Parametric study of the critical pairs of linear differential systems with commensurate delays

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Abstract:
This paper aims at studying the stability of linear differential systems with commensurate delays and arbitrary real parameters. Parameters naturally arise in numerous problems such as, for instance, the design of stabilizing controllers. It is well-known that the asymptotic stability of a purely retarded linear differential system is related to the condition that all the complex roots of the corresponding quasipolynomial have negative real parts. In different approaches, the stability analysis strongly relies on the computation of the critical pairs of a quasipolynomial, i.e., the amplitudes of the delay and the frequencies which are roots of the quasipolynomial. The delays which correspond to the critical zeros/frequencies define a boundary for which the behavior of the system regarding to its stability can change. The number of the critical zeros as well as their multiplicities give important information about the stability and usually reflect the difficulty of the stability analysis of the system. In this work, using standard computer algebra techniques, particularly on the resolution of algebraic systems with parameters, we propose a new method that characterizes the set of critical pairs of a quasipolynomial in terms of the system parameters. More precisely, starting from a quasipolynomial $p(s, e^{-\tau\, s}, u) \in k[s, u][e^{-\tau\, s}]$, where $u = \{u_1, \ldots, u_r\}$ are r real parameters, our method decomposes the parameter space $\mathbb{R}^r$ into disjoint regions (semi-algebraic sets) on which the number of the critical zeros is constant. As a consequence, we can choose values of the parameters u which reduce the number/multiplicities of the critical zeros of the quasipolynomial, which can substantially simplify the stability analysis of the corresponding system.

Keywords: Time-delay systems, systems with parameters, stability analysis, quasipolynomials, critical pairs, discriminant variety.

1. INTRODUCTION

In this extended abstract, we develop computer algebra methods towards the stability analysis of linear time-invariant differential systems with commensurate time-delays and whose coefficients depend polynomially on a finite set of real parameters $\{u_1, \ldots, u_r\}$. More precisely, we consider systems whose dynamics in the frequency domain are defined by quasipolynomials of the form

$$f(s, \tau, u) = \sum_{k=0}^{n} a_k(s, u)\, e^{-k\,\tau\, s}, \qquad (1)$$

where the a_k's are polynomials in the complex variable s with coefficients belonging to the commutative polynomial ring $A = \mathbb{Q}[u_1, \ldots, u_s]$, i.e., $a_k \in A[s]$ for $k = 0, \ldots, n$.

In this extended abstract, we investigate the stability of linear time-invariant differential commensurate time-delay systems whose dynamics are defined by quasipolynomials of the form of (1). We recall that the (asymptotically) stability of a quasipolynomial of the form of (1) is (partially) related to the real part of the complex zeros s of $f(s, \tau, u) = 0$. Clearly, the real part of such complex zeros can change with the system parameters u, and so does the behavior of the corresponding system. For instance, the quasipolynomial (1) can be the dynamics of a closed-loop system where the parameters u appear in the controller, and they have to be tuned in a way to achieve the stability of the global system. Thus, the study of the location of the complex solutions s of a quasipolynomial $f(s, \tau, u)$ with parameters u is an important issue in stability analysis.

In the absence of parameters, a classical approach for analyzing the asymptotic stability of purely retarded systems is based on the computation of the so-called *critical pairs* of $f(s, \tau)$, i.e., the pairs $(\omega, \tau) \in \mathbb{R} \times \mathbb{R}_+$ such that $f(i\,\omega, \tau) = 0$. For instance, see Li et al. (2015); Gu et al. (2003); Marshall et al. (1992); Niculescu (2001) and the references therein. If such critical pairs exist, the stability is then derived from the asymptotic behavior of the coor-

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dinates s of these pairs, called *critical imaginary roots* of $f(s,\tau)$, that is to say the way these critical imaginary roots behave under a small variation of the time-delay τ. For more details, see Niculescu (2001); Bouzidi et al. (2016) and the references therein. As a consequence, the number of the critical pairs has a direct and significant impact on the difficulty of analyzing the asymptotic stability of the system.

In the case of systems with parameters as (1), the number of the critical pairs obviously depends on the parameters values. A challenging problem consists in analyzing the variation of the number of critical pairs with respect to the variation of the parameters u. We can then select parameter values that reduce this number and thus eases the analysis of stability.

In what follows, we propose a method for studying the critical pair of $f(s,\tau,u)$ with respect to the parameter values u. More precisely, given a quasi-polynomial of the form of (1) that depends on a set of parameters u, we show how to compute regions $\mathcal{U}_1,\ldots,\mathcal{U}_s$ in the parameter space $\mathbb{R}^r$ such that the number of critical pairs of $f(s,\tau,\mu)$ is constant for all $\mu=(\mu_1,\ldots,\mu_r)\in\mathcal{U}_j$ and for all $j=1,\ldots,s$. Note that the number of critical pairs can change from a region $\mathcal{U}_j$ to another but stays constant inside a given $\mathcal{U}_j$.

The paper is organized as follows, in Section 2, we introduce the concept of a *discriminant variety* associated to an algebraic set, which will be our main tool in what follows. In Section 3, we present our approach for studying the critical pairs of a quasipolynomial that depends on a set of parameters. The idea behind this approach is to first construct an algebraic system that encodes these critical pairs, and then to use the discriminant variety in order to obtain a suitable characterization of the latters. Finally, in Section 4, we illustrate our method through simple examples.

2. DISCRIMINANT VARIETY

The main tool we use is the so-called *discriminant variety* associated to an algebraic set, which was introduced in Lazard and Rouillier (2007). Before recalling the definition of this object, let us start with some useful notations.

For polynomials $p_1,\ldots,p_m\in\mathbb{Q}[x_1,\ldots,x_{n-r},u_1,\ldots,u_r]$, we can consider the following corresponding algebraic set:

$$\mathcal{S}=\{\alpha\in\mathbb{C}^n \mid p_1(\alpha)=0,\ldots,p_m(\alpha)=0\}. \qquad (2)$$

We can also consider the canonical projection onto the parameter space $\mathbb{C}^r$, namely, the following map:

$$\begin{array}{cccc}\Pi_u: & \mathbb{C}^n & \longrightarrow & \mathbb{C}^r\\ & (x_1,\ldots,x_{n-r},u_1,\ldots,u_r) & \longmapsto & (u_1,\ldots,u_r).\end{array}$$

Finally, we denote by $\overline{\Pi_u(\mathcal{S})}$ the so-called *Zariski closure* of the projection of $\mathcal{S}$ onto the parameter space $\mathbb{C}^r$.

Definition 1. (Lazard and Rouillier (2007)). With the above notations, an algebraic variety $V\subset\mathbb{C}^r$ is called a *discriminant variety* of $\mathcal{S}$ if the following conditions are satisfied:

(1) V is contained in $\overline{\Pi_u(\mathcal{S})}$.
(2) The connected components $\mathcal{U}_1,\ldots,\mathcal{U}_s$ of
$$\overline{\Pi_u(\mathcal{S})}\setminus V$$
are analytic submanifolds (note that if $\overline{\Pi_u(\mathcal{S})}$ is connected, there is only one component).
(3) For $j=1,\ldots,s$, $(\Pi_u^{-1}(\mathcal{U}_j)\cap\mathcal{S},\Pi_u)$ is an analytic covering of $\mathcal{U}_j$.

A consequence of Definition 1 is a fundamental property of the discriminant variety which is stated in the next theorem. In this theorem, we assume that the polynomial system $\mathcal{S}$ defined by (2) is *generically zero-dimensional*, namely, for almost all values of the parameters $\mu\in\mathbb{C}^r$, the polynomial system $\mathcal{S}_{u=\mu}$, obtained by substituting the parameters u to μ, admits a finite number of complex solutions.

Theorem 1. [Lazard and Rouillier (2007)] Let $\mathcal{S}$ be an algebraic system and $\mathcal{U}_1,\ldots,\mathcal{U}_s$ defined as in Definition 1. Then, for two vectors of parameters $\mu,\nu\in\mathcal{U}_j$, the specialized polynomial systems $\mathcal{S}_{u=\mu}$ and $\mathcal{S}_{u=\nu}$ have exactly the same number of zeros.

Given a system S defined by a set of polynomials $\{p_1,\ldots,p_m\}\subset\mathbb{Q}[x_1,\ldots,x_{n-r},u_1,\ldots,u_r]$, we can compute a set of polynomials $\{h_1,\ldots,h_s\}\subset\mathbb{Q}[u_1,\ldots,u_r]$ whose zeros define a discriminant variety associated to S. The polynomials $\{h_1,\ldots,h_s\}$ are computed by means of variable eliminations using, for instance, standard *Gröbner bases computations* (see, e.g., Lazard and Rouillier (2007)). Once we have computed the discriminant variety, the complementary of this algebraic variety in $\mathbb{C}^s$ can be partitioned into a set of connected components using, for instance, the classical *Cylindrical Algebraic Decomposition* (CAD) algorithm (Arnon et al. (1984)). Given a set of polynomials $F=\{h_1,\ldots,h_s\}\subset\mathbb{Q}[u_1,\ldots,u_r]^s$, a cylindrical algebraic decomposition adapted to F is, roughly speaking, a disjoint union of cells in $\mathbb{R}^r$ (these cells are described by *semi-algebraic sets*, namely, a set of polynomial equations and inequalities) in which the signs of all the polynomials h_k's are constant.

For more details, see Lazard and Rouillier (2007).

Example: To better grasp the concept of discriminant variety, let's describe it through a standard example. Consider the quadratic polynomial whose coefficients are given as parameters, $f:=a\,x^2+b\,x+c$. A discriminant variety of the polynomial f is nothing but the zeros of its discriminant $b^2-4\,a\,c$. Indeed for any a_0,b_0,c_0 such that $b_0^2-4\,a_0\,c_0\neq 0$, the polynomial $a_0\,x^2+b_0\,x+c_0$ has exactly two distinct roots. Furthermore, this discriminant is computed by eliminating the variable x, in the system defined by f and its derivative with respect to x, $\frac{\partial f}{\partial x}$.

3. CRITICAL PAIR CHARACTERIZATION

Theorem 1 can be used to directly study the zeros s of the quasipolynomial $f(s,\tau,u)$ with respect to the parameter values u. But, due to the presence of transcendental terms, this quasipolynomial usually admits, for a generic value of parameters u, an infinite number of complex zeros, a fact which contradicts the assumption of Theorem 1. Following the approach developed in Niculescu (2001), we can reduce the problem of studying the zeros s of $f(s,\tau,u)$ to the study of the real solutions of a generically zero-dimensional polynomial system. To do that, the standard transformation, called *Rekasius transformation*, is used.

This transformation, which appears in Rekasius (1980), has been used in the context of time-delay systems in a series of papers (see, e.g., Niculescu (2001) and the references therein).

Rekasius transformation. This transformation consists in replacing in the quasipolynomial $f(i\,\omega, \tau, u)$ the term $e^{-\tau\, i\, \omega}$ by the rational fraction $\frac{1-T\,i\,\omega}{1+T\,i\,\omega}$, where $T \in \mathbb{R}$. Cleaning the denominators, we then obtain a polynomial of the form $\mathcal{R}(\omega, T, u) + i\,\mathcal{I}(\omega, T, u)$. One can notice that the above transformation yields a one-to-one mapping between the zeros (ω, τ) of $f(\omega, \tau, u)$ that satisfy $\tau\,\omega \neq (2\,k+1)\,\pi$ for $k \in \mathbb{Z}$ (the roots (ω, τ) such that $e^{-\tau\, i\,\omega} = -1$), and the solutions of the following polynomial system:

$$\begin{cases} \mathcal{R}(\omega, T, u) = 0, \\ \mathcal{I}(\omega, T, u) = 0. \end{cases} \tag{3}$$

Moreover, given a solution $(\omega, T) \in \mathbb{R}^2$ of (3), the critical delays can then be obtained by:

$$\tau_k = \frac{2}{\omega}\,(\arctan(\omega\,T) + k\,\pi), \quad k \in \mathbb{Z}. \tag{4}$$

In order to catch the remaining zeros of $f(\omega, \tau, u)$ (i.e., (ω, τ) such that $\omega\,\tau = (2\,k+1)\,\pi$), we also need to consider the polynomial $f_0(\omega, u)$ resulting from $f(\omega, \tau, u)$ after substituting $e^{-\tau\, i\,\omega}$ by -1. Similalry as above, this polynomial yields the following polynomial system:

$$\begin{cases} \mathcal{R}_0(\omega, u) = 0, \\ \mathcal{I}_0(\omega, u) = 0, \end{cases} \tag{5}$$

and the critical delays are then deduced from the solutions of (5) as:

$$\tau_k = \frac{(2\,k+1)}{\omega}\,\pi, \quad k \in \mathbb{Z}. \tag{6}$$

Combining the previous results, the critical pairs of the polynomial $f(s, \tau, u)$ can be deduced from the union of the solutions of both the systems (3) and (5).

we are now in position to apply Theorem 1 in order to characterize the zeros of each of the systems (3) and (5) with respect to the parameters u. Precisely, we first compute discriminant varieties for the zeros of $\{\mathcal{R}, \mathcal{I}\}$ and the zeros of $\{\mathcal{R}_0, \mathcal{I}_0\}$, which yields a set of polynomials $h_1, \ldots, h_s \in \mathbb{Q}[u]$. Then, we compute a cylindrical algebraic decomposition of this set of polynomials. This yields a set of disjoint cells in $\mathbb{R}^r$ from which we only keep those that do not intersect the variety of $h_1, \ldots, h_s$, i.e., $\mathcal{U}_1, \ldots, \mathcal{U}_s$. Finally, for the computation of the (constant) number of critical pairs over each $\mathcal{U}_j$, it suffices to take one vector μ of parameter values in $\mathcal{U}_j$ and to solve the zero-dimensional systems resulting from (3) and (5) after substituting u by μ. For instance, this can easily be done by first computating *Rational Univariate Representations* of the corresponding zero-dimensional polynomial systems $\mathcal{S}_{u=\mu}$ (Rouillier (1999)) and then use a very efficient algorithm algorithm for the numerical isolation of roots of univariate polynomials (Kobel et al. (2016)). These two steps can be done, e.g., by means of the `RationalUnivariateRepresentation` command of the Maple package `Groebner`.

4. ILLUSTRATIVE EXAMPLES

We now illustrate our approach on the following examples.

Example 1. As a first example, we consider the following quasipolynomial, which appearcs in several work, e.g.,: Kamen (1980); Thowsen (1981); Hertz et al. (1984):

$$f(s, \tau) = s + u_1 + u_2\, e^{-\tau\, s} \tag{7}$$

Following the approach descibed in 3, we first construct the set of critical pairs of $f(s, \tau)$. The latter is given by the two following systems:

$$\begin{cases} -T\,\omega^2 + u_1 + u_2 = 0, \\ u_1\,T\,\omega - u_2\,T\,\omega + \omega = 0, \end{cases} \quad \begin{cases} \omega = 0, \\ u_1 - u_2 = 0, \end{cases}. \tag{8}$$

Then, using the Maple routine `CellDecomposition` of the `RootFinding[Parametric]` package, we compute a set of polynomials whose zeros define a discriminant variety of the systems (8),

$$d_1 := u_1 - u_2, \quad d_2 := u_1 + u_2,$$

as well as a cylindrical algebraic decomposition of the complementary of this discriminant variety, which yields the six following cells, depicted in Figure 1.

$c_1 : u_2 < 0, u_1 < u_2$

$c_2 : u_2 < 0, u_2 < u_1 < -u_2$

$c_3 : u_2 < 0, u_1 > -u_2$

$c_4 : u_2 > 0, u_1 < -u_2$

$c_5 : u_2 > 0, -u_2 < u_1 < u_2$

$c_6 : u_2 > 0, u_1 > u_2$

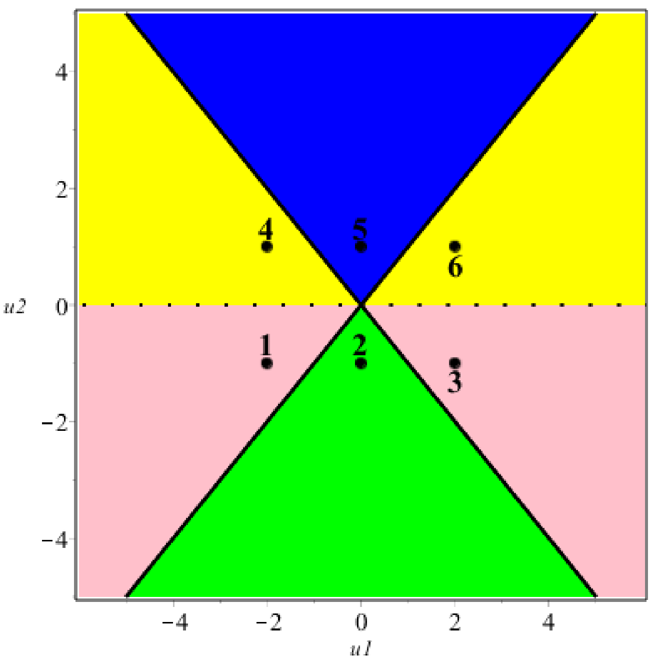

Fig. 1. Output of `CellDecomposition` of (8)

Now, for each cell, we choose an arbitrary point, we substitute it in the systems (8) and then we compute the corresponding number of solutions (the sum of the solutions of S_1 and S_2). Doing so, we can remark that the cells c_1, c_3, c_4, c_6 correspond to regions of the parameters where there is no critical pair of $f(s, \tau)$, which means that the asymptotic stability is independent of the delay. Thus, to conclude on the stability, it suffices to replace the delay in $f(s, \tau)$ by an arbitrary value τ_0 (e.g., $\tau_0 = 0$) and to check the stability of the resulting polynomial. Doing so, we remark that the stability cells are c_3 and c_6.

For the cells c_2, c_5, the number of critical pairs of $f(s, \tau)$ is equal to two, and we can easily find values for the delay τ_0 such that the polynomial $f(s, \tau_0)$ is unstable which implie that the system is asymptotically unstable. Note however that for general systems, further computations might be needed in order to conclude.

Finally, in order to complete the analysis, we need to check for the stability on the discriminant variety itself, i.e. for the parameters that satisfy $u_1 + u_2 = 0$ and $u_1 - u_1 = 0$.

Setting $u_2 = -u_1$ in the systems (8), we obtain the following system $\{-T\,\omega = 0, 2\,u_1\,T\,\omega + \omega = 0\}$ which we analyze the zeros by means of a descriminant variety. The latter yields two cells $u_1 > 0$ and $u_1 < 0$ inside which, the previous system does not admit any solutions. Thus, a simple substitution by an arbitrary point in each cell and an arbitrary delay value τ_0 (e.g. $\tau_0 = 0$) shows that the system is unstable in each of these two cells.

We now set $u_2 = u_1$ in (8), and obtain the system $\{-T\,\omega^2 = 0, \omega(u_1\,T + 1) = 0\}$. We use again the descriminant variety which yields the two cells $u_1 > 0$ and $u_1 < 0$ with the same property as above regarding the number of solutions. After substitution, we obtain that the system is stable only for the cell $u_1 > 0$.

To summarize, the system (7) is asymptotically stable if and only if ($u_2 < 0$ and $u_1 > -u_2$) or ($u_2 > 0$ and $u_1 > u_2$) or ($u_1 = u_2 > 0$).

Example 2. We now consider the following quasipolynomial which depends on two parameters u_1 and u_2:

$$f(s, \tau) = (u_1^2 + u_2)\,e^{-\tau\,s} + 3\,u_2\,e^{-2\,\tau\,s} + 2\,u_1\,s^2 + u_1^2$$

The systems corresponding to the critical pairs are

$$\begin{cases} 2\,T^2\,u_1\,\omega^4 - 4\,T^2\,u_2\,\omega^2 - 2\,u_1\,\omega^2 + 2\,u_1^2 + 2\,u_2 = 0, \\ -4\,T\,u_1\,\omega^3 + 2\,T\,u_1^2\,\omega - 6\,T\,u_2\,\omega = 0, \end{cases} \tag{9}$$

and

$$-2\,u_1\,\omega^2 + 4\,u_2 = 0, \tag{10}$$

whose a discriminant variety consists of $u_1 = 0$, $u_2 = 0$ and the zeros of the following polynomials:

$$d_1 := u_1^2 - 7\,u_2, d_2 := u_1^2 - 3\,u_2, d_3 := u_1^2 + u_2, d_4 := u_1^2 + 5\,u_2$$

Computing a cylindrical algebraic decomposition of the complementary of this discriminant variety yields 12 cells (see Figure 2), among which only the cells $1, 2, 7$ and 8 represente the regions of the parameters for which the stability is independent of the delay. By simple substitutions and isolation of the roots, we can remark that for these four cells, the system is unstable independently of the delay.

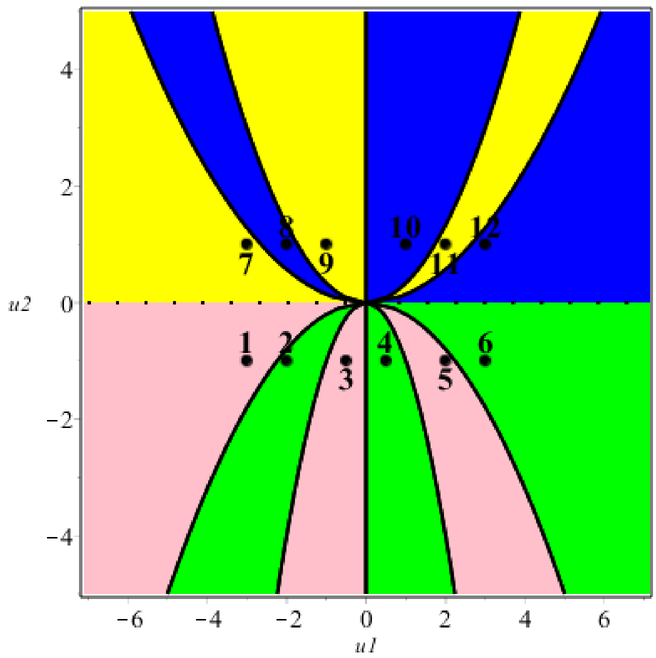

Fig. 2. Output of `CellDecomposition` of (9) and (10)

REFERENCES

Arnon, D.S., Collins, G.E., and McCallum, S. (1984). Cylindrical algebraic decomposition i: The basic algorithm. *SIAM Journal on Computing*, 13(4), 865–877.

Bouzidi, Y., Poteaux, A., and Quadrat, A. (2016). Computer algebra methods for the stability analysis of differential systems with commensurate time-delays. *IFAC-PapersOnLine*, 49(10), 194–199.

Gu, K., Kharitonov, V.L., and Chen, J. (2003). *Stability of Time-Delay Systems.* Birkhäuser.

Hertz, D., Jury, E.I., and Zeheb, E. (1984). Simplified analytic stability test for systems with commensurate time delays. In *IEE Proceedings D-Control Theory and Applications*, volume 1, 52–56.

Kamen, E. (1980). On the relationship between zero criteria for two variable polynomials and asymptotic stability of delay differential equations. *IEEE Transactions on Automatic Control*, 25(5), 983.

Kobel, A., Rouillier, F., and Sagraloff, M. (2016). Computing real roots of real polynomials... and now for real! *arXiv preprint arXiv:1605.00410.*

Lazard, D. and Rouillier, F. (2007). Solving parametric polynomial systems. *Journal of Symbolic Computation*, 42(6), 636–667.

Li, X.G., Niculescu, S.I., and Çela, A. (2015). *Analytic Curve Frequency-Sweeping Stability Tests for Systems with Commensurate Delays.* Springer.

Marshall, J.E., Gorecki, H., Korytowski, A., and Walton, K. (1992). *Time-Delay. Systems: Stability and Performance Criteria with Applications.* London: Ellis Horwood.

Niculescu, S.I. (2001). *Delay Effects on Stability: A Robust Control Approach.* Number 269 in Lecture Notes in Control and Information Sciences. Springer.

Rekasius, Z.V. (1980). A stability test for systems with delays. In *Proc. Joint Automatic Control Conf.*

Rouillier, F. (1999). Solving zero-dimensional systems through the rational univariate representation. *J. of Applicable Algebra in Engineering, Communication and Computing*, 9(5), 433–461.

Thowsen, A. (1981). The routh hurwitz method for stability determination of linear differential difference systems. *International Journal of Control*, 33(5), 991–995.