Centrohermitian Solutions of a Factorization Problem Arising in Vibration Analysis. Part I: Lee's Transformation

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***Abstract*— Motivated by an application of vibration analysis to gearbox fault surveillance, a new demodulation approach for gearbox vibration signals has recently been developed. Within this approach, the demodulation problem yields the study of a rank factorization problem for centrohermitian matrices. In this paper, using the properties of centrohermitian matrices, we first show that the rank factorization problem for centrohermitian matrices can be transformed into a rank factorization problem for real matrices. Based on previous works, we then show how to parametrize a class of centrohermitian solutions of the rank factorization problem that is important in practice.**

Notation. In what follows, let $\Bbbk$ denote a field (e.g., $\Bbbk = \mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$), $\Bbbk^{n\times m}$ the $\Bbbk$-vector space formed by all the $n \times m$ matrices with entries in $\Bbbk$ and I_n the identity matrix of $\Bbbk^{n\times n}$. We also denote by J_n the $n \times n$ *exchange matrix*, namely, the matrix formed by 1 on the second diagonal (i.e., the anti-diagonal) and 0 elsewhere. If $\Bbbk = \mathbb{C}$, then $\overline{M}$ (resp., $M^\star$) denotes the *conjugate matrix* (resp., the *adjoint*, namely, the *conjugate transpose*) of M. Finally, if $M \in \Bbbk^{n\times m}$, then we can consider the following $\Bbbk$-linear maps

$$\begin{array}{rcl} M.: \Bbbk^{m\times 1} & \longrightarrow & \Bbbk^{n\times 1} \\ \eta & \longmapsto & M\,\eta, \end{array} \qquad \begin{array}{rcl} .M: \Bbbk^{1\times n} & \longrightarrow & \Bbbk^{1\times m} \\ \lambda & \longmapsto & \lambda\, M, \end{array}$$

and denote their kernels (resp., images) respectively by $\ker_{\Bbbk}(M.)$ and $\ker_{\Bbbk}(.M)$ (resp., $\mathrm{im}_{\Bbbk}(M.)$ and $\mathrm{im}_{\Bbbk}(.M)$).

I. Statement of the problem

We first introduce the concept of a *centrohermitian* matrix.

Definition 1 ([12], [5]): A matrix $M \in \mathbb{C}^{n\times m}$ is called *centrohermitian* if $J_n\,\overline{M}\,J_m = M$. The set of all the centrohermitian matrices of $\mathbb{C}^{n\times m}$ is denoted by $\mathrm{CH}_{n,m}$.

Using $J_1 = 1$, a vector $u \in \mathbb{C}^{n\times 1}$ (resp., $v \in \mathbb{C}^{1\times m}$) is centrohermitian if $J_n\,\overline{u} = u$ (resp., $\overline{v}\,J_m = v$).

The next examples show that centrohermitian vectors and matrices naturally appear while considering Fourier analysis of real signals (e.g., gearbox vibration signals) [6], [7].

Example 1: If $s : \mathbb{R} \to \mathbb{R}$ is a T-periodic integrable function, then s can be expressed by its *Fourier series* [13]

$$\forall\, t \in \mathbb{R}, \quad s(t) = \sum_{j\in\mathbb{Z}} c_j(s)\, e^{\frac{2\pi i j t}{T}},$$

whose *Fourier coefficients* are defined by:

$$\forall\, j \in \mathbb{Z}, \quad c_j(s) = \frac{1}{T}\int_0^T s(t)\, e^{-\frac{2\pi i j t}{T}}\, dt. \tag{1}$$

If $r \in \mathbb{Z} \setminus \{0\}$, then we can define the column vector

$$c = (c_r(s)\ \ldots\ c_0(s)\ \ldots\ c_{-r}(s))^T \in \mathbb{C}^{(2\,r+1)\times 1}$$

formed with the first Fourier coefficients $c_j(s)$ of s centered around 0. Since s is supposed to be real, using (1), we have

$$\forall\, j \in \mathbb{Z}, \quad \overline{c_j(s)} = c_{-j}(\overline{s}) = c_{-j}(s), \tag{2}$$

which shows that c is a centrohermitian column vector.

By Definition 1, a matrix M is centrohermitian if M is equal to the matrix obtained by reversing the rows and columns of $\overline{M}$. For instance, the following matrix

$$M = \begin{pmatrix} a & b & c \\ d & e & \overline{d} \\ \overline{c} & \overline{b} & \overline{a} \end{pmatrix},$$

where a, b, c, $d \in \mathbb{C}$ and $e \in \mathbb{R}$, is centrohermitian.

Example 2: We consider again Example 1. If we simply denote by c_j the j^{th} Fourier coefficient $c_j(s)$ of s and fix p, $q \in \mathbb{Z}_{>0}$, then we can form the following complex matrix

$$M = \begin{pmatrix} c_{q\,(2\,p+1)+p} & \cdots & c_p & \cdots & c_{-q\,(2\,p+1)+p} \\ \vdots & & \vdots & & \vdots \\ c_{q\,(2\,p+1)} & \cdots & c_0 & \cdots & c_{-q\,(2\,p+1)} \\ \vdots & & \vdots & & \vdots \\ c_{q\,(2\,p+1)-p} & \cdots & c_{-p} & \cdots & c_{-q\,(2\,p+1)-p} \end{pmatrix}$$

with the Fourier coefficients c_j for $0 \le |j| \le q\,(2\,p+1)+p$. Using (2), we can check that M is a centrohermitian matrix.

In [6], [7], it was shown that the *gearbox spectrum* can be represented by a centrohermitian matrix M and the problem of separating the time vibration signal into its main components amounts to estimating centrohermitian vectors u and $v_1, \ldots, v_r$ that solve the following problem.

The centrohermitian rank factorization problem: Let $D_1, \ldots, D_r \in \mathrm{CH}_{n,n} \setminus \{0\}$ and $M \in \mathrm{CH}_{n,m} \setminus \{0\}$. Determine – when they exist – vectors $u \in \mathrm{CH}_{n,1}$ and $v_1, \ldots, v_r \in \mathrm{CH}_{1,m}$ satisfying:

$$M = \sum_{i=1}^{r} D_i\, u\, v_i. \tag{3}$$

Remark 1: If a solution of (3) exists, then we have $M\,w = \sum_{i=1}^{r} D_i\, u\,(v_i\, w)$ for all $w \in \Bbbk^{m\times 1}$, which implies that $\mathrm{im}_{\Bbbk}(M.) \subseteq \mathrm{span}_{\Bbbk}\{D_1\,u, \ldots, D_r\,u\}$, and thus, we get:

$$\mathrm{rank}_{\Bbbk}(M) \le \dim_{\Bbbk}(\mathrm{span}_{\Bbbk}\{D_1\,u, \ldots, D_r\,u\}) \le \min\{r, n\}.$$

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The centrohermitian rank factorization problem can be generalized by considering general matrices M, $D_1, \ldots, D_r$ and general vectors u and $v_1, \ldots, v_r$ as follows.

The rank factorization problem:
Let $D_1, \ldots, D_r \in \Bbbk^{n \times n} \setminus \{0\}$ and $M \in \Bbbk^{n \times m} \setminus \{0\}$. Determine – when they exist – vectors $u \in \Bbbk^{n \times 1}$ and $v_1, \ldots, v_r \in \Bbbk^{1 \times m}$ satisfying (3).

The rank factorization problem was solved for $r = 1$ and $D_1 = I_n$ in [7], and for $r = 2$ and $D_1 = I_n$ in [8]. In [9], the problem was solved with the assumption that the row vectors v_i's are $\Bbbk$-linearly independent, i.e., that the matrix $v := (v_1^T \ \ldots \ v_r^T)^T$ has full row rank. In our application, the vector $v_i \in \Bbbk^{1 \times n}$ contains the first $(n/2)^{\text{th}}$ harmonics of a time signal (see Example 1). Hence, the assumption that v has full row rank means that the truncation of the underlying time signals at their first $(n/2)^{\text{th}}$ harmonics are $\Bbbk$-linear independent. Based on *module theory* and *computer algebra*, local closed-form solutions were obtained in [10].

The goal of this paper is to show that the centrohermitian rank factorization problem can be reduced to the rank factorization for real matrices and real vectors. Hence, using the results obtained in [7], [9], [10], we can effectively solve the centrohermitian rank factorization problem when the row vectors v_i's are supposed to be $\mathbb{C}$-linearly independent.

II. Centrohermitian matrices

If M_1, $M_2 \in \mathbb{C}^{t \times s}$, $m_1 \in \mathbb{C}^{t \times 1}$, $m_2 \in \mathbb{C}^{1 \times s}$ and $m_3 \in \mathbb{R}$, then we can check that a centrohermitian matrix $M \in \mathbb{C}^{n \times m}$ has one of the following 4 forms [12]:

1) If $n = 2\,t$ and $m = 2\,s$, then:
$$M = \begin{pmatrix} M_1 & M_2\,J_s \\ J_t\,\overline{M_2} & J_t\,\overline{M_1}\,J_s \end{pmatrix}.$$
2) If $n = 2\,t$ and $m = 2\,s+1$, then:
$$M = \begin{pmatrix} M_1 & m_1 & M_2\,J_s \\ J_t\,\overline{M_2} & J_t\,\overline{m_1} & J_t\,\overline{M_1}\,J_s \end{pmatrix}.$$
3) If $n = 2\,t+1$ and $m = 2\,s$, then:
$$M = \begin{pmatrix} M_1 & M_2\,J_s \\ m_2 & \overline{m_2}\,J_s \\ J_t\,\overline{M_2} & J_t\,\overline{M_1}\,J_s \end{pmatrix}.$$
4) If $n = 2\,t+1$ and $m = 2\,s+1$, then:
$$M = \begin{pmatrix} M_1 & m_1 & M_2\,J_s \\ m_2 & m_3 & \overline{m_2}\,J_s \\ J_t\,\overline{M_2} & J_t\,\overline{m_1} & J_s\,\overline{M_1}\,J_s \end{pmatrix}.$$

We state again a few results on centrohermitian matrices. Clearly, if $M \in \mathbb{C}^{n \times m}$ is a centrohermitian matrix, so are:
$$\overline{M},\ M^T,\ M^\star,\ J_n\,M,\ M\,J_m.$$
Using $J_m^2 = I_m$, the product of two centrohermitian matrices $A \in \mathbb{C}^{n \times m}$ and $B \in \mathbb{C}^{m \times p}$ is then also centrohermitian:
$$A\,B = J_n\,\overline{A}\,J_m^2\,\overline{B}\,J_p = J_n\,\overline{A\,B}\,J_p.$$
Hence, the set $\mathrm{CH}_{n,n}$ of all the centrohermitian matrices of $\mathbb{C}^{n \times n}$ is stable under addition, multiplication by reals and multiplication, i.e., $\mathrm{CH}_{n,n}$ has an $\mathbb{R}$-algebra structure. Finally, if $M \in \mathrm{CH}_{n,n}$ is non-singular, then $M^{-1} \in \mathrm{CH}_{n,n}$.

Theorem 1 ([5], [12]): If M is a centrohermitian, then so is its *Moore-Penrose inverse* M^+.

In what follows, we state Lee's theorem [12] which explicitly characterizes $\mathrm{CH}_{n,m}$. To do that, we first introduce a few more definitions and simple results.

Definition 2 ([12]): The matrix $Q \in \mathbb{C}^{n \times m}$ is called *J-real* if we have:
$$J_n\,\overline{Q} = Q.$$
Hence, Q is J-real iff its columns are centrohermitian.

Example 3: The following complex matrices are J-real:
$$Q_{2t} = \frac{1}{\sqrt{2}} \begin{pmatrix} I_t & i\,I_t \\ J_t & -i\,J_t \end{pmatrix} \in \mathbb{C}^{2t \times 2t}, \quad Q_1 = 1,$$
$$Q_{2t+1} = \frac{1}{\sqrt{2}} \begin{pmatrix} I_t & 0 & i\,I_t \\ 0 & \sqrt{2} & 0 \\ J_t & 0 & -i\,J_t \end{pmatrix} \in \mathbb{C}^{(2t+1) \times (2t+1)}.$$
We note that both matrices Q_{2t} and Q_{2t+1} are also *unitary*:
$$Q_{2t}^\star\,Q_{2t} = Q_{2t}\,Q_{2t}^\star = I_{2t},$$
$$Q_{2t+1}^\star\,Q_{2t+1} = Q_{2t+1}\,Q_{2t+1}^\star = I_{2t+1}.$$
Let $U \in \mathbb{C}^{m \times m}$ and $V \in \mathbb{C}^{n \times n}$ be two non-singular J-real matrices. Using $J_m^2 = I_m$, then we have:
$$\overline{U} = J_m\,(J_m\,\overline{U}) = J_m\,U,$$
$$J_n\,\overline{V} = V \ \Leftrightarrow\ V^{-1}\,J_n\,\overline{V} = I_n \ \Leftrightarrow\ \overline{V^{-1}} = \overline{V}^{-1} = V^{-1}\,J_n.$$
If $M \in \mathrm{CH}_{n,m}$ and $\varphi(M) := V^{-1}\,M\,U$, then we have
$$\begin{aligned}\overline{\varphi(M)} &= \overline{V^{-1}}\,\overline{M}\,\overline{U} = (V^{-1}\,J_n)\,(J_n\,M\,J_m)\,(J_m\,U) \\ &= V^{-1}\,M\,U = \varphi(M),\end{aligned}$$
which shows that $\varphi(M) \in \mathbb{R}^{n \times m}$, i.e., $\varphi(M)$ is a real matrix.

Now, if $N \in \mathbb{R}^{n \times m}$, then we have
$$J_n\,\overline{(V\,N\,U^{-1})}\,J_m = J_n\,\overline{V}\,N\,\overline{U}^{-1}\,J_m = V\,N\,U^{-1},$$
which shows that $V\,N\,U^{-1}$ is a centrohermitian matrix.

These two results show the following important result.

Theorem 2 ([12]): Let $U \in \mathbb{C}^{m \times m}$ and $V \in \mathbb{C}^{n \times n}$ be two non-singular J-real matrices. Then, the $\mathbb{R}$-linear application
$$\begin{array}{rccc} \varphi : & \mathbb{C}^{n \times m} & \longrightarrow & \mathbb{C}^{n \times m} \\ & M & \longmapsto & V^{-1}\,M\,U, \end{array} \tag{4}$$
bijectively maps $\mathrm{CH}_{n,m}$ onto $\mathbb{R}^{n \times m}$, i.e.:

- $\varphi(\mathrm{CH}_{n,m}) = \mathbb{R}^{n \times m}$,
- $\forall\, N \in \mathbb{R}^{n \times m},\ \exists!\, M \in \mathrm{CH}_{n,m}(\mathbb{C}) :\ N = V^{-1}\,M\,U.$

Hence, $\mathrm{CH}_{n,m} = \varphi^{-1}(\mathbb{R}^{n \times m})$, i.e., $M \in \mathrm{CH}_{n,m}$ iff there exists a unique $N \in \mathbb{R}^{n \times m}$ such that $N = V^{-1}\,M\,U$.

Finally, if $m = n$, then φ isomorphically maps the ring $\mathrm{CH}_{n,n}$ onto the ring $\mathbb{R}^{n \times n}$ and φ is a *ring automorphism*.

Example 4: Let Q_n (resp., Q_m) be defined as in Example 3. Then, $Q_n^{-1} = Q_n^\star$ since Q_n is unitary. If $u \in \mathbb{C}^{n \times 1}$ is a centrohermitian column vector, i.e., $u \in \mathrm{CH}_{n,1}$, then, by

Theorem 2, $u_\varphi := Q_n^\star\, u$ is a real vector, i.e., $u_\varphi \in \mathbb{R}^{n\times 1}$. Similarly, if $v_i \in \mathrm{CH}_{1,m}$, then $v_{i,\varphi} := v_i\, Q_m \in \mathbb{R}^{1\times m}$.

Example 5: Let us consider the following matrix:

$$M = \left(\begin{array}{cccc} \frac{45}{2}+5\,i & 10-15\,i & 15\,i & \frac{45}{2}+10\,i \\ \frac{5}{2}\sqrt{2}-\frac{15}{2}\sqrt{2}\,i & 10\sqrt{2} & 10\sqrt{2} & \frac{5}{2}\sqrt{2}+\frac{15}{2}\sqrt{2}\,i \\ \frac{45}{2}-10\,i & -15\,i & 10+15\,i & \frac{45}{2}-5\,i \end{array}\right).$$

Clearly, $M \in \mathrm{CH}_{3,4}$. With the notations of Example 3, i.e.,

$$Q_3 = \frac{1}{\sqrt{2}}\left(\begin{array}{ccc} 1 & 0 & i \\ 0 & \sqrt{2} & 0 \\ 1 & 0 & -i \end{array}\right), \qquad Q_4 = \frac{1}{\sqrt{2}}\left(\begin{array}{cc} I_2 & i\, I_2 \\ J_2 & -i\, J_2 \end{array}\right),$$

we then have $Q_3^{-1} = Q_3^\star$ and:

$$\varphi(M) = Q_3^\star\, M\, Q_4 = \left(\begin{array}{cccc} 45 & 10 & 5 & 30 \\ 5 & 20 & 15 & 0 \\ 15 & 0 & 0 & 10 \end{array}\right) \in \mathbb{R}^{3\times 4}.$$

Note that Theorem 2 yields $\dim_{\mathbb{R}}(\mathrm{CH}_{n,m}) = n\, m$.

If λ is an eigenvalue of $M \in \mathrm{CH}_{n,n}$ with as x an associated eigenvector, then $M\, x = \lambda\, x$ yields $\overline{M}\,\overline{x} = \overline{\lambda}\,\overline{x}$, and thus, $M\,(J_n\,\overline{x}) = \overline{\lambda}\,(J_n\,\overline{x})$, which shows that $\overline{\lambda}$ is also an eigenvalue of M with $J_n\,\overline{x}$ as an associated eigenvector.

Theorem 2 shows that a centrohermitian matrix is similar to a real matrix. Hence, the spectral theory of centrohermitian matrices corresponds to the one of real matrices.

Theorem 3 ([12]): If M is centrohermitian and λ is an eigenvalue of M of algebraic multiplicity k, then $\overline{\lambda}$ is an eigenvalue of M of algebraic multiplicity k. In particular, if $M \in \mathrm{CH}_{n,n}$ with n odd, then M always has a real eigenvalue. The characteristic polynomial of $M \in \mathrm{CH}_{n,n}$ has thus all real coefficients, and thus, $\det(M) \in \mathbb{R}$.

Example 6: Let us consider the centrohermitian matrix:

$$M = \left(\begin{array}{ccc} 9+18\,i & -225 & 9+198\,i \\ 0 & 0 & 0 \\ 9-198\,i & -225 & 9-18\,i \end{array}\right) \in \mathbb{C}^{3\times 3}.$$

Let us consider the J-real matrix Q_3 defined in Example 5. Then, we have $Q_3^{-1} = Q_3^\star$ and:

$$\varphi(M) = Q_3^\star\, M\, Q_3 = \left(\begin{array}{ccc} 18 & -225\sqrt{2} & 180 \\ 0 & 0 & 0 \\ 216 & 0 & 0 \end{array}\right) \in \mathbb{R}^{3\times 3}.$$

The characteristic polynomial $p(\lambda) = \lambda\,(\lambda^2 - 18\,\lambda - 38880)$ of M has real coefficients and $\det(M) = 0 \in \mathbb{R}$.

Using Theorem 2, we obtain the following consequence.

Proposition 1 ([4]): Let $M \in \mathrm{CH}_{n,m}$, $Q_m \in \mathbb{C}^{m\times m}$ and $Q_n \in \mathbb{C}^{n\times n}$ be two J-real and unitary matrices (e.g., as defined in Example 3), and $M_\varphi := Q_n^\star\, M\, Q_m \in \mathbb{R}^{n\times m}$. Let $M_\varphi = U_\varphi\, \Sigma\, V_\varphi^T$ be a *Singular Value Decomposition* (SVD) of M_φ, where $U_\varphi \in \mathbb{R}^{n\times n}$ and $V_\varphi \in \mathbb{R}^{m\times m}$ are two orthogonal matrices and Σ is the diagonal matrix formed by the singular values of M_φ. Then, a SVD of M is given by:

$$M = (Q_n\, U_\varphi)\, \Sigma\, (Q_m\, V_\varphi)^\star.$$

Hence, the singular values of M are equal to those of M_φ and the left and right singular vectors of M are centrohermitian.

III. Transforming the centrohermitian rank factorization problem into a real rank factorization problem

In this section, we use Theorem 2 to study the centrohermitian rank factorization problem stated in Section I.

Let $Q_m \in \mathbb{C}^{m\times m}$ and $Q_n \in \mathbb{C}^{n\times n}$ be two J-real and unitary matrices (e.g., as defined in Example 3). Since M and $D_1, \ldots, D_r$ are centrohermitian matrices, using Theorem 2, we can define the following real matrices:

$$\left\{\begin{array}{l} M_\varphi := Q_n^\star\, M\, Q_m \in \mathbb{R}^{n\times m}, \\ D_{i\varphi} := Q_n^\star\, D_i\, Q_n \in \mathbb{R}^{n\times n}, \quad i = 1, \ldots, r. \end{array}\right.$$

Note that we have:

$$\mathrm{rank}_{\mathbb{C}}(M) = \mathrm{rank}_{\mathbb{C}}(M_\varphi) = \mathrm{rank}_{\mathbb{R}}(M_\varphi).$$

Let us suppose that the centrohermitian rank factorization problem (3) admits a centrohermitian solution $u, v_1, \ldots, v_r$. Again, by Theorem 2, we have the following real vectors:

$$\left\{\begin{array}{l} u_\varphi := Q_n^\star\, u \in \mathbb{R}^{n\times 1}, \\ v_{i\varphi} := v_i\, Q_m \in \mathbb{R}^{1\times m}, \quad i = 1, \ldots, r. \end{array}\right.$$

See also Example 4. Substituting $M = Q_n\, M_\varphi\, Q_m^\star$ and $D_i = Q_n\, D_{i\varphi}\, Q_n^\star$ for $i = 1, \ldots, r$ into (3), we then obtain

$$\begin{array}{ll} & Q_n\, M_\varphi\, Q_m^\star = \sum_{i=1}^r Q_n\, D_{i\varphi}\, Q_n^\star\, u\, v_i \\ \Leftrightarrow & M_\varphi = \sum_{i=1}^r D_{i\varphi}\, (Q_n^\star\, u)\, (v_i\, Q_m) \\ \Leftrightarrow & M_\varphi = \sum_{i=1}^r D_{i\varphi}\, u_\varphi\, v_{i\varphi}, \end{array}$$

which shows that the rank factorization problem for the real matrices M_φ and $D_{i\varphi}$, $i = 1, \ldots, r$, then admits a real solution $(u_\varphi, v_{1\varphi}, \ldots, v_{r\varphi})$.

Conversely, if there exists a real solution $u_\varphi \in \mathbb{R}^{n\times 1}$, $v_{i\varphi} \in \mathbb{R}^{1\times m}$, $i = 1, \ldots, r$, of the rank factorization problem

$$M_\varphi = \sum_{i=1}^r D_{i\varphi}\, u_\varphi\, v_{i\varphi}, \tag{5}$$

then, by Theorem 2 (see also Example 4), the vectors

$$\left\{\begin{array}{l} u := Q_n\, u_\varphi \in \mathrm{CH}_{n,1}, \\ v_i := v_{i\varphi}\, Q_m^\star \in \mathrm{CH}_{1,m}, \quad i = 1, \ldots, r, \end{array}\right.$$

satisfy (3), i.e., they define a centrohermitian solution of the centrohermitian rank factorization problem (3).

The above results prove the following new result.

Theorem 4: Let $M \in \mathrm{CH}_{n,m}$, $D_i \in \mathrm{CH}_{n,n}$, $i = 1, \ldots, r$, $Q_m \in \mathbb{C}^{m\times m}$ and $Q_n \in \mathbb{C}^{n\times n}$ be two J-real and unitary matrices (e.g., as defined in Example 3), and:

$$\left\{\begin{array}{l} M_\varphi := Q_n^\star\, M\, Q_m \in \mathbb{R}^{n\times m}, \\ D_{i\varphi} := Q_n^\star\, D_i\, Q_n \in \mathbb{R}^{n\times n}, \quad i = 1, \ldots, r. \end{array}\right. \tag{6}$$

Then, the centrohermitian rank factorization problem (3) admits a solution $(u, v_1, \ldots, v_r) \in \mathrm{CH}_{n,1} \times \mathrm{CH}_{1,m}^r$ if and only if the rank factorization problem (5) admits a solution $(u_\varphi, v_{1\varphi}, \ldots, v_{r\varphi}) \in \mathbb{R}^{n\times 1} \times (\mathbb{R}^{1\times m})^r$. Finally, the bijections between the two sets of solutions is given by:

$$\begin{array}{rcl} \mathrm{CH}_{n,1} \times \mathrm{CH}_{1,m}^r & \longmapsto & \mathbb{R}^{n\times 1} \times \left(\mathbb{R}^{1\times m}\right)^r \\ (u, v_1, \ldots, v_r) & \longmapsto & (Q_n^\star\, u, v_1\, Q_m, \ldots, v_r\, Q_m), \end{array}$$

$$\begin{array}{rcl} \mathbb{R}^{n\times 1} \times \left(\mathbb{R}^{1\times m}\right)^r & \longmapsto & \mathrm{CH}_{n,1} \times \mathrm{CH}_{1,m}^r \\ (u_\varphi, v_{1\varphi}, \ldots, v_{r\varphi}) & \longmapsto & (Q_n\, u_\varphi, v_{1\varphi}\, Q_m^\star, \ldots, v_{r\varphi}\, Q_m^\star)\,. \end{array}$$

IV. A CLASS OF SOLUTIONS OF THE CENTROHERMITIAN RANK FACTORIZATION PROBLEM

Theorem 4 shows that centrohermitian solutions of the centrohermitian rank factorization problem are in a one-to-one correspondence with real solutions of the rank factorization problem (i.e., $\Bbbk = \mathbb{R}$). As stated in Section I, a class of solutions of the rank factorization problem (namely, the solutions such that $v = (v_1^T\ \ldots\ v_r^T)^T$ has full row rank) $-$ important in practice $-$ can be explicitly parametrized [9], [10]. Hence, we can parametrize the class of centrohermitian solutions of the centrohermitian rank factorization problem for which v has full row rank. Let us explain how to do it.

Let us state again results on the rank factorization problem (see Section I) obtained in [9], [10]. Let $\Bbbk$ be a field (e.g., $k = \mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$), $M \in \Bbbk^{n\times m}$ and $l := \mathrm{rank}_\Bbbk(M)$. Stacking a basis of $\mathrm{im}(M.) := M\, \Bbbk^{m\times 1}$ into a full column rank matrix $X \in \Bbbk^{n\times l}$, there exists a unique $Y \in \Bbbk^{l\times m}$ such that:

$$M = X\, Y. \tag{7}$$

Denoting

$$\left\{ \begin{array}{l} A(u) := (D_1\, u\ \ldots\ D_r\, u) \in \Bbbk^{n\times r}, \\ v := (v_1^T\ \ldots\ v_r^T)^T \in \Bbbk^{r\times m}, \end{array} \right.$$

then (3) can be rewritten as follows:

$$A(u)\, v = M. \tag{8}$$

The bilinear structure of (8) implies that $(\lambda\, u,\, \lambda^{-1}\, v)$ is a solution for all solutions $(u,\, v)$ and for all $\lambda \in \Bbbk\backslash\{0\}$. Hence, if a solution exists for the rank factorization problem, then it is not unique. Moreover, we note that (8) is equivalent to the existence of $u \in \Bbbk^{n\times 1}$ such that $\mathrm{im}_\Bbbk(M.) \subseteq \mathrm{im}_K(A(u).)$.

In what follows, we shall investigate the class of solutions $(u,\, v)$ of (8) defined by a full row rank matrix v. If such a solution of (8) exists, then v admits a right inverse since v has full row rank, i.e., there exists $t \in \Bbbk^{m\times r}$ such that $v\, t = I_r$. Hence, (8) yields $A(u) = M\, t$, which shows that $\mathrm{im}_K(A(u).) \subseteq \mathrm{im}_\Bbbk(M.)$, and thus, we obtain:

$$\mathrm{im}_\Bbbk(A(u).) = \mathrm{im}_\Bbbk(M.). \tag{9}$$

Thus, the existence of a solution $(u,\, v)$ of (8), where v has full row rank, yields (9). Hence, let us focus on the existence of $u \in \Bbbk^{n\times 1}$ satisfying (9). We note that (9) is equivalent to:

$$\exists\, u \in \Bbbk^{n\times 1}\ \left\{ \begin{array}{l} D_i\, u \in \mathrm{im}_\Bbbk(M.),\ i = 1, \ldots, r, \\ \mathrm{rank}_\Bbbk(D_1\, u\ \ldots\ D_r\, u) = \mathrm{rank}_\Bbbk(M) = l. \end{array} \right. \tag{10}$$

We note that v has been eliminated, i.e., we have transformed the polynomial problem (8) into a pure linear algebra problem (10). This last problem can easily be solved as follows.

If $\mathrm{im}_\Bbbk(M.) = \Bbbk^{n\times 1}$, then we let $L := 0$. Else let $L \in \Bbbk^{p\times n}$ be such that $\ker_\Bbbk(L.) = \mathrm{im}_\Bbbk(M.)$. The first condition of (10) is equivalent to the following linear system on u:

$$\left\{ \begin{array}{l} (L\, D_1)\, u = 0, \\ \qquad \vdots \\ (L\, D_r)\, u = 0. \end{array} \right. \tag{11}$$

If the only solution of (11) is $u = 0$, then (9) has no solution since $A(0) = 0$ and $M \neq 0$. Hence, let $Z \in \Bbbk^{n\times d}$ be a full column matrix whose columns form a basis of the $\Bbbk$-vector space (11). Note that if $\mathrm{im}_\Bbbk(M.) = \Bbbk^{n\times 1}$, and thus, $L = 0$, then we get $Z = I_n$. Hence, the vectors $u = Z\, \psi$ satisfy the first condition of (10) for all $\psi \in \Bbbk^{d\times 1}$. Now, we note that $D_i\, Z\, \psi \in \mathrm{im}_\Bbbk(M.) = \mathrm{im}_\Bbbk(X.)$ for all $\psi \in \Bbbk^{d\times 1}$ and $i = 1, \ldots, r$, which shows the existence of unique matrices $W_i \in \Bbbk^{l\times d}$ satisfying the following identities:

$$D_i\, Z = X\, W_i, \quad i = 1, \ldots, r.$$

Let us note:

$$\forall\, \psi \in \Bbbk^{d\times 1}, \quad B(\psi) := (W_1\, \psi\ \ldots\ W_r\, \psi) \in \Bbbk^{l\times r}.$$

By assumption, $l := \mathrm{rank}_\Bbbk(M) \le r$ (see Remark 1), which shows that B is a wide matrix. The second condition of (10) then becomes $A(Z\, \psi) = (D_1\, Z\, \psi\ \ldots\ D_r\, Z\, \psi) = X\, B(\psi)$ for all $\psi \in \Bbbk^{d\times 1}$. Since X has full column rank, we get:

$$\mathrm{rank}_\Bbbk A(Z\, \psi) = \mathrm{rank}_\Bbbk B(\psi).$$

The second condition of (10) is equivalent to $\psi \in \mathcal{P}$, where:

$$\mathcal{P} := \{\psi \in \Bbbk^{d\times 1} \mid \mathrm{rank}_\Bbbk B(\psi) = l\}. \tag{12}$$

If $\psi \in \mathcal{P}$, then using (7) and the fact that X has full column rank, then we obtain:

$$A(Z\, \psi)\, v = M\ \Leftrightarrow\ X\, B(\psi)\, v = X\, Y\ \Leftrightarrow\ B(\psi)\, v = Y. \tag{13}$$

Moreover, since $\mathrm{rank}_\Bbbk B(\psi) = l$, $B(\psi) \in \Bbbk^{l\times r}$ and $l \le r$, $B(\psi)$ has full row rank, which shows that $B(\psi)$ admits a right inverse $E_\psi \in \Bbbk^{r\times l}$, i.e., $B(\psi)\, E_\psi = I_l$. Hence, $v = E_\psi\, Y$ is a particular solution of (13). If $C_\psi \in \Bbbk^{r\times(r-l)}$ is a full column matrix whose columns form a basis of $\ker_\Bbbk(B(\psi).)$, then the general solution of (13) is given by:

$$\forall\, Y' \in \Bbbk^{(r-l)\times m}, \quad v = (E_\psi \quad C_\psi)\, \left(\begin{array}{c} Y \\ Y' \end{array}\right).$$

Finally, we note that $(E_\psi \quad C_\psi)$ is invertible, i.e., is a non-singular matrix, which shows that v has full row rank iff so has the matrix $(Y^T \quad {Y'}^T)^T \in \Bbbk^{r\times m}$. We get the theorem.

Theorem 5 ([9]): With the above notations, (9) holds iff (12) is not empty. If so, then for every $\psi \in \mathcal{P}$,

$$\forall\, Y' \in \Bbbk^{(r-l)\times m}, \quad \left\{ \begin{array}{l} u = Z\, \psi, \\ v = (E_\psi \quad C_\psi)\, \left(\begin{array}{c} Y \\ Y' \end{array}\right), \end{array} \right.$$

are solutions of (8), and thus, solutions of (3). Finally, v has full row rank iff so has $(Y^T \quad Y'^T)^T \in \Bbbk^{r\times m}$.

Set $C_r^l := r!/(l!\,(r-l)!)$ and let $\{\mathbb{m}_k(\psi)\}_{k=1,\dots,C_r^l}$ denote the set of all $l \times l$-minors of $B(\psi)$. Note that the $\mathbb{m}_k$'s are either 0 or *homogeneous polynomials of degree* l, namely:

$$\forall\, \lambda \in \Bbbk \setminus \{0\}, \quad \mathbb{m}_k(\lambda\, x) = \lambda^l\, \mathbb{m}_k(x), \quad k = 1,\dots,C_r^l.$$

Hence, we clearly have:

$$\mathcal{P} = \Bbbk^{d\times 1} \setminus \left\{\psi \in \Bbbk^{d\times 1} \mid \mathbb{m}_k(\psi) = 0,\, k = 1,\dots,C_r^l\right\}.$$

Since C_r^l grows exponentially with r, a more efficient characterization of $\mathcal{P}$ was obtained in [10] based on module theory and computer algebra (*Gröbner bases* [3], [10]).

According to Theorem 4, a particular class of solutions of the centrohermitian rank factorization problem can be obtained by applying Theorem 5 to (6) with $\Bbbk = \mathbb{R}$. Hence, we obtain the following new result.

Corollary 1: Let Q_n and Q_m be two J-real unitary matrices (e.g., as defined in Example 3), $M \in \mathrm{CH}_{n,m}$ and $D_i \in \mathrm{CH}_{n,n}$ for $i = 1,\dots,r$, and $M_\varphi \in \mathbb{R}^{n\times m}$ and $D_{i\varphi} \in \mathbb{R}^{n\times n}$, $i = 1,\dots,r$, defined by (6). Moreover, if

$$\left\{\begin{array}{l} u_\varphi = Z_\varphi\, \psi, \\ v_\varphi = (E_{\varphi\psi} \quad C_{\varphi\psi}) \left(\begin{array}{c} Y \\ Y' \end{array}\right), \end{array}\right.$$

denotes the solutions of the rank factorization problem (5) for $\Bbbk = \mathbb{R}$ − defined by Theorem 5 − where $Y' \in \mathbb{R}^{(r-l)\times m}$ and $\psi \in \mathcal{P} := \{\psi \in \mathbb{R}^{d\times 1} \mid \mathrm{rank}_{\mathbb{R}} B_\varphi(\psi) = l\}$, then

$$\left\{\begin{array}{l} u = Q_n\, u_\varphi, \\ v = v_\varphi\, Q_m^\star, \end{array}\right. \tag{14}$$

is a centrohermitian solution of (3). Finally, v has full row rank iff so has $(Y^T \quad Y'^T)^T \in \mathbb{R}^{r\times m}$.

The last point of Corollary 1 comes from the fact that $\lambda\, v = \lambda\, v_\varphi\, Q_m^\star = 0$ iff $\lambda\, v_\varphi = 0$, and thus, we get $\lambda = 0$ iff v_φ has full row rank, which is characterized in Theorem 5.

Example 7: We consider again the matrix $M \in \mathrm{CH}_{3,4}$ defined in Example 5. Let us consider the following matrices:

$$D_1 = \frac{1}{2}\left(\begin{array}{ccc} -i & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & i \end{array}\right) \in \mathrm{CH}_{3,3},$$

$$D_2 = \frac{3\,i}{\sqrt{2}}\left(\begin{array}{ccc} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{array}\right) \in \mathrm{CH}_{3,3},$$

$$D_3 = \frac{i}{\sqrt{2}}\left(\begin{array}{ccc} 0 & 0 & 0 \\ -1 & 0 & 1 \\ 0 & 0 & 0 \end{array}\right) \in \mathrm{CH}_{3,3},$$

$$D_4 = \frac{1}{2}\left(\begin{array}{ccc} 1-3\,i & 0 & -1+3\,i \\ 0 & 0 & 0 \\ -1-3\,i & 0 & 1+3\,i \end{array}\right) \in \mathrm{CH}_{3,3}.$$

Using (6), we have:

$$D_{1\varphi} = Q_3^\star\, D_1\, Q_3 = \left(\begin{array}{ccc} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right) \in \mathbb{R}^{3\times 3},$$

$$D_{2\varphi} = Q_3^\star\, D_2\, Q_3 = \left(\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 3 & 0 \end{array}\right) \in \mathbb{R}^{3\times 3},$$

$$D_{3\varphi} = Q_3^\star\, D_3\, Q_3 = \left(\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{array}\right) \in \mathbb{R}^{3\times 3},$$

$$D_{4\varphi} = Q_3^\star\, D_4\, Q_3 = \left(\begin{array}{ccc} 0 & 0 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{array}\right) \in \mathbb{R}^{3\times 3}.$$

Let M_φ denote the matrix $\varphi(M)$ defined in Example 5. Then, we can solve (5) using Theorem 5. We can easily check that $l := \mathrm{rank}_{\mathbb{R}}(M_\varphi) = \mathrm{rank}_{\mathbb{C}}(M) = 3 < r = 4$. We then get:

$$X_\varphi = \left(\begin{array}{ccc} 45 & 10 & 5 \\ 5 & 20 & 15 \\ 15 & 0 & 0 \end{array}\right) \in \mathbb{R}^{3\times 3},$$

$$Y_\varphi = \frac{1}{3}\left(\begin{array}{cccc} 3 & 0 & 0 & 2 \\ 0 & 3 & 0 & 1 \\ 0 & 0 & 3 & -2 \end{array}\right) \in \mathbb{R}^{3\times 4},$$

$$L_\varphi = 0, \quad Z_\varphi = I_3, \quad d_\varphi = 3, \quad \psi = \left(\begin{array}{c} \psi_1 \\ \psi_2 \\ \psi_3 \end{array}\right) \in \mathbb{R}^{3\times 1},$$

$$W_{1\varphi} = \frac{1}{10}\left(\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & -4 \end{array}\right), W_{2\varphi} = \frac{1}{5}\left(\begin{array}{ccc} 0 & 1 & 0 \\ 0 & -13 & 0 \\ 0 & 17 & 0 \end{array}\right),$$

$$W_{3\varphi} = \frac{1}{10}\left(\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 2 \end{array}\right), W_{4\varphi} = \frac{1}{30}\left(\begin{array}{ccc} 0 & 0 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & -2 \end{array}\right),$$

$$B_\varphi(\psi) = \frac{1}{30}\left(\begin{array}{cccc} 0 & 6\ \psi_2 & 0 & 2\,\psi_3 \\ 9\,\psi_3 & -78\,\psi_2 & -3\,\psi_3 & \psi_3 \\ -12\,\psi_3 & 102\,\psi_2 & 6\,\psi_3 & -2\,\psi_3 \end{array}\right),$$

$$\begin{aligned} \mathcal{P} &= \mathbb{R}^{3\times 1} \setminus \{(\psi_1\ \psi_2\ \psi_3)^T \in \mathbb{R}^{3\times 1} \mid \psi_3 = 0\} \\ &= \mathbb{R} \times \mathbb{R} \times (\mathbb{R} \setminus \{0\}), \end{aligned}$$

$$E_{\psi_3} = \frac{1}{\psi_3}\left(\begin{array}{ccc} 0 & 10 & 5 \\ 0 & 0 & 0 \\ 5 & 20 & 15 \\ 15 & 0 & 0 \end{array}\right), \quad C = \left(\begin{array}{c} 9\,\psi_2 \\ \psi_3 \\ 0 \\ -3\,\psi_2 \end{array}\right).$$

Real solutions of $M_\varphi = \sum_{i=1}^4 D_{i\varphi}\, u_\varphi\, v_{i\varphi}$ are then given by:

$$\forall\, \psi \in \mathcal{P}, \quad \forall\, Y' = (y_1' \quad y_2' \quad y_3' \quad y_4') \in \mathbb{R}^{1\times 4},$$

$$u_\varphi = Z_\varphi\, \psi = \psi,$$

$$v_\varphi = E_{\psi_3}\, Y + C\, Y' =$$

$$\begin{pmatrix} 9\,\psi_2\, y_1' & \dfrac{10}{\psi_3} + 9\,\psi_2\, y_2' & \dfrac{5}{\psi_3} + 9\,\psi_2\, y_3' & 9\,\psi_2\, y_4' \\ \psi_3\, y_1' & \psi_3\, y_2' & \psi_3\, y_3' & \psi_3\, y_4' \\ \dfrac{5}{\psi_3} & \dfrac{20}{\psi_3} & \dfrac{15}{\psi_3} & 0 \\ \dfrac{15}{\psi_3} - 3\,\psi_2\, y_1' & -3\,\psi_2\, y_2' & -3\,\psi_2\, y_3' & \dfrac{10}{\psi_3} - 3\,\psi_2\, y_4' \end{pmatrix}.$$

Moreover, v_φ has full row rank iff $\det \begin{pmatrix} Y^T & {Y'}^T \end{pmatrix} \neq 0$, i.e., iff $Y' = (y_1' \quad y_2' \quad y_3' \quad y_4') \in \mathbb{R}^{1\times 4}$ is chosen such that:

$$2\, y_1' + y_2' - 2\, y_3' - 3\, y_4' \neq 0. \tag{15}$$

Now, using (14) (see Corollary 1), centrohermitian solutions of $M = \sum_{i=1}^4 D_i\, u\, v_i$ are then defined by

$$\forall\, \psi \in \mathcal{P}, \quad \forall\, Y' = (y_1' \quad y_2' \quad y_3' \quad y_4') \in \mathbb{R}^{1\times 4},$$

$$\begin{cases} u = Q_3\, \psi = \begin{pmatrix} \dfrac{\sqrt{2}}{2}\, \psi_1 + \dfrac{\sqrt{2}}{2}\, i\, \psi_3 \\ \psi_2 \\ \dfrac{\sqrt{2}}{2}\, \psi_1 - \dfrac{\sqrt{2}}{2}\, i\, \psi_3 \end{pmatrix}, \\ v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} = v_\varphi\, Q_4^\star, \end{cases}$$

$$v_1 = \frac{1}{\sqrt{2}\, \psi_3}$$
$$(9\,\psi_2\,\psi_3\, y_1' - 9\,\psi_2\,\psi_3\, y_3'\, i - 5\, i \quad 10 + 9\,\psi_2\,\psi_3\, y_2' - 9\,\psi_2\,\psi_3\, y_4'\, i$$
$$10 + 9\,\psi_2\,\psi_3\, y_2' + 9\,\psi_2\,\psi_3\, y_4'\, i \quad 9\,\psi_2\,\psi_3\, y_1' + 9\,\psi_2\,\psi_3\, y_3'\, i + 5\, i),$$

$$\begin{aligned} v_2 &= \frac{1}{2}\, \psi_3\, (y_1'\, \sqrt{2} - y_3'\, \sqrt{2}\, i \quad y_2'\, \sqrt{2} - y_4'\, \sqrt{2}\, i \\ &\qquad y_2'\, \sqrt{2} + y_4\, \sqrt{2}\, i \quad y_1'\, \sqrt{2} + y_3'\, \sqrt{2}\, i), \\ v_3 &= \frac{5}{2\, \psi_3}\, \left((1 - 3\, i)\, \sqrt{2} \quad 4\, \sqrt{2} \quad 4\, \sqrt{2} \quad (1 + 3\, i)\, \sqrt{2}\right), \\ v_4 &= \frac{1}{\sqrt{2}\, \psi_3}\, (15 - 3\,\psi_2\,\psi_3\, y_1' + 3\,\psi_2\,\psi_3\, y_3'\, i \\ &\qquad -3\,\psi_2\,\psi_3\, y_2' + 3\,\psi_2\,\psi_3\, y_4'\, i - 10\, i \\ &\qquad -3\,\psi_2\,\psi_3\, y_2' - 3\,\psi_2\,\psi_3\, y_4'\, i + 10\, i \\ &\qquad 15 - 3\,\psi_2\,\psi_3\, y_1' - 3\,\psi_2\,\psi_3\, y_3'\, i). \end{aligned}$$

Finally, the matrix v has full row rank iff the row vector $Y' = (y_1' \quad y_2' \quad y_3' \quad y_4') \in \mathbb{R}^{1\times 4}$ satisfies (15).

All the results developed in this paper can be easily implemented using, e.g., the OREMODULES package [3].

V. CONCLUSION

In this paper, using the structure of centrohermitian matrices and Lee's theorem [12], we proved that the centrohermitian rank factorization problem can be reduced to a real rank factorization one. Using results of [9], [10], we then show how to parametrize a class of solutions which are important in practice. In [11], we develop an alternative approach to Lee's transformation based on the so-called *coinvolutory matrices*. This alternative approach brings new insights.

An important issue that will be investigated in a future publication is the following minimization problem

$$\min_{u \in \mathrm{CH}_{n,1}(\mathbb{C}),\, v_i \in \mathrm{CH}_{1,m}(\mathbb{C})} \left\| \sum_{i=1}^r D_i\, u\, v_i - M \right\|_{\mathrm{Frob}}, \tag{16}$$

where $\|A\|_{\mathrm{Frob}} = \sqrt{\mathrm{trace}(A^\star A)}$ stands for the standard *Frobenius norm*. If U and V are two unitary matrices, using the *cyclic property* of the trace, then we have:

$$\begin{aligned} \|U\, A\, V\|_{\mathrm{Frob}} &= \sqrt{\mathrm{trace}(V^\star\, A^\star\, U^\star\, U\, A\, V)} \\ &= \sqrt{\mathrm{trace}(V^\star\, A^\star\, A\, V)} \\ &= \sqrt{\mathrm{trace}(V\, V^\star\, A^\star\, A)} \\ &= \sqrt{\mathrm{trace}(A^\star\, A)} = \|A\|_{\mathrm{Frob}}\, . \end{aligned}$$

Hence, using the fact that the matrices Q_n and Q_m in Theorem 4 are unitary, we then obtain

$$(16) \Leftrightarrow \min_{u_\varphi \in \mathbb{R}^{n\times 1},\, v_{i\varphi} \in \mathbb{R}^{1\times m}} \left\| \textstyle\sum_{i=1}^r D_{i\varphi}\, u_\varphi\, v_{i\varphi} - M_\varphi \right\|_{\mathrm{Frob}},$$

which shows that (16) can be reduced to the search for real solutions to a polynomial optimization problem.

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