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Algebraic analysis of linear multidimensional control systems

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The purpose of this paper is to show how to use the modern methods of algebraic analysis in partial differential control theory, when the input–output relations are defined by systems of partial differential equations in the continuous case or by multi-shift difference equations in the discrete case. The essential tool is the duality existing between the theory of differential modules or D-modules and the formal theory of systems of partial differential equations. We reformulate and generalize many formal results that can be found in the extensive literature on multidimensional systems (controllability, primeness concepts, poles and zeros,...). All the results are presented through effective algorithms.

Keywords: Control theory; primeness; poles and zeros; multidimensional systems; algebraic analysis; extension functor; Janet conjecture; formal theory of partial differential equations; duality; homological algebra; commutative algebra; noncommutative algebra.

1. Introduction

In 1963, Kalman (Kalman *et al.* 1963) related the controllability of a linear ordinary differential control system, with constant coefficients, of the form $\dot{y} = Ay + Bu$, to the full row rank of the controllability matrix $[B, AB, \ldots, A^{m-1}B]$, where m is the number of outputs y. Hautus (1969) showed this criterion to be equivalent to the full row rank of the matrix $[A - \chi I, B]$ for all values of the indeterminate χ, in an attempt to study the transfer matrix $(\chi I - A)^{-1}B$. Then more general polynomial systems of the form $D(\chi)y = N(\chi)u$, with D a nondegenerate square matrix, were considered in an attempt to study the transfer matrix $D(\chi)^{-1}N(\chi)$. In particular, left-coprimeness conditions for the matrices D and N were given for multi-input–multi-output (MIMO) systems generalizing the case of single-input–single-output (SISO) systems where common factors of D (denominator) and N (numerator) could disappear in the transfer function (Kailath 1980). One should notice that the Kalman criterion came from an explicit integration by means of exponentials of matrices, which are not easily available in the general case. Little by little, the preceding separate conditions for D and N have been reformulated for the full matrix $[D, -N]$ in terms of a Bezout identity, a result showing that controllability is a built-in property of the control system, not depending on the separation of the variables between inputs and outputs. Meanwhile, a few people tried to extend these results to matrices over the polynomial ring $k[\chi] = k[\chi_1, \ldots, \chi_n]$ in n indeterminates over a field k of constants, or to operator matrices with variable coefficients (Fornasini & Valcher 1997; Kleon & Oberst

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1998; Oberst 1990, 1996; Youla & Gnavi 1979; Youla & Pickel 1984; Wood *et al.* 1998; Zerz 1996). It was soon discovered that the case $n = 1$, where $k[\chi]$ is a principal-ideal ring, should be distinguished with care from the cases $n = 2$ and $n \geqslant 3$ (Youla & Gnavi 1979; Zerz 1996). It is only recently that people paid attention to algebraic analysis, pionneered by Palamodov (1970) for the constant-coefficients case and by Kashiwara (1995) for the general case. We quote in particular the work of Oberst (1990: Thm 21c, p. 142) showing, in the first place, that a control system is controllable if and only if the corresponding differential module is torsion-free. For the one-dimensional case, one may also refer to Fliess (1989).

In this paper, the mathematical results are not new, and we provide all corresponding references since their homological proofs are often very delicate. However, the applications to control are quite new. In particular, the main purpose of this paper is to combine the formal theory of differential operators with that of differential modules and a description by extension functors in order to avoid the introduction of signal spaces, while recovering and generalizing most of the results previously quoted. Since the two main ingredients are (a) the construction of differential sequences, for which symbolic computer packages will soon be available, and (b) the construction of a formal adjoint of an operator, we can say that the techniques presented in this paper lead to effective algorithms. All these results will be developed in a forthcoming book (Pommaret 2000).

In view of the amount of mathematical equipment needed in order to understand algebraic analysis, we suppose that the reader has a basic familiarity with differential sequences or resolutions and their use for defining the extension functor (Hu 1968; Northcott 1966; Rotman 1979).

2. Algebraic analysis

Let us start by recalling and giving some algebraic results which will be useful for the rest of the paper. For more details, we refer the reader to Hu (1968), Northcott (1966), Ritt (1966), Rotman (1979), and Serre (1989).

2.1 *D-modules*

DEFINITION 1 A *differential ring* A with n commuting derivations $\partial_1,\dots,\partial_n$ is a ring which satisfies, for all $a, b \in A$, and for all $i, j = 1,\dots,n$:

- $\partial_i a \in A$,
- $\partial_i (a + b) = \partial_i a + \partial_i b$,
- $\partial_i (ab) = (\partial_i a)\, b + a\, \partial_i b$,
- $\partial_i \partial_j = \partial_j \partial_i = \partial_{ij}$.

For applications, the differential ring A will either be a differential field K containing $\mathbb{Q}$ or its subfield of constants $k = \mathrm{cst}(K) = \{a \in K \mid \forall i = 1,\dots,n : \partial_i a = 0\}$. If $d_1,\dots,d_n$ are n commuting formal derivative operators, we shall introduce the noetherian ring $D = A[d] = A[d_1,\dots,d_n]$ of differential operators. Any element of D has the form $P = \sum_{\text{finite}} a^\mu d_\mu$, where $\mu = (\mu_1,\dots,\mu_n)$ is a multi-index with length $|\mu| = \mu_1 + \cdots + \mu_n$, with $a^{\mu} \in A$ and $d_{\mu} = (d_1)^{\mu_1} ... (d_n)^{\mu_n}$. The ring D is a non-commutative integral domain which satisfies

$$\forall\, a, b \in A : a d_i\,(b\, d_j) = ab\, d_i\, d_j + a\,(\partial_i\, b)\, d_j,$$

and possesses the left (right) Ore property:

$$\forall\,(P, Q) \in D^2,\ \exists\,(U, V) \in (D\backslash\{0\})^2 \text{ such that } UP = VQ\ (PU = QV).$$

EXAMPLE 1 The field of rational functions $\mathbb{R}(t)$ is a differential field with derivation $\mathrm{d}/\mathrm{d}t$. Indeed, for all $a(t)$ and nonzero $b(t)$ in $\mathbb{R}(t)$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{a(t)}{b(t)} = \frac{\dot{a}(t)\, b(t) - a(t)\, \dot{b}(t)}{b^2(t)} \in \mathbb{R}(t).$$

Let $D = \mathbb{R}(t)[\mathrm{d}/\mathrm{d}t]$ be the non-commutative ring of linear operators with coefficients in $\mathbb{R}(t)$. Any element $P \in D$ has the form $P = \sum_{\text{finite}} a_i(t)(\mathrm{d}/\mathrm{d}t)^i$, with $a_i \in \mathbb{R}(t)$.

In the constant-coefficients case, one can identify the ring of multi-shift difference operators and the ring of differential operators with the commutative ring of polynomials in many indeterminates. This is the reason why we shall no longer refer to the multi-shift difference situation.

In the general case, since D is a non-commutative ring, we define the notion of filtration and gradation in order to pass from the non-commutative ring D to the commutative ring $\mathrm{gr}(D)$ and thus to use all the results and techniques developed in the commutative case for the non-commutative one (Bjork 1979; Maisonobe & Sabbah 1993; Pham 1980). Moreover, the ring of differential operators $D = A[d]$ looks like a polynomial ring, and thus we may like to generalize the well-known notion of degree of a polynomial to a differential operator in D. This can be done by introducing the notion of graded ring. For more details, see Bjork (1979), Maisonobe & Sabah (1993), and Pham (1980).

Throughout the rest of the paper, we denote the A-module $\{0\}$ by the abbreviation 0.

DEFINITION 2 A *filtration* of an A-algebra D is a sequence $\{D_r\}_{r\in\mathbb{N}}$ of A-modules satisfying:

- $0 = D_{-1} \subseteq D_0 \subseteq D_1 \subseteq ... \subseteq D$,
- $\bigcup_{r\geqslant 0} D_r = D$,
- $D_r D_s \subseteq D_{r+s}$.

The *associated graded* A-algebra $\mathrm{gr}\,(D)$ of D is defined by:

- $\mathrm{gr}\,(D) = \bigoplus_{r\in\mathbb{N}} D_r/D_{r-1}$,
- $\forall\,\overline{P} \in D_r/D_{r-1}, \forall\,\overline{Q} \in D_s/D_{s-1} : \overline{P}\,\overline{Q} = \overline{PQ} \in D_{r+s}/D_{r+s-1}$.

D_r/D_{r-1} is called the *homogeneous component of degree r* of D.

EXAMPLE 2 The sequence $\{D_r\}_{r\in\mathbb{N}}$ of A-modules, where $D_r = \{\sum_{0\leqslant|\mu|\leqslant r} a^{\mu}\, d_{\mu} \mid a^{\mu} \in A\}$, is a filtration of $D = A[d_1, ..., d_n]$. In particular, we have $D_0 = A \subset D$, and thus D_r is a free left A-module with basis $\{d_{\mu} \mid 0 \leqslant |\mu| \leqslant r\}$.

In the next sections, D_r will always refer to the filtration of Example 2, and we shall endow $T = D_1/D_0$ with a bracket induced by the filtration of D, namely $[P, Q] = PQ - QP$, for $P, Q \in D$.

PROPOSITION 1 The natural morphism

$$\begin{array}{rcl} \operatorname{gr}(D) & \longrightarrow & A[\chi_1, \ldots, \chi_n] \\ D_r/D_{r-1} \ni \sum_{|\mu|=r} a^\mu d_\mu & \longmapsto & \sum_{|\mu|=r} a^\mu \chi_\mu, \end{array}$$

is an isomorphism of A-algebras.

DEFINITION 3 Let M be a D-module, where D admits the filtration $\{D_r\}_{r\in\mathbb{N}}$. A family $\{M_q\}_{q\in\mathbb{N}}$ of A-modules is a *filtration* of M if

- $0 = M_{-1} \subseteq M_0 \subseteq M_1 \subseteq \ldots \subseteq M$,
- $\bigcup_{q\in\mathbb{N}} M_q = M$,
- $D_r M_q \subseteq M_{q+r}$.

The *associated* graded gr (D)-module $G = \operatorname{gr}(M)$ is then defined by:

- $G = \bigoplus_{q\in\mathbb{N}} G_q$, with $G_q = M_q/M_{q-1}$,
- $\forall \overline{P} \in D_r/D_{r-1}, \forall \overline{m} \in M_q/M_{q-1} : \overline{P}\,\overline{m} = \overline{Pm} \in M_{q+r}/M_{q+r-1}$.

We have the short exact sequence:

$$0 \longrightarrow M_{q-1} \longrightarrow M_q \longrightarrow G_q \longrightarrow 0. \tag{1}$$

DEFINITION 4 A filtration $\{M_q\}_{q\in\mathbb{N}}$ of a D-module M is called a *good filtration* if it satisfies one of the following equivalent conditions:

(i) For all $q \in \mathbb{N}$, M_q is finitely generated over A, and there exists $p \in \mathbb{N}$ such that

$$D_r M_p = M_{p+r} \quad \text{for } r \geqslant 0,$$

(ii) G is a finitely generated gr (D)-module.

EXAMPLE 3 (i) Let M be a finitely generated D-module with the set of generators $\{e_1, \ldots, e_m\}$. Then the filtration $M_q = \sum_{i=1}^m D_q e_i$ is good, since $G = \sum_{i=1}^m \operatorname{gr}(D)\, e_i$, and thus G is finitely generated over gr(D) by $\{e_1, \ldots, e_m\}$.

(ii) Let $M = D$ be the left D-module and let $\{M_q = D_{2q}\}_{q\in\mathbb{N}}$ be a filtration of M. Then, we have $D_r M_q = D_r D_{2q} \subsetneq D_{2q+r}$, which is in general only a strict submodule of $D_{2(q+r)} = M_{r+q}$, and thus $\{D_{2q}\}_{q\in\mathbb{N}}$ is not a good filtration of D.

PROPOSITION 2 Let M be a left D-module. Then M admits a good filtration if and only if M is finitely generated over D. Moreover, if M has a good filtration, then:

(i) any submodule M' of M has a good filtration, defined by $M'_q = M_q \cap M'$, and M' is finitely generated over D;

(ii) any quotient M'' of M has a good filtration, defined by the image of the filtration of M under the projection $M \longrightarrow M'' \longrightarrow 0$.

(iii) if $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ is a short exact sequence of filtred modules, then we have the short exact sequence $0 \longrightarrow G' \longrightarrow G \longrightarrow G'' \longrightarrow 0$, where the associated graded modules $G' = \mathrm{gr}(M')$, $G = \mathrm{gr}(M)$, and $G'' = \mathrm{gr}(M'')$ are defined with respect to the above induced filtrations.

Now, using the graded module gr(M) over the commutative ring gr(D), instead of the D-module M, we can use the results of algebraic geometry to give an intrinsic definition of the dimension of a module M.

DEFINITION 5 Let M be a finitely generated D-module with a good filtration, and let $G = \mathrm{gr}(M)$ be its associated graded gr(D)-module; then the ideal $I(M) = \sqrt{\mathrm{ann}(G)} = \{a \in \mathrm{gr}(D) \mid \exists n \in \mathbb{N} : a^n G = 0\}$ does not depend on the filtration of M, and we introduce the *characteristic set* $\mathrm{char}(M) = V(I(M)) = \{\mathfrak{p} \in \mathrm{spec}(\mathrm{gr}(D)) \mid \sqrt{\mathrm{ann}(G)} \subseteq \mathfrak{p}\}$, where spec(gr($D$)) is the set of proper prime ideals of gr(D).

The previous definition and proposition lead to the following result (Bjork 1979; Maisonobe & Sabah 1993; Pham 1980).

PROPOSITION 3 Let M be a finitely generated left D-module admitting a good filtration.

(i) For r large enough, there exists a unique Hilbert polynomial H_M such that $\dim(M_r) = H_M(r) = (m/d!)\, r^d + \ldots$, where d is the degree of the polynomial. The degree $d = \mathrm{d}(M)$, called the *dimension* of M, and the coefficient $m = \mathrm{m}(M)$, called the *multiplicity* of M, do not depend on the good filtration.

(ii) If $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ is an exact sequence of finitely generated filtered modules, then

  (a) $\mathrm{char}(M) = \mathrm{char}(M') \cup \mathrm{char}(M'')$,
  (b) $H_M = H_{M'} + H_{M''}$ and thus $\mathrm{d}(M) = \max\{\mathrm{d}(M'), \mathrm{d}(M'')\}$ and, if $\mathrm{d}(M') = \mathrm{d}(M'')$, then $\mathrm{m}(M) = \mathrm{m}(M') + \mathrm{m}(M'')$.

We shall see, in the next section, how to compute effectively the Hilbert polynomial and thus the dimension and the multiplicity of a D-module M. Now, let us give some basic definitions of properties of modules that will be at the core of this paper (Bjork 1993; Kashiwara 1995).

DEFINITION 6
- A D-module M is *free* if there are elements of M which generate M and which are independent over D.
- A D-module M is *projective* if there exist a free D-module F and a D-module N such that $F = M \oplus N$. Hence the module N is also a projective D-module.
- A D-module M is *reflexive* if $M \cong \hom_D(\hom_D(M, D), D)$.
- A D-module M is *torsion-free* if $\mathrm{t}(M) = \{m \in M \mid \exists P \neq 0 : P m = 0\} = 0$. We call t($M$) the *torsion submodule* of M. In any case, $M/\mathrm{t}(M)$ is a torsion-free D-module.

If D is a principal ideal ring, then any torsion-free module is free and, if $A = k$, then the Quillen–Suslin theorem (Rotman 1979; Youla & Pickel 1984) shows that any projective module is free. Moreover, it follows immediately from (1) that M is torsion-free (reflexive, projective, free) whenever G is torsion-free (reflexive, projective, free). The converse is not true.

EXAMPLE 4 The SISO system $\dot{y} - y - u = 0$ is torsion-free, while the graded part $\dot{y} = 0$ is not torsion-free.

2.2 *Differential operators*

Let X be a differential manifold of dimension n with local coordinates $x = (x^1, \ldots, x^n)$ and structural ring $A = C^\infty(X)$. We denote by $T = T(X)$ the tangent bundle of X and by $T^* = T^*(X)$ the cotangent bundle of X. By $S_q T^*$ and $\bigwedge^r T^*$ we shall mean the qth symetric product of T^* and the rth exterior product of T^*. Let E be a vector bundle of fibre dimension m over X, with local coordinates (x, y), where $y = (y^1, \ldots, y^m)$; we notice that E is a left A-module. Following Malgrange (1966) and Spencer (1965), we shall use the same notation E for a vector bundle and for its sheaf of germs of sections. We consider the vector bundle $J_q(E)$ of q-jets of E. Its fibre at $x \in X$ is the quotient of the space of germs of sections of E at x by the subspace of germs of sections which vanish up to order q at x ($f, g \in J_q(E)_x \Leftrightarrow \partial_\mu f(x) = \partial_\mu g(x)$ for $0 \leqslant |\mu| \leqslant q$). We identify $J_0(E)$ with E and denote the projection of $J_q(E)$ onto X by π and the projection of $J_q(E)$ onto $J_{q-1}(E)$ by π^q_{q-1}. If ξ is a section of E, we denote by $j_q(\xi)(x)$ the equivalence class of germs of ξ at x. We have the following exact sequence (Goldschmidt 1968; Pommaret 1994; Spencer 1965):

$$0 \longrightarrow S_q T^* \otimes E \longrightarrow J_q(E) \xrightarrow{\pi^q_{q-1}} J_{q-1}(E) \longrightarrow 0. \tag{2}$$

Let us recall a few definitions; for more details, see Goldschmidt (1968), Malgrange (1966), Pommaret (1994), and Spencer (1965).

DEFINITION 7 Let F be a vector bundle over X, of fibre dimension l.

- A *differential operator* $\mathcal{D} = \Phi \circ j_q : E \to F$ of order q is given by a bundle morphism $\Phi : J_q(E) \to F$, where we may suppose that Φ is surjective.
- The *r-prolongation* of Φ is the unique bundle morphism $\rho_r(\Phi) : J_{q+r}(E) \to J_r(F)$ such that $\rho_r(\Phi) \circ j_{q+r} = j_r \circ \mathcal{D} = j_r \circ \Phi \circ j_q$.
- If the kernel $R_q \subseteq J_q(E)$ of Φ is a vector bundle, we say that the system of partial differential (PD) equations R_q is *defined by* $\mathcal{D}$ and a solution of R_q is a *local section* ξ of E, over an open set $U \subset X$, such that $j_q(\xi)(x) \in R_q$ for all $x \in U$.
- The rth *prolongation* R_{q+r} of R_q is the kernel of $\rho_r(\Phi)$.
- We denote by $R^{(s)}_{q+r}$ the projection of R_{q+r+s} onto $J_{q+r}(E)$, i.e. $R^{(s)}_{q+r} = \pi^{q+r+s}_{q+r}(R_{q+r+s})$.
- Using the sequence (2) and the definition of the r-prolongation of Φ, we obtain the induced map $\sigma_r(\Phi) : S_{q+r} T^* \otimes E \to S_r T^* \otimes F$. We denote by g_{q+r} the kernel of $\sigma_r(\Phi)$ and call it *the symbol* of R_{q+r}. We easily see that $g_{q+r} = R_{q+r} \cap S_{q+r} T^* \otimes E$.

To help the reader, we provide the local-coordinate expressions of the above concepts. The bundle morphism Φ defined by

$$\begin{array}{rlcl} \Phi : & J_q(E) & \longrightarrow & F, \\ & (x, y^k_\mu) & \longmapsto & \left(x, \sum_{0 \leqslant |\mu| \leqslant q, 1 \leqslant k \leqslant m} a^{\tau\mu}_k(x)\, y^k_\mu : \tau = 1, \ldots, l\right), \end{array}$$

gives rise to the differential operator $\mathcal{D}$ defined by

$$\begin{array}{rcl} \mathcal{D}: E & \longrightarrow & F, \\ (x, \xi^k(x)) & \longmapsto & (x, \eta(x)^\tau = \left(\sum_{0\leqslant|\mu|\leqslant q, 1\leqslant k\leqslant m} a_k^{\tau\mu}(x)\, \partial_\mu \xi^k(x) : \tau = 1,\dots, l\right). \end{array}$$

The system R_q, defined by the differential operator $\mathcal{D}$, is given by

$$\sum_{0\leqslant|\mu|\leqslant q, 1\leqslant k\leqslant m} a_k^{\tau\mu}(x)\, y_\mu^k = 0 \quad \text{for } 1 \leqslant \tau \leqslant l.$$

The map $\sigma_r(\Phi)$ is then defined by

$$\begin{array}{rcl} \sigma_r(\Phi): S_{q+r}T^* \otimes E & \longrightarrow & S_r T^* \otimes F, \\ (x, y_\mu^k, |\mu| = q + r) & \longmapsto & \left(x, \sum_{|\mu|=q, |\nu|=r, 1\leqslant k\leqslant m} a_k^{\tau\mu}(x)\, y_{\mu+\nu}^k : \tau = 1,\dots, l\right). \end{array}$$

Now, the Spencer *δ-sequence* is defined by

$$\Lambda^s T^* \otimes g_{q+r+1} \xrightarrow{\delta} \Lambda^{s+1} T^* \otimes g_{q+r},$$

with $(\delta(\omega))_\mu^k = \mathrm{d}x^i \wedge \omega_{\mu+1_i}^k$ where $\omega = v_{\nu,I}^k\, \mathrm{d}x^I \in \Lambda^s T^* \otimes g_{q+r+1}$; here we use the summation convention with $\mathrm{d}x^I = \mathrm{d}x^{i_1} \wedge \dots \wedge \mathrm{d}x^{i_s}$, for $I = \{i_1,\dots, i_s\}$ with $i_1 < \dots < i_s$, and $|\mu| = q + r$; see Goldschmidt (1968), Pommaret (1994), and Spencer (1965) for more details. We easily verify that $\delta \circ \delta = 0$. The resulting cohomology at $\Lambda^s T^* \otimes g_{q+r}$ is denoted by $H_{q+r}^s(g_q)$, since it depends only on g_q.

DEFINITION 8 The symbol g_q of R_q is said to be *s-acyclic* if $H_{q+r}^1 = \dots = H_{q+r}^s = 0$ for all $r \geqslant 0$. The symbol g_q is *involutive* if it is n-acyclic. In particular, every system R_q of ordinary differential equations (ODE) has an involutive symbol. A symbol g_q is *of finite type* if there exists $r \geqslant 0$ such that $g_{q+r} = 0$.

One can prove that the symbol g_q of a system R_q is such that g_{q+r} becomes involutive for r large enough. If g_q is an involutive symbol, we may define integers α_q^i, called *characters* of g_q such that

$$\dim g_{q+r} = \sum_{i=1}^{n} \frac{(r + i - 1)!}{r!\,(i - 1)!}\, \alpha_q^i \quad \text{for } r \geqslant 0,$$

and the following relations are satisfied:

- $\dim g_q = \alpha_q^1 + \dots + \alpha_q^n$,
- $\alpha_q^1 \geqslant \alpha_q^2 \geqslant \dots \geqslant \alpha_q^n \geqslant 0$,
- $0 \leqslant \alpha_q^n \leqslant m$.

DEFINITION 9 A system R_q is said to be *formally integrable* if, for all $r, s \geqslant 0$, R_{q+r} is a vector bundle and the projection $\pi_{q+r}^{q+r+s} : R_{q+r+s} \to R_{q+r}$ is surjective. A system R_q is *involutive* if R_q is formally integrable and has an involutive symbol g_q.

If Φ is sufficiently regular, then R_{q+r} is a vector bundle for any $r \geqslant 0$; if furthermore R_q is formally integrable, then we have the exact sequences

$$0 \longrightarrow g_{q+r} \longrightarrow R_{q+r} \xrightarrow{\pi^{q+r}_{q+r-1}} R_{q+r-1} \longrightarrow 0. \tag{3}$$

COROLLARY 1 If the system R_q is an involutive system, then

$$\dim(R_{q+r}) = \dim(R_{q-1}) + \sum_{i=1}^{n} \frac{(r+i)!}{r!\,i!}\alpha_q^i = \frac{\alpha_q^n}{n!}r^n + \dots,$$

where R_{q-1} is the projection of R_q onto $J_{q-1}(E)$.

Accordingly, formal solutions of the system R_q depend on α_q^1 functions of x^1, α_q^2 functions of $(x^1, x^2), \dots,$ and α_q^n functions of $(x^1, \dots, x^n)$—a result leading to the famous Cartan–Khäler–Janet theorem in the analytic case (Janet 1921; Pommaret 1994).

If the system R_q is not formally integrable, then by adding enough equations, we can extend the system R_q to a formally integrable system $R^{(s)}_{q+r}$ with the same solutions, by means of a finite algorithm (Goldschmidt 1968; Pommaret 1994). Knowledge of the latter system, which is the finite substitute for $R_\infty = \rho_\infty(R_q)$, is essential for studying the formal properties of the given system and of the corresponding differential module.

DEFINITION 10 Let $\mathcal{D} : E \to F$ be an involutive operator; then there exists n new first-order involutive operators $\mathcal{D}_i : F_{i-1} \to F_i$, with $F_0 = F$, such that the sequence

$$E \xrightarrow{\mathcal{D}} F_0 \xrightarrow{\mathcal{D}_1} F_1 \xrightarrow{\mathcal{D}_2} \dots \xrightarrow{\mathcal{D}_n} F_n \longrightarrow 0,$$

is *stricly exact*, i.e. the operator $\mathcal{D}_i$ generates all the compatibility conditions of $\mathcal{D}_{i-1}$, and the sequence is exact at any order on the jet level. This sequence is called the *Janet sequence* of $\mathcal{D}$.

2.3 *Duality*

Using coordinates (x, y) for E, we may identify $Dy = Dy^1 + \dots + Dy^m$ with D^m. We shall denote by J_q the bundle of q-jets of the trivial vector bundle $X \times \mathbb{R}$. Because the transition laws of $J_q(E)$ are obtained from the coordinate changes $\bar{y} = a(x)y$ of E by differentiating up to order q, in case of a single y and $a \in A$, we may obtain a right A-module structure on J_q. Comparing now the two formal relations at first order, namely

$$\bar{y}_i = a y_i + (\partial_i a) y,$$
$$d_i \otimes \bar{y} = d_i \otimes ay = d_i a \otimes y = (a d_i + \partial_i a) \otimes y = a d_i \otimes y + \partial_i a \otimes y,$$

we obtain therefore by prolongation the identification $J_q(E) = J_q \otimes_A E$, where E is equipped with its left A-module structure, while J_q is equipped with the above right A-module structure (Malgrange 1966). The duality between differential geometry and differential algebra is obtained by setting

$$J_q^* = \hom_A(J_q, A) = D_q \quad \Rightarrow \quad J_q(E)^* = D_q(E) = D_q \otimes_A E^*,$$

whenever (X, A) is a ringed space—see Malgrange (1966) and also Kashiwara (1995: Introduction) and Goldschmidt (1970). Such a result explains the multi-index contraction $\sum a^{\mu} y_{\mu}$ whenever the operator $\sum a^{\mu} d_{\mu}$ is applied to the single differential indeterminate y. Accordingly, we can define a differential module M by the cokernel in the exact sequence of modules:

$$D \otimes_A F^{*} \longrightarrow D \otimes_A E^{*} \longrightarrow M \longrightarrow 0,$$

or simply

$$D^{l} \longrightarrow D^{m} \longrightarrow M \longrightarrow 0,$$

in the trivial case if $\dim(E) = m$ and $\dim(F) = l$. Hence, from the exactness of the contravariant functor $\hom_A(\cdot, A)$, we obtain the exact sequence $0 \to R_{\infty} \to \mathrm{J}_{\infty}(E) \to \mathrm{J}_{\infty}(F)$, where $R_{\infty} = \rho_{\infty}(R_q) = \hom_A(M, A)$ (Goldschmidt 1970; Malgrange 1966), and the main difficulty is that certain properties of M, using injective limits, are not easily interpreted as properties of R_{∞} using projective limits, and vice versa. When $\mathcal{D}$ is involutive and sufficiently regular, we notice that a canonical finite resolution of the sheaf Θ of solutions of $\mathcal{D}$ is of the form of the Janet sequence with $F = F_0$, $\dim(F_r) = l_r$, and $\dim(E) = m$:

$$0 \longrightarrow \Theta \longrightarrow E \xrightarrow{\mathcal{D}} F_0 \xrightarrow{\mathcal{D}_1} F_1 \xrightarrow{\mathcal{D}_2} \dots \xrightarrow{\mathcal{D}_n} F_n \longrightarrow 0, \tag{4}$$

where $\mathcal{D}_i$ represents all the compatibility conditions of $\mathcal{D}_{i-1}$. The sequence (4) provides, by duality, a finite free resolution of M (Hu 1968; Northcott 1966; Rotman 1979):

$$0 \longleftarrow M \longleftarrow D^{m} \xleftarrow{\mathcal{D}} D^{l_0} \xleftarrow{\mathcal{D}_1} D^{l_1} \xleftarrow{\mathcal{D}_2} \dots \xleftarrow{\mathcal{D}_n} D^{l_n} \longleftarrow 0. \tag{5}$$

The problem is to study the properties of an operator $(m \times l)$-matrix, acting on column vectors on the right, in the operator sense, or on row vectors on the left in the module sense. Accordingly, a preliminary problem for being able to deal equivalently with $\mathcal{D}$ or with M is to bring effectively $\mathcal{D}$ or R_q to formal integrability or even to involutiveness, in such a way that $R_q = M_q^{*} = \hom_A(M_q, A)$ and $g_q = G_q^{*} = \hom_A(G_q, A)$. In that case, the sequence (3) for $r = 0$ is the dual with respect to $\hom_A(\cdot, A)$ of the sequence (1) if $\mathcal{D}$ has coefficients in A while E and F are trivial bundles. This result provides an effective way for computing the Hilbert polynomial and justifies the comment after Proposition 3. Indeed, we have defined an algebraic set over k or K, namely the characteristic set $\mathrm{char}(M) = \mathrm{supp}(G)$ of M, as the *support* of G, namely the set of prime ideals of $\mathrm{gr}(D)$, containing the annihilator $\mathrm{ann}(G)$ of $G = \mathrm{gr}(M)$. Keeping the word *variety* for an irreductible algebraic set, we notice that the dimension $\dim(M) = \mathrm{d}(M)$ of the D-module M is the maximum dimension over an algebraic closure of k or K of the varieties corresponding to the minimum prime ideals in $\mathrm{char}(M)$, i.e. the degree d of the Hilbert polynomial H_M. Equivalently, allowing us to avoid dealing with many irreductible components, the Hilbert–Serre theorem says that $\mathrm{d}(M)$ is equal to the maximum number of non-zero characters α_q^i (Serre 1989). We denote by $\mathrm{cd}(M) = n - \mathrm{d}(M)$ the *codimension* of $\mathrm{char}(M)$; cf. Bjork (1993).

It is important to notice that the dualities $\hom_A(\cdot, A)$ and $\hom_D(\cdot, D)$ that will be systematically used in this paper can lead to effective computations and provide a formal interpretation of the behavioural approach used by Oberst (1990,1996) and Willems (1991) thus avoiding any signal space in the definition of a system.

We present the *extension functor* in the operator language; see Hu (1968), Northcott (1966), and Rotman (1979) for a module approach. If $\mathcal{D} : E \longrightarrow F$ is a differential operator of order q, we denote by $\mathrm{ad}(\mathcal{D}) = \tilde{\mathcal{D}} : \tilde{F} = \bigwedge^n T^* \otimes F^* \longrightarrow \tilde{E} = \bigwedge^n T^* \otimes E^*$ the *formal adjoint* of $\mathcal{D}$. The operator $\tilde{\mathcal{D}}$ is of the same order as $\mathcal{D}$, with coefficients in A. The formal adjoint $\tilde{\mathcal{D}} : \tilde{F} \to \tilde{E}$ can be easily computed by using the following three rules.

- The adjoint of a matrix (zeroth-order operator) is the transposed matrix.
- The adjoint of ∂_i is $-\partial_i$.
- For two linear PD operators P and Q that can be composed: $\widetilde{PQ} = \tilde{Q}\tilde{P}$.

Moreover, we have the following relation

$$\langle \mu, \mathcal{D}\xi \rangle = \langle \tilde{\mathcal{D}}\mu, \xi \rangle + d(\cdot),$$

with d the exterior derivative. We compute the adjoint of an operator P by multiplying P by test functions on the left and integrating the result by parts, as we could do for distributions.

It is important to notice firstly that $\tilde{\mathcal{D}}$ may not be formally integrable when $\mathcal{D}$ is, and secondly that $\tilde{\mathcal{D}}_r$ may not generate at all the compatibility conditions of $\tilde{\mathcal{D}}_{r+1}$ in the adjoint of the Janet sequence (4). Let us give an example.

EXAMPLE 5 We take the operator $\mathcal{D} : \xi \mapsto \eta$, defined on sections, by

$$\partial_{12}\xi = \eta^1, \qquad \partial_{22}\xi = \eta^2,$$

and easily see that the compatibility condition of $\mathcal{D}$ is the operator $\mathcal{D}_1 : \eta \mapsto \zeta$, defined by $\partial_1\eta^2 - \partial_2\eta^1 = \zeta$. Then the adjoint $\tilde{\mathcal{D}}_1 : \lambda \mapsto \mu$ of $\mathcal{D}_1$ is given by

$$\partial_2\lambda = \mu_1, \qquad -\partial_1\lambda = \mu_2.$$

The compatibility condition of $\tilde{\mathcal{D}}_1 : \mu \mapsto \nu$ is defined by the operator $\partial_1\mu_1 + \partial_2\mu_2 = \nu$, which is not the adjoint $\tilde{\mathcal{D}}$ of the operator $\mathcal{D}$ (defined by $\partial_{12}\,\mu_1 + \partial_{22}\,\mu_2 = \nu$).

One can roughly say that $\mathrm{ext}^r_D(M, D)$ measures the defect of exactness at $\tilde{F}_{r-1}$ in the adjoint sequence. However, since $\mathrm{ext}^r_D(M, D)$ does not depend on the presentation of M, the previous definition by means of the Janet sequence is by far the best, though one could use the second Spencer sequence too (another finite free resolution of the sheaf Θ of solutions of $\mathcal{D}$); see Malgrange (1966), Pommaret (1994), and Spencer (1965). Namely one can use (do not confuse with standard notations)

$$0 \longrightarrow \Theta \xrightarrow{j_q} C_0 \xrightarrow{D_1} C_1 \xrightarrow{D_2} \dots \xrightarrow{D_n} C_n \longrightarrow 0, \tag{6}$$

and measure the defect of exactness at $\tilde{C}_r$ by dealing only with first-order operators D_r—though with many more unknowns (take for example $E = T$, $F = \bigwedge^n T^*$, and for $\mathcal{D}$ the divergence operator expressed as the Lie derivative of a given n-form).

The first key result of algebraic analysis is the following theorem relating the vanishing of the extension functor to the codimension of the characteristic set (Kashiwara 1995; Palamodov 1970).

THEOREM 1 $\mathrm{cd}(M) \geqslant r$ if and only if $\mathrm{ext}^i_D(M, D) = 0$ for $i < r$.

The second key result, instead of looking for the compatibility condition $\mathcal{D}_1$ of a differential operator $\mathcal{D}$, deals with the converse problem of looking for a potential-like expression of $\mathcal{D}$, namely seeking to determine whether one can find an operator $\mathcal{D}_{-1} : E_{-1} \longrightarrow E_0 = E$ such that $\mathcal{D}$ generates all the compatibility conditions of $\mathcal{D}_{-1}$. For example, one may keep in mind the Poincaré sequence for the exterior derivative. If there exists such an operator $\mathcal{D}_{-1}$, we say that the operator $\mathcal{D}$ is *parametrized* by $\mathcal{D}_{-1}$.

THEOREM 2 There exists a sequence of differential operators

$$E_{-r} \xrightarrow{\mathcal{D}_{-r}} E_{-r+1} \xrightarrow{\mathcal{D}_{-r+1}} \dots \xrightarrow{\mathcal{D}_{-2}} E_{-1} \xrightarrow{\mathcal{D}_{-1}} E_0 \xrightarrow{\mathcal{D}} F,$$

where each operator generates all the compatibility conditions of the preceding one, if and only if $\mathrm{ext}^i_D(N, D) = 0$ for $i = 1,\dots, r$ whenever N is the differential module determined by the operator $\tilde{\mathcal{D}}$, exactly as M was determined by $\mathcal{D}$.

Taking into account the fact that the ext modules do not depend on the chosen resolution, the above conditions can be checked effectively, since we just need to construct the adjoint operator, find a sequence of compatibility conditions with length r, dualize it, and check whether the adjoint sequence is formally exact—i.e. whether each operator generates exactly the compatibility conditions of the preceding one. The global dimension of D is n because, using the Spencer sequence (6), we obtain at once: $\mathrm{ext}^i_D(M, D) = 0$ for $i > n$.

EXAMPLE 6 Let us take the divergence operator $\mathcal{D} : \xi \rightarrow \eta$, in $\mathbb{R}^3$, defined by

$$\partial_1 \xi^1 + \partial_2 \xi^2 + \partial_3 \xi^3 = \eta.$$

Dualizing the divergence operator, we obtain the operator $\tilde{\mathcal{D}} : \mu \rightarrow \nu$, defined by

$$-\partial_1 \mu = \nu_1, \qquad -\partial_2 \mu = \nu_2, \qquad -\partial_3 \mu = \nu_3,$$

which is nothing other than minus the gradient operator. We let the reader check that the compatibility condition $\tilde{\mathcal{D}}_{-1}$ of $\tilde{\mathcal{D}}$ is the curl operator, and the adjoint of $\tilde{\mathcal{D}}_{-1}$ is still the curl operator, i.e. the curl is a *self-adjoint* operator. The compatibility condition of the curl operator $\mathcal{D}_{-1}$ is the divergence, and thus $\mathcal{D}$ is parametrized by the curl operator $\mathcal{D}_{-1}$. In other words, if M is the D-module defined by $\mathcal{D}$, we have $\mathrm{ext}^1_D(N, D) = 0$, where N is the D-module defined by $\tilde{\mathcal{D}}$. Moreover, we can check that the compatibility condition $\tilde{\mathcal{D}}_{-2}$ of $\tilde{\mathcal{D}}_{-1}$ is minus the divergence operator, and thus its adjoint $\mathcal{D}_{-2}$ is the gradient which parametrizes the curl, i.e. $\mathrm{ext}^2_D(N, D) = 0$. We shall see in the next section that, if $\mathcal{D}$ is a formally surjective operator—that is, without any compatibility conditions—then N is a torsion D-module and thus $\hom_D(N, D) = \mathrm{ext}^0_D(N, D) = 0$. Using Theorem 1, we obtain $\mathrm{cd}(N) > 2$, or equivalently $\mathrm{d}(N) = 0$, i.e. $\alpha^i_1(\tilde{\mathcal{D}}) = 0$ for $i = 1, 2, 3$, and we find that the solutions of the gradient operator depend only on constants.

It is essential to notice that the right D-module $N_r = \bigwedge^n T^* \otimes_A N$, obtained from the left D-module $N = N_l$ by the side-changing functor (Bjork 1993; Maisonobe & Sabah 1993), must not be confused with $\tilde{M} = \hom_D(M, D)$, since we have the exact sequence

$$0 \longrightarrow \tilde{M} \longrightarrow E \otimes_A D \longrightarrow F \otimes_A D \longrightarrow N_r \longrightarrow 0, \tag{7}$$

and thus the relation: $\mathrm{ext}^i_D(N_r, D) = \mathrm{ext}^{i-2}_D(\mathrm{hom}_D(M, D), D)$ for $i \geqslant 3$. Finally, the result of subsection 3.1 below will prove that $\mathrm{ext}^i_D(N, D)$ depends only on M for $i \geqslant 1$. In fact, as a much stronger (but delicate) result, one can prove that N and $\mathrm{ext}^0_D(N, D) = \mathrm{hom}_D(N, D) = \tilde{N}$ are only determined up to a projective equivalence, according to Schanuel's lemma (Hu 1968; Northcott 1966; Rotman 1979) and the exact sequence of left D-modules:

$$0 \longrightarrow \tilde{N}_r \longrightarrow D \otimes F^* \longrightarrow D \otimes E^* \longrightarrow M \longrightarrow 0. \qquad (8)$$ 5

We shall now divide the properties of control systems into two categories, depending on whether they do or do not depend on a separation of the variables of the control system between input and output.

3. Applications to control theory (I): structural properties

We first study the properties that do not depend on such a separation.

3.1 *Primeness*

The key idea, *not at all intuitively evident*, is to use $\tilde{\mathcal{D}}$ or N instead of $\mathcal{D}$ or M in order to achieve a classification of modules:

$$\text{free} \subseteq \text{projective} \subseteq \ldots \subseteq \text{reflexive} \subsetneq \text{torsion-free}.$$

First of all, we recall that M is torsion-free (reflexive) if and only if the central morphisn in the long exact sequence of left D-modules

$$0 \longrightarrow \mathrm{ext}^1_D(N_r, D) \longrightarrow M \xrightarrow{\epsilon} \mathrm{hom}_D(\mathrm{hom}_D(M, D), D) \longrightarrow \mathrm{ext}^2_D(N_r, D) \longrightarrow 0,$$
$$m \mapsto \epsilon(m),$$

with $\epsilon(m)(f) = f(m)$ for all $f \in \mathrm{hom}_D(M, D)$, is injective (bijective); see Kashiwara (1995) and Palamodov (1970).

DEFINITION 11 We shall say that a control system, defined by an operator $\mathcal{D}$, is *controllable*, if one cannot find locally any *autonomous elements*, namely any scalar differential combination $\zeta = \mathcal{A}\xi$, satisfying at least one PDE of the form $\mathcal{B}\zeta = 0$ whenever $\mathcal{D}\xi = 0$.

Then we have the following corollary.

COROLLARY 2. The following assertions are equivalent (Kashiwara 1995; Palamodov 1970; Pommaret 1994, 2000).

(i) The control system defined by $\mathcal{D}$ is controllable.
(ii) The operator $\mathcal{D}$ is parametrizable by a $\mathcal{D}_{-1}$.
(iii) The D-module M is torsion-free.
(iv) $\mathrm{ext}^1_D(N_r, D) = \bigwedge^n T^* \otimes_A \mathrm{ext}^1_D(N, D) = 0$.

REMARK 1 Moreover, if $\mathcal{D}$ is formally surjective, i.e. $\mathcal{D}_1 = 0$, then $\hom_D(N, D) = \mathrm{ext}^0_D(N, D) = 0$. By Theorem 1, this means $\mathrm{cd}(N) \geqslant 1$ or equivalently $\mathrm{d}(N) \leqslant n - 1$. That is, $\alpha^n_q(N) = 0$; namely N is a torsion module. Thus, if M is torsion-free, then $\mathrm{ext}^i_D(N, D) = 0$ for $i \leqslant 1$. Hence $\mathrm{cd}(N) \geqslant 2$, and thus $\mathrm{d}(N) \leqslant n - 2$. We therefore reestablish the concept of *minor left-primeness* (MLP: Fornasini & Valcher 1997; Oberst 1990; Youla & Gnavi 1979; Wood *et al.* 1998; Zerz 1996) for the operator matrix representing $\mathcal{D}$. Note that, in the variable-coefficient case, the matrix of $\tilde{\mathcal{D}}$ is not just the transpose of the matrix of $\mathcal{D}$. In the particular case $n = 1$, we obtain the Hautus test (Hautus 1969) and the fact that the system is controllable if and only if $\tilde{\mathcal{D}}$ is injective (Pommaret 1995). In that case, there is a lift operator $\tilde{\mathcal{P}} : \tilde{E} \to \tilde{F}$ such that $\tilde{\mathcal{P}} \circ \tilde{\mathcal{D}} = \mathrm{id}_{\tilde{F}}$ and thus $\mathcal{D} \circ \mathcal{P} = \mathrm{id}_F$, a result amounting to the forward and reversed generalized Bezout identities (Kailath 1980; Pommaret & Quadrat 1998). When $\mathcal{D}$ is not surjective, the above result amounts to *generalized factor left primeness* (Oberst 1990; Youla & Gnavi 1979; Zerz 1996; Wood *et al.* 1998).

COROLLARY 3 *M is reflexive if and only if* $\mathrm{ext}^i_D(N, D) = 0$ *for* $i = 1, 2$.

REMARK 2 Moreover, if $\mathcal{D}$ is surjective, reasoning as before, we get $\mathrm{d}(N) \leqslant n - 3$. The divergence operator provides a good example of a reflexive module which is nevertheless not projective, since $\mathrm{ext}^i_D(N, D) = 0$.

Setting $r = n - 1$ in Theorem 2 yields the case $\mathrm{ext}^i_D(N, D) = 0$ for $i = 1, \ldots, n - 1$; that is, $\mathrm{d}(N) = 0$ when $\mathcal{D}$ is surjective, and this is the concept of *weakly zero left-primeness* (Oberst 1990; Wood *et al.* 1998; Zerz 1996). A particular example is provided by a system of finite type, or *holonomic* module N such that $\mathrm{I}(N) = (\chi_1, \ldots, \chi_n)$, implying that the algebraic set $\mathrm{char}(N)$ is reduced to the origin and $\mathrm{ext}^i_D(N, D) = 0$ for $i \neq n$. Since N is a differential module too, we have $\mathrm{ext}^i_D(N, D) = 0$ for $i > n$. We now consider the case when $\mathrm{ext}^i_D(N, D)$ for $i \geqslant 1$.
$=0$

COROLLARY 4 *M is projective if and only if* $\mathrm{ext}^i_D(N, D) = 0$ *for* $i \geqslant 1$.

Proof. In general, when $\mathcal{D}$ is not surjective, since the $\mathrm{ext}^i_D(N, D)$ do not depend on the resolution of N, we may bring $\tilde{\mathcal{D}}$ to involutiveness and use the corresponding Spencer sequence (6) to construct inductively the lift operators P_r of the Spencer operators D_r in such a way that $D_r P_r D_r = D_r$. More precisely, if $\tilde{\mathcal{D}} = D_1$ is involutive, it follows from $\mathrm{ext}^n_D(N, D) = 0$ that $\tilde{D}_n$ is injective and admits therefore a left-inverse $\tilde{P}_n$ such that $\tilde{P}_n \tilde{D}_n = \mathrm{id}_{\tilde{C}_n}$ or $D_n P_n = \mathrm{id}_{C_n}$. However, we have $(\mathrm{id}_{\tilde{C}_{n-1}} - \tilde{D}_n \tilde{P}_n)\tilde{D}_n = 0$ while $\tilde{D}_{n-1}\tilde{D}_n = 0$, and $\tilde{D}_{n-1}$ represents all the compatibility conditions of $\tilde{D}_n$ because $\mathrm{ext}^{n-1}_D(N, D) = 0$. Hence there is an operator P_{n-1} such that $\tilde{D}_n \tilde{P}_n + \tilde{P}_{n-1}\tilde{D}_{n-1} = \mathrm{id}_{\tilde{C}_{n-1}}$ or equivalently $P_n D_n + D_{n-1} P_{n-1} = \mathrm{id}_{C_{n-1}}$, leading to $D_{n-1} P_{n-1} D_{n-1} = D_{n-1}$ and so on; see Pommaret & Quadrat (1998) for more details. Accordingly, N itself is projective and $\tilde{M}$ is projective in (7). Since M is already reflexive, we have $M \cong \tilde{\tilde{M}}$ and M is projective too. Conversely, if M is projective, the exact sequence (8) splits and thus, applying $\hom_D(\cdot, D)$, it follows that the exact sequence (7) splits too; that is, N_r and N are projective D-modules, a fact leading to $\mathrm{ext}^i_D(N, D) = 0$ for $i \geqslant 1$. We notice that choosing A to be K or k is essential in the proof in order to allow for the existence of a finite free resolution of length n. □

REMARK 3 Moreover, when $\mathcal{D}$ is surjective, we obtain $\mathrm{ext}^i_D(N, D) = 0$ for $i \geqslant 0$, and thus $\mathrm{d}(N) = -1$. That is, $\mathrm{char}(N) = \emptyset$, and this is only possible if $N = 0$. Hence $\tilde{\mathcal{D}}$ admits a lift, and M is a projective module (Pommaret & Quadrat 1998). We find a generalization of *zero left-primeness* (Fornasini & Valcher 1997; Oberst 1990; Youla & Gnavi 1979; Zerz 1996; Wood *et al.* 1998), since we are now dealing with variable coefficients.

In the commutative case, one may use $\mathrm{ann}(M)$ instead of $\mathrm{ann}(G)$ (Oberst 1990,1996). Finally, when $D = k[d]$ is commutative, it is known that $\mathrm{ann}(M)$ and ann(G) define algebraic sets with the same dimension, according to the Hilbert–Serre theorem (Serre 1989). Hence, such a generalization of all existing results explains the existence of a whole range of 'possible types of primeness' conjectured by Wood *et al.* (1998).

EXAMPLE 7 When $n = 1$, only one type of primeness is left. Dealing with a formally integrable Kalman system $-\dot{y}+Ay+Bu = 0$ and multiplying it on the left by a row vector of test functions λ, we find for the kernel of $\tilde{\mathcal{D}}$:

$$\dot{\lambda} + \lambda A = 0 \quad \text{and} \quad \lambda B = 0$$
$$\Rightarrow \quad \dot{\lambda} B = 0 \quad \Rightarrow \quad \lambda AB = 0 \quad \Rightarrow \quad \lambda A^2 B = 0 \quad \Rightarrow \ldots \Rightarrow \quad \lambda A^{m-1} B = 0,$$

and the Kalman test surprisingly amounts to the injectivity of the non-formally integrable operator $\tilde{\mathcal{D}}$—a result also equivalent to the lack of first integrals (Pommaret 1995; Pommaret & Quadrat 1999); cf. Oberst (1990: Ex. 56, p. 152).

EXAMPLE 8 The system $\partial_1\xi^1 + \partial_2\xi^2 = 0$ defines a torsion-free D-module with a first-order parametrization, which is nevertheless not projective, whereas $\partial_1\xi^1+\partial_2\xi^2-x^2\xi^1 = 0$ defines a projective (but not free) and thus reflexive D-module which is automatically torsion-free and admits a second-order parametrization.

EXAMPLE 9 The last operator $\mathcal{D}_n$ in a Janet sequence always provides a projective module.

EXAMPLE 10 With $n = 3$, let us consider the second-order system

$$\partial_{33}\xi - \partial_{13}\xi - \partial_3\xi = 0, \qquad \partial_{23}\xi - \partial_{12}\xi - \partial_2\xi = 0, \qquad \partial_{22}\xi - \partial_{12}\xi = 0,$$

with characters $\alpha_2^1 = 3$, $\alpha_2^2 = 0$, and $\alpha_2^3 = 0$. The algebraic sets defined by $\mathrm{ann}(M)$ and $\mathrm{ann}(G)$ are different, though they are both unions of three varieties of dimension 1, and thus have the same dimension.

EXAMPLE 11 The case of a surjective operator $\mathcal{D} : E \to F$ with $\dim(E) = \dim(F)$ is standard in physics (wave equations in elasticity, electromagnetism,...). Indeed, M is a torsion module, and thus $\mathrm{ext}^0_D(M, D) = \hom_D(M, D) = 0$, so that $\mathrm{cd}(M) \geqslant 1$. If $\mathrm{cd}(M) = 1$, then $\mathrm{ext}^1_D(M, D) \neq 0$ and $\mathcal{D}$ is a determined operator which is therefore always formally integrable. A good example is the Cauchy–Riemann system defining holomorphic transformations. However, if $\mathrm{cd}(M) \geqslant 2$, then $\mathrm{ext}^1_D(M, D) = 0$, implying that $N = 0$. It follows that $\tilde{\mathcal{D}}$ is invertible, showing that $\mathcal{D}$ is not formally integrable. This result implies that $M = 0$. We have thus obtained a simple proof of the conjecture of Janet (1921) first solved by Johnson (1978), saying that, for a system of this kind, there is a gap in the possible dimensions of the corresponding modules. The following second-order system, with $n = 2$, $\dim(E) = \dim(F) = 2$, provides a good example (Janet 1929; Pommaret 1994)

$$\partial_{11}\xi^1 + \partial_{12}\xi^2 - \xi^2 = \eta^1, \qquad \partial_{12}\xi^1 + \partial_{22}\xi^2 + \xi^1 = \eta^2,$$

and one can easily check that the square matrix of $\mathcal{D}$ is unimodular with determinant equal to 1.

3.2 *Pure modules*

We end this section with a generalization of the torsion-free property of a module, and follow Bjork (1993). In order to explain this useful new direction for applications, we first provide a few examples.

EXAMPLE 12 Starting with the system

$$y_{22} = 0, \qquad y_{12} = 0,$$

we notice that $z = y_1$ satisfies just $z_2 = 0$, while $z = y_2$ satisfies both $z_1 = 0$ and $z_2 = 0$. Therefore $\mathrm{cd}(Dy_1) = 1$ while $\mathrm{cd}(Dy_2) = 2$. Hence we may distinguish the torsion elements of a differential module according to the properties of the system of PDEs they satisfy. Two examples from engineering science will particularly well illustrate the different behaviour of various torsion elements.

EXAMPLE 13 In the linearized system of Euler equations for an incompressible fluid (Pommaret 1992), namely

$$\vec{\nabla}\cdot\vec{v} = 0, \qquad \frac{\partial \vec{v}}{\partial t} + \vec{\nabla} p = 0,$$

where $\vec{v}$ is the speed and p the pressure of the fluid, one notices that we have the PDE system

$$\Delta p = 0, \qquad \frac{\partial(\Delta \vec{v})}{\partial t} = 0.$$

Similarly, the Boussinesq stationary system for the Benard problem (Pommaret 1992, 1994), namely

$$\vec{\nabla}\cdot\vec{v} = 0, \qquad \Delta\vec{v} - \theta\vec{g} - \vec{\nabla}\pi = 0, \qquad \Delta\theta - \vec{g}.\vec{v} = 0,$$

where $\vec{g} = (0, 0, -g)$ is gravity while π and θ are perturbations of pressure and temperature, we obtain from vector analysis that

$$\Delta\Delta\Delta\theta - g^2(\partial_{11} + \partial_{22})\theta = 0,$$

though, setting $w = \partial_1 v_2 - \partial_2 v_1$, we only get $\Delta w = 0$, and thus $\mathrm{cd}(D\theta) = \mathrm{cd}(Dw) = 1$.

Accordingly, among the elements of a differential module, one can find the elements which are *free*, i.e. they do not satisfy any PDE, and the others (torsion elements) which are *constrained* by at least one PDE.

DEFINITION 12 (i) We introduce the D-submodules $t_r(M) = \{m \in M \mid cd(Dm) > r\}$, with $t_0(M) = t(M)$, the torsion submodule of M.

(ii) A D-module is said to be *r-pure* if $t_r(M) = 0$ and $t_{r-1}(M) = M$.

The chain of inclusions

$$0 = t_n(M) \subseteq t_{n-1}(M) \subseteq .... \subseteq t_1(M) \subseteq t_0(M) = t(M) \subseteq M,$$

will be particularly useful for studying the specific properties of engineering quantities that can be observed experimentally by decoupling them from other quantities. Of course, $t_{r-1}(M)/t_r(M)$ is r-pure, and one has the following delicate criterion for knowing whether a differential module is r-pure or not (Bjork 1993).

THEOREM 3 M is r-pure if and only if $M \subseteq ext^r_D(ext^r_D(M, D), D)$, with $cd(M) = r$.

COROLLARY 5 When M is r-pure, then char(M) is *r-equidimensional*; namely it can be decomposed into irreductible components of the same dimension r.

We notice that the above criterion generalizes the situation of the torsion-free modules described in Corollary 2 for the case $r = 0$.

EXAMPLE 14 Without the previous criterion, it is not evident that the differential module provided by Example 10 is 2-pure and thus that the corresponding adjoint operator is torsion-free. More generally, any differential module defined by a finite-type system is automatically n-pure. This is particulary clear in 2-dimensional elasticity, with $\mathcal{D}$: $(\xi_1, \xi_2) \mapsto (\partial_1\xi_1 = \epsilon_{11}, \frac{1}{2}(\partial_1\xi_2 + \partial_2\xi_1) = \epsilon_{12}, \partial_2\xi_2 = \epsilon_{22})$ and $\mathcal{D}_1 : \epsilon \mapsto \partial_{11}\epsilon_{22} + \partial_{22}\epsilon_{11} - 2\partial_{12}\epsilon_{12} = 0$, defining the strain tensor and its compatibility condition, while the adjoint sequence allows us to parametrize the stress equation by $\tilde{\mathcal{D}}_1$ acting on the Airy function, with $\sigma^{11} = \partial_{22}\lambda, \sigma^{12} = \sigma^{21} = -\partial_{12}\lambda$, and $\sigma^{22} = \partial_{11}\lambda$.

To our knowledge, it does not seem that such a classification of systems/modules has ever been applied. The following nontrivial theorem (Kashiwara 1995; Palamodov 1970) is particularly useful.

THEOREM 4 char$(M) = \bigcup_{i=0}^{n}$ char $(ext^i_D(M, D))$.

EXAMPLE 15 If $\mathcal{D}_1$ denotes the compatibility conditions of $\mathcal{D}$, and $\tilde{\mathcal{D}}$ generates the compatibility conditions of $\tilde{\mathcal{D}}_1$, in such a way that both the module M determined by $\mathcal{D}$ and the module N determined by $\tilde{\mathcal{D}}_1$ are torsion modules, then both $\mathcal{D}_1$ and $\tilde{\mathcal{D}}$ are surjective and char(M) = char(N). This result generalizes the equality of the primeness degrees of left and right factor matrix descriptions of a given transfer matrix (Wood *et al.* 1998: p. 74). A typical example of this situation is provided by Examples 10 and 14.

4. Applications to control theory (II): input–output properties

We now turn to the properties involving inputs and outputs. First of all, contrary to tradition, there is no especial reason for choosing the inputs as determining a maximum free differential submodule of M, though it is a possible choice. Accordingly, many concepts in control theory are based upon the two types of exact sequences that can be constructed from M:

$$0 \longrightarrow t(M) \longrightarrow M \longrightarrow M/t(M) \longrightarrow 0, \tag{9}$$

$$0 \longrightarrow F \longrightarrow M \longrightarrow M/F \longrightarrow 0, \tag{10}$$

where $\mathrm{t}(M)$ is the torsion submodule of M, and F is a maximum free submodule of M. We notice that $M/\mathrm{t}(M)$ is torsion-free while M/F is a torsion module. Setting $S = D\backslash\{0\}$, we may construct the field $Q(D) = S^{-1}D = DS^{-1}$ of quotients of D and tensor by $Q(D)$ the previous sequences in order to kill their torsion modules (Kashiwara 1995; Oberst 1990; Pommaret & Quadrat 1999). Such a construction, which is basic in algebraic analysis, gives the way to generalize the transfer-matrix approach, even for variable coefficients, by considering the localization $S^{-1}M = Q(D) \otimes_D M$, without any reference to the Laplace transform (Oberst 1990; Pommaret & Quadrat 1999). If we already know that M is torsion-free, it may provide a parametrization of $\mathcal{D}$ generalizing the *controller form* in the OD case (Kailath 1980). For more details, see Pommaret & Quadrat (1999). We notice that $M/\mathrm{t}(M)$ and M/F are two specializations of M giving rise to two subsystems R'_∞ and R''_∞ of R_∞. Taking into account that $\mathrm{t}(M) \cap F = 0$, we obtain the following commutative and exact diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & 0 & \longrightarrow & F & = & F & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{t}(M) & \longrightarrow & M & \longrightarrow & M/\mathrm{t}(M) & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{t}(M) & \longrightarrow & M/F & \longrightarrow & M/(\mathrm{t}(M) \oplus F) & \longrightarrow & 0, \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array} \tag{11}$$

and, dualizing it, we obtain the following commutative and exact diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \uparrow & & \uparrow & & \\
 & & 0 & \longleftarrow & R_\infty/R''_\infty & = & R_\infty/R''_\infty & \longleftarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longleftarrow & R_\infty/R'_\infty & \longleftarrow & R_\infty & \longleftarrow & R'_\infty & \longleftarrow & 0 \\
 & & \| & & \uparrow & & \uparrow & & \\
0 & \longleftarrow & R_\infty/R'_\infty & \longleftarrow & R''_\infty & \longleftarrow & R'_\infty \cap R''_\infty & \longleftarrow & 0, \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array} \tag{12}$$

which provides at once the relation $R_\infty = R'_\infty + R''_\infty$. This very basic reason, hidden in the possible underlying confusion concerning the choice of input and output, comes from the fact that, when $n = 1$, any torsion-free module is free and the first of the two preceding sequences splits. However, the resulting backward sequence should not be confused in general with the second sequence, and the two sequences should be distinguished with care. In particular, only the first one depends entirely on M and provides the so-called minimum realization (Pommaret & Quadrat 1999); see also Oberst (1990).

Because input and output always play a reciprocal role and are made by elements of M, we shall consider two different differential submodules M_{in} and M_{out} of M such that $M_{\text{in}} + M_{\text{out}}$ may be a strict differential submodule of M if there are latent variables. There is no general reason for supposing that M/M_{in} is a torsion module, because M/M_{out} is not a torsion module in general. The main construction is to introduce $\mathrm{t}(M)$ and set $M'_{\text{in}} = M_{\text{in}} + \mathrm{t}(M)$ and $M'_{\text{out}} = M_{\text{out}} + \mathrm{t}(M)$ in M. Then, the idea of the minimal realization is to replace M_{in}, M_{out}, and M by $M'_{\text{in}}/\mathrm{t}(M) = M_{\text{in}}/(M_{\text{in}} \cap \mathrm{t}(M))$, $M'_{\text{out}}/\mathrm{t}(M) = M_{\text{out}}/(M_{\text{out}} \cap \mathrm{t}(M))$, and $M/\mathrm{t}(M)$ in order to deal only with torsion-free modules, always keeping in mind that the differential rank $\mathrm{rk}_D(M)$ of M, namely the last character, is intrinsically defined, does not depend on the presentation, and is additive; that is to say, if $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ is a short exact sequence of differential modules, then $\mathrm{rk}_D(M)$=$\mathrm{rk}_D(M')$+$\mathrm{rk}_D(M'')$. This is exactly the module analogue of the differential transcendence degree in differential algebra (Kolchin 1973; Ritt 1966), and one can prove that it is equal to the Euler characteristic of M. If one chooses $M_{\text{in}} = F$ as already defined, then $F \cap \mathrm{t}(M) = 0$ and $M'_{\text{in}}/\mathrm{t}(M) \cong F$ can always be considered as a submodule of $M/\mathrm{t}(M)$.

The final idea is to define poles and zeros for multidimensional systems (Oberst 1990, 1996; Wood *et al.* 1998, to appear). First of all, we have seen (Proposition 2) that, if

$$0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$$

is a short exact sequence of modules, and if M is filtered, we can endow M' and M'' with the induced filtrations $M'_q = M' \cap M_q$ and $M''_q = g(M_q)$ and obtain, for these filtrations, the short exact sequence

$$0 \longrightarrow G' \longrightarrow G \longrightarrow G'' \longrightarrow 0$$

of associated graded modules. Taking the radicals of the respective annhililators, we get

$$\sqrt{\mathrm{ann}(G)} = \sqrt{\mathrm{ann}(G')} \cap \sqrt{\mathrm{ann}(G'')},$$

and thus

$$\mathrm{char}(M) = \mathrm{char}(M') \cup \mathrm{char}(M''),$$

since the characteristic set does not depend on the filtration (see Proposition 3). Because we are dealing with finitely generated modules, we also recall that, in the commutative case, the support $\mathrm{supp}(M)$ of a module M is the set of proper prime ideals, of the corresponding ring, that contain the annihilator of M over the ring. The key point, in order to generalize the concept of the transfer-matrix approach, is to localize the graded sequence with respect to a prime ideal and get the short exact sequence

$$0 \longrightarrow G'_{\mathfrak{p}} \longrightarrow G_{\mathfrak{p}} \longrightarrow G''_{\mathfrak{p}} \longrightarrow 0,$$

with $\mathfrak{p} \in \operatorname{spec}(A[\chi])$, but we can also localize the filtered sequence when D is commutative. In the case of the SISO system defined in Example 4, we get $(\chi - 1)y = u$, and we can divide by $\chi - 1$, provided that $\chi \neq 1$. Hence the trick is to notice that $G'_{\mathfrak{p}} \cong G_{\mathfrak{p}}$ if and only if $G''_{\mathfrak{p}} = 0$, i.e. iff $\mathfrak{p} \notin \operatorname{supp}(G'')$, the true reason for looking at $\operatorname{char}(M'')$.

If N is any submodule of M, then setting $N' = N + \mathrm{t}(M)$, we have the commutative and exact diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{t}(M) \cap N & \longrightarrow & N & \longrightarrow & N'/\mathrm{t}(M) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{t}(M) & \longrightarrow & M & \longrightarrow & M/\mathrm{t}(M) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{t}(M)/(\mathrm{t}(M) \cap N) & \longrightarrow & M/N & \longrightarrow & M/N' & \longrightarrow & 0, \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

with both the isomorphisms

$$\mathrm{t}(M)/(\mathrm{t}(M) \cap N) \cong N'/N, \tag{13}$$

$$N/(\mathrm{t}(M) \cap N) \cong N'/\mathrm{t}(M). \tag{14}$$

Setting M_{in}, M_{out}, and $M_{\text{in}} + M_{\text{out}}$ in place of N, we get similar commutative and exact diagrams, both with short exact sequences of the type

$$0 \longrightarrow M_{\text{in}} \longrightarrow M_{\text{in}} + M_{\text{out}} \longrightarrow (M_{\text{in}} + M_{\text{out}})/M_{\text{in}} \longrightarrow 0, \tag{15}$$

$$0 \longrightarrow M'_{\text{in}} \longrightarrow M'_{\text{in}} + M'_{\text{out}} \longrightarrow (M'_{\text{in}} + M'_{\text{out}})/M'_{\text{in}} \longrightarrow 0, \tag{16}$$

and similar sequences with 'in' and 'out' interchanged.

Now, we have in general an exact sequence of the form

$$0 \longrightarrow N \longrightarrow N' \longrightarrow \mathrm{t}(M)/(\mathrm{t}(M) \cap N) \longrightarrow 0, \tag{17}$$

and similar sequences with M_{in}, M_{out}, and $M_{\text{in}} + M_{\text{out}}$ in place of N. Combining the two preceding sequences starting respectively with M_{in} and M'_{in}, we obtain the short exact sequence

$$0 \longrightarrow (\mathrm{t}(M) \cap (M_{\text{in}} + M_{\text{out}}))/(\mathrm{t}(M) \cap M_{\text{in}}) \longrightarrow (M_{\text{in}} + M_{\text{out}})/M_{\text{in}} \longrightarrow (M'_{\text{in}} + M'_{\text{out}})/M'_{\text{in}} \longrightarrow 0, \tag{18}$$

which is not evident at first sight and where many of the previous modules appear.

We claim that all poles and zeros considered in classical control theory are only examples of the characteristic sets of the modules introduced above, and all the relations among poles and zeros come from the preceding exact diagrams/sequences, by using the additive property of $\operatorname{char}(\cdot)$ (see Proposition 3). Of course, it is essential to notice that the identification of $\operatorname{char}(M)$ with $\operatorname{supp}(G)$, when $G = \operatorname{gr}(M)$, only allows the use of proper prime ideals of $A[\chi]$—a reason for setting $\operatorname{char}(0)=\emptyset$. Also, if $A = k$, then we can use supp(M) instead of supp(G), exactly as proposed by Bourlès & Fliess (1997) and Wood *et al.* (to appear). The advantage of our definition is that it covers the variable-coefficient case as well by introducing another types of algebraic sets having the same dimension as the one that could be introduced in the constant-coefficient case. Hence both sets provide the same dimension numbers in the constant-coefficient case, though only the characteristic sets can be used in the variable-coefficient case. For this reason, we shall provide the following definitions for the general situation but refer, for simplicity, between braces, to the standard names that could be used with 'supp' instead of 'char':

- { observables poles } = $\text{char}((M_{\text{in}} + M_{\text{out}})/M_{\text{in}})$,
- { transmission poles } = $\text{char}((M'_{\text{in}} + M'_{\text{out}})/M'_{\text{in}})$,
- { input decoupling zeros} = $\text{char}(\text{t}(M))$,
- { input-output decoupling zeros } = $\text{char}(\text{t}(M)/(\text{t}(M) \cap (M_{\text{in}} + M_{\text{out}})))$.

We obtain from the last exact sequence with evident notations:

$$\{\text{ob.p.}\} \cup \{\text{i.o.d.z.}\} \cup \text{char}(\text{t}(M) \cap M_{\text{in}}) = \{\text{tr.p.}\} + \{\text{i.d.z.}\}.$$

If M_{in} is identifed with F, we obtain therefore $\text{t}(M) \cap M_{\text{in}} = 0$ and recover the formula (23) of Bourlès & Fliess (1997).

We may recapitulate the various modules involved on the following picture, explaining all the situations that can be met in the range of applications.

$$\begin{array}{ccccc}
 & & M & & \\
 & & \uparrow & & \\
 & & M'_{\text{in}} + M'_{\text{out}} & & \\
 & \nearrow & & \nwarrow & \\
M'_{\text{in}} & & \uparrow & & M'_{\text{out}} \\
 & \nwarrow & & \nearrow & \\
\uparrow & & \text{t}(M) & & \uparrow \\
M_{\text{in}} & & \uparrow & & M_{\text{out}}. \\
 & \nwarrow & & \nearrow & \\
 & & 0 & &
\end{array}$$

Introducing also the sets

- {system poles} = $\text{char}(M/M_{\text{in}})$,
- {output decoupling zeros} = $\text{char}(M/(M_{\text{in}} + M_{\text{out}}))$,

and using the short exact sequence

$$0 \longrightarrow (M_{\text{in}} + M_{\text{out}})/M_{\text{in}} \longrightarrow M/M_{\text{in}} \longrightarrow M/(M_{\text{in}} + M_{\text{out}}) \longrightarrow 0,$$

we obtain, with evident notations,

$$\{\text{sys.p.}\} = \{\text{ob.p.}\} + \{\text{o.d.z.}\}.$$

However, in pratice, there is no loss of generality in supposing $M = M_{\text{in}} + M_{\text{out}}$. In such a simple situation, assuming moreover $\text{t}(M) \cap M_{\text{in}} = 0$ and combining the preceding results, we get

$$\{\text{syst.p.}\} = \{\text{ob.p}\} = \{\text{tr.p.}\} + \{\text{i.d.z.}\}.$$

Also, in general, we may thus introduce the set

$$\{\text{hidden modes}\} = \text{char}(M/(M'_{\text{in}} + M'_{\text{out}})) \cup \text{char}(\text{t}(M) \cap (M_{\text{in}} + M_{\text{out}})),$$

and obtain the relation (cf. Bourlès & Fliess 1997)

$$\{\text{h.m.}\} \cup \{\text{i.o.d.z.}\} = \{\text{i.d.z.}\} \cup \{\text{o.d.z.}\}.$$

In particular, if $M = M_{\text{in}} + M_{\text{out}}$, then $\{\text{i.o.d.z.}\} = \{\text{o.d.z.}\} = \emptyset$, and we only get $\{\text{h.m.}\} = \{\text{i.d.z.}\}$ in a coherent way.

The preceding results prove that input and output play a similar role, and that it is thus better to use only the words 'zero', 'supp', or 'char' for the corresponding modules and not the word 'pole'. Also, despite duality seeming to appear only in the diagrams (11) and (12), it is in fact of constant use for constructing the characteristic sets of the various modules involved by looking at the corresponding systems.

EXAMPLE 16 If we have a SISO system $\dot{y} - y = u$ with input u satisfying $\dot{u} + u = 0$, we obtain $\ddot{y} - y = 0$ and thus $\text{supp}(M) = \{\chi - 1, \chi + 1\}$ while $\text{supp}(\text{t}(M) \cap M_{\text{in}}) = \{\chi + 1\}$, and we find the hidden mode $\chi + 1$. Such a situation can happen in an electrical LCR circuit if we suppose conditions on a voltage input.

5. Conclusion

We hope to have convinced the reader that, despite the difficulty of the underlying mathematical tools, the formal methods of algebraic analysis allow us to clarify and unify many existing results on multidimensional control systems. In most cases, the corresponding algorithms are effective and can easily be checked. Finally, this approach is the only one which can separate the intrinsic/built-in properties of a control system, such as torsion-freeness or pureness, from the other properties that depend on the choice of input and output. Meanwhile, another essential aspect is the possibility of bringing the study of modules over non-commutative rings to the simpler study of modules over commutative rings. We believe that none of the results presented here could be obtained without the use of the extension functor and/or duality, a fact explaining why it took such a long time to estabish a link between algebraic analysis and control theory.

REFERENCES

BJORK, J. E. 1979 *Rings of differential operators*. North Holland, Amsterdam.

BJORK, J. E. 1993 *Analytic $\mathcal{D}$-modules and applications*. Kluwer.

BOURLÈS, H. & FLIESS, M. 1997 Finite poles and zeros of linear systems: an intrinsic approach. *International Journal of Control* **68**, 897–922.

FLIESS, M. 1989 Commandabilité, matrices de transfert et modes cachés. *C. R. Acad. Sci. Paris Série I* **309**, 847–851.

FORNASINI, E. & VALCHER, M. E. 1997 *n*D polynomial matrices with applications to multidimensional signal analysis. *Multidimensional Systems and Signal Processing* **8**, 387–408.

GOLDSCHMIDT, H. 1968 Prolongation of linear partial differential equations: I. A conjecture of Elie Cartan. *Ann. Sci. Ecole. Norm. Sup.* IV **1**, 417–444.

GOLDSCHMIDT, H. 1970 Points singuliers d'un opérateurs différentiel analytique. *Inventiones Math.* **9**, 165–174.

HAUTUS, M. L. J. 1968 Controllability and observability conditions of linear autonomous systems. *Proc. Ned. Akad. Wetensch.* A **72**, 443–448.

HU, S. T. 1968 *Introduction to homological algebra.* Holden Day.

JANET, M. 1929 *Leçons sur les systèmes d'équations aux dérivées partielles.* Cahiers Scientifiques IV. Gauthier-villars, Paris.

JANET, M. 1921 Sur les systèmes aux dérivées partielles comprenant autant d'équations que de fonctions inconnues. *Comptes Rendus Hebdomadaires des Séances de L'Académie des Sciences* **172**, 1637–1639.

JOHNSON, J. 1978 Systems of n partial differential equations in n unknown functions: the conjecture of M. Janet. *Trans AMS* **242**, August.

KAILATH, T. 1980 *Linear systems.* Prentice-Hall, Englewood Cliffs NJ.

KALMAN, R. E., HO, Y. C., & NARENDRA, K. S. 1963 Controllability of linear dynamical systems *Contrib. Differential Equations* **1**, 189–213.

KASHIWARA, M. 1995 Algebraic study of systems of partial differential equations. *Mémoires de la Société Mathématiques de France* **63**. (Transl. from Japanese of his 1970 Master's Thesis.)

KLEON, S. & OBERST, U. 1998 *Transfer operators and state spaces for discrete multidimensional linear systems.* Seminary Report, University of Hagen, 63. pp. 409–426.

KOLCHIN, E. R. 1973 *Differential algebra and algebraic groups.* Pure and Applied Mathematics 54. Academic Press, New York.

MAISONOBE, P. & SABBAH, C. 1993 *$\mathcal{D}$-modules cohérents et holonomes.* Travaux en cours 45. Hermann, Paris.

MALGRANGE, B. 1966 *Cohomologie de Spencer (d'après Quillen). Sém. Math. Orsay*, 1966.

NORTHCOTT, D. G. 1966 *An introduction to homological algebra.* Cambridge University Press.

OBERST, U. 1990 Multidimensional constant linear systems. *Acta Appl. Math.* **20**, 1–175.

OBERST, U. 1996 Finite dimensional systems of partial differential or difference equations. *Advances in Applied Mathematics* **17**, 337–356.

PALAMODOV, V. P. 1970 *Linear differential operators with constant coefficients.* Grundlehren der Mathematischen Wissenschaften 168. Springer.

PHAM, F. 1980 *Singularités des systèmes différentiels de Gauss–Manin.* Progress in Mathematics 2. Birkhaüser, Boston.

POMMARET, J. F. 1992 New perspectives in control theory for partial differential equations. *IMA Journal of Mathematical Control & Information* **9**, 305–330.

POMMARET, J. F. 1994 *Partial differential equations and group theory: new perspectives for applications.* Kluwer, Dordrecht.

POMMARET, J. F. 1995 Duality for control systems revisited. *Proceedings of the IFAC Conference 'System Structure and Control'*, Nantes, 5–7 July 1995, Ecole Centrale de Nantes, France. pp. 438–443.

POMMARET, J. F. 2000 *Partial differential control theory.* Kluwer, Dordrecht. (To appear.)

POMMARET, J. F. & QUADRAT, A. 1997 Formal obstructions of the controllability of partial differential control systems. In: *Proceedings of the IMACS World Congress*, Berlin, August 1997.

POMMARET, J. F. & QUADRAT, A. 1998 Generalized Bezout identity. *AAECC* **9**, 91–116.

POMMARET, J. F. & QUADRAT, A. 1999 Localization and parametrization of differential operators in control theory. Accepted for publication in *Systems and Control Letters.*

RITT, J. F. 1950, 1966 *Differential algebra.* Dover.

ROTMAN, J. J. 1979 *An introduction to homological algebra*. Academic Press.

SERRE, J.-P. 1989 *Algèbre locale multiplicités*. Lecture Notes in Mathematics 11. Springer.

SPENCER, D. C. 1965 Overdetermined systems of partial differential equations. *Bull. Amer. Math. Soc.* **75**, 1–114.

YOULA, D. C. & GNAVI, G. 1979 Notes on n-dimensional system theory. *IEEE Trans* CAS **26** (2).

YOULA, D. C. & PICKEL, P. F. 1984 The Quillen–Suslin theorem and the structure of n-dimensional elementary polynomial matrices. *IEEE Trans* CAS **31**.

WILLEMS, J. C. 1991 Paradigms and puzzles in theory of dynamical systems. *IEEE Trans* AC **36**, 259–294.

WOOD, J., ROGERS, E., & OWENS, D. 1998 Formal theory of matrix primeness. *Mathematics of Control, Signals, and Systems* **11**, 40—78.

WOOD, J., OBERST, U., ROGERS, E., & OWENS, D., to appear. A behaviour approach to the pole structure of 1D and nD linear systems. *SIAM J. Control & Optimization*.

ZERZ, E. 1996 Primeness of multivariate polynomial matrices. *Systems & Control Letters* **29**, 139–145.