An Introduction to Control Theory

Alban Quadrat

INRIA Saclay - Île-de-France,
projet DISCO, Supélec, L2S,
3 rue Joliot Curie,
91192 Gif-sur-Yvette, France.

Alban.Quadrat@inria.fr

http://pages.saclay.inria.fr/alban.quadrat/

SUNY Cortland University

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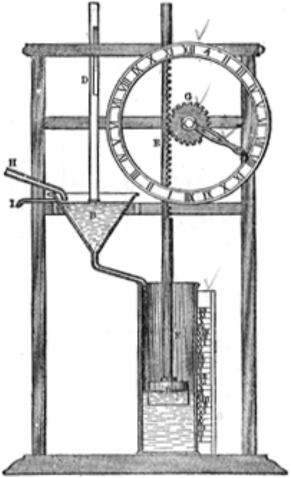
Outline of the talk

- A brief history of control theory
- Input-output representation
- State-space representation
- Stability and stabilizability
- Controllability and observability
- Pole placement and observers
- Extensions (research part)

A brief history of control theory

- Water clocks (clepsydra) (Egypt, -1400): Time measurement



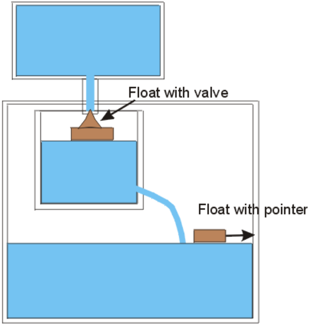
(http://www.youtube.com/watch?v=s9i5ny9NBOU)

Water clocks

- The pressure of the outflow drops with the height of the water.
- **Problem:** Design of a mechanism to keep the pressure constant.

⇒ Escapement mechanism (valve).

⇒ Concepts of self-regulation and feedback.



Oil lamps

- Hero of Alexandria (I A.D.): *Pneumatica*.
- Oil lamp : control of the position of the wick/the height of oil.

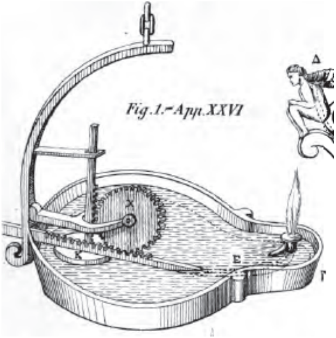
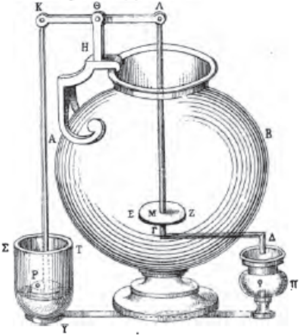
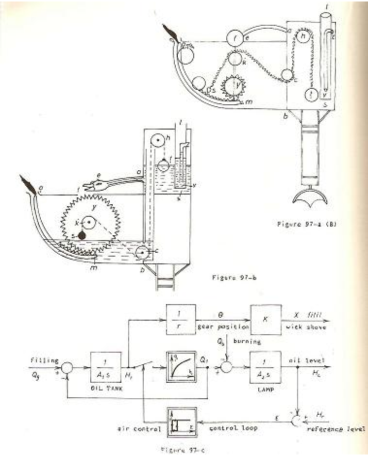


Fig.4.- App.LVIII



Elephant clock

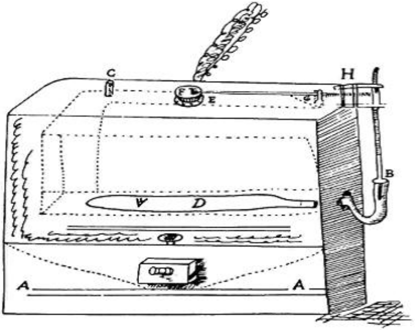
- Banou Moussa (9 A.D.), Al-Djazari (1136-1206):



(http://www.youtube.com/watch?v=doYPp-gaJ0o)

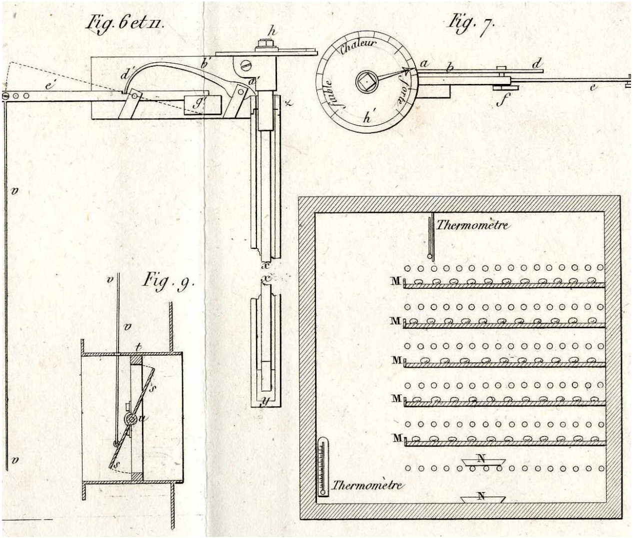
Thermostat: heat control

- Cornelis Drebbel (1572-1633)



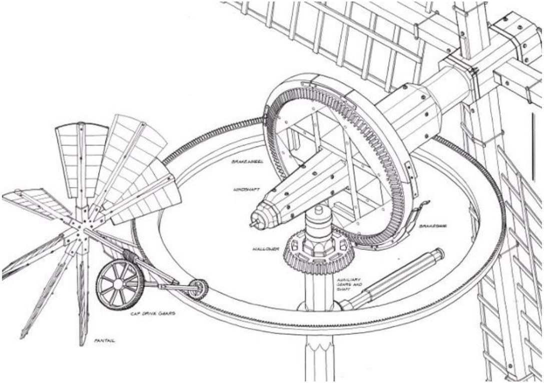
Thermostat: heat control

- Bonnemain (1783)



Windmills

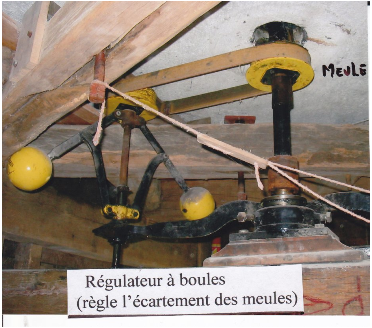
- Self-regulating wind machine, E. Lee (1745).



- Control of the distance and the pressure between millstones, . . .

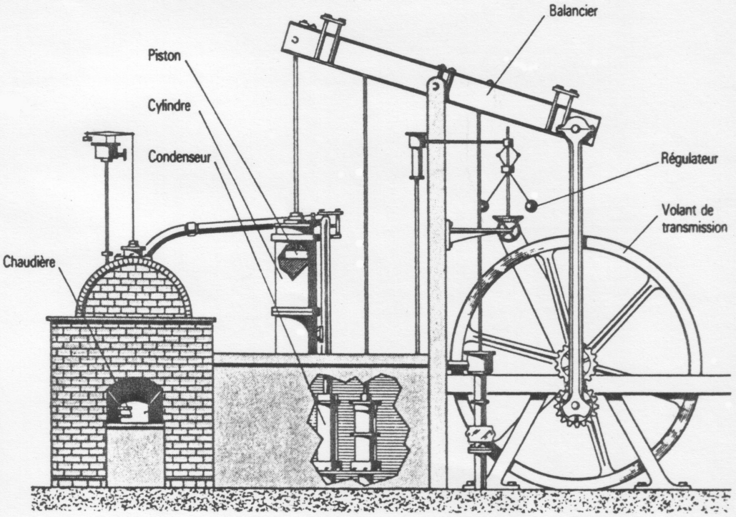
Flyball governors for windmills

- Control of the distance and the pressure between millstones



Steam engines

- Thomas Newcomen (1664-1729): pumping water out of mines.



Watt governor

- Flyball governor (∼ 1783) : James Watt (1736-1819)

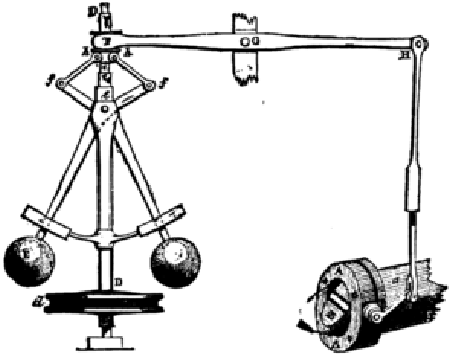
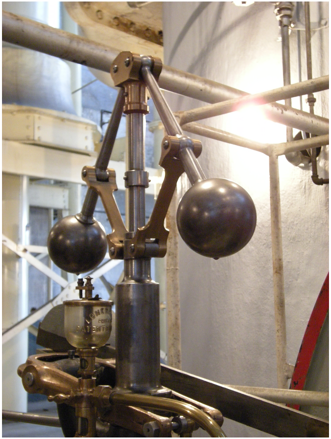


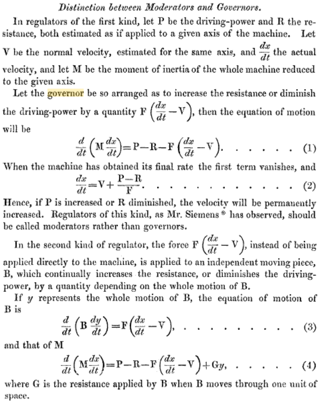
Fig. 4.—Governor and Throttle-Valve.



(http://www.youtube.com/watch?v=M2LsvSQ--wc
http://www.youtube.com/watch?v=3x3Mo6_8zGc)

Mathematical foundation of control theory

- James Clerk Maxwell (1831-1879): "On Governors", Proceedings of the Royal Society of London 16 (1868), 270-283.



Distinction between Moderators and Governors.

In regulators of the first kind, let P be the driving-power and R the resistance, both estimated as if applied to a given axis of the machine. Let V be the normal velocity, estimated for the same axis, and $\frac{dx}{dt}$ the actual velocity, and let M be the moment of inertia of the whole machine reduced to the given axis.

Let the governor be so arranged as to increase the resistance or diminish the driving-power by a quantity $F\left(\frac{dx}{dt}-V\right)$, then the equation of motion will be

$$\frac{d}{dt}\left(M\frac{dx}{dt}\right)=P-R-F\left(\frac{dx}{dt}-V\right). \quad (1)$$

When the machine has obtained its final rate the first term vanishes, and

$$\frac{dx}{dt}=V+\frac{P-R}{F}. \quad (2)$$

Hence, if P is increased or R diminished, the velocity will be permanently increased. Regulators of this kind, as Mr. Siemens* has observed, should be called moderators rather than governors.

In the second kind of regulator, the force $F\left(\frac{dx}{dt}-V\right)$, instead of being applied directly to the machine, is applied to an independent moving piece, B, which continually increases the resistance, or diminishes the driving-power, by a quantity depending on the whole motion of B.

If y represents the whole motion of B, the equation of motion of B is

$$\frac{d}{dt}\left(B\frac{dy}{dt}\right)=F\left(\frac{dx}{dt}-V\right), \quad (3)$$

and that of M

$$\frac{d}{dt}\left(M\frac{dx}{dt}\right)=P-R-F\left(\frac{dx}{dt}-V\right)+Gy, \quad (4)$$

where G is the resistance applied by B when B moves through one unit of space.

"A GOVERNOR is a part of a machine by means of which the velocity of the machine is kept nearly uniform, notwithstanding variations in the driving-power or the resistance".

The first telecommunications network

- First telegraph: Samuel Morse (1791-1872), 1837, USA.
- First telegraph line between Baltimore and Washington: 1844.
- Transatlantic telegraph cable : 1866.
- The invention of the phone is patented by A. G. Bell (1847-1922) in 1876.
- First phone call using Bell's telephone : 1878.
- First long distance phone call between Boston and Salem : 1881.

The first telecommunications network

- East coast: American Telephone & Telegraph Company (1891).

Fight the distance and the noise!

- Harold Black (1898-1983) found again the negative feedback law used in the water clocks and Watt governor (Bell Laboratories).

- Black's negative feedback stabilises an amplifier by sacrificing gain of the amplifier.

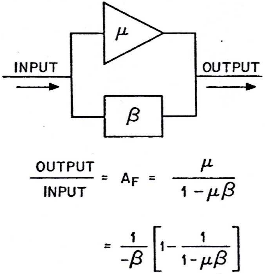
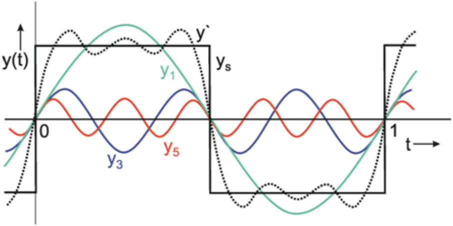


Fig. 2. A copy of the famous Lackawanna Ferryboat sketch made in 1927 by H. S. Black. (Provided to J. E. Brittain by Black on Feb. 14, 1977.)

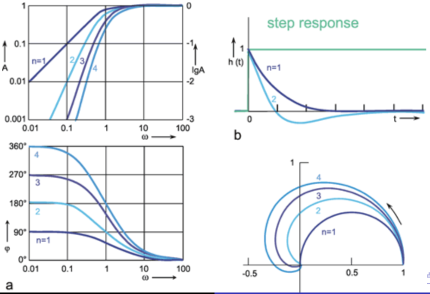
- Black's works were developed by Harry Nyquist (1889-1976) and Hendrik Wade Bode (1905-1982) (Bell Labs).

Rebirth of control theory

- They developed the frequency domain approach of control theory



the block diagrams and Black, Nyquist and Bode plots.



What is control theory ?

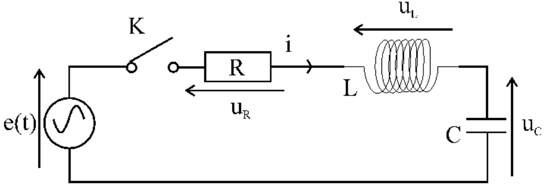
- Control theory is a branch of the mathematical systems theory which studies the concepts of inputs, outputs, feedback laws, ...

- Main goals:
  - Study the stability of systems.
  - Stabilize systems by means of feedback laws.
  - Track desired trajectories independently from the perturbations.
  - Optimize the performances of the closed-loop system.
  - Consider the errors of the mathematical model (robust control), . . .

(http://www.dailymotion.com/video/x2xdah_xpark-creneau-automatise-op-ii_auto)

A simple example: RLC circuit

- e voltage of the power source, R resistor, L inductor, C capacitor



- We have $e = u_R + u_L + u_C$ (Kirchhoff's voltage law), where:

$$\begin{cases} u_R(t) = R\, i(t), \\ u_L(t) = L\, \dfrac{di(t)}{dt}, \\ u_C(t) = \frac{1}{C} \int_0^t i(\tau)\, d\tau, \end{cases} \Rightarrow \begin{cases} i(t) = C\, \dfrac{du_C(t)}{dt} \\ u_R(t) = R\, C\, \dfrac{du_C(t)}{dt}, \\ u_L(t) = L\, C\, \dfrac{d^2 u_C(t)}{dt^2} \end{cases}$$

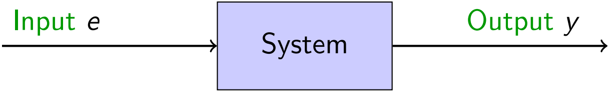
$$\Rightarrow\ L\, C\, \frac{d^2 u_C(t)}{dt^2} + R\, C\, \frac{du_C(t)}{dt} + u_C(t) = e(t).$$

Linear systems

- The system is formed by the RLC circuit.
- Let us suppose that we measure $y(t) := u_C(t)$ with a voltmeter.
- We obtain the following system of ODEs:

$$\begin{cases} L\,C\,\dfrac{d^2 u_C(t)}{dt^2} + R\,C\,\dfrac{d u_C(t)}{dt} + u_C(t) = e(t), \\ y(t) = u_C(t). \end{cases} \qquad (\star)$$

- Initial conditions: $u_C(0) = c_1, \; \dfrac{d u_C}{dt}(0) = c_2.$
- Cauchy theorem $\Rightarrow$ unique smooth solution.



input-output behavior

Realization of an input-output behavior

$$\ddot{u}_C(t) = -\frac{R}{L}\,\dot{u}_C(t) - \frac{1}{L\,C}\,u_C(t) + \frac{1}{L\,C}\,e(t) \quad \left(\dot{u}_c := \frac{du_C(t)}{dt}, \ldots\right)$$

- Let us call state of the system the following functions:

$$\begin{cases} x_1 := u_C, \\ x_2 := \dot{x}_1 = \dot{u}_C, \end{cases}$$

$$\Rightarrow \quad \begin{cases} \dot{x}_1(t) = x_2(t), \\ \dot{x}_2(t) = -\frac{R}{L}\,x_2 - \frac{1}{L\,C}\,x_1(t) + \frac{1}{L\,C}\,e(t). \end{cases}$$

- Equivalent first order linear OD system called a realization of $(\star)$:

$$\begin{cases} \frac{d}{dt}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{1}{L\,C} & -\frac{R}{L} \end{pmatrix}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{1}{L\,C} \end{pmatrix} e(t). \\ y = x_1. \end{cases}$$

The polynomial and state-state representations

- The state-space representation of a linear system is defined by

$$\begin{cases} \dot{x}(t) = A\,x(t) + B\,u(t), \\ y(t) = C\,x(t) + D\,u(t), \end{cases}$$

where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{p \times n}$ and $D \in \mathbb{R}^{p \times m}$.

- The polynomial representation of a linear system is defined by

$$P\left(\frac{d}{dt}\right)\, y(t) = Q\left(\frac{d}{dt}\right)\, u(t),$$

where $y := (y_1 \ \ldots \ y_p)^T$, $u := (u_1 \ \ldots \ u_m)^T$ and

$$P \in D^{p \times p}, \quad \det P \neq 0, \quad Q \in D^{p \times m},$$

where $D := \mathbb{R}\left[\frac{d}{dt}\right]$ is the commutative polynomial ring in $\frac{d}{dt}$.

$$\left(\text{e.g., } \left(L\,C\,\frac{d^2}{dt^2} + R\,C\,\frac{d}{dt} + 1\right)\, y(t) = e(t)\right)$$

Stability analysis

- If $y(t) := \exp(\lambda\, t) = e^{\lambda\, t}$, then we have:

$$\dot{y}(t) = \lambda\, e^{\lambda\, t} = \lambda\, y(t) \;\;\Rightarrow\;\; \ddot{y}(t) = \lambda^2\, y(t) \;\;\Rightarrow\;\; y^{(i)}(t) = \lambda^i\, y(t).$$

- $a_m\, y^{(m)}(t) + a_{m-1}\, y^{(m-1)}(t) + \ldots + a_1\, \dot{y}(t) + a_0\, y(t) = 0.$

$$y(t) = e^{\lambda\, t} \quad \Rightarrow \quad (a_m\, \lambda^m + \ldots + a_0)\, e^{\lambda\, t} = 0.$$

- The exponential solutions of the ODE are of the form $p_\lambda(t)\, e^{\lambda\, t}$ where $\lambda \in \mathbb{C}$ satisfies the characteristic equation

$$a_m\, \lambda^m + \ldots + a_0 = 0,$$

and the p_λ's are certain polynomials (multiplicities of the λ's).

- If $\lambda \in \mathbb{C}$, then $\lambda = \mathrm{Re}(\lambda) + i\,\mathrm{Im}(\lambda)$ and:

$$\begin{aligned} e^{\lambda\,t} &= e^{(\mathrm{Re}(\lambda)+i\,\mathrm{Im}(\lambda))\,t} = e^{\mathrm{Re}(\lambda)\,t}\,e^{i\,\mathrm{Im}(\lambda)\,t} \\ &= e^{\mathrm{Re}(\lambda)\,t}\,(\cos(\mathrm{Im}(\lambda)\,t) + i\,\sin(\mathrm{Im}(\lambda)\,t)). \end{aligned}$$

We get: $|e^{\lambda\,t}| = e^{\mathrm{Re}(\lambda)\,t}$.

1. If $\mathrm{Re}(\lambda) = 0$, then $|e^{\lambda\,t}| = 1$ for all $t \in \mathbb{R}$.
2. If $\mathrm{Re}(\lambda) > 0$, then $\lim_{t \to +\infty} |e^{\lambda\,t}| = +\infty$.
3. If $\mathrm{Re}(\lambda) < 0$, then $\lim_{t \to +\infty} |e^{\lambda\,t}| = 0$.

- **Definition:** A linear ODE with constant coefficients is said to be exponentially stable if all roots $\lambda \in \mathbb{C}$ of the characteristic equation lie in the left half plane $\mathbb{C}_- := \{s \in \mathbb{C} \mid \mathrm{Re}(s) < 0\}$.

Stability analysis

- Let us consider again the RLC circuit with $e = 0$ (no input).

$$\frac{d^2 u_C(t)}{dt^2} + \frac{R}{L}\,\frac{d u_C(t)}{dt} + \frac{1}{L\,C}\,u_C(t) = 0.$$

- The associated characteristic polynomial is then:

$$\lambda^2 + \frac{R}{L}\,\lambda + \frac{1}{L\,C} = 0.$$

- If we note

$$\alpha := \frac{R}{2\,L}, \quad \omega_0 := \frac{1}{\sqrt{L\,C}}, \quad \zeta := \frac{\alpha}{\omega_0} = \frac{R}{2}\sqrt{\frac{C}{L}},$$

then we get $\lambda^2 + 2\,\alpha\,\lambda + \omega_0^2 = 0$, which yields:

$$\lambda_1 = -\omega_0\,\left(\zeta + \sqrt{\zeta^2 - 1}\right), \quad \lambda_2 = -\omega_0\,\left(\zeta - \sqrt{\zeta^2 - 1}\right).$$

Stability analysis

- If $\zeta = 1$, then we have:

$$\lambda_1 = \lambda_2 = -\omega_0 < 0 \quad \Rightarrow \quad u_C(t) = (A_1 + A_2\, t)\, e^{-\omega_0\, t}.$$

- If $\zeta > 1$, then we have:

$$\lambda_1,\, \lambda_2 \in \mathbb{R}_- := \{x \in \mathbb{R} \mid x < 0\}, \quad u_C(t) = A_1\, e^{\lambda_1\, t} + A_2\, e^{\lambda_2\, t}.$$

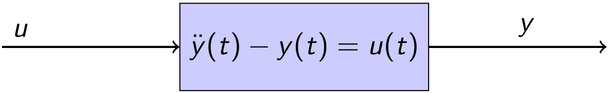
- If $\zeta < 1$ and if we note $\omega_1 := \omega_0\, \sqrt{1-\zeta^2}$, then we have:

$$\begin{cases} \lambda_1 = -\omega_0\, \left(\zeta + i\, \sqrt{1-\zeta^2}\right) = -\alpha - i\, \omega_d, \\ \lambda_2 = -\omega_0\, \left(\zeta - i\, \sqrt{1-\zeta^2}\right) = -\alpha + i\, \omega_d, \end{cases}$$

$$u_C(t) = e^{-\alpha\, t}\, (A_1\, \cos(\omega_1\, t) + A_2\, \sin(\omega_1\, t)).$$

- Since $-\omega_0,\, \lambda_1,\, \lambda_2,\, -\alpha \in \mathbb{R}_-$, we have:

$$\lim_{t \to +\infty} u_C(t) = 0 \quad \text{(stable)}.$$



- If $u = 0$ (no input), then the system is unstable:

$$\ddot{y}(t) - y(t) = 0 \quad \Rightarrow \quad \lambda^2 - 1 = 0 \quad \Leftrightarrow \quad \lambda = \{-1, 1\}.$$

- **Problem:** Can we find a stabilizing output feedback law?

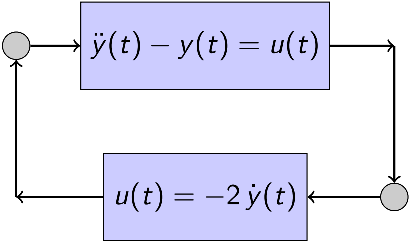
- Find a feeback law $u(t) := k_1\, y(t) + k_2\, \dot{y}(t)$ s.t. the closed-loop

$$\ddot{y}(t) - k_2\, \dot{y}(t) - (k_1 - 1)\, y(t) = 0 \text{ is stable.}$$

- **Example:** If we take $k_1 = 0$, $k_2 = -2$, we get:

$$\lambda^2 - k_2\, \lambda - (k_1 - 1) = 0 \;\Rightarrow\; \lambda^2 + 2\,\lambda + 1 = (\lambda + 1)^2 = 0 \;\Leftrightarrow\; \lambda = \{-1, -1\}.$$

Stabilization by output feedbacks



Stability analysis for state-space representations

$$\dot{x}(t) = A\,x(t) \quad \Leftrightarrow \quad x(t) = e^{A\,t}\,x(0), \quad e^{A\,t} := \sum_{i=0}^{+\infty} A^i\,\frac{t^i}{i!}.$$

$$A' := \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & \dots & \lambda_n \end{pmatrix} \Rightarrow e^{A'\,t} = \begin{pmatrix} e^{\lambda_1\,t} & 0 & \dots & 0 \\ 0 & e^{\lambda_2\,t} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & \dots & e^{\lambda_n\,t} \end{pmatrix}.$$

- If we can diagonalize A, there exists a non-singular matrix T s.t.:

$$T^{-1}\,A\,T = A'.$$

- $A = T\,A'\,T^{-1} \;\Rightarrow\; e^{A\,t} = T\,e^{A'\,t}\,T^{-1}$.

- **Theorem**: The linear OD system $\dot{x}(t) = A\,x(t)$ is exponentially stable iff all complex roots λ of $\det(s\,I - A)$ lie in $\mathbb{C}_-$.

Stabilization by state feedbacks

- Let us consider the following system:

$$\dot{x}(t) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(t).$$

- We have $\det(s\, I_2 - A) = (s+1)(s-1) = 0 \quad \Leftrightarrow \quad s \in \{-1, 1\}.$

- **Problem:** Can we find a stabilizing state feedback law?

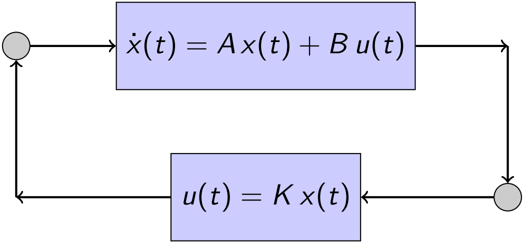
- If we consider $u(t) := (0 \quad -2)\, x(t)$, then we get:

$$\begin{aligned} \dot{x}(t) &= \left(\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} (0 \quad -2) \right) x(t) \\ &= \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} x(t) \quad (\text{stable}) \end{aligned}$$

Stabilization by state feedbacks

- **Definition**: A matrix A is called Hurwitz if all the roots $\lambda \in \mathbb{C}$ of $\det(s\, I_n - A)$ lie in $\mathbb{C}_- := \{s \in \mathbb{C} \mid \mathrm{Re}(s) < 0\}$.

- **General problem:** Given $\dot{x}(t) = A\, x(t) + B\, u(t)$, can we find a state feedback $u = K\, x$ such that $A + B\, K$ is Hurwitz?



- **Definition**: The system is (state-feedback) stabilizable if there exists $K \in \mathbb{R}^{m \times n}$ such that $A + B\, K$ is Hurwitz.

General solution of the state-space representation

- Let us solve:

$$\dot{x}(t) = A\,x(t) + B\,u(t), \quad x(0) = x_0. \quad (\star)$$

- The general solution of $(\star)$ is the sum of:
  - the general solution of $\dot{x}(t) = A\,x(t)$, i.e., $x(t) = e^{A\,t}\,x(0)$.
  - a particular solution of $(\star)$.
- The variation of constants method with $x(t) = e^{A\,t}\,C(t)$:

$$\dot{x}(t) - A\,x(t) = e^{A\,t}\,(A\,C(t) + \dot{C}(t) - A\,C(t)) = e^{A\,t}\,\dot{C}(t).$$

$$e^{A\,t}\,\dot{C}(t) = B\,u(t) \quad \Rightarrow \quad \dot{C}(t) = e^{-A\,t}\,B\,u(t)$$

$$\Rightarrow \quad C(t) = \int_0^t e^{-A\,\tau}\,B\,u(\tau)\,d\tau + x_0$$

$$\Rightarrow \quad x(t) = e^{A\,t}\,x_0 + \int_0^t e^{A\,(t-\tau)}\,B\,u(\tau)\,d\tau.$$

Controllability

- **Definition** (Kalman): The state-space system

$$\dot{x}(t) = A\,x(t) + B\,u(t) \quad (\star)$$

is controllable if for all $T > 0$, $x_0 \in \mathbb{R}^n$ and $x_1 \in \mathbb{R}^n$, there exists an input $u : [0, T] \longrightarrow \mathbb{R}^m$ such that $(\star)$ with $x(0) = x_0$ yields:

$$x(T) = x_1. \qquad (x_0 \xrightarrow{u} x_1 = x(T))$$

- **Definition:** The controllability gramian is defined by:

$$\forall\ T > 0, \quad W_c(T) := \int_0^T e^{A\,(T-\tau)}\,B\,B^T\,e^{A^T\,(T-\tau)}\,d\tau \in \mathbb{R}^{n \times n}.$$

- **Theorem** (Kalman): $(\star)$ is controllable iff $W_c(T)$ is non-singular, i.e., $\det(W_c(T)) \neq 0$ for all $T > 0$, iff we have:

$$\mathrm{rank}(B \quad A\,B \quad A^2\,B \quad \ldots \quad A^{n-1}\,B) = n.$$

Example

- We consider the RLC circuit $\dot{x}(t) = A\,x(t) + B\,u(t)$, where:

$$A := \begin{pmatrix} 0 & 1 \\ -\frac{1}{L\,C} & -\frac{R}{L} \end{pmatrix}, \quad B := \begin{pmatrix} 0 \\ \frac{1}{L\,C} \end{pmatrix}.$$

- Since $n = 2$, we have:

$$\mathcal{C} := (B \quad A\,B) = \begin{pmatrix} 0 & \frac{1}{L\,C} \\ \frac{1}{L\,C} & -\frac{R}{L^2\,C} \end{pmatrix}.$$

- We have

$$\det(\mathcal{C}) = -\frac{1}{L^2\,C^2} \neq 0,$$

which proves that the RLC circuit is controllable.

Example

- If we fix $L = C = 1$ and $R = 2$, then

$$A = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

and we have:

$$e^{A\,t} = \begin{pmatrix} e^{-t} + t\,e^{-t} & t\,e^{-t} \\ -t\,e^{-t} & e^{-t} - t\,e^{-t} \end{pmatrix},$$

$$e^{A\,(T-\tau)}\,B = \begin{pmatrix} (T-\tau)\,e^{\tau-T} \\ (\tau+1-T)\,e^{\tau-T} \end{pmatrix},$$

$$W_C(T) = \int_0^T \left((T-\tau)^2\,e^{2\,(\tau-T)} + (\tau+1-T)^2\,e^{2\,(\tau-T)}\right) d\tau$$
$$= \frac{1}{2} - \left(T^2 + \frac{1}{2}\right) e^{-2\,T} > 0, \quad \forall\; T > 0.$$

Sketch of the proof

- If $W_c(T)$ is non singular for all $T > 0$ and if we define

$$u(\tau) := -B^T \, e^{A^T (T-\tau)} \, W_c(T)^{-1} \, (e^{A\,T} \, x_0 - x_1), \quad \tau \in [0, \, T],$$

then we get:

$$\begin{aligned}
&\int_0^t e^{A\,(T-\tau)} \, B \, u(\tau) \, d\tau \\
&= -\left(\int_0^T e^{A\,(T-\tau)} \, B \, B^T \, e^{A^T (T-\tau)} \, d\tau\right) \, W_c(T)^{-1} \, (e^{A\,T} \, x_0 - x_1) \\
&= -W_c(T) \, W_c(T)^{-1} \, (e^{A\,T} \, x_0 - x_1) = x_1 - e^{A\,T} \, x_0,
\end{aligned}$$

$$\Rightarrow \quad x(T) = e^{A\,T} \, x_0 + \int_0^T e^{A\,(T-\tau)} \, B \, u(\tau) \, d\tau = x_1.$$

Sketch of the proof

- Let us suppose that $\dot{x}(t) = A\,x(t) + B\,u(t)$ is controllable.
- (contradiction) Suppose that $\exists\; T > 0$ such that $\det W_c(T) = 0$.

Then, there exists $v \in \mathbb{R}^n \setminus \{0\}$ such that $W_c(T)\,v = 0$

$$\Rightarrow \quad v^T\,W_c(T)\,v = \int_0^T v^T\,e^{A\,(T-\tau)}\,B\,B^T\,e^{A^T\,(T-\tau)}\,v\,d\tau = 0$$

$$= \int_0^T \|\,v^T\,e^{A\,(T-\tau)}\,B\,\|\;d\tau = 0$$

$$\Rightarrow \quad v^T\,e^{A\,t}\,B = 0, \quad \forall\, t \in [0,\,\tau].$$

- Let us choose $x_1 := 0$.
- Since the system is controllable, for all $x_0 \in \mathbb{R}^n$, there exists u s.t.

$$e^{A\,T}\,x_0 + \int_0^T e^{A\,(T-\tau)}\,B\,u(\tau)\,d\tau = x(T) = x_1 = 0. \quad (\star)$$

Pre-multiplying $(\star)$ by v^T, we get $v^T\,e^{A\,T}\,x_0 = 0$.

- If we take $x_0 = e^{-A\,T}\,v$, then $\|\,v\,\| = v^T\,v = 0$, i.e., $v = 0$!.

Sketch of the proof

- Let us note $\mathcal{C} := (B \quad A\,B \quad A^2\,B \quad \ldots \quad A^{n-1}\,B)$.
- Let us suppose that $\mathrm{rank}\,(\mathcal{C}) = n$.
- (contradiction) Let us prove that $\exists\ T > 0$ s.t. $W_c(T)$ is singular.
- As before, we get $v^T\,e^{A\,t}\,B = 0$ for $t \in [0, T]$ for $v \neq 0$.
- For $t = 0$, we get $v^T\,B = 0$. Morever, we have:

$$\frac{d}{dt}\,(v^T\,e^{A\,t}\,B) = v^T\,A\,e^{A\,t}\,B = 0 \quad \stackrel{t=0}{\Longrightarrow} \quad v^T\,A\,B = 0.$$

- Similarly, for $i = 2, \ldots, n-1$, we obtain:

$$\frac{d^i}{dt^i}\,(v^T\,e^{A\,t}\,B) = v^T\,A^i\,e^{A\,t}\,B = 0 \quad \stackrel{t=0}{\Longrightarrow} \quad v^T\,A^i\,B = 0.$$

$$\Rightarrow \quad v^T\,\mathcal{C} = 0 \quad \Rightarrow \quad v = 0!$$

Sketch of the proof

- Let us suppose that $W_c(T)$ is non-singular for all $T > 0$.
- (contradiction) Let us prove that $\mathrm{rank}(\mathcal{C}) < n$, i.e., $\mathcal{C}$ does not have full row rank.
- Then, there exists $v \in \mathbb{R}^n \setminus \{0\}$ such that:

$$\begin{aligned} & v^T \, (B \quad A\,B \quad A^2\,B \quad \ldots \quad A^{n-1}\,B) = 0 \\ \Leftrightarrow \quad & v^T \, A^i \, B = 0, \; i = 0, \ldots, n-1. \end{aligned}$$

- The Cayley-Hamilton theorem says that $A^n = \sum_{i=0}^{n-1} a_i \, A^i$, where $a_i \in \mathbb{R}$, $i = 1, \ldots, n-1$.
- Thus, we have $v^T \, e^{A\,t} \, B = 0$ for all $t > 0$.
- We then get the contradiction:

$$v^T \, W_c(T) = \int_0^T v^T \, e^{A\,(T-\tau)} \, B \, B^T \, e^{A^T\,(T-\tau)} \, d\tau = 0.$$

Observability

- **Definition** (Kalman): The state-space representation

$$\begin{cases} \dot{x}(t) = A\,x(t) + B\,u(t), \\ y(t) = C\,x(t) + D\,u(t), \end{cases} \qquad (\star)$$

is observable if for any $T > 0$, then the initial condition $x(0) = x_0 \in \mathbb{R}^n$ can be determined from the history of the input u and the output y in the interval $[0,\ T]$.

- **Definition:** The observability gramian is defined by:

$$\forall\ T > 0, \quad W_o(T) := \int_0^T e^{A^T\,(T-\tau)}\,C^T\,C\,e^{A^T\,(T-\tau)}\,d\tau \in \mathbb{R}^{n \times n}.$$

- **Theorem**: $(\star)$ is observable iff $W_o(T)$ is non-singular $\forall\ T > 0$

$$\Leftrightarrow \quad \mathrm{rank}\,(\mathcal{O}) = n, \quad \mathcal{O} := \begin{pmatrix} C \\ C\,A \\ \vdots \\ C\,A^{n-1} \end{pmatrix}.$$

Observability

$$\left\{\begin{array}{l}\dot{x}(t)=A\,x(t)+B\,u(t),\\ y(t)=C\,x(t)+D\,u(t),\end{array}\right. \quad\Leftrightarrow\quad \left\{\begin{array}{l}\dot{x}(t)-A\,x(t)=B\,u(t),\\ C\,x(t)=y(t)-D\,u(t).\end{array}\right.$$

- Differentiating once the last equation, we get:

$$\left\{\begin{array}{l}\dot{x}(t)-A\,x(t)=B\,u(t),\\ C\,x(t)=y(t)-D\,u(t),\\ C\,\dot{x}(t)=\dot{y}(t)-D\,\dot{u}(t),\end{array}\right. \Leftrightarrow \left\{\begin{array}{l}\dot{x}(t)-A\,x(t)=B\,u(t),\\ C\,x(t)=y(t)-D\,u(t),\\ C\,A\,x(t)=\dot{y}(t)-D\,\dot{u}(t)-C\,B\,u(t).\end{array}\right.$$

- Repeating the same procedure $n-2$ more times, we obtain:

$$\left(\begin{array}{c} C \\ C\,A \\ \vdots \\ C\,A^{n-1}\end{array}\right) x(t)=\left(\begin{array}{c} y(t)-D\,u(t) \\ \dot{y}(t)-D\,\dot{u}(t)-C\,B\,u(t) \\ \vdots \\ \vdots \end{array}\right).$$

- If $\mathrm{rank}\,(\mathcal{O})=n$, then we have $x=F(y,\dot{y},\ldots,u,\dot{u},\ldots)$.

State transformations

- Let us consider the state-space representation:

$$\begin{cases} \dot{x}(t) = A\,x(t) + B\,u(t), \\ y(t) = C\,x(t) + D\,u(t). \end{cases}$$

- Let $T \in \mathbb{R}^{n \times n}$ be a non-singular matrix.
- Let us consider $\overline{x} := T\,x$, i.e., $\overline{x} = T^{-1}\,x$. Then, we have:

$$\begin{cases} \dot{\overline{x}}(t) = T\,\dot{x}(t) = T\,A\,x(t) + T\,B\,u(t) = T\,A\,T^{-1}\,\overline{x}(t) + T\,B\,u(t), \\ y(t) = C\,T^{-1}\,\overline{x}(t) + D\,u(t), \end{cases}$$

i.e., $$\begin{cases} \dot{\overline{x}}(t) = A'\,\overline{x}(t) + B'\,u(t), \\ y(t) = C'\,\overline{x}(t) + D\,u(t), \end{cases}$$

where $A' := T\,A\,T^{-1}$, $B' := T\,B$ and $C' := C\,T^{-1}$.

Invariance

- Let $T \in \mathbb{R}^{n\times n}$ be a non-singular matrix.

$$\mathcal{C} := (B \quad A\,B \quad A^2\,B \quad \ldots \quad A^{n-1}\,B), \quad \mathcal{O} := \begin{pmatrix} C \\ C\,A \\ \vdots \\ C\,A^{n-1} \end{pmatrix}$$

- $A' := T\,A\,T^{-1}$, $B' := T\,B$, $C' := C\,T^{-1}$.

$$\mathcal{C}' := (B' \quad A'\,B' \quad A'^2\,B' \quad \ldots \quad A'^{n-1}\,B'), \quad \mathcal{O}' := \begin{pmatrix} C' \\ C'\,A' \\ \vdots \\ C'\,A'^{n-1} \end{pmatrix}$$

$$\begin{cases} A'\,B' = T\,A\,T^{-1}\,T\,B = T\,(A\,B), \\ C'\,A' = C\,T^{-1}\,T\,A\,T^{-1} = (C\,A)\,T^{-1}, \end{cases} \ldots \Rightarrow \begin{cases} \mathcal{C}' = T\,\mathcal{C}, \\ \mathcal{O}' = \mathcal{O}\,T^{-1}. \end{cases}$$

Canonical decomposition

- **Theorem** (Kalman): Let $\mathrm{rank}\,(\mathcal{C}) = k < n$. Then, there exists a non-singular matrix $T \in \mathbb{R}^{n \times n}$ such that the equivalent system

$$\begin{cases} \dot{\overline{x}}(t) = A'\,\overline{x}(t) + B'\,u(t), & \overline{x} = \begin{pmatrix} \overline{x}_1 \\ \overline{x}_2 \end{pmatrix} := T\,x, \\ y(t) = C'\,\overline{x}(t) + D\,u(t), & \overline{x}_1 \in \mathbb{R}^k, \quad \overline{x}_2 \in \mathbb{R}^{n-k}, \end{cases}$$

is such that

$$A' := T\,A\,T^{-1} = \begin{pmatrix} \overline{A}_{11} & \overline{A}_{12} \\ 0 & \overline{A}_{22} \end{pmatrix}, \quad B' := T\,B = \begin{pmatrix} \overline{B}_1 \\ 0 \end{pmatrix},$$

where $\overline{A}_{11} \in \mathbb{R}^{k \times k}$, $\overline{A}_{12} \in \mathbb{R}^{k \times (n-k)}$, $\overline{A}_{22} \in \mathbb{R}^{(n-k) \times (n-k)}$ and:

$$\dot{\overline{x}}_1 = \overline{A}_{11}\,\overline{x}_1 + \overline{B}_1\,u \ \text{ is controllable.}$$

Sketch of the proof

- We have $\mathrm{rank}\,(\mathcal{C}) = k < n$, where:

$$\mathcal{C} := (B \quad A\,B \quad A^2\,B \quad \ldots \quad A^{n-1}\,B).$$

- Let us consider k linearly independent columns $\{s_i\}_{i=1,\ldots,k}$ of $\mathcal{C}$.
- Complete $\{s_i\}_{i=1,\ldots,k}$ to a basis $\{s_i\}_{i=1,\ldots,n}$ of $\mathbb{R}^n$. Then,

$$S := (s_1 \quad s_2 \quad \ldots \quad s_k \quad s_{k+1} \quad \ldots \quad s_n) \in \mathbb{R}^{n \times n}$$

is a non-singular matrix and we define:

$$T := S^{-1}.$$

- Let $\mathcal{V} := \mathcal{C}\,\mathbb{R}^{n \times n\,m}$ be the $\mathbb{R}$-vector subspace of $\mathbb{R}^n$. The Cayley-Hamilton theorem yields:

$$A\,\mathcal{C} = \left(A\,B \quad A^2\,B \quad \ldots \quad A^{n-1}\,B \quad \sum_{i=1}^{n-1} a_i\,A^{i-1}\,B\right) \Rightarrow A\,\mathcal{V} \subseteq \mathcal{V}.$$

Sketch of the proof

- For $i = 1, \ldots, k$, $A\, s_i$ is a $\mathbb{R}$-linear combinations of $\{s_i\}_{i=1,\ldots,k}$.

$$\begin{aligned} A\,T^{-1} = A\,S &= (A\,s_1 \quad \ldots \quad A\,s_k \quad A\,s_{k+1} \quad \ldots \quad A\,s_n) \\ &= (s_1 \quad \ldots \quad s_k \quad s_{k+1} \quad \ldots \quad s_n) \begin{pmatrix} \overline{A}_{11} & \overline{A}_{12} \\ 0 & \overline{A}_{22} \end{pmatrix} \\ &= S\,A', \end{aligned}$$

which yields $A' = T\,A\,T^{-1}$.

- Each column of B is a $\mathbb{R}$-linear combination of $\{s_i\}_{i=1,\ldots,k}$.

$$\begin{aligned} B &= (s_1 \quad \ldots \quad s_k \quad s_{k+1} \quad \ldots \quad s_n) \begin{pmatrix} \overline{B}_1 \\ 0 \end{pmatrix} \\ &= S\,B' \end{aligned}$$

which yields $B' = T\,B$.

Sketch of the proof

- In the new basis of $\mathbb{R}^n$, we have

$$\left\{ \begin{array}{l} \dot{\overline{x}}(t) = A'\,\overline{x}(t) + B'\,u(t), \\ y(t) = C'\,\overline{x}(t) + D\,u(t), \end{array} \right.$$

where:

$$A' := \left(\begin{array}{cc} \overline{A}_{11} & \overline{A}_{12} \\ 0 & \overline{A}_{22} \end{array} \right), \quad B' = \left(\begin{array}{c} \overline{B}_1 \\ 0 \end{array} \right).$$

- Now, we have:

$$A'\,B' = \left(\begin{array}{c} \overline{A}_{11}\,\overline{B}_1 \\ 0 \end{array} \right), \quad \ldots \quad A'^{i}\,B' = \left(\begin{array}{c} \overline{A}_{11}^{i}\,\overline{B}_1 \\ 0 \end{array} \right).$$

$$\Rightarrow \quad \mathcal{C}' := \left(B' \quad A'\,B' \quad A'^2\,B' \quad \ldots \quad A'^{n-1}\,B'\right)$$

$$= \left(\begin{array}{cccc} \overline{B}_1 & \overline{A}_{11}\,\overline{B}_1 & \ldots & \overline{A}_{11}^{n-1}\,\overline{B}_1 \\ 0 & 0 & \ldots & 0 \end{array} \right).$$

Sketch of the proof

- Since $\mathcal{C}' = T\,\mathcal{C}$ and T is non-singular, $\operatorname{rank}(\mathcal{C}') = \operatorname{rank}(\mathcal{C}) = k$.
- Since $\overline{A}_{11} \in \mathbb{R}^{k \times k}$, the Cayley-Hamilton theorem yields:

$$A_{11}^k = \sum_{j=0}^{k-1} \beta_j\, A_{11}^j, \quad \beta_j \in \mathbb{R}.$$

- Thus, we obtain

$$\begin{aligned} & \operatorname{rank}\begin{pmatrix} \overline{B}_1 & \overline{A}_{11}\,\overline{B}_1 & \dots & \overline{A}_{11}^{n-1}\,\overline{B}_1 \end{pmatrix} \\ = \; & \operatorname{rank}\begin{pmatrix} \overline{B}_1 & \overline{A}_{11}\,\overline{B}_1 & \dots & \overline{A}_{11}^{k}\,\overline{B}_1 \end{pmatrix} = k, \end{aligned}$$

which proves that the system $\dot{\overline{x}}_1 = \overline{A}_{11}\,\overline{x}_1 + \overline{B}_1\,u$ is controllable.

Pole placement

- **General problem:** Given $\dot{x}(t) = A\,x(t) + B\,u(t)$, can we find a state feedback $u = K\,x$ such that $A + B\,K$ is Hurwitz?

- **Definition:** $\{\lambda_1, \ldots, \lambda_n\}$ is called a symmetric set of complex numbers if $\lambda_i \in \mathbb{C} \setminus \mathbb{R}$, then there exists $j = 1, \ldots, n$ such that:

$$\lambda_j := \overline{\lambda}_i.$$

- **Theorem** (Kalman): The following assertions are equivalent:

  1. $\dot{x}(t) = A\,x(t) + B\,u(t)$ is controllable.
  2. For any symmetric set of complex numbers $\{\lambda_1, \ldots, \lambda_n\}$, there exists $K \in \mathbb{R}^{m \times n}$ such that the zeros of the closed-loop

$$\det(s\,I_n - (A + K\,B))$$

  are exactly $\{\lambda_1, \ldots, \lambda_n\}$.

- A similar result holds for observable system.

Stabilizability

- **Definition:** The system $\dot{x}(t) = A\,x(t) + B\,u(t)$ is state-feedback stabilizable if there exists $K \in \mathbb{R}^{m \times n}$ such that $A + B\,K$ is Hurwitz.

- **Corollary:** A controllable system is stabilizable.

- If $T \in \mathbb{R}^{n \times n}$ is non-singular and $\overline{x} = T\,x$, then

$$\dot{x}(t) = A\,x(t) + B\,u(t) \quad \Leftrightarrow \quad \dot{\overline{x}}(t) = A'\,\overline{x}(t) + B'\,u(t)$$

$$A' := \begin{pmatrix} \overline{A}_{11} & \overline{A}_{12} \\ 0 & \overline{A}_{22} \end{pmatrix}, \quad B' = \begin{pmatrix} \overline{B}_1 \\ 0 \end{pmatrix},$$

where $\dot{\overline{x}}_1 = \overline{A}_{11}\,\overline{x}_1 + \overline{B}_1\,u$ is controllable, $\dot{\overline{x}}_2 = \overline{A}_{22}\,\overline{x}_2$ no inputs!

- $\det(s\,I_n - A') = \det(s\,I_k - \overline{A}_{11})\,\det(s\,I_{n-k} - \overline{A}_{22})$.

- **Corollary:** The system $\dot{x}(t) = A\,x(t) + B\,u(t)$ is state-feedback stabilizable iff $\overline{A}_{22}$ is Hurwitz.

Canonical decomposition (dual result)

- **Theorem** (Kalman): Let $\mathrm{rank}\,(\mathcal{O}) = k < n$. Then, there exists a non-singular matrix $T \in \mathbb{R}^{n \times n}$ such that the equivalent system

$$\begin{cases} \dot{\overline{x}}(t) = A'\,\overline{x}(t) + B'\,u(t), \\ y(t) = C'\,\overline{x}(t) + D\,u(t), \end{cases} \qquad \overline{x} = \begin{pmatrix} \overline{x}_1 \\ \overline{x}_2 \end{pmatrix} := T\,x, \quad \overline{x}_1 \in \mathbb{R}^k, \quad \overline{x}_2 \in \mathbb{R}^{n-k},$$

is such that

$$A' := T\,A\,T^{-1} = \begin{pmatrix} \overline{A}_{11} & 0 \\ \overline{A}_{21} & \overline{A}_{22} \end{pmatrix}, \quad C' := C\,T^{-1} = (\overline{C}_1 \quad 0),$$

where $\overline{A}_{11} \in \mathbb{R}^{k \times k}$, $\overline{A}_{12} \in \mathbb{R}^{k \times (n-k)}$, $\overline{A}_{22} \in \mathbb{R}^{(n-k) \times (n-k)}$ and:

$$\begin{cases} \dot{\overline{x}}_2 = \overline{A}_{21}\,\overline{x}_2, \\ C_1\,\overline{x}_2 = z(t), \end{cases} \quad \text{is observable.}$$

Approximation $\widehat{x}$ of the state x

- **Problem:** How can we compute the state x for $u = K\,x$?

- Indeed, the state x is not directly accessible!

- We need to reconstruct x from u and y ($\Rightarrow$ observability).

- **Definition:** An observer of the following linear system

$$\left\{ \begin{array}{l} \dot{x}(t) = A\,x(t) + B\,u(t), \\ y(t) = C\,x(t) + D\,u(t), \end{array} \right. \qquad (\star)$$

is a linear system of the form

$$\left\{ \begin{array}{l} \dot{z}(t) = M\,z(t) + N\,u(t) + M\,y(t), \\ \widehat{x} = P\,z(t) + Q\,u(t) + S\,y(t), \end{array} \right.$$

which is such that $\lim_{t \to +\infty} (\widehat{x}(t) - x(t)) = 0, \ \forall\ x(0),\ z(0),\ u(\cdot)$.

Luenberger observers

- **Definition**: The system is detectable if there exists $L \in \mathbb{R}^{n \times p}$ such that $A + L\,C$ is Hurwitz.

- **Theorem:** An observer exists iff $(\star)$ is detectable and

$$\begin{cases} \dot{z}(t) = (A + L\,C)\,z(t) + (B + L\,D)\,u(t) - L\,y(t), \\ \hat{x} = z(t), \end{cases}$$

is an observer where $L \in \mathbb{R}^{n \times p}$ is s.t. $A + L\,C$ is Hurwitz.

- **Proof:** Define the error by $e := \widehat{x} - x$. Then, we have:

$$\begin{aligned} \dot{e} &= \dot{\widehat{x}} - \dot{x} = (A + L\,C)\,\widehat{x} + (B + L\,D)\,u - L\,y - A\,x - B\,u \\ &= A\,(\widehat{x} - x) + L\,C\,\widehat{x} + L\,D\,u - L\,(C\,x + D\,u) \\ &= (A + L\,C)\,(\widehat{x} - x) = (A + L\,C)\,e. \end{aligned}$$

Then, we have $\lim_{t \to +\infty}(\widehat{x}(t) - x(t)) = 0,\ \forall\ x(0),\ z(0),\ u(\cdot)$.

Polynomial systems

- Let $\mathbb{R}\left[\frac{d}{dt}\right]$ be a commutative polynomial ring in $\frac{d}{dt}$.

$$P := \sum_{i=0}^{n} a_i \frac{d}{dt}^{\,i} \in D, \quad a_i \in \mathbb{R}, \quad \frac{d}{dt}^{\,i} = \frac{d}{dt} \circ \ldots \circ \frac{d}{dt} = \frac{d^i}{dt^i}.$$

- The polynomial representation of a linear system is defined by

$$P\left(\frac{d}{dt}\right) y(t) = Q\left(\frac{d}{dt}\right) u(t),$$

where $y := (y_1 \ \ldots \ y_p)^T$, $u := (u_1 \ \ldots \ u_m)^T$ and:

$$P \in D^{p \times p}, \quad \det P \neq 0, \quad Q \in D^{p \times m}.$$

- **Example:** $P\left(\frac{d}{dt}\right) := \frac{d}{dt} I_n - A, \quad Q\left(\frac{d}{dt}\right) := B\, u.$
- **Problem:** Extension of the concept of controllability (no states!)?

Willems' controllability

- **Definition** (Willems): The polynomial system

$$\left\{ (y^T \quad u^T)^T \in C^\infty(\mathbb{R}_+)^{p+m} \mid P\left(\frac{d}{dt}\right) y(t) = Q\left(\frac{d}{dt}\right) u(t) \right\} \quad (\star)$$

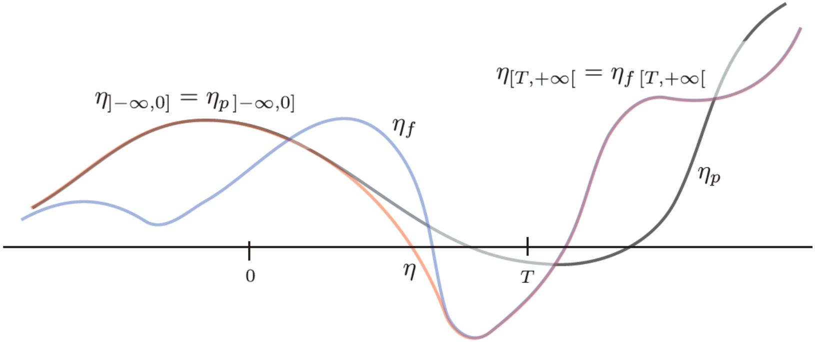
is called controllable if for all

1. $T > 0$,
2. $\eta_p := (y_p^T \quad u_p^T)^T$ a trajectory of $(\star)$ called past trajectory,
3. $\eta_f := (y_f^T \quad u_f^T)^T$ a trajectory of $(\star)$ called future trajectory,

then a trajectory $\eta := (y^T \quad u^T)^T$ of $(\star)$ exists which satisfies:

$$\begin{cases} \eta_{]-\infty,0[} = \eta_p, \\ \eta_{]T,+\infty[} = \eta_f. \end{cases}$$

Willems' controllability



Smith normal forms

- Let D be a principal ideal domain (e.g., $D := \mathbb{R}\left[\frac{d}{dt}\right]$).
- $\mathrm{GL}_p(D) = \{U \in D^{p \times p} \mid \exists\, V \in D^{p \times p} :\ U\,V = V\,U = I_p\}$.
- **Theorem**: $\forall\, R \in D^{q \times p}, \quad \exists\, V \in \mathrm{GL}_q(D), \quad \exists\, U \in \mathrm{GL}_p(D)$ s.t.

$$S := V\,R\,U = \begin{pmatrix} \alpha_1 & 0 & \dots & \dots & 0 & \dots & 0 \\ 0 & \alpha_2 & \ddots & & \vdots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots & & \vdots \\ 0 & \dots & 0 & \alpha_r & 0 & \dots & 0 \\ 0 & \dots & \dots & \dots & 0 & \dots & 0 \\ \vdots & & & & \vdots & \ddots & \vdots \\ 0 & \dots & \dots & \dots & 0 & \dots & 0 \end{pmatrix},$$

where $\alpha_1 \,|\, \alpha_2 \,|\, \dots \,|\, \alpha_r \neq 0$. S is called the Smith normal form of R.

- Implementations in Maple, Mathematica, Singular, ...

Example: Smith normal form

- 2 pendulum of the same length mounted on a car: $\alpha = \frac{g}{l}$

$$\left\{ \begin{array}{l} \ddot{y}_1(t) + \alpha\, y_1(t) - \alpha\, u(t) = 0, \\ \ddot{y}_2(t) + \alpha\, y_2(t) - \alpha\, u(t) = 0, \end{array} \right. \qquad (\star)$$

$$\begin{pmatrix} -\alpha & 0 \\ -\alpha & 1 \end{pmatrix} \begin{pmatrix} \frac{d^2}{dt^2} + \alpha & 0 & -\alpha \\ 0 & \frac{d^2}{dt^2} + \alpha & -\alpha \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & \frac{1}{\alpha}\frac{d^2}{dt^2} + 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{d^2}{dt^2} + \alpha & 0 \end{pmatrix}.$$

- Then, $(\star)$ is equivalent to:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{d^2}{dt^2} + \alpha & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ v \end{pmatrix} = 0 \quad \Leftrightarrow \quad \left\{ \begin{array}{l} z_1 = 0, \\ \left(\frac{d^2}{dt^2} + \alpha\right) z_2(t) = 0, \\ z_3 \text{ arbitrary}. \end{array} \right.$$

Example: Smith normal form

$$\begin{cases} z_1 = 0, \\ \left(\frac{d^2}{dt^2} + \alpha\right) z_2(t) = 0, \\ z_3 \text{ arbitrary}, \end{cases} \Leftrightarrow \begin{cases} z_1 = 0, \\ z_2 = A \cos(\sqrt{\alpha}\, t) + B \sin(\sqrt{\alpha}\, t), \\ z_3 \text{ arbitrary}. \end{cases}$$

- Using $S\, z = 0 \;\Leftrightarrow\; R\,(U\, z) = 0$, we obtain the parametrization:

$$\begin{cases} y_1(t) = z_3(t), \\ y_2(t) = A \cos(\sqrt{\alpha}\, t) + B \sin(\sqrt{\alpha}\, t) + z_3(t), \quad \forall\, z_3 \in C^\infty(\mathbb{R}_+). \\ u(t) = \frac{1}{\alpha}\, \ddot{z}_3(t) + z_3(t), \end{cases}$$

- If $A_p \neq A_f$ or $B_p \neq B_f$, there is no y_2 solution of $(\star)$ which is s.t.

$$\forall\, t \in]-\infty, 0] : \; y_2(t) = y_{2\,p}(t), \quad \forall\, t \in [T; +\infty[: \; y_2(t) = y_{2\,f}(t),$$

which proves that $(\star)$ is not controllable.

Willems' controllability

- **Theorem:** The polynomial system

$$\left\{ (y^T \quad u^T)^T \in C^\infty(\mathbb{R}_+)^{p+m} \mid P\left(\frac{d}{dt}\right) y(t) = Q\left(\frac{d}{dt}\right) u(t) \right\}$$

is controllable iff the Smith normal form of the matrix

$$R := \left(P\left(\frac{d}{dt}\right) \quad -Q\left(\frac{d}{dt}\right) \right) \in D^{p\times(p+m)}$$

is equal to $S = (I_p \quad 0)$, i.e., $\alpha_1 = \ldots = \alpha_p = 1$.

- Let $U := (U_1 \quad U_2)$, $W := U^{-1} = (W_1^T \quad W_2^T)^T$. Then, we get:

$$R = V^{-1}\,(I_p \quad 0)\,U^{-1} \quad \Leftrightarrow \quad \left(\begin{array}{c} R \\ W_2 \end{array} \right) (U_1\,V^{-1} \quad U_2) = I_{p+m}.$$

- If $T := U_1\,V^{-1}$ and $\Pi := T\,R$, then $\Pi^2 = T\,(R\,T)\,R = \Pi$.

Parametrization: $R \left(\begin{array}{c} y \\ u \end{array} \right) = 0 \Leftrightarrow \left(\begin{array}{c} y \\ u \end{array} \right) = U_2\,\xi, \;\; \forall\,\xi \in C^\infty(\mathbb{R}_+)^m.$

Extensions (research part)

- Explicit first order non-linear OD systems:

$$\begin{cases} \dot{x}(t) = f(x(t), u(t)), \\ y(t) = h(x(t), u(t)). \end{cases}$$

- Implicit first order non-linear OD systems:

$$\begin{cases} F(\dot{x}(t), x(t), u(t)) = 0, \\ G(y(t), x(t), u(t)) = 0. \end{cases}$$

- Implicit non-linear OD systems:

$$F(y(t), \dot{y}(t), \ddot{y}(t), \ldots, u(t), \dot{u}(t), \ldots) = 0.$$

- Extensions to difference, time-delay, partial differential equations.

Example

- Stirred tank model (Kwakernaak-Sivan 72): $h \in \mathbb{R}_+$

$$\left\{ \begin{array}{l} \dot{x}_1(t) + \dfrac{1}{2\,\theta}\, x_1(t) - u_1(t) - u_2(t) = 0, \\ \dot{x}_2(t) + \dfrac{1}{\theta}\, x_2(t) - \left(\dfrac{c_1 - c_0}{V_0}\right) u_1(t-\tau) - \left(\dfrac{c_2 - c_0}{V_0}\right) u_2(t-\tau) = 0. \end{array} \right. \qquad (\star)$$

- Let $D = \mathbb{Q}(\theta, c_0, c_1, c_2, V_0)[\partial_1, \partial_2]$ be the commutative polynomial ring:

$$\partial_1\, x(t) = \dot{x}(t), \quad \partial_2\, x(t) = x(t-h).$$

$$(\partial_1\,\partial_2)(x(t)) = \partial_1(x(t-h)) = \dot{x}(t-h) = \partial_2(\dot{x}(t)) = (\partial_2\,\partial_1)(x(t)).$$

- The differential time-delay system $(\star)$ can be rewritten as:

$$\left(\begin{array}{cccc} \partial_1 + \dfrac{1}{2\,\theta} & 0 & -1 & -1 \\ 0 & \partial_1 + \dfrac{1}{\theta} & -\left(\dfrac{c_1 - c_0}{V_0}\right)\partial_2 & -\left(\dfrac{c_2 - c_0}{V_0}\right)\partial_2 \end{array} \right) \left(\begin{array}{c} x_1(t) \\ x_2(t) \\ u_1(t) \\ u_2(t) \end{array} \right) = 0.$$

Example

- Linearization of the Navier-Stokes $\sim$ a parabolic Poiseuille profile

$$\begin{cases} \partial_t\, u_1 + 4\, y\, (1-y)\, \partial_x\, u_1 - 4\, (2\, y - 1)\, u_2 - \nu\, (\partial_x^2 + \partial_y^2)\, u_1 + \partial_x\, p = 0, \\ \partial_t\, u_2 + 4\, y\, (1-y)\, \partial_x\, u_2 - \nu\, (\partial_x^2 + \partial_y^2)\, u_2 + \partial_y\, p = 0, \\ \partial_x\, u_1 + \partial_y\, u_2 = 0. \end{cases}$$

- Let $D = \mathbb{Q}(\nu)\langle \partial_t, \partial_x, \partial_y, y\rangle$ be the associative algebra defined by

$$\partial_t\, \partial_x = \partial_x\, \partial_t,\ \partial_t\, y = y\, \partial_t,\ \partial_t\, \partial_y = \partial_y\, \partial_t,\ \partial_x\, \partial_y = \partial_y\, \partial_x,\ \partial_x\, y = y\, \partial_x,$$

$$\partial_y\, y = y\, \partial_y + 1 \quad (\partial_y(y\, f(y)) = y\, \partial_y\, f(y) + f(y)).$$

- The PD linear system can be rewritten as:

$$\begin{pmatrix} \partial_t + 4\, y\, (1-y)\, \partial_x - \nu\, (\partial_x^2 + \partial_y^2) & -4\, (2\, y - 1) & \partial_x \\ 0 & \partial_t + 4\, y\, (1-y)\, \partial_x - \nu\, (\partial_x^2 + \partial_y^2) & \partial_y \\ \partial_x & \partial_y & 0 \end{pmatrix} \begin{pmatrix} u_1(x_1, x_2, t) \\ u_2(x_1, x_2, t) \\ p(x_1, x_2, t) \end{pmatrix} = 0.$$

Finitely presented modules

- Let D be a noetherian domain, $R \in D^{q \times p}$.
- We consider the left D-homomorphism (i.e., left D-linear map):

$$\begin{array}{rcl} D^{1 \times q} & \xrightarrow{.R} & D^{1 \times p} \\ \lambda = (\lambda_1 \ \ldots \ \lambda_q) & \longmapsto & \lambda\, R. \end{array}$$

- We introduce the finitely presented left D-module:

$$M := \operatorname{coker}_D(.R) = D^{1 \times p}/(D^{1 \times q}\, R).$$

- $M = D^{1 \times p}/(D^{1 \times q}\, R)$ is formed by the classes $\pi(\lambda)$ of $\lambda \in D^{1 \times p}$:

$$\pi(\lambda) = \pi(\lambda') \quad \Leftrightarrow \quad \exists\, \mu \in D^{1 \times q} : \ \lambda = \lambda' + \mu\, R.$$

- M is a left D-module: $\forall\, \lambda,\, \lambda' \in D^{1 \times p}, \quad \forall\, d \in D$:

$$\pi(\lambda) + \pi(\lambda') := \pi(\lambda + \lambda'), \quad d\, \pi(\lambda) := \pi(d\, \lambda).$$

Duality: modules — linear systems

- Let $\{f_k\}_{k=1,\ldots,p}$ the standard basis of $D^{1\times p}$ ($f_k = (0\ldots1\ldots0)$).
- Let $\pi : D^{1\times p} \longrightarrow M$ be the D-homomorphism sending μ to $\pi(\mu)$.
- $M = D^{1\times p}/(D^{1\times q}\, R)$ is generated by $\{y_k = \pi(f_k)\}_{k=1,\ldots,p}$ and:

$$l = 1, \ldots, q, \quad \sum_{k=1}^{p} R_{lk}\, y_k = 0 \;\Leftrightarrow\; R\, y = 0, \quad y = (y_1 \;\ldots\; y_p)^T.$$

- Let $\mathcal{F}$ be a left D-module and the linear system:

$$\ker_{\mathcal{F}}(R.) := \{\eta \in \mathcal{F}^p \mid R\,\eta = 0\} \quad \text{(Willems' behaviour)}.$$

- Let $\hom_D(M, \mathcal{F})$ be the abelian group of D-homomorphisms:

$$\hom_D(M, \mathcal{F}) := \{f : M \to \mathcal{F} \mid f(d_1\, m_1 + d_2\, m_2) = d_1\, f(m_1) + d_2\, f(m_2)\}.$$

- **Theorem** (Malgrange): $\hom_D(M, \mathcal{F}) \cong \ker_{\mathcal{F}}(R.)$.

Module theory

- **Definition**: 1. M is free if $\exists\, r \in \mathbb{Z}_+$ such that $M \cong D^{1\times r}$.

2. M is stably free if $\exists\, r,\, s \in \mathbb{Z}_+$ such that $M \oplus D^{1\times s} \cong D^{1\times r}$.

3. M is projective if $\exists\, r \in \mathbb{Z}_+$ and a D-module P such that:

$$M \oplus P \cong D^{1\times r}.$$

4. M is reflexive if $\varepsilon : M \longrightarrow \hom_D(\hom_D(M, D), D)$ is an isomorphism, where:

$$\varepsilon(m)(f) = f(m), \quad \forall\, m \in M, \quad f \in \hom_D(M, D).$$

5. M is torsion-free if:

$$t(M) = \{m \in M \mid \exists\, 0 \neq d \in D : d\, m = 0\} = 0.$$

6. M is torsion if $t(M) = M$.

Classification of modules

- **Theorem**: 1. We have the following implications:

free $\Rightarrow$ stably free $\Rightarrow$ projective $\Rightarrow$ reflexive $\Rightarrow$ torsion-free.

2. If D is a principal domain (e.g., $B_1(\mathbb{Q}) := \mathbb{Q}(t)\langle\partial\rangle$), then:

torsion-free $=$ free.

3. If D is a hereditary ring (e.g., $A_1(\mathbb{Q}) := \mathbb{Q}[t]\langle\partial\rangle$), then:

torsion-free $=$ projective.

4. If $D = k[\partial_1, \ldots, \partial_n]$ and k a field, then:

projective $=$ free (Quillen-Suslin theorem).

4. If $D = A_n(k)$ or $B_n(k)$, k is a field of characteristic 0, then

projective $=$ free (Stafford theorem),

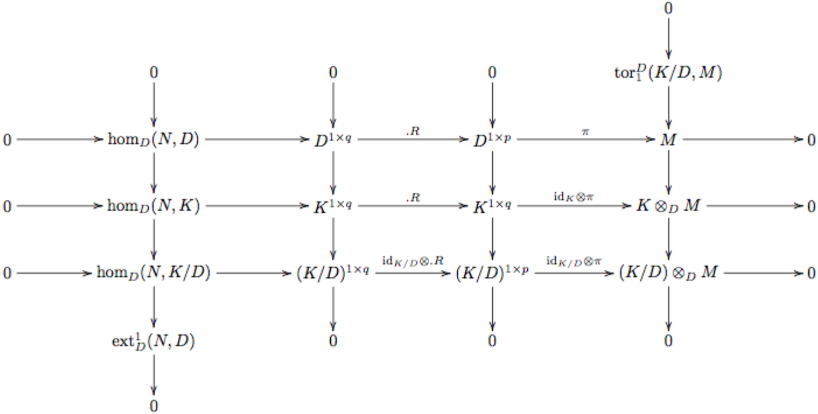
for modules of rank at least 2.

Module M	**Homological algebra**	$\mathcal{F}$ **injective**
with torsion	$t(M) \cong \mathrm{ext}^1_D(N, D)$	$\emptyset$
torsion-free	$\mathrm{ext}^1_D(N, D) = 0$	$\ker_{\mathcal{F}}(R.) = Q\, \mathcal{F}^{l_1}$
reflexive	$\mathrm{ext}^i_D(N, D) = 0$ $i = 1, 2$	$\ker_{\mathcal{F}}(R.) = Q_1\, \mathcal{F}^{l_1}$ $\ker_{\mathcal{F}}(Q_1.) = Q_2\, \mathcal{F}^{l_2}$
projective = stably free	$\mathrm{ext}^i_D(N, D) = 0$ $1 \leq i \leq n = \mathrm{gld}(D)$	$\ker_{\mathcal{F}}(R.) = Q_1\, \mathcal{F}^{l_1}$ $\ker_{\mathcal{F}}(Q_1.) = Q_2\, \mathcal{F}^{l_2}$ $\dots$ $\ker_{\mathcal{F}}(Q_{n-1}.) = Q_n\, \mathcal{F}^{l_n}$
free	$\exists\ Q \in D^{p \times m},\ T \in D^{m \times p}$, $\ker_D(.Q) = D^{1 \times q} R,\ T\, Q = I_m$	$\ker_{\mathcal{F}}(R.) = Q\, \mathcal{F}^m$, $\exists\ T \in D^{m \times p}:\ T\, Q = I_m$

OreModules packages (Maple, Mathematica, GAP, Singular): Gröbner basis techniques for n.c. polynomial rings

Homological algebra techniques

- $K := Q(D)$ the left and right quotient field of D.
- $N := D^{q\times 1}/(R\,D^{p\times 1})$ the transpose of $M = D^{1\times p}/(D^{1\times q}\,R)$.



$$t(M) := \{m \in M \mid \exists\, d \in D\backslash\{0\} : \; d\, m = 0\} \cong \mathrm{tor}_1^D(K/D, M) \cong \mathrm{ext}_D^1(N, D).$$

- Are the linearized Einstein's equations parametrizable? (Wheeler)

$$
R = \left(\begin{array}{cccccccccc}
\partial_2^2+\partial_3^2-\partial_t^2 & \partial_1^2 & \partial_1^2 & -\partial_1^2 & -2\,\partial_1\,\partial_2 & 0 & 0 & -2\,\partial_1\,\partial_3 & 0 & 2\,\partial_1\,\partial_t \\
\partial_2^2 & \partial_1^2+\partial_3^2-\partial_t^2 & \partial_2^2 & -\partial_2^2 & -2\,\partial_1\,\partial_2 & -2\,\partial_2\,\partial_3 & 0 & 0 & 2\,\partial_2\,\partial_t & 0 \\
\partial_3^2 & \partial_3^2 & \partial_1^2+\partial_2^2-\partial_t^2 & -\partial_3^2 & 0 & -2\,\partial_2\,\partial_3 & 2\,\partial_3\,\partial_t & -2\,\partial_1\,\partial_3 & 0 & 0 \\
\partial_t^2 & \partial_t^2 & \partial_t^2 & \partial_1^2+\partial_2^2+\partial_3^3 & 0 & 0 & -2\,\partial_3\,\partial_t & 0 & -2\,\partial_2\,\partial_t & -2\,\partial_1\,\partial_t \\
0 & 0 & \partial_1\,\partial_2 & -\partial_1\,\partial_2 & \partial_2^3-\partial_t^2 & -\partial_1\,\partial_3 & 0 & -\partial_2\,\partial_3 & \partial_1\,\partial_t & \partial_2\,\partial_t \\
\partial_2\,\partial_3 & 0 & 0 & -\partial_2\,\partial_3 & -\partial_1\,\partial_3 & \partial_1^2-\partial_t^2 & \partial_2\,\partial_t & -\partial_1\,\partial_2 & \partial_3\,\partial_t & 0 \\
\partial_3\,\partial_t & \partial_3\,\partial_t & 0 & 0 & 0 & -\partial_2\,\partial_t & \partial_1^2+\partial_2^2 & -\partial_1\,\partial_t & -\partial_2\,\partial_3 & -\partial_1\,\partial_3 \\
0 & \partial_1\,\partial_3 & 0 & -\partial_1\,\partial_3 & -\partial_2\,\partial_3 & -\partial_1\,\partial_2 & \partial_1\,\partial_t & \partial_2^2-\partial_t^2 & 0 & \partial_3\,\partial_t \\
\partial_2\,\partial_t & 0 & \partial_2\,\partial_t & 0 & -\partial_1\,\partial_t & -\partial_3\,\partial_t & -\partial_2\,\partial_3 & 0 & \partial_1^2+\partial_3^2 & -\partial_1\,\partial_3 \\
0 & \partial_1\,\partial_t & \partial_1\,\partial_t & 0 & -\partial_2\,\partial_t & 0 & -\partial_1\,\partial_3 & -\partial_3\,\partial_t & -\partial_1\,\partial_3 & \partial_2^2+\partial_3^2
\end{array}\right)
$$

Example: torsion-free and reflexive modules

- Wind tunnel model (Manitius 84): torsion-free but not reflexive

$$\left\{\begin{array}{l} \dot{x}_1(t) + a\,x_1(t) - k\,a\,x_2(t-h) = 0, \\ \dot{x}_2(t) - x_3(t) = 0, \\ \dot{x}_3(t) + \omega^2\,x_2(t) + 2\,\zeta\,\omega\,x_3(t) - \omega^2\,u(t) = 0, \end{array}\right.$$

$$\Leftrightarrow \left\{\begin{array}{ll} x_1(t) = \omega^2\,k\,a\,z(t-h), & \\ x_2(t) = \omega^2\,\dot{z}(t) - a\,\omega^2\,z(t), & \text{(motion planning} \\ x_3(t) = \omega^2\,\ddot{z}(t) + \omega^2\,a\,\dot{z}(t), & \text{(Mounier et al 95))} \\ u(t) = z(t)^{(3)} + (2\,\zeta\,\omega + a)\,\ddot{z}(t) + (\omega^2 + 2\,a\,\omega\,\zeta)\,\dot{z}(t) + a\,\omega\,z(t). & \end{array}\right.$$

- First group of Maxwell equations: reflexive but not projective

$$\left\{\begin{array}{l} \dfrac{\partial \vec{B}}{\partial t} + \vec{\nabla} \wedge \vec{E} = \vec{0}, \\ \vec{\nabla}.\,\vec{B} = 0, \end{array}\right. \quad \Leftrightarrow \quad \left\{\begin{array}{l} \vec{E} = -\dfrac{\partial \vec{A}}{\partial t} - \vec{\nabla}\,V, \\ \vec{B} = \vec{\nabla} \wedge \vec{A}. \end{array}\right.$$

$$\left\{\begin{array}{l} -\dfrac{\partial \vec{A}}{\partial t} - \vec{\nabla}\,V = \vec{0}, \\ \vec{\nabla} \wedge \vec{A} = \vec{0}, \end{array}\right. \quad \Leftrightarrow \quad \left\{\begin{array}{l} \vec{A} = \vec{\nabla}\,\xi, \\ V = -\dfrac{\partial \xi}{\partial t}. \end{array}\right. \quad \text{(gauge)}$$

Example: reflexive modules

- Equilibrium of the stress tensor (3D elasticity):

$$\left\{\begin{array}{l} \dfrac{\partial \sigma_x}{\partial x} + \dfrac{\partial \tau_{xy}}{\partial y} + \dfrac{\partial \tau_{zx}}{\partial z} = 0, \\ \dfrac{\partial \tau_{xy}}{\partial x} + \dfrac{\partial \sigma_y}{\partial y} + \dfrac{\partial \tau_{yz}}{\partial z} = 0, \\ \dfrac{\partial \tau_{zx}}{\partial x} + \dfrac{\partial \tau_{yz}}{\partial y} + \dfrac{\partial \sigma_z}{\partial z} = 0, \end{array}\right. \qquad \Leftrightarrow$$

$$\left\{\begin{array}{ll} \sigma_x = \dfrac{\partial^2 \chi_3}{\partial y^2} + \dfrac{\partial^2 \chi_2}{\partial z^2} + \dfrac{\partial^2 \psi_1}{\partial y\, \partial z}, & \tau_{yz} = -\dfrac{\partial^2 \chi_1}{\partial y\, \partial z} - \dfrac{1}{2}\dfrac{\partial}{\partial x}\left(-\dfrac{\partial \psi_1}{\partial x} + \dfrac{\partial \psi_2}{\partial y} + \dfrac{\partial \psi_3}{\partial z}\right), \\ \sigma_y = \dfrac{\partial^2 \chi_1}{\partial z^2} + \dfrac{\partial^2 \chi_3}{\partial x^2} + \dfrac{\partial^2 \psi_2}{\partial z\, \partial x}, & \tau_{zx} = -\dfrac{\partial^2 \chi_2}{\partial z\, \partial x} - \dfrac{1}{2}\dfrac{\partial}{\partial y}\left(\dfrac{\partial \psi_1}{\partial x} - \dfrac{\partial \psi_2}{\partial y} + \dfrac{\partial \psi_3}{\partial z}\right), \\ \sigma_z = \dfrac{\partial^2 \chi_2}{\partial x^2} + \dfrac{\partial^2 \chi_1}{\partial y^2} + \dfrac{\partial^2 \psi_3}{\partial x\, \partial y}, & \tau_{xy} = -\dfrac{\partial^2 \chi_3}{\partial x\, \partial y} - \dfrac{1}{2}\dfrac{\partial}{\partial z}\left(\dfrac{\partial \psi_1}{\partial x} + \dfrac{\partial \psi_2}{\partial y} - \dfrac{\partial \psi_3}{\partial z}\right). \end{array}\right.$$

Maxwell parametrization, Morera parametrization.

Constructive version of Stafford's theorem

- The time-varying linear control system

$$\begin{cases} \dot{x}_1(t) - t\, u_1(t) = 0, \\ \dot{x}_2(t) - u_2(t) = 0, \end{cases}$$

is injectively parametrized by (STAFFORD (Robertz, Q.))

$$\begin{cases} x_1(t) = t^2\, \xi_1(t) - t\, \dot{\xi}_2(t) + \xi_2(t), \\ x_2(t) = t\,(t+1)\, \xi_1(t) - (t+1)\, \dot{\xi}_2(t) + \xi_2(t), \\ u_1(t) = t\, \dot{\xi}_1(t) + 2\, \xi_1(t) - \ddot{\xi}_2(t), \\ u_2(t) = t\,(t+1)\, \dot{\xi}_1(t) + (2\,t+1)\, \xi_1(t) - (t+1)\, \ddot{\xi}_2(t), \end{cases}$$

and $\{\xi_1, \xi_2\}$ is a basis of the free left $A_1(\mathbb{Q})$-module M as:

$$\begin{cases} \xi_1(t) = (t+1)\, u_1(t) - u_2(t), \\ \xi_2(t) = (t+1)\, x_1(t) - t\, x_2(t). \end{cases}$$

- Idem for $\partial_1\, y_1 + \partial_2\, y_2 + \partial_3\, y_3 + x_3\, y_1 = 0$.

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