affine algebraic set $V(\mathrm{Fitt}_0(\mathcal{N})) = \{t \in \mathbb{k} \mid \forall F \in \mathrm{Fitt}_0(\mathcal{N}),\ F(t) = 0\}$ is empty, which, by the Nullstellensatz, shows that $\mathrm{Fitt}_0(\mathcal{N}) = \mathbb{k}[t]$. Thus, Theorem 4.14 implies $\mathcal{N} = 0$. □

The proof of Proposition 4.12 shows that $\mathcal{N} = 0$ yields the existence of a left inverse U of C, i.e., $U\, C = I_{r+1}$. The converse result also holds. If $U\, C = I_{r+1}$, then $\mathrm{im}_{\mathbb{k}[t]}(.C) = \mathbb{k}[t]^{1\times(r+1)}$, i.e., $\mathcal{N} = 0$. Thus, for $\mathbb{k}$-linearly independent a_i's, Proposition 4.16 states the existence of a left inverse of C.

Corollary 4.17. *Let $a = \sum_{i=0}^r a_i\, e\, \partial^i \in \langle e\rangle$, where $a_i \in \mathbb{k}[t]$, for $i = 0, \ldots, r$. Moreover, we assume that the a_i's are $\mathbb{k}$-linearly independent. Then, we have $\mathrm{ann}_{\mathbb{I}_1}(.a) = \sum_{j=1}^u \mathbb{I}_1\, f_j$, where the f_j's are the differential operators defined in Lemma 4.5 and $u = m-r+1$.*

Proof. Theorem 4.9, Proposition 4.12, and Proposition 4.16 imply the first part of the result. Using the fact that C has a left inverse, i.e., $.C$ is surjective, we have the following standard exact sequence

$$0 \longrightarrow \ker_{\mathbb{k}[t]}(.C) \longrightarrow \mathbb{k}[t]^{1\times(m+2)} \xrightarrow{.C} \mathbb{k}[t]^{1\times(r+1)} \longrightarrow 0,$$

which yields $\mathrm{rank}_{\mathbb{k}[t]}\ker_{\mathbb{k}[t]}(.C) = m+2-(r+1) = m-r+1$ [18]. Since $\ker_{\mathbb{k}[t]}(.C)$ is a $\mathbb{k}[t]$-submodule of a free $\mathbb{k}[t]$-module and $\mathbb{k}[t]$ is a principal ideal domain, $\ker_{\mathbb{k}[t]}(.C)$ is then a free $\mathbb{k}[t]$-module of rank $m-r+1$ [18], i.e., $\ker_{\mathbb{k}[t]}(.C) \cong \mathbb{k}[t]^{1\times(m-r+1)}$. Thus, we can take $u = m-r+1$ in Lemma 4.5, which completes the result. □

Example 4.18. Let us consider again Examples 4.6 and 4.8. Using, e.g., `Maple`, we can see that the matrix C admits the left inverse

$$U = \begin{pmatrix} 0 & 1 & -t & \frac{1}{2}\,t^2 + \frac{1}{3} & 0\\ 0 & 0 & \frac{1}{2} & -\frac{1}{2}\,t & 0\\ 0 & 0 & 0 & \frac{1}{6} & 0\end{pmatrix}.$$

This yields $\mathcal{N} = \mathrm{coker}_{\mathbb{k}[t]}(.C) = 0$ and we can use the result of Proposition 4.12. In Examples 4.6 and 4.8, we found that $e(D) \neq E$. But the matrices E and $e(D)$ generate the same $\mathbb{k}$-vector space since we have $E = V\, e(D)$, where $V = \begin{pmatrix} -\frac{1}{3} & 0\\ 0 & 1\end{pmatrix}$. Proposition 4.12 then implies that we can choose E to be $e(D)$ so that $g_1 = e\, f_1$ and $g_2 = e\, f_2$. Finally, Corollary 4.17 yields $\mathrm{ann}_{\mathbb{I}_1}(.a) = \mathbb{I}_1\, f_1 + \mathbb{I}_1\, f_2$, where f_1 and f_2 are defined in Example 4.6.

Note that Proposition 4.4 is a direct consequence of Corollary 4.17. Indeed, in the case $a = p\, e$, we have $C = (p\ \ldots\ p^{(m)}\ 0)^T$, where $p^{(m)} = m!\,\mathrm{lc}(p) \in \mathbb{k}\setminus\{0\}$, and thus, $\mathcal{N} = \mathbb{k}[t]/\langle p, \ldots, p^{(m)}\rangle = 0$. In this case, $r = 0$ yields $u = m+1 = \deg_t(p)+1$, i.e., $\mathrm{ann}_{\mathbb{I}_1}(.p\, e)$ can be generated by $m+1$ elements. However Proposition 4.4 shows that $\mathrm{ann}_{\mathbb{I}_1}(.p\, e)$ can be generated by two elements. Indeed, this result comes from Stafford's theorem [19] stating that every left ideal of $\mathbb{A}_1$ can be generated by two elements and the fact that [5, Proposition 3.2] gives two explicit generators for $\mathcal{K}_{\mathbb{A}_1}(p)$. An extension of Stafford's theorem, namely, that every finitely generated left/right ideal of $\mathbb{I}_1$ (e.g., $\mathrm{ann}_{\mathbb{I}_1}(.a)$ for $a \in \mathbb{I}_1$) can be generated by two elements of $\mathbb{I}_1$, was proved in [1]. Reducing the set of generators of $\mathrm{ann}_{\mathbb{I}_1}(.a)$ obtained in Proposition 4.4 to two elements is out of the scope of the present paper. For the case of $\mathbb{A}_1$, see [16] and the references therein.

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