On a Rank factorisation Problem Arising in Gearbox Vibration Analysis

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Abstract: Given a field $\mathbb{k}$, r matrices $D_i \in \mathbb{k}^{n\times n}$, a matrix $M \in \mathbb{k}^{n\times m}$ of rank at most r, in this paper, we study the problem of factoring M as follows $M = \sum_{i=1}^{r} D_i\, u\, v_i$, where $u \in \mathbb{k}^{n\times 1}$ and $v_i \in \mathbb{k}^{1\times m}$ for $i = 1, \ldots, r$. This problem arises in modulation-based mechanical models studied in gearbox vibration analysis (e.g., amplitude and phase modulation). We show how linear algebra methods combined with linear system theory ideas can be used to characterize when this polynomial problem is solvable and if so, how to explicitly compute the solutions.

Keywords: factorisation methods, linear algebra, modulation, demodulation, vibration analysis.

1. INTRODUCTION

Notation. In what follows, $\mathbb{k}$ will denote a field (e.g., $\mathbb{k} = \mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$), $\mathbb{k}^{n\times m}$ the $\mathbb{k}$-vector space formed by all the $n \times m$ matrices with entries in $\mathbb{k}$,

$$\mathrm{GL}_n(\mathbb{k}) = \{U \in \mathbb{k}^{n\times n} \mid \det(U) \neq 0\}$$

the general linear group of invertible $n \times n$ matrices and I_n the unit of $\mathrm{GL}_n(\mathbb{k})$, i.e., the identity matrix of $\mathbb{k}^{n\times n}$.

Let $A \in \mathbb{k}^{r\times s}$ and let us consider the two $\mathbb{k}$-linear maps:

$$\begin{array}{rcl} .A : \mathbb{k}^{1\times r} & \longrightarrow & \mathbb{k}^{1\times s} \\ \lambda & \longmapsto & \lambda\, A, \end{array} \qquad \begin{array}{rcl} A. : \mathbb{k}^{s\times 1} & \longrightarrow & \mathbb{k}^{r\times 1} \\ \eta & \longmapsto & A\, \eta. \end{array}$$

Then, we can define the finite-dimensional $\mathbb{k}$-vector spaces

$$\mathrm{im}_{\mathbb{k}}(.A) := \mathbb{k}^{1\times r} A = \{\mu \in \mathbb{k}^{1\times s} \mid \exists\, \lambda \in \mathbb{k}^{1\times r} :\ \mu = \lambda\, A\},$$
$$\ker_{\mathbb{k}}(.A) := \{\lambda \in \mathbb{k}^{1\times r} \mid \lambda\, A = 0\}.$$

Similarly, we can define the following $\mathbb{k}$-vector spaces:

$$\mathrm{im}_{\mathbb{k}}(A.) := A\, \mathbb{k}^{s\times 1} = \{\zeta \in \mathbb{k}^{r\times 1} \mid \exists\, \eta \in \mathbb{k}^{s\times 1} :\ \zeta = A\, \eta\},$$
$$\ker_{\mathbb{k}}(A.) := \{\eta \in \mathbb{k}^{s\times 1} \mid A\, \eta = 0\}.$$

We recall that A is *full row rank* (resp., *full column rank*) if $\ker_{\mathbb{k}}(.A) = 0$ (resp., $\ker_{\mathbb{k}}(A.) = 0$) and a full row (resp., column) rank matrix $A \in \mathbb{k}^{r\times s}$ admits a right (resp., left) inverse $B \in \mathbb{k}^{s\times r}$, i.e., $A\, B = I_r$ (resp., $B\, A = I_s$).

Let us state the main problem studied in this paper.

The rank factorisation problem: Let $M \in \mathbb{k}^{n\times m} \setminus \{0\}$ and $D_1, \ldots, D_r \in \mathbb{k}^{n\times n} \setminus \{0\}$. Determine – if they exist – $u \in \mathbb{k}^{n\times 1}$ and $v_1, \ldots, v_r \in \mathbb{k}^{1\times m}$ satisfying:

$$M = \sum_{i=1}^{r} D_i\, u\, v_i. \tag{1}$$

Within the framework of *vibration analysis* applied to *gearbox fault surveillance*, a new *demodulation* approach of the gearbox vibration signal was proposed in Hubert et al. (2018); Hubert (2019). It first states that the gearbox spectrum can be represented as a (structured) matrix M and then that separating the time vibration signal into its two main components amounts to estimating vectors u and v set in Problem (1). The exact case of the above mentioned factorisation problem was studied in Hubert et al. (2019) and solved in Hubert et al. (2018) for $r = 1$ and in Hubert et al. (2019) for $r = 2$. In this paper, based on linear algebra, we give a proof of the general case ($r \geq 1$).

Note that Problem (1) corresponds to a polynomial system formed by $n \times m$ equations in the $n + r\, m$ unknowns $\{u, v_1, \ldots, v_r\}$. Hence, methods from algebraic geometry and symbolic computation (e.g., *Gröbner bases*, *resultants*) can be used to study Problem (1). In this paper, exploiting algebraic structures of (1) and combining them with ideas of linear systems theory, we shall show that linear algebra methods are sufficient to solve Problem (1) for the case of $\mathrm{rank}_{\mathbb{k}}(M) = r$ and for the case where $\mathrm{rank}_{\mathbb{k}}(M) \leq r$ and $v := (v_1^T\ \ldots\ v_r^T)^T \in \mathbb{k}^{r\times m}$ is a full row rank matrix. The case of a non full row rank matrix v will be studied in a future publication. Indeed, in this case, it seems that we cannot avoid the use of elimination methods (e.g., Gröbner or *Janet bases*).

2. THE FACTORISATION PROBLEM

2.1 Introductory remarks

We first note that if we set

$$A(u) := (D_1\, u \;\dots\; D_r\, u) \in \Bbbk^{n \times r},$$
$$v := (v_1^T \;\dots\; v_r^T)^T \in \Bbbk^{r \times m},$$

then Problem (1) can be rewritten as follow:

$$A(u)\, v = M. \tag{2}$$

Remark 1. If (u, v) is a solution of (2), then so is $(\lambda\, u, \lambda^{-1}\, v)$ for all $\lambda \in \Bbbk \setminus \{0\}$. Hence, if a solution exists, then it is not unique.

If $M = A(u)\, v$, then $M\, w = A(u)\,(v\, w)$ for all $w \in \Bbbk^{m \times 1}$, which shows the following inclusion of $\Bbbk$-vector spaces:

$$\mathrm{im}_{\Bbbk}(M.) \subseteq \mathrm{im}_{\Bbbk}(A(u).). \tag{3}$$

Conversely, if there exists $u \in \Bbbk^{n \times 1}$ such that (3) holds, then, denoting by $M_{\bullet i}$ the i^{th} column of M, we have $M_{\bullet i} \in \mathrm{im}_{\Bbbk}(A(u).)$ for $i = 1, \dots, m$, and thus, there exists $w_i \in \Bbbk^{r \times 1}$ such that $M_{\bullet i} = A(u)\, w_i$ for $i = 1, \dots, m$, i.e., with the notation $v := (w_1 \;\dots\; w_m) \in \Bbbk^{r \times m}$, we then get:

$$M = (M_{\bullet 1} \;\dots\; M_{\bullet m}) = (A(u)\, w_1 \;\dots\; A(u)\, w_m) = A(u)\, v.$$

Lemma 1. A necessary and sufficient condition for the existence of a solution $(u, v_1, \dots, v_r)$ of Problem (1), i.e., of (2), is the existence of $u \in \Bbbk^{n \times 1}$ satisfying (3).

In spite of the simplicity of the statement of Lemma 1, as noticed in Introduction, Problem (1) corresponds to a polynomial system in the unknowns $\{u, v_1, \dots, v_r\}$. Hence, we cannot hope to get a simple answer for the general case. Since the $\Bbbk$-vector space $\mathrm{im}_{\Bbbk}(A(u).) = \sum_{i=1}^{r} (D_i\, u)\, \Bbbk$ is generated by the r vectors $D_i\, u$'s, we have:

$$\mathrm{rank}_{\Bbbk}(A(u)) := \dim_{\Bbbk}(\mathrm{im}_{\Bbbk}(A(u).)) \leq r.$$

Hence, using (3) and $\mathrm{rank}_{\Bbbk}(M) = \dim_{\Bbbk}(\mathrm{im}_{\Bbbk}(M.))$, a necessary condition for the solvability of (1) is then:

$$\mathrm{rank}_{\Bbbk}(M) \leq r. \tag{4}$$

If Problem (2) is solvable and if v has not full row rank, then there exists $(\alpha_1 \;\dots\; \alpha_r) \in \Bbbk^{1 \times r}$, with $\alpha_k \neq 0$ for a certain $k \in \{1, \dots, r\}$, such that $\sum_{i=1}^{r} \alpha_i\, v_i = 0$, which yields $M = \sum_{i=1}^{r} \left(D_i - \alpha_i\, \alpha_k^{-1}\, D_k\right) u\, v_i$. Hence, we get $M\, w = \sum_{i=1}^{r} \left(D_i - \alpha_i\, \alpha_k^{-1}\, D_k\right) u\, (v_i\, w)$ for all $w \in \Bbbk^{m \times 1}$, where $v_i\, w \in \Bbbk$. This sum contains at most $r - 1$ non-zero vectors which yields $\mathrm{rank}_{\Bbbk}(M) \leq r - 1$. Hence, a necessary condition for the solvability of Problem (2) for a matrix M of rank r is that v has full row rank, i.e., that v admits a right inverse $t \in \Bbbk^{m \times r}$, i.e., $v\, t = I_r$. Then, (2) yields:

$$A(u) = M\, t. \tag{5}$$

Then, (5) implies the following equality of $\Bbbk$-vector spaces:

$$\mathrm{im}_{\Bbbk}(M.) = \mathrm{im}_{\Bbbk}(A(u).). \tag{6}$$

Note that (6) can also be obtained by noticing that $\mathrm{im}_{\Bbbk}(M.) \subseteq \mathrm{im}_{\Bbbk}(A(u).)$ and $\mathrm{rank}_{\Bbbk}(A(u)) \leq r = \mathrm{rank}_{\Bbbk}(M)$. Finally, we also note that (6) is equivalent to (2) and (5), which yields $A(u)\,(I_r - v\, t) = 0$, and thus, $v\, t = I_r$ since the r columns of $A(u)$ are $\Bbbk$-linearly independent.

In what follows, we shall suppose that v has full row rank since it is a necessary condition for handling the case of $\mathrm{rank}_{\Bbbk}(M) = r$, an important case in practice. If $\mathrm{rank}_{\Bbbk}(M) \leq r$, then the results presented here will only yield solutions with a full row rank matrix v. The case of non full row rank matrix v will be studied later.

Writing $t = (t_1 \;\dots\; t_r)$, $t_i \in \Bbbk^{m \times 1}$, $i = 1, \dots, r$, (5) yields

$$A(u) = (D_1\, u \;\dots\; D_r\, u) = (M\, t_1 \;\dots\; M\, t_r), \tag{7}$$

which shows that necessary conditions for the solvability of Problem (2) are given by:

$$D_i\, u \in \mathrm{im}_{\Bbbk}(M.), \quad i = 1, \dots, r. \tag{8}$$

These conditions yield the following inclusion:

$$\mathrm{im}_{\Bbbk}(A(u).) \subseteq \mathrm{im}_{\Bbbk}(M.). \tag{9}$$

Moreover, if $u \in \Bbbk^{n \times 1}$ can be chosen such that

$$\mathrm{rank}_{\Bbbk}(A(u)) = \mathrm{rank}_{\Bbbk}(M), \tag{10}$$

then we get (6), which proves the existence of a solution $(u,\, v)$ by Lemma 1. This is the approach developed in this paper: find $u \in \Bbbk^{n \times 1}$ such that (8) and (10) holds.

Let $X \in \Bbbk^{n \times l}$ be a full column rank matrix defining a basis of $\mathrm{im}_{\Bbbk}(M.)$, where $l := \mathrm{rank}_{\Bbbk}(M)$. Now, using $\mathrm{im}_{\Bbbk}(M.) = \mathrm{im}_{\Bbbk}(X.)$ and the fact that X has full column rank, there exist a unique matrix $Y \in \Bbbk^{l \times m}$ and a (non necessarily unique) matrix $T \in \Bbbk^{m \times l}$ such that:

$$\begin{cases} M = X\, Y, \\ X = M\, T. \end{cases} \tag{11}$$

Since X has full column rank, $M\, \eta = X\,(Y\, \eta) = 0$ yields $Y\, \eta = 0$, and thus $\ker_{\Bbbk}(M.) = \ker_{\Bbbk}(Y.)$. Moreover, combining the two identities of (11), we get $X = X\, Y\, T$, i.e., $X\,(Y\, T - I_l) = 0$, which yields $Y\, T = I_l$ since X has full column rank. Hence, we have $\mathrm{im}_{\Bbbk}(Y.) = \Bbbk^{l \times 1}$. We have $\ker_{\Bbbk}(.Y) = 0$ since $\lambda = (\lambda\, Y)\, T = 0$ for all $\lambda \in \ker_{\Bbbk}(.Y)$, which shows that Y is a full row rank matrix. Hence, if $\mu \in \ker_{\Bbbk}(.M)$, then $(\mu\, X)\, Y = 0$, which yields $\mu\, X = 0$ and shows that $\ker_{\Bbbk}(.M) \subseteq \ker_{\Bbbk}(.X)$ and proves $\ker_{\Bbbk}(.M) = \ker_{\Bbbk}(.X)$. Finally, since X has full column rank, there exists a left inverse $Z \in \Bbbk^{l \times n}$ of X, i.e., $Z\, X = I_l$. Hence, we have $\mathrm{im}_{\Bbbk}(.X) = \Bbbk^{1 \times l}$ and $\mathrm{im}_{\Bbbk}(.M) = \mathrm{im}_{\Bbbk}(.X\, Y) = \mathrm{im}_{\Bbbk}(.Y)$.

If $M = X'\, Y'$ with $X' \in \Bbbk^{n \times l}$, $Y' \in \Bbbk^{l \times m}$ and $\mathrm{im}_{\Bbbk}(M.) = \mathrm{im}_{\Bbbk}(X'.)$, then there exists $V \in \mathrm{GL}_l(\Bbbk)$ such that $X = X'\, V$, i.e., $X' = X\, V^{-1}$, and thus, $X'\,(Y' - V\, Y) = 0$, which yields $Y' = V\, Y$ since X' has full column rank.

If $\Bbbk = \mathbb{R}$ or $\mathbb{C}$, a way to get such a factorisation $M = X\, Y$ is to compute a *Singular Value Decomposition* of M to get $M = U\, \Sigma\, V$, where $U \in \Bbbk^{n \times n}$ (resp., $V \in \Bbbk^{m \times m}$) is a unitary matrix, i.e., $U\, U^\star = U^\star\, U = I_n$ (resp., $V\, V^\star = V^\star\, V = I_m$), where $U^\star = \overline{U}^T$ denotes the *adjoint* (i.e., the conjugate transposed) of U and $\overline{U}$ the conjugate of U, and $\Sigma \in \Bbbk^{n \times m}$ is a diagonal matrix whose diagonal entries are the *singular values* $\sigma_i(M)$ of M listed in descending order. Then, l is the number of non-zero singular values, i.e.:

$$\sigma_i(M) \neq 0,\; i = 1, \dots, l, \quad \sigma_i(M) = 0,\; i \geq l + 1.$$

Let $U = (U_1 \quad U_2)$, where $U_1 \in \Bbbk^{n \times l}$ and $U_2 \in \Bbbk^{n \times (n-l)}$, $V = (V_1^T \quad V_2^T)^T$, where $V_1 \in \Bbbk^{l \times m}$ and $V_2 \in \Bbbk^{(m-l) \times m}$, and $\sigma \in \Bbbk^{l \times l}$ be the diagonal matrix formed by the non-zero singular values of M. Then, we have

$$M = (U_1 \quad U_2) \begin{pmatrix} \sigma & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = U_1\, \sigma\, V_1,$$

which shows that we can take:

$$X = U_1 \in \Bbbk^{n \times l}, \quad Y = \sigma\, V_1 \in \Bbbk^{l \times m}. \tag{12}$$

Any factorisation $M = X' Y'$ with $\mathrm{im}_{\Bbbk}(M.) = \mathrm{im}_{\Bbbk}(X'.)$ is then of the form $X' = X\,W$, $Y' = W^{-1}\,Y$ for $W \in \mathrm{GL}_l(\Bbbk)$.

2.2 Necessary and sufficient solvability condition

Before stating an effective necessary and sufficient condition for Problem (1) and an explicit characterization of the solutions, we introduce matrices derived from M and D_i.

Let $L \in \Bbbk^{p\times n}$ be a matrix whose rows generate a basis of $\ker_{\Bbbk}(.M)$, i.e., L is a full row rank matrix satisfying

$$\ker_{\Bbbk}(.M) = \mathrm{im}_{\Bbbk}(.L) = \Bbbk^{1\times p}\,L,$$

where $p := \dim_{\Bbbk}(\ker_{\Bbbk}(.M))$. Using

$$l := \mathrm{rank}_{\Bbbk}(M) = \dim_{\Bbbk}(\mathrm{im}_{\Bbbk}(M.)) = \dim_{\Bbbk}(\mathrm{im}_{\Bbbk}(.M)),$$

we then obtain:

$$p = \dim_{\Bbbk}(\ker_{\Bbbk}(.M)) = n - \dim_{\Bbbk}(\mathrm{im}_{\Bbbk}(.M)) = n - \mathrm{rank}_{\Bbbk}(M). \tag{13}$$

Remark 2. With the notations of the end of Section 2.1,

$$U^{-1} = U^{\star} = \begin{pmatrix} U_1^{\star} \\ U_2^{\star} \end{pmatrix} \Rightarrow \begin{pmatrix} U_1^{\star} \\ U_2^{\star} \end{pmatrix} (U_1 \quad U_2) = I_n,$$

which shows that $\ker_{\Bbbk}(.U_1) = \mathrm{im}_{\Bbbk}(.U_2^{\star}) = \Bbbk^{1\times(n-l)}\,U_2^{\star}$. Hence, using (12), we get

$$\ker_{\Bbbk}(.M) = \ker_{\Bbbk}(.X) = \ker_{\Bbbk}(.U_1) = \mathrm{im}_{\Bbbk}(.U_2^{\star}),$$

which shows that we can take $L := U_2^{\star}$.

Let us now consider the following linear system:

$$\begin{cases} L\,D_1\,x = 0, \\ \quad\vdots \\ L\,D_r\,x = 0. \end{cases} \tag{14}$$

Computing a basis of the $\Bbbk$-vector space (14), i.e., of

$$\mathcal{V} := \ker_{\Bbbk}\left(\begin{pmatrix} L\,D_1 \\ \vdots \\ L\,D_r \end{pmatrix} .\right), \tag{15}$$

we obtain a full column rank matrix $Z \in \Bbbk^{n\times d}$, where $d := \dim_{\Bbbk}(\mathcal{V})$, such that $\mathcal{V} = \mathrm{im}_{\Bbbk}(Z.)$. All the solutions $x \in \Bbbk^{n\times 1}$ of (14) are then of the form:

$$\forall\,\psi \in \Bbbk^{d\times 1}, \quad x = Z\,\psi. \tag{16}$$

Example 1. Let us consider the following matrices:

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad D_1 = I_3, \quad D_2 = \begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{pmatrix}.$$

We can check that $\ker_{\Bbbk}(.M) = \Bbbk\,L$ with $L = (1 \quad 0 \quad -1)$.

$$(14) \Leftrightarrow \begin{cases} x_1 - x_3 = 0, \\ d_1\,x_1 - d_3\,x_3 = 0, \end{cases} \Leftrightarrow \begin{cases} x_3 = x_1, \\ (d_1 - d_3)\,x_1 = 0. \end{cases}$$

- If $d_1 \neq d_3$, then $x_1 = 0$ and $x = (0 \quad x_2 \quad 0)^T$ for $x_2 \in \Bbbk$, i.e., we have $Z = (0 \quad 1 \quad 0)^T$ and $\psi = x_2 \in \Bbbk$.
- If $d_1 = d_3$, then $x = (x_1 \quad x_2 \quad x_1)^T$ for $x_1, x_2 \in \Bbbk$:

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \psi = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

By definition of the matrix $L \in \Bbbk^{p\times n}$, its rows define a basis of $\ker_{\Bbbk}(.M)$. Hence, L has full row rank, which shows that $\mathrm{im}_{\Bbbk}(L.) = \Bbbk^{p\times 1}$, where p satisfies (13), and $L\,M = 0$. Thus, we have $\mathrm{im}_{\Bbbk}(M.) \subseteq \ker_{\Bbbk}(L.)$. Using (13), we obtain

$$\begin{aligned} \dim_{\Bbbk}(\ker_{\Bbbk}(L.)) &= n - \mathrm{rang}(L) = n - p = \mathrm{rank}_{\Bbbk}(M) \\ &= \dim_{\Bbbk}(\mathrm{im}_{\Bbbk}(M.)), \end{aligned}$$

which proves $\ker_{\Bbbk}(L.) = \mathrm{im}_{\Bbbk}(M.)$. Then, using the factorisation of $M = X\,Y$ defined in Section 2 (see (11)), we obtain $\ker_{\Bbbk}(L.) = \mathrm{im}_{\Bbbk}(M.) = \mathrm{im}_{\Bbbk}(X.)$.

Substituting (16) into (14), we get $L\,D_i\,Z\,\psi = 0$ for all $\psi \in \Bbbk^{d\times 1}$, i.e., $D_i\,Z\,\psi \in \ker_{\Bbbk}(L.) = \mathrm{im}_{\Bbbk}(X.)$ for all $\psi \in \Bbbk^{d\times 1}$ and $i = 1, \ldots, r$. Since X has full column rank, there exists a unique matrix $W_i \in \Bbbk^{l\times d}$ such that

$$D_i\,Z = X\,W_i, \quad i = 1, \ldots, r, \tag{17}$$

and let us note:

$$\forall\,\psi \in \Bbbk^{d\times 1}, \quad B(\psi) := (W_1\,\psi \;\ldots\; W_r\,\psi) \in \Bbbk^{l\times r}. \tag{18}$$

If $l := \mathrm{rank}_{\Bbbk}(M) = r$, then $B(\psi) \in \Bbbk^{l\times l}$, i.e., B is square.

Example 2. We continue Example 1. We first can check that we have the factorisation $M = X\,Y$, where:

$$X = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

Then, using (17), we obtain:

- If $d_1 \neq d_3$, then we have:
$$W_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \; W_2 = \begin{pmatrix} 0 \\ d_2 \end{pmatrix}, \; B(\psi) = \begin{pmatrix} 0 & 0 \\ \psi & d_2\,\psi \end{pmatrix}.$$
- If $d_1 = d_3$, then we have:
$$W_1 = I_2, \; W_2 = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}, \; B(\psi) = \begin{pmatrix} \psi_1 & d_1\,\psi_1 \\ \psi_2 & d_2\,\psi_2 \end{pmatrix}.$$

Example 3. Let us consider the case where $D_i := D^{i-1}$ for $i = 1, \ldots, r$, where $D \in \Bbbk^{n\times n}$ satisfies $D^r = I_n$. As explained in Hubert (2019), this case corresponds to a so-called *planetary gearbox*. Clearly, if x is a solution of (14), then so is $D\,x$, i.e., we have the following map:

$$\begin{aligned} D : \mathcal{V} &\longrightarrow \mathcal{V} \\ x &\longmapsto D\,x. \end{aligned}$$

Hence, if $x = Z\,\psi \in \mathcal{V}$, then there exists $\psi' \in \Bbbk^{d\times 1}$ such that $D\,x = Z\,\psi'$, i.e., $(D\,Z)\,\psi = Z\,\psi'$ for all $\psi \in \Bbbk^{d\times 1}$. Thus, there exists $D' \in \Bbbk^{d\times d}$ such that $D\,Z = Z\,D'$. Then, we get $D^2\,Z = D\,Z\,D' = Z\,{D'}^2$, which yields:

$$D^i\,Z = Z\,{D'}^i, \quad i = 0, \ldots, r-1.$$

Moreover, using $D^r = I_n$, then $Z = Z\,{D'}^r$, which yields

$${D'}^r = I_d$$

since Z has full column rank. Using (17) with $D_1 = I_n$, there exists $W_1 \in \Bbbk^{l\times d}$ such that $Z = X\,W_1$. Since both Z and X have full column rank, so has W_1. Hence, we have:

$$D^i\,Z = X\,\left(W_1\,{D'}^i\right), \quad i = 0, \ldots, r-1.$$

Since the matrices W_i's defined by (17) are unique, we get:

$$W_i = W_1\,{D'}^i, \quad i = 0, \ldots, r-1.$$

Hence, if we note

$$F(\psi) := \left(\psi \quad D'\,\psi \;\ldots\; {D'}^{r-1}\,\psi\right) \in \Bbbk^{d\times r}, \tag{19}$$

then we finally obtain $B(\psi) = W_1\,F(\psi)$.

We can now state the main result of the paper.

Theorem 3. Let $\Bbbk$ be a field, $D_i \in \Bbbk^{n\times n} \setminus \{0\}$, $i = 1, \ldots, r$, and $M \in \Bbbk^{n\times m} \setminus \{0\}$ be such that $l := \operatorname{rank}_{\Bbbk}(M) \leq r$.

Let us introduce the following matrices:

(1) Let $X \in \Bbbk^{n\times l}$ a full column matrix whose columns form a basis of $\operatorname{im}_{\Bbbk}(M.)$ and $Y \in \Bbbk^{l\times m}$ a full row rank matrix such that $M = X\,Y$ (see Section 2).
(2) Let $L \in \Bbbk^{(n-l)\times n}$ be a full row rank matrix whose rows define a basis of $\ker_{\Bbbk}(.M)$.
(3) Let $Z \in \Bbbk^{n\times d}$ be full column rank matrix whose columns define a basis of:
$$\ker_{\Bbbk}\left(\begin{pmatrix} L\,D_1 \\ \vdots \\ L\,D_r \end{pmatrix}.\right).$$
(4) Let $W_i \in \Bbbk^{l\times d}$ be the unique matrix satisfying:
$$D_i\,Z = X\,W_i, \quad i = 1, \ldots, r.$$
(5) Let $B(\psi) = (W_1\,\psi \ \ldots \ W_r\,\psi) \in \Bbbk^{l\times r}$ for all $\psi \in \Bbbk^{d\times 1}$.

Then, Problem (2) admits a solution (u, v), where v has full row rank, iff there exists $\psi \in \Bbbk^{d\times 1} \setminus \{0\}$ such that
$$\operatorname{rank}_{\Bbbk}(B(\psi)) = l, \tag{20}$$
i.e., such that $B(\psi)$ admits a right inverse $E(\psi) \in \Bbbk^{r\times l}$.

If we consider the following set
$$\mathcal{P} := \left\{\psi \in \Bbbk^{d\times 1} \setminus \{0\} \mid \operatorname{rank}_{\Bbbk}(B(\psi)) = l\right\},$$
then solutions of Problem (2) are of the form
$$\forall\, \psi \in \mathcal{P}, \quad \forall\, z \in \Bbbk^{(r-l)\times m}, \quad \begin{cases} u = Z\,\psi, \\ v = (E(\psi) \quad C(\psi)) \begin{pmatrix} Y \\ z \end{pmatrix}, \end{cases} \tag{21}$$
where $C(\psi) \in \Bbbk^{r\times(r-l)}$ is a full column rank matrix whose columns define a basis of $\ker_{\Bbbk}(B(\psi).)$. Then, v has full row rank iff z is chosen such that so has $(Y^T \quad z^T)^T \in \Bbbk^{r\times m}$.

Finally, if $l = r$, then we have $E(\psi) = B(\psi)^{-1}$, $C(\psi) = 0$, and all the solutions of Problem (2) are given by
$$\begin{cases} u = Z\,\psi, \\ v = B(\psi)^{-1}\,Y, \end{cases} \tag{22}$$
for all $\psi \in \mathcal{P} = \left\{\psi \in \Bbbk^{d\times 1} \setminus \{0\} \mid \det(B(\psi)) \neq 0\right\}$.

Proof. Let us first suppose that there exists a solution (u, v) of Problem (2), where v has full row rank. Then, multiplying (2) by L, we get:
$$L\,A(u)\,v = L\,M = 0. \tag{23}$$
Since v has full row rank, (23) is equivalent to $L\,A(u) = 0$, i.e., u satisfies the linear system (14), i.e., u is of the form $u = Z\,\psi$ for a certain $\psi \in \Bbbk^{d\times 1} \setminus \{0\}$. Since Z has full column rank, ψ is unique. Now, using (17), we get:
$$\begin{aligned} A(Z\,\psi) &= (D_1\,Z\,\psi \ \ldots \ D_r\,Z\,\psi) = X\,(W_1\,\psi \ \ldots \ W_r\,\psi) \\ &= X\,B(\psi). \end{aligned} \tag{24}$$
Hence, we have $A(Z\,\psi)\,v = X\,B(\psi)\,v = M = X\,Y$ which, using the fact that X has full row rank, yields:
$$B(\psi)\,v = Y. \tag{25}$$
Since both v and Y have full row rank, so is $B(\psi)$, i.e., $\ker_{\Bbbk}(.B(\psi)) = 0$, which yields $\operatorname{rank}_{\Bbbk}(B(\psi)) = l$, which proves (20). This condition is equivalent to the existence of a right inverse $E(\psi) \in \Bbbk^{r\times l}$ of $B(\psi)$. Then, we have
$$B(\psi)\,(E(\psi)\,Y) = Y,$$
and thus, $B(\psi)\,(v - E(\psi)\,Y) = 0$, which shows that $v - E(\psi)\,Y \in \ker_{\Bbbk}(B(\psi).) = \operatorname{im}_{\Bbbk}(C(\psi).)$, i.e., we have $v = E(\psi)\,Y + C(\psi)\,z$ for a unique $z \in \Bbbk^{(r-l)\times m}$, which shows (21) for a certain $z \in \Bbbk^{(r-l)\times m}$. Note that the matrix $U(\psi) := (E(\psi) \quad C(\psi)) \in \Bbbk^{r\times r}$ satisfies $U(\psi) \in \mathrm{GL}_r(\Bbbk)$ for all $\psi \in \mathcal{P}$ and $U(\psi)^{-1} = (B(\psi)^T \quad F(\psi)^T)^T$, where $F(\psi) \in \Bbbk^{(r-l)\times r}$ is a left inverse of $C(\psi)$. Hence, we get $z = F(\psi)\,v$. Finally, if $l = r$, then $E(\psi) = B(\psi)^{-1}$ and $\ker_{\Bbbk}(B(\psi).) = 0$, i.e., $C(\psi) = 0$, which proves (22).

Conversely, let us suppose that $\mathcal{P} \neq 0$ and let $\psi \in \mathcal{P}$. Let us define (u, v) as (21) if $l < r$ or as (22) if $l = r$. Then, using (17), $B(\psi)\,E(\psi) = I_l$ and $B(\psi)\,C(\psi) = 0$, we have
$$\begin{aligned} A(u)\,v &= (D_1\,Z\,\psi \ \ldots \ D_r\,Z\,\psi)\,v = X\,B(\psi)\,v \\ &= X\,B(\psi)\,(E(\psi)\,Y + C(\psi)\,z) = X\,Y = M, \end{aligned}$$
which proves that (u, v) is a solution of Problem (2). Finally, using $U(\psi) \in \mathrm{GL}_r(\Bbbk)$, v defined by (21) has full row rank iff $(Y^T \quad z^T)^T \in \Bbbk^{r\times m}$ has full row rank.

Remark 4. The fact that (21) gives solutions (u, v) such that the matrix v has not necessarily full row rank comes from the fact that u is chosen such as $L\,A(u) = 0$, which automatically yields $L\,A(u)\,v = 0$ independently of v.

The next remark proves that Theorem 3 does not depend on arbitrary choices of the matrices X, Y, L and Z.

Remark 5. If $M = X'\,Y'$, then it was shown in Section 2.1 that $X = X'\,V$ and $Y' = V\,Y$ for a certain $V \in \mathrm{GL}_l(\Bbbk)$, which yields $D_i\,Z = X'\,(V\,W_i)$ and shows that $W_i' = V\,W_i$ satisfies $D_i\,Z = X'\,W_i'$ since W_i' is unique, and thus,
$$B'(\psi) = (V\,W_1\,\psi \ \ldots \ V\,W_r\,\psi) = V\,B(\psi),$$
which yields $\operatorname{rank}_{\Bbbk}(B'(\psi)) = \operatorname{rank}_{\Bbbk}(B(\psi))$ and proves that (20) does not depend on arbitrary choices for X and Y.

If $L' \in K^{(n-l)\times n}$ is a full row rank matrix whose rows define a basis of $\ker_{\Bbbk}(.M)$, then there exists $U \in \mathrm{GL}_{n-l}(\Bbbk)$ such that $L' = U\,L$, which yields
$$\begin{pmatrix} L'\,D_1 \\ \vdots \\ L'\,D_r \end{pmatrix} = U \begin{pmatrix} L\,D_1 \\ \vdots \\ L\,D_r \end{pmatrix},$$
and shows that the $\Bbbk$-vector space $\mathcal{V}$ defined by (15) does not depend on an arbitrary choice for L.

If $Z' \in \Bbbk^{n\times d}$ defines another basis of (14), then there exists $U \in \mathrm{GL}_d(\Bbbk)$ such that $Z' = Z\,U$. Then, (17) yields $D_i\,Z' = D_i\,Z\,U = X\,(W_i\,U)$ and shows that $W_i' = W_i\,U$ is the unique matrix which satisfies $D_i\,Z' = X\,W_i'$ for $i = 1, \ldots, r$, which yields $B'(\psi) = (W_1'\,\psi \ \ldots \ W_r'\,\psi) = B(U\,\psi')$, and thus, $\operatorname{rank}_{\Bbbk}(B'(\psi)) = \operatorname{rank}_{\Bbbk}(B(\psi))$ and (20) does not depend on a particular choice of a basis of (14).

Example 4. Let $M \in \Bbbk^{n\times m}$ be of rank 1 and $D_1 = I_n$. Let $M = X\,Y$ be a factorisation defined by (11), where $X \in \Bbbk^{n\times 1}$ and $Y \in \Bbbk^{1\times m}$. Then, we get $Z = X$, and thus $u = X\,\psi$, where $\psi \in \Bbbk \setminus \{0\}$, $W_1 = 1$, which yields $B(\psi) = \psi$. Hence, we have $\mathcal{P} = \Bbbk \setminus \{0\}$ and $u = X\,\psi$ and $v = \psi^{-1}\,Y$ define a solution of Problem (2) for all $\psi \in \mathcal{P}$. The factorisation $M = X\,Y$ is a solution of Problem (2).

Example 5. We continue Examples 1 and 2.

- If $d_1 \neq d_3$, then $\det(B(\psi)) = 0$ for all $\psi \in \Bbbk \setminus \{0\}$, which proves that Problem (2) is not solvable.
- If $d_1 = d_3$, then $\det(B(\psi)) = (d_1 - d_2)\,\psi_1\,\psi_2$, which proves that Problem (2) is solvable iff $d_1 \neq d_2$ and

$\psi_1 \neq 0$ and $\psi_2 \neq 0$. If so, using $Z = X$, then we get:

$$u = X\,\psi = (\psi_1 \quad \psi_2 \quad \psi_3)^T,$$

$$v = B(\psi)^{-1}\,Y = \frac{1}{d_1 - d_2}\begin{pmatrix} -\dfrac{d_2}{\psi_1} & \dfrac{d_1}{\psi_2} & \dfrac{d_2}{\psi_2} \\ \dfrac{1}{\psi_1} & -\dfrac{1}{\psi_2} & -\dfrac{1}{\psi_2} \end{pmatrix}.$$

Example 6. Let us consider the following matrices:

$$M = \begin{pmatrix} 1 & 0 & 1 \\ -2 & 0 & -2 \\ 1 & 0 & 1 \end{pmatrix}, \quad D_1 = I_3, \quad D_2 = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}.$$

We can easily check that $\operatorname{rank}_{\Bbbk}(M) = 1$ and:

$$X = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, \; Y = (1 \quad 0 \quad 1), \; Z = X, \; W_1 = 1, \; W_2 = 0.$$

Hence, we obtain $B(\psi) = (\psi \quad 0)$, $\mathcal{P} = \Bbbk \setminus \{0\}$ and $E(\psi) = \psi^{-1}\,(1 \quad 0)^T$ is a right inverse of $B(\psi)$ for all $\psi \in \mathcal{P}$. Since $\ker_{\Bbbk}(B(\psi).) = (0 \quad 1)^T\,\Bbbk$, the solutions of Problem (2) for a full row row matrix v are of the form:

$$u = (1 \quad -2 \quad 1)^T\,\psi, \quad \forall\,\psi \in \mathcal{P},$$

$$v = \begin{pmatrix} \psi^{-1} & 0 & \psi^{-1} \\ z_1 & z_2 & z_3 \end{pmatrix}, \quad \forall\, z = (z_1 \quad z_2 \quad z_3) \in \Bbbk^{1\times 3}.$$

Finally, the matrix v has full row rank iff the matrix $(Y^T \quad z^T)^T$ has full row rank, i.e., iff $z_1 \neq z_3$ or $z_2 \neq 0$.

Example 7. We continue Example 3. By Theorem 3, Problem (2) is solvable iff the matrix $B(\psi) = W_1\,F(\psi)$ admits a right inverse, where the matrix C is defined by (19).

If $f : \mathcal{E} \to \mathcal{F}$ and $g : \mathcal{F} \to \mathcal{G}$ are two $\Bbbk$-linear maps between finite-dimensional $\Bbbk$-vector spaces with g injective, then we can prove that $g \circ f$ is surjective iff so are f and g (and so g is invertible). Hence, using the fact that W_1 has full column rank, i.e., $\ker_{\Bbbk}(W_1) = 0$, we obtain that $\operatorname{rank}_{\Bbbk}(B(\psi)) = l$ iff $\operatorname{rank}_{\Bbbk}(F(\psi)) = d$ and $\operatorname{rank}_{\Bbbk}(W_1) = l$, i.e., iff $\operatorname{rank}_{\Bbbk}(F(\psi)) = d$ and $W_1 \in \mathrm{GL}_l(\Bbbk)$. We obtain that Problem (2) is solvable iff the following 3 conditions hold:

(1) $d := \dim_{\Bbbk}(\mathcal{V}) = l := \operatorname{rank}_{\Bbbk}(M)$,

(2) W_1 is invertible, i.e., $\det(W_1) \neq 0$,

(3) there exists $\psi \in \Bbbk^{d\times 1} \setminus \{0\}$ such that $\operatorname{rank}_{\Bbbk}(F(\psi)) = l$.

In particular, if $l = r$, then $F(\psi) \in \Bbbk^{r\times r}$ and the above Condition 3 is reduced to the existence of $\psi \in \Bbbk^{r\times 1} \setminus \{0\}$ such that $F(\psi)$ is invertible, i.e., $\det F(\psi) \neq 0$.

For instance, if we consider the following matrices

$$M = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad D^2 = I_3,$$

then we obtain $L = L\,D = (1 \quad -2 \quad 1)$ and

$$Z = \begin{pmatrix} 2 & -1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad X = \begin{pmatrix} 1 & 2 \\ 4 & 5 \\ 7 & 8 \end{pmatrix}, \quad Y = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{pmatrix},$$

$$W_1 = \frac{1}{3}\begin{pmatrix} -8 & 5 \\ 7 & -4 \end{pmatrix}, \; W_2 = \frac{1}{3}\begin{pmatrix} 2 & -5 \\ -1 & 4 \end{pmatrix}, \; D' = \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}.$$

We can check that $D'^2 = I_2$ and $\det(W_1) = -1/3 \neq 0$,

$$u = \begin{pmatrix} 2\,\psi_1 - \psi_2 \\ \psi_1 \\ \psi_2 \end{pmatrix}, \quad F(\psi) = \begin{pmatrix} \psi_1 & \psi_1 \\ \psi_2 & 2\,\psi_1 - \psi_2 \end{pmatrix},$$

for all $\psi = (\psi_1 \quad \psi_2)^T \in \Bbbk^{2\times 1}$, and:

$$\det(F(\psi)) = 2\,\psi_1\,(\psi_1 - \psi_2).$$

Hence, if we consider $\psi_1 \neq 0$ and $\psi_1 \neq \psi_2$, then we have:

$$E(\psi) = B(\psi)^{-1} = \det(F(\psi))^{-1}\begin{pmatrix} \psi_1 - 4\,\psi_2 & 2\,\psi_1 - 5\,\psi_2 \\ 7\,\psi_1 - 4\,\psi_2 & 8\,\psi_1 - 5\,\psi_2 \end{pmatrix},$$

$$v = \det(F(\psi))^{-1}\begin{pmatrix} \psi_1 - 4\,\psi_2 & 2\,\psi_1 - 5\,\psi_2 & 3\,(\psi_1 - 2\,\psi_2) \\ 7\,\psi_1 - 4\,\psi_2 & 8\,\psi_1 - 5\,\psi_2 & 3\,(3\,\psi_1 - 2\,\psi_2) \end{pmatrix}.$$

Remark 6. If $\psi \in \mathcal{P}$, then $B(\psi)$ admits a right inverse $E(\psi)$. Then, using $B(\lambda\,\psi) = \lambda\,B(\psi)$ for $\lambda \in \Bbbk \setminus \{0\}$, $B(\lambda\,\psi)$ admits a right inverse $E(\lambda\,\psi)$ (e.g., $\lambda^{-1}\,E(\psi)$), i.e., $B(\lambda\,\psi)\,E(\lambda\,\psi) = I_l$ for $\lambda \in \Bbbk \setminus \{0\}$. Hence, we get $B(\psi)\,(\lambda\,E(\lambda\,\psi) - E(\psi)) = 0$, which shows that there exists a unique $q(\psi, \lambda) \in \Bbbk^{(r-l)\times l}$ such that:

$$\lambda\,E(\lambda\,\psi) = E(\psi) + C(\psi)\,q(\psi, \lambda).$$

Hence, we obtain $E(\lambda\,\psi) = \lambda^{-1}\,E(\psi) + C(\psi)\,\lambda^{-1}\,q(\psi, \lambda)$. In particular, setting $\lambda = 1$, we get $C(\psi)\,q(\psi, 1) = 0$. Now, for $\lambda \in \Bbbk \setminus \{0\}$ and $\psi \in \mathcal{P}$, let us consider a solution $(u = Z\,\psi, \; v = E(\psi)\,Y + C(\psi)\,z)$ of Problem (2). We get:

$$\begin{cases} \lambda\,u = Z\,(\lambda\,\psi), \\ \lambda^{-1}\,v = E(\lambda\,\psi)\,Y + \lambda^{-1}\,C(\psi)\,(z - q(\psi, \lambda)\,Y). \end{cases}$$

Using (24) and $B(\lambda\,\psi)\,C(\psi) = \lambda\,B(\psi)\,C(\psi) = 0$, we get

$$\begin{aligned} & A(\lambda\,u)\,(\lambda^{-1}\,v) \\ &= A(Z\,\lambda\,\psi)\,(\lambda^{-1}\,v) = X\,B(\lambda\,\psi)\,(\lambda^{-1}\,v) \\ &= X\,B(\lambda\,\psi)\,(E(\lambda\,\psi)\,Y + \lambda^{-1}\,C(\psi)\,(z - q(\psi, \lambda)\,Y)) = X\,Y, \end{aligned}$$

which shows that the solutions (21) and (22) of Problem (2) are stable under the following transformations:

$$\forall\,\lambda \in \Bbbk \setminus \{0\}, \quad \varphi_\lambda : (u, \, v) \mapsto (\lambda\,u, \, \lambda^{-1}\,v).$$

We explicitly find again a result stated in Remark 1. This result comes from the bilinear structure of Problem (2), which is a *multi-homogeneous polynomial system* and whose natural geometric framework is *multi-projective algebraic geometry*. The geometric structures of Problem (1) will be studied in a forthcoming publication.

Finally, let us study how (20) can be effectively checked. Let $A = \Bbbk[\psi_1, \ldots, \psi_r]$ denote the commutative polynomial ring in $\psi_1, \ldots, \psi_r$ with coefficients in $\Bbbk$. Then, $B(\psi)$ can be considered as an element $B \in A^{l\times r}$. We note that (20) is equivalent to the existence of $\psi \in \Bbbk^{d\times 1}$ such that $\ker_{\Bbbk}(.B(\psi)) = 0$. Using elimination methods, we can test whether or not $\ker_A(.B) = \{\lambda \in A^{1\times l} \mid \lambda\,B = 0\}$ is 0. We can use the command SyzygyModule of the OreModules package. If $\ker_A(.B) \neq 0$, then (20) is never satisfied and Problem (1) is not solvable. If $\ker_A(.B) = 0$, then the A-module $N := \operatorname{coker}_A(B.) = A^{l\times 1}/\left(B\,A^{r\times 1}\right)$ is a *torsion* A-module, i.e., for all $n \in N$, there exists $0 \neq a \in A$ such that $a\,n = 0$. We can compute its *annihilator* $\operatorname{ann}_A(N) := \{a \in A \mid a\,N = 0\}$, i.e., compute $g_1, \ldots, g_t \in A$ such that $\operatorname{ann}_A(N) = \left\{\sum_{i=1}^t a_i\,g_i \mid a_i \in A\right\}$. We can use the command PiPolynomial of the OreModules package Considering the ring $S_{g_i}^{-1}\,A := \{a/g_i^r \mid r \in \mathbb{Z}_{\geq 0}\}$, we have $S_{g_i}^{-1}\,N = 0$, i.e., $B(\psi)\,(S_{g_i}^{-1}\,A)^{r\times 1} = (S_{g_i}^{-1}\,A)^{l\times 1}$, which shows that B admits a right inverse $E \in (S_{g_i}^{-1}\,A)^{r\times l}$.

Hence, $\mathrm{ann}_A(N)$ defines the obstructions for (20), i.e., (20) is not satisfied for all $\psi \in \Bbbk^{d\times 1}$ satisfying $g_i(\psi) = 0$. For instance, if $l = r$, then $\mathrm{ann}_A(N) = A\, g_1$ and $g_1 = \det(B)$.

3. A FEW MORE RESULTS

Using the factorisation (24), i.e., $A(Z\,\psi) = X\,B(\psi)$, where X has full column rank, we obtain:

$$\forall\, \psi \in \Bbbk^{d\times 1}, \quad \ker_{\Bbbk}(A(Z\,\psi).) = \ker_{\Bbbk}(B(\psi).).$$

Hence, for $\psi \in \Bbbk^{d\times 1}$, we have:

$$\dim_{\Bbbk}(\ker_{\Bbbk}(A(Z\,\psi).)) = \dim_{\Bbbk}(\ker_{\Bbbk}(B(\psi).)) = r - \mathrm{rank}_{\Bbbk}(B(\psi)).$$

Using (20) of Theorem 3, we obtain the following corollary.

Corollary 7. Problem (2) is solvable with a full row rank matrix V iff there exists $\psi \in \Bbbk^{d\times 1} \setminus \{0\}$ such that:

$$\dim_{\Bbbk}(\ker_{\Bbbk}(A(Z\,\psi).)) = r - l. \qquad (26)$$

Corollary 7 is a generalization of a result obtained in Hubert et al. (2019) for $r = 2$.

To end this section, we focus on the particular case

$$\mathrm{rank}_{\Bbbk}(M) = r = 2, \quad D_1 = I_2,$$

which is interesting in practice (Hubert (2019)). Using $D_1 = I_n$, we get $\mathcal{V} \subseteq \ker_{\Bbbk}(L.)$, which, using (13), yields:

$$\dim_{\Bbbk}(\mathcal{V}) \leq \dim_{\Bbbk}(\ker_{\Bbbk}(L.)) = n - p = \mathrm{rank}_{\Bbbk}(M) = 2. \quad (27)$$

Moreover, (26) is equivalent to $\ker_{\Bbbk}(A(Z\,\psi).) = 0$, where:

$$\ker_{\Bbbk}(A(u).) = \{(\alpha_1 \quad \alpha_2)^T \in \Bbbk^{2\times 1} \mid u\,\alpha_1 + D_2\,u\,\alpha_2 = 0\}.$$

If $\alpha_1 = 0$, then $D_2\,u = 0$, i.e., u is an eigenvector of D_2 with 0 as associated eigenvalue. If $\alpha_2 \neq 0$, then $D_2\,u = (-\alpha_1/\alpha_2)\,u$, i.e., u is an eigenvalue of D_2 with $-\alpha_1/\alpha_2$ as associated eigenvalue.

Corollary 8. If $\mathrm{rank}_{\Bbbk}(M) = r = 2$ and $D_1 = I_2$, then Problem (2) is solvable iff there exists $\psi \in \Bbbk^{d\times 1} \setminus \{0\}$ such that $u = Z\,\psi$ is not an eigenvalue of the matrix D_2.

Example 8. We consider again Example 1. If $d_1 \neq d_3$, then $u = (0 \quad \psi \quad 0)^T$ is an eigenvector of D_2 for all $\psi \in \Bbbk \setminus \{0\}$, which proves that Problem (2) is not solvable. If $d_1 = d_3$, then $D_2\,u = (d_1\,\psi_1 \quad d_2\,\psi_2 \quad d_1\,\psi_1)^T$, where $u = (\psi_1 \quad \psi_2 \quad \psi_1)^T$. If $d_2 = d_1$, then u is an eigenvalue of D_2, which proves that Problem (2) is not solvable. If $d_2 \neq d_1$, u is not an eigenvalue of D_2 and Problem (2) is solvable. We find again the results obtained in Example 5.

Example 9. We consider again Example 7. The vector $u = (2\,\psi_1 - \psi_2 \quad \psi_1 \quad \psi_2)^T$ is an eigenvector of D iff

$$\exists\, \lambda \in \Bbbk, \quad D\,u = \begin{pmatrix} \psi_2 \\ \psi_1 \\ 2\,\psi_1 - \psi_2 \end{pmatrix} = \lambda \begin{pmatrix} 2\,\psi_1 - \psi_2 \\ \psi_1 \\ \psi_2 \end{pmatrix},$$

i.e., iff $\lambda = 1$ and $\psi_2 = \psi_1$ (compare with Example 7).

Example 10. We consider the case of a diagonal matrix $D_2 = \mathrm{diag}(d_1, \ldots, d_n)$ whose diagonal elements are the d_i's. Then, using (27), $u \in \Bbbk^{n\times 1} \setminus \{0\}$ must belong to the solution space of the linear system (14) of dimension less than or equal to 2. According to Corollary 8, to solve Problem (2), we must find $0 \neq u \in \Bbbk^{n\times 1}$ satisfying (14) and such that u is not an eigenvector of D_2. Let $\{f_j\}_{j\in J}$ be a basis of $\mathcal{V}$, where $J = \{1\}$ if $\dim_{\Bbbk}(\mathcal{V}) = 1$ and $J = \{1, 2\}$ if $\dim_{\Bbbk}(\mathcal{V}) = 2$. The vectors e_i's of the standard basis of $\Bbbk^{n\times 1}$ – where e_i is the column vector with 1 at the i^{th} position and 0 elsewhere – are the eigenvectors of D_2 with d_i as an associated eigenvalue.

If all the d_i's are distinct, then Problem (1) is solvable iff

$$\forall\, j \in J,\ \exists\, \gamma_j \in \Bbbk: \ u := \sum_{j\in J} \gamma_j\, f_j \notin \Bbbk\, e_i, \quad i = 1, \ldots, n,$$

i.e., iff there exists a $\Bbbk$-linear combination of the f_i's with at least two non-zero components.

If the multiplicity of certain of the d_i's is strictly larger than 1, denoting by $\{d'_1, \ldots, d'_r\}$ the set of distinct d_i's and by $\mathcal{E}_i$ the $\Bbbk$-vector space generated by the e_j's whose indices j correspond to the position j of d'_i in the set $\{d_1, \ldots, d_n\}$, then a solution of Problem (1) exists iff:

$$\forall\, j \in J,\ \exists\, \gamma_j \in \Bbbk: \ u := \sum_{j\in I} \gamma_j\, f_j \notin \mathcal{E}_i, \quad i = 1, \ldots, r.$$

In other words, Problem (1) admits a solution iff there exists a $\Bbbk$-linear combination of the f_i's with a non-zero entry at a position different from the j^{th} positions corresponding to the e_j's defining the $\mathcal{E}_i$'s for $i = 1, \ldots, r$.

Example 11. If u and $D_2\,u$ can be chosen to be orthogonal, then they cannot be collinear, and thus, u is not an eigenvector of D_2 and Problem (2) is solvable.

4. CONCLUSION

In this paper, we give an effective necessary and sufficient condition for the solvability of the factorisation problem $M = \sum_{i=1}^{r} D_i\,u\,v_i$ in the case where $\mathrm{rank}_{\Bbbk}(M) \leq r$ and $v := (v_1^T \ \ldots \ v_r^T)^T$ is a full row rank matrix. Moreover, we give an explicit form of all the solutions.

In future works, we shall study the case of non full row rank matrices v as well as the (multi-projective) geometric structures of Problem (1) based on the use of the multi-homogeneity of the corresponding polynomial system.

Finally, the following minimization problem

$$\min_{u\in\Bbbk^{n\times 1},\, v\in\Bbbk^{r\times m}} \left\| \sum_{i=1}^{r} D_i\,u\,v_i - M \right\|_{\text{Frob}}, \qquad (28)$$

where $\|\cdot\|_{\text{Frob}}$ is the *Frobenius norm*, will be studied, particularly based on symbolic-numeric methods for $\Bbbk = \mathbb{R}$. As shown in Hubert (2019), the demodulation problem corresponds to (28) where the matrices M and $D_1, \ldots, D_r$ are *centrohermitian* and the vectors u and $v_1, \ldots, v_r$ are *centrohermitian*. Using Hubert et al. (2020), this problem can be transformed into (28) for $k = \mathbb{R}$. This study corresponds to the demodulation problem for noisy data.

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