Algebraic estimation of a biased and noisy continuous signal via orthogonal polynomials

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***Abstract*— Many important problems in signal processing and control engineering concern the reconstitution of a noisy biased signal. For this issue, in this paper, we consider the signal written as an orthogonal polynomial series expansion and we provide an algebraic estimation of its coefficients. We specialize in Hermite polynomials. On the other hand, the dynamical system described by the noisy biased signal may be given by an ordinary differential equation associated with classical orthogonal polynomials. The signal may be recovered through the coefficients identification. As an example, we illustrate our algebraic method on the parameter estimation in the case of Hermite polynomials.**

I. Introduction

Is is a widely known fact that parameter estimation is an important topic in various practical domains, such as in control engineering and signal processing. The list of applications concerning this question is extensively long.

Most traditional methods for solving this problem concern statistical approaches. From the last 10 years, an algebraic framework started to become more popular in the study of parameter identification. Algebraic approaches are mainly based on differential algebra concepts, operational calculus and module theory. An important paper in this subject is [3] where a closed-loop parametric identification procedure for continuous-time constant linear systems is introduced. A very complete survey on algebraic identification can be found in [12]. Interesting applications within the algebraic context were provided in [7], [6], [11], [2], [10].

A longstanding essential problem in signal processing and control engineering consists in recovering a signal from a noisy biased measurement. One way to approach this question is to use a Taylor series expansion of the signal or, in a practical manner, to approximate the signal by a truncation of its Taylor series. It is in this way that numerical differentiation, *i.e.* the derivative estimation of the signal, has been the center of attention in countless papers. A worth-to-mention work is the algebraic framework started with [8]. More details on this algebraic numerical differentiation technique can be found, for instance, in [4], [5], [6].

It seems of interest to study an alternative to numerical differentiation by considering the signal in another functional basis. A common signal decomposition, notably used in signal processing, originates from an orthogonal polynomial basis. In other words, the signal is written as an infinite sum of orthogonal polynomials with coefficients given by its respective basis projections which must then be identified. Hence, the question of the reconstitution of the original signal fits in the category of parameter estimation issues.

In some problems, the dynamical system described by the signal x can be defined by a second-order ordinary differential equation (ODE). In particular, the coefficients can be given by low-order polynomials in the time t. In this case, the identification of the ODE coefficients will allow the reconstitution of the noisy signal. Classical orthogonal polynomials, such as Jacobi, Legendre, Laguerre and Hermite polynomials, do satisfy such ODEs. For more details on orthogonal polynomials, see [1] and the references therein.

The algebraic method considered here is strongly based on *Weyl algebra* structural properties and one of its main advantages is to provide closed formulas for the parameter estimates. The reader may refer to works in [13], [14].

II. General approach

In this section, we present the generic approaches aforementioned in the introduction. More precisely, the reconstitution of the signal via an orthogonal polynomial basis extension and the particular case of second-order time-varying systems are studied in this paper.

Throughout the text, $\mathbb{K}$ denotes a field of characteristic zero (*e.g.* $\mathbb{K} = \mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$). To formalize our problem, we consider a signal x that has to be recovered form a noisy biased signal y defined by

$$y(t) = x(t) + \gamma + \varpi(t),$$

where γ is an unknown constant bias and $\varpi(t)$ is a zero-mean noise.

A. Orthogonal bases

The classical decomposition of a continuous signal y in a basis of orthogonal polynomials $\mathcal{P} = \{P_n(t)\}_{n\geq 0}$ can be described by

$$y(t) = \sum_{n\geq 0} \lambda_n P_n(t),$$

where $\lambda_n \in \mathbb{R}$ corresponds to the projection of y onto the orthogonal basis $\mathcal{P}$. Remark that P_n might depend of unknown parameters, such as, *e.g.* the Jacobi polynomials $P_n^{(\alpha,\beta)}$ (see Appendix).

Similar to the case of the Taylor expansion [8], an approximation of y will be given by truncating the above series

$$y(t) \approx y_N(t) = \sum_{n=0}^{N} \lambda_n P_n(t), \tag{1}$$

for some $N > 0$. We wish to identify the coefficients λ_0, ..., λ_N. The algebraic method proposed in this paper involves computations in the operational domain, so we apply the

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Laplace transform $\mathcal{L}$. First, recall that the action of $\mathcal{L}$ on a continuous function f with a positive support is given by

$$\mathcal{L}(f)(s) = \int_0^{+\infty} e^{-st} f(t)\, dt,$$

where s denotes the Laplace variable. Applying $\mathcal{L}$ on (1) yields

$$Y_N(s) = \sum_{n=0}^{N} \lambda_n \mathcal{L}(P_n)(s),$$

where Y_N denotes the Laplace transform of y_N. Then the idea is to individually estimate the constants λ_i, $i = 0, \ldots, N$. Our algebraic method proposes the elimination of all other coefficients except the one to be estimated. This elimination is realized through the action of differential operators called *annihilators*. After the elimination, the resulting equation in the time domain obtained with the action of the inverse Laplace transform provides closed formulas for estimating the λ_i's.

B. Second-order time-varying systems

Now, we consider that the signal x satisfies the following differential equation

$$A(t)\ddot{x}(t) + B(t)\dot{x}(t) + C(t)x(t) = 0, \tag{2}$$

where A, B and C are elements of the polynomial ring $\mathbb{K}[t]$ in t with coefficients in $\mathbb{K}$. Consider a signal z by setting

$$z(t) = x(t) + \gamma,$$

then using (2), z satisfies:

$$A(t)\ddot{z}(t) + B(t)\dot{z}(t) + C(t)z(t) - C(t)\gamma = 0. \tag{3}$$

Our goal is to identify parameters appearing as the unknown coefficients of A, B and C in (3), since the bias will not be estimated. As before, we will then compute the Laplace transform of (3).

Prior to applying $\mathcal{L}$ to the above equation, few properties of the Laplace transform $\mathcal{L}$ of a signal f are reviewed in the proposition below. We use the notation $\partial_s := \frac{d}{ds}$.

Proposition 1:

1) $\mathcal{L}\left(f^{(n)}\right)(s) = s^n \mathcal{L}(f)(s) - \sum_{i=0}^{n-1} s^{n-i-1} f^{(i)}(0)$.
2) $\mathcal{L}(t^n \,.\, f)(s) = (-1)^n \partial_s^n (\mathcal{L}(f)(s))$.
3) $\mathcal{L}(\gamma) = \dfrac{\gamma}{s}$, for all $\gamma \in \mathbb{K}$.

An immediate corollary can be derived from the above proposition.

Corollary 2: Let $R = \sum_{k=0}^{n} a_k t^k \in \mathbb{K}[t]$ and $f \in \mathcal{C}^\infty(\mathbb{R}_+)$. Then, we have:

$$\mathcal{L}(R \,.\, f)(s) = R(-\partial_s)(\mathcal{L}(f)(s)) = \sum_{k=0}^{n} (-1)^k a_k \partial_s^k (\mathcal{L}(f)(s)).$$

Hence, taking the Laplace transform of (3) and using the above properties, we obtain

$$A(-\partial_s)\left(s^2 Z(s) - s z(0) - \dot{z}(0)\right) + B(-\partial_s)(sZ(s) - z(0)) + C(-\partial_s) Z(s) - C(-\partial_s)\frac{\gamma}{s} = 0,$$

where Z denotes the Laplace transform $\mathcal{L}(z)$ of z. Using the following notation

$$\begin{cases} \theta_1 := -x(0) = -z(0) + \gamma, \\ \theta_2 := -\dot{x}(0) = -\dot{z}(0), \\ \theta_3 := -\gamma, \end{cases}$$

and thus $-z(0) = \theta_1 + \theta_3$, we obtain:

$$\begin{aligned} &\left(A(-\partial_s)\, s^2 + B(-\partial_s)\, s + C(-\partial_s)\right) Z(s) \\ &+ (A(-\partial_s)\, s + B(-\partial_s))\, (\theta_1 + \theta_3) \\ &+ A(-\partial_s)\, \theta_2 + C(-\partial_s)\, \frac{\theta_3}{s} = 0. \end{aligned} \tag{4}$$

In this paper, we shall only consider the following case

$$\begin{cases} A(t) = a_2 t^2 + a_1 t + a_0, \\ B(t) = b_1 t + b_0, \\ C(t) = c, \end{cases}$$

where $a_2, a_1, a_0, b_1, b_0, c \in \mathbb{K}$. Replacing these expressions in (4) yields

$$\begin{aligned} &\left(\widetilde{A}(s)\, \partial_s^2 + \widetilde{B}(s)\, \partial_s + \widetilde{C}(s)\right) Z(s) \\ &+ T_1(s)\, \theta_1 + T_2(s)\, \theta_2 + T_3(s)\, \theta_3 = 0, \end{aligned} \tag{5}$$

$$\text{where } \begin{cases} \widetilde{A}(s) = a_2 s^3, \\ \widetilde{B}(s) = s^2\left(-s a_1 + (4 a_2 - b_1)\right), \\ \widetilde{C}(s) = s\left(a_0 s^2 + (-2 a_1 + b_0)\, s + (2 a_2 - b_1 + c)\right), \\ T_1(s) = a_0 s^2 + (-a_1 + b_0)\, s, \\ T_2(s) = a_0 s, \\ T_3(s) = T_1(s) + c. \end{cases} \tag{6}$$

To estimate the coefficients of polynomials A, B and C, algebraic manipulations are necessary to eliminate the undesired terms in (5). For that, algebraic operators called annihilators are then applied to (5). The resulting expressions will contain only terms that are sought, allowing their identification in the time domain. To return to the time domain, the inverse Laplace transform $\mathcal{L}^{-1}$ is then applied. Recall that the inverse Laplace transform is given by

$$\mathcal{L}^{-1}\left(\frac{1}{s^m}\partial_s^p Z(s)\right) = \frac{(-1)^p}{(m-1)!}\int_0^t v_{m-1,p}(\tau)\, z(\tau)\, d\tau, \tag{7}$$

with the following notation:

$$\forall\, p, \in \mathbb{N}, \quad m \geq 1, \quad v_{m,p} = v_{m,p}(\tau) = (t - \tau)^m \tau^p. \tag{8}$$

Let us consider classical examples.

Example 1: 1) Assume that $x(t) = A\sin(\omega t + \phi)$, the sinusoidal signal [13]. In this case, the polynomials A, B and C are given by:

$$A(t) = 1, \quad B(t) = 0, \quad C(t) = \omega^2.$$

Then using (6), we have (5), where:

$$\begin{aligned} &\widetilde{A}(s) = \widetilde{B}(s) = 0, \quad \widetilde{C}(s) = s(s^2 + \omega^2), \\ &T_1(s) = s^2, \quad T_2(s) = s, \quad T_3(s) = s^2 + \omega^2. \end{aligned} \tag{9}$$

2) Assume that $x(t) = P_n^{\alpha,\beta}(t)$, the *n-th Jacobi polynomial* depending on parameters α and β. In this case, the polynomials A, B and C are given by:

$$\begin{cases} A(t) = -t^2 + 1, \\ B(t) = -(\alpha+\beta+2)\,t + \beta - \alpha, \\ C(t) = n\,(n+\alpha+\beta+1). \end{cases}$$

Then using (6), we have (5), where:

$$\begin{cases} \widetilde{A}(s) = -s^3, \\ \widetilde{B}(s) = s^2\,(\alpha+\beta-2), \\ \widetilde{C}(s) = s\left(s^2 + (\beta-\alpha))\,s + (n+1)\,(n+\alpha+\beta)\right), \\ T_1(s) = s^2 + (\beta-\alpha)\,s, \\ T_2(s) = s, \\ T_3(s) = s^2 + (\beta-\alpha)\,s + n\,(n+\alpha+\beta+1). \end{cases} \tag{10}$$

3) Assume that $x(t) = P_n(t)$, the *n-th Legendre* which is a particular case of the n-th Jacobi polynomial for $\alpha = \beta = 0$. In this case, the A, B and C are given by:

$$A(t) = -t^2 + 1, \quad B(t) = -2t, \quad C(t) = n\,(n+1).$$

Then using (6), we have (5), where:

$$\begin{cases} \widetilde{A}(s) = -s^3, \\ \widetilde{B}(s) = -2s^2, \\ \widetilde{C}(s) = s\,(s^2 + n\,(n+1)), \\ T_1(s) = s^2, \\ T_2(s) = s, \\ T_3(s) = s^2 + n\,(n+1). \end{cases} \tag{11}$$

4) Assume that $x(t) = L_n^{(\alpha)}(t)$, the *n-th Laguerre polynomial* depending on the parameter α. Polynomials A, B and C in (3) are given by:

$$A(t) = t, \quad B(t) = -t + \alpha + 1, \quad C(t) = n.$$

Then using (6), we have (5), where:

$$\begin{cases} \widetilde{A}(s) = 0, \\ \widetilde{B}(s) = s^2\,(-s+1), \\ \widetilde{C}(s) = s\,((\alpha-1)\,s + (n+1)), \\ T_1(s) = \alpha\,s, \\ T_2(s) = 0, \\ T_3(s) = \alpha\,s + n. \end{cases} \tag{12}$$

5) Assume that $x(t) = H_n(t)$, the *n-th Hermite polynomial*. Polynomials A, B and C in (3) are given by:

$$A(t) = 1, \quad B(t) = -2t, \quad C(t) = 2n.$$

Then, using (6), we have (5), where:

$$\begin{cases} \widetilde{A}(s) = 0, \\ \widetilde{B}(s) = 2s^2, \\ \widetilde{C}(s) = s\,(s^2 + 2\,(n+1)), \\ T_1(s) = s^2, \\ T_2(s) = s, \\ T_3(s) = s^2 + 2n. \end{cases} \tag{13}$$

III. Motivational examples

A. Hermite polynomials

In this work, to illustrate one application of our algebraic estimation method, we will be particularly concerned with the Hermite polynomial series expansion of a continuous signal x. Hermite polynomials H_n form an orthogonal set for $t \in \mathbb{R}$ with respect to the weight function e^{-t^2} (see Appendix). So any continuous function y can be written as

$$y(t) = \sum_{n \geq 0} \lambda_n H_n(t),$$

where:

$$\lambda_n = \frac{1}{2^n\,n!\,\sqrt{\pi}} \int_{-\infty}^{\infty} y(\tau)\,H_n(\tau)\,\mathrm{e}^{-\tau^2}\,d\tau.$$

An approximation of the function y is provided by selecting a constant $N > 0$ such that:

$$y(t) \approx \sum_{n=0}^{N} \lambda_n H_n(t). \tag{14}$$

We denote this polynomial approximation by $y_N(t)$.

The aim is to estimate the terms λ_n, for $n = 0, \ldots, N$. Notice that y represents the measured signal from a signal x with some negligible noise, hence we may consider only y. As we mentioned in the above subsection, the first step is to apply the Laplace transform on (14) and we obtain

$$Y_N(s) = \sum_{n=0}^{N} \lambda_n \mathcal{L}\,(H_n)\,(s), \tag{15}$$

where Y_N denotes the Laplace transform of y_N. From the definition of Hermite polynomials (see Appendix), it follows

$$H_n(t) = 2^n t^n + h_n(t) = 2^n t^n + \eta_{n,n-2} t^{n-2} + \cdots + \eta_{n,m} t^m$$

with $m = n \bmod 2$ (*i.e.*, $m = 0$ if n is even and $m = 1$ if n is odd). So, denoting $n = 2j$ or $n = 2j+1$, it results that

$$P_n(s) := \mathcal{L}\,(H_n)\,(s) = 2^n \frac{n!}{s^{n+1}} + \sum_{k=1}^{j} \eta_{n,n-2k} \frac{(n-2k)!}{s^{n-2k+1}}.$$

Multiplying (15) by the highest power of s to eliminate denominators gives:

$$\begin{aligned} s^{N+1} Y_N(s) = \lambda_N \left(2^N N! + \textstyle\sum_{k=1}^{j} \eta_{N,N-2k} (N-2k)!\,s^{2k}\right) \\ + \textstyle\sum_{n=0}^{N-1} \lambda_n s^{N+1} P_n(s). \end{aligned} \tag{16}$$

The parameters λ_i up to order N will be estimated individually. Denote the set of parameters to be estimated by:

$$\Theta = \{\lambda_1, \ldots, \lambda_N\}.$$

We start with the dominant coefficient λ_N and use the notation $\Theta_{\text{est}} = \{\lambda_N\}$. Then, accordingly separate the terms in (16) to rewrite the equation into the following relation

$$(\mathcal{R}) \qquad P(s)\, Y_N(s) + Q(s) + \overline{Q}(s) = 0, \tag{17}$$

where P is a differential operator on the Laplace variable s with coefficients in the field $\mathbb{R}_\Theta := \mathbb{R}(\Theta)$ (*i.e.* an algebraic extension of $\mathbb{R}$ containing the set Θ), so $P \in \mathbb{R}_\Theta[s]$, Q is a polynomial in s with coefficients in $\mathbb{R}_{\Theta_{\text{est}}} := \mathbb{R}(\Theta_{\text{est}})$, and $\overline{Q}$ is a polynomial in s with coefficients in $\mathbb{R}_\Theta$:

$$\begin{cases} P(s) = s^{N+1}, \\ Q(s) = -\lambda_N\ s^{N+1} P_N(s), \\ \overline{Q}(s) = -\sum_{n=0}^{N-1} \lambda_n\ s^{N+1} P_n(s). \end{cases}$$

To determine a closed formula for the estimation of λ_N, the polynomial $\overline{Q}$ must be eliminated from (17). This elimination is realized by the action of differential operators, providing an expression containing only λ_N, Y_N and its derivatives. A time-domain expression for λ_N can then be obtained by applying the inverse Laplace transform. The same procedure is applied to estimate all remaining λ_i.

For instance, if $N = 3$, then we obtain from (16) that:

$$s^4\, Y_N(s) - s^3 \lambda_0 - 2 s^2 \lambda_1 + 2 s \left(s^2 - 4\right) \lambda_2 + 12 \left(s^2 - 4\right) \lambda_3 = 0. \tag{18}$$

Starting with $\Theta_{\text{est}} = \{\lambda_3\}$, the relation obtained is:

$$P(s) = s^4, \quad Q(s) = 12 \left(s^2 - 4\right) \lambda_3,$$
$$\overline{Q}(s) = -s^3 \lambda_0 - 2 s^2 \lambda_1 + 2 s \left(s^2 - 4\right) \lambda_2.$$

B. Second-order time-varying systems

Particular cases of time-varying second-order differential equations are orthogonal polynomials as seen in Section II-B.

Let Θ be the set of all parameters in the operational equations (9), (10), (11), (12) and (13). Since we do not wish to identify the bias $\gamma = -\theta_3$, the parameters to be estimated are then θ_1 and θ_2 and in some cases α and/or β and n.

In the next section, to illustrate our algebraic method for parameter estimation, we will focus on Example 4 and identify parameters in the case of *Hermite polynomials*.

After the passage from the time domain to the operational domain via the Laplace transform, the next step in the estimation procedure consists in rewriting (13) according to the parameters to be identified. For instance, if we wish to estimate θ_1, and θ_2, then we define a set Θ_{est} by:

$$\Theta_{\text{est}} = \{\theta_1, \theta_2\}.$$

So $\Theta_{\text{est}} \subset \Theta = \{\theta_1, \theta_2, \theta_3\}$. From (13), we can rewrite the relation (5) as follows

$$(\mathcal{R}) \qquad P(s, \partial_s)\ Z(s) + Q(s) + \overline{Q}(s) = 0, \tag{19}$$

where:

$$\begin{cases} P(s,\partial_s) &=& 2 s^2 \partial_s + s\left(s^2 + 2(n+1)\right), \\ Q(s) &=& s^2 \theta_1 + s \theta_2, \\ \overline{Q}(s) &=& \left(s^2 + 2n\right)\ \theta_3. \end{cases}$$

Annihilators are then applied on the relation (19) to eliminate $\overline{Q}$. The remaining terms provide a system of equations in Θ_{est}. A short description on the algebraic framework used to design annihilators can be found in the next section.

IV. Annihilators

Algebraic concepts and some structural properties can be found in the Appendix VII-B and [9].

For the sake of simplicity, from now on we consider $\mathbb{K} = \mathbb{Q}$ or $\mathbb{R}$. We also set $\mathsf{B} := \mathsf{B}_1(\mathbb{K}) = \mathbb{K}(s)[\partial_s]$.

Definition 1: Let $R \in \mathbb{K}_\Theta[s]$. A *R-annihilator w.r.t.* B is an element of $\text{Ann}_{\mathsf{B}}(R) = \{F \in \mathsf{B} \mid F(R) = 0\}$.

By Proposition 9, Appendix VII-B, B is a left principal domain. Therefore $\text{Ann}_{\mathsf{B}}(R)$ is a left principal ideal (*i.e.* it is generated by a unique $\Pi_{\min} \in \mathsf{B}$, up to multiplication by a nonzero polynomial in B). That means $\text{Ann}_{\mathsf{B}}(R) = \mathsf{B}\ \Pi_{\min}$. We call $\Pi_{\min}$ a *minimal Q-annihilator w.r.t.* B. Remark that $\text{Ann}_{\mathsf{B}}(R)$ contains annihilators in finite integral form, *i.e.* differential operators with coefficients in $\mathbb{K}\left[\frac{1}{s}\right]$. Indeed, we can always multiply $\Pi_{\min}$ by $p \in \mathbb{K}[s]$ so that $p\Pi_{\min}$ is a differential operator in ∂_s with polynomial coefficients in s, and then multiply $p\Pi_{\min}$ by s^{-N}, where N is the maximal degree of the polynomial in s of $p\Pi_{\min}$. The following lemmas are useful:

Lemma 3: Consider $R = s^n$ for $n \in \mathbb{N}$. A minimal R-annihilator is given by $\Pi_n = s\,\partial_s - n$.

For m, $n \in \mathbb{N}$, the operators Π_m and Π_n commute. Thus, one has the following Lemma

Lemma 4: Let R_1, $R_2 \in \mathbb{K}_\Theta[s]$. Let F_i be a R_i-annihilator for $i = 1$, 2, such that $F_1 F_2 = F_2 F_1$. Then $F_1 F_2$ is a $(\mu R_1 + \eta R_2)$-annihilator for all $\mu, \eta \in \mathbb{K}_\Theta$.

Lemma 5: Let $R \in \mathbb{K}_\Theta[s]$. Then, a minimal R-annihilator w.r.t. $\mathsf{B}_{\Theta_{\text{est}}}$ is given by $\Pi_{\min} = R\,\partial_s - \partial_s(R)$.

It may happen that a $\overline{Q}$-annihilator eliminates all terms in the relation $\mathcal{R}$ (see (17) and (19)) that contain the parameters to be estimated. Hence, another important concept lies in the definition of an estimator.

Definition 2: An *estimator* $\Pi \in \mathsf{B}$ is a $\overline{Q}$-annihilator satisfying $\text{coeffs}(\Pi((\mathcal{R})) \cap \mathbb{K}_\Theta = \emptyset$.

Now, from (19) we have:

$$\overline{Q}(s) = \left(s^2 + 2n\right) \theta_3.$$

Since the degree of $\overline{Q}$ in s is equal to 2, then $\Pi = \partial_s^3$ is a $\overline{Q}$-annihilator. But the action of Π on the relation $(\mathcal{R})$ defined by (19) also eliminates the polynomial Q that contains the parameters to be identified. So Π is not an estimator.

Lemma 3 gives a minimal $\overline{Q}$-annihilator for s^2:

$$\pi_1 = s\,\partial_s - 2.$$

To complete annihilate $\overline{Q}$, it is enough to apply $\pi_2 = \partial_s$ on $\pi_1\left(\overline{Q}\right)$. That gives a $\overline{Q}$-annihilator:

$$\Pi = \pi_2\,\pi_1 = s\,\partial_s^2 - \partial_s.$$

Moreover, Lemma 5 provides a minimal $\overline{Q}$-annihilator w.r.t. B:

$$\Phi = \left(s^2 + 2n\right)\,\partial_s - 2s.$$

V. EXAMPLES

Example 1: Hermite expansion series of x

In this subsection, we work with a truncate series expansion of order 3 and illustrate our method with (18) by giving the estimation of λ_3.

From (18), we have:

$$P(s) = s^4,\ Q(s) = \left(12\,s^2 - 48\right)\lambda_3,$$
$$\overline{Q}(s) = -\lambda_0\,s^3 - 2\,\lambda_1\,s^2 + \left(2\,s^3 - 8\,s\right)\lambda_2.$$

To annihilate $\overline{Q}$, we begin by eliminating λ_0. From Lemma 3, we apply $\pi_1 = s\,\partial_s - 3$ on $\overline{Q}$ and obtain $\pi_1\left(\overline{Q}\right) = 2\,s^2\,\lambda_1 + 16\,\lambda_2\,s$. Using the same Lemma twice, we apply subsequently $\pi_2 = s\,\partial_s - 2$ and $\pi_3 = s\,\partial_s - 1$ to completely annihilate $\overline{Q}$.

The annihilator $\pi = \pi_3\,\pi_2\,\pi_1$ can be rewritten in the canonical as follows:

$$\pi = s^3\,\partial_s^3 - 3\,s^2\,\partial_s^2 + 6\,s\,\partial_s - 6.$$

We apply π on (18) and it follows:

$$s^4\,(s^3\,\partial_s^3 + 9\,s^2\,\partial_s^2 + 18\,s\,\partial_s + 6)\,Y(s) + 288\,\lambda_3 = 0.$$

Using the notation (8), the inverse Laplace transform (7) helps to return to the time domain:

$$\frac{2\,\lambda_3\,t^7}{35} + \int_0^t \left(-9\,\nu_{2,1} + 9\,\nu_{1,2} + \nu_{3,0} - \nu_{0,3}\right)\,y(\tau)\,d\tau = 0.$$

Finally, solving (20) and changing variables, we then obtain:

$$\lambda_3 = \frac{35\,\int_0^1 y(t\,\nu)\,\left(20\,\nu^3 - 30\,\nu^2 + 12\,\nu - 1\right)\,d\nu}{2\,t^3}.$$

Example 2: Hermite polynomial

We have seen in Section III-B that the parameters to be estimated can be θ_1 and θ_2 and in some cases α and/or β and n. In this paper, to illustrate our method we will focus on the estimation of n since the closed formulas are simpler than in the other cases. We consider the following time-varying second-order differential equation:

$$\ddot{z}(t) - 2\,t\,\dot{z}(t) + 2\,n\,z(t) - 2\,n\,\gamma = 0. \qquad (20)$$

As seen in (13), Section II, the above equation yields in the operational domain:

$$2\,s^2\,\partial_s Z(s) + s\,\left(s^2 + 2\,(n+1)\right)\,Z(s) + \theta_1\,s^2 + \theta_2\,s + \left(s^2 + 2\,n\right)\,\theta_3 = 0.$$

The aim is to identify the parameter n in (20). So the relation $\mathcal{R}$ that will be considered is

$$P(s,\partial_s)\,Z(s) + Q(s) + \overline{Q}(s) = 0,$$

with $P(s,\partial_s) = 2\,s^2\,\partial_s + \left(s^3 + 2\,(n+1)\,s\right)$, $Q(s) = 0$ and $\overline{Q}(s) = s^2\,\theta_1 + s\,\theta_2 + \left(s^2 + 2\,n\right)\,\theta_3$. Using Lemma 3 three times, a $\overline{Q}$-annihilator in the canonical form is given by

$$\pi = s\,\partial_s^3,$$

and provides in the operational domain:

$$(2\,s^3\,\partial_s^4 + s^2\,(s^2+14)\,\partial_s^3 + 9\,s\,(s^2+2)\,\partial_s^2 + 18\,s^2\,\partial_s + 6\,s)\,Z(s) + n\,2\,s\,(s\,\partial_s^2 + 3)\,\partial_s Z(s) = 0.$$

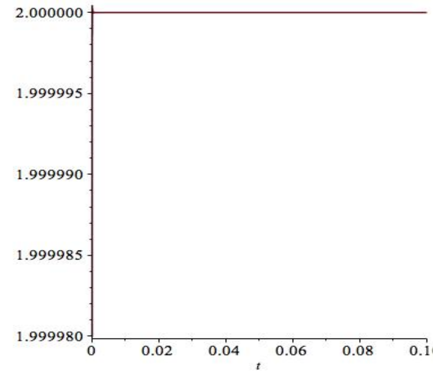

Fig. 1. The result of the identification of n

The equation allowing the identification of n is provided by:

$$n = \frac{\int_0^t\ \mathcal{A}\ z(\tau)\,d\tau}{\int_0^t\ \mathcal{B}\ z(\tau)\,d\tau},$$

$$\mathcal{A} = 9\,(\nu_{2,1} - \nu_{1,2}) + \nu_{0,3} - \nu_{3,0} - 2\nu_{1,4} - 3\nu_{3,2} + 7\nu_{2,3},$$
$$\mathcal{B} = \nu_{3,2} - \nu_{2,3}.$$

In the case of the second-order Hermite Polynomial $H_2(t)$, we consider $z(t) = H_2(t) + \varpi(t)$, with $\varpi(t)$ some noise. Fig. 1 shows the simulation using the above identification of n for $n = 2$.

VI. CONCLUSION

In this paper, we addressed the problem of the reconstitution of a noisy biased signal that is involved in many important problems in signal processing and control engineering. Two approaches were studied for this issue. One of them was the use of an orthogonal polynomial series expansion for the signal. In this approach, thanks to an algebraic framework we provided an estimation of its coefficients in the particular case of Hermite polynomials. This choice was motivated by the common use of these polynomials in the domain of signal processing. Future work will include a broader study of other classical polynomials, such as Jacobi, Legendre and Laguerre. Errors arising from truncating the series expansion should be also analyzed.

The second approach presented in this paper concerns the case where the dynamical system described by the noisy biased signal is given by a differential equation satisfied by classical orthogonal polynomials. The ODE coefficients identification allowed the signal to be recovered. An example for second-degree Hermite polynomial was chosen to illustrate our algebraic methods. Further research will focus on different choices of parameters to be estimated, as well as on different orders for the ODE polynomial coefficients.

It should be stressed that the choice of differential operators called annihilators is a crucial step in the algebraic procedure allowing a better posed problem and consequently, better estimates.

VII. Appendix

A. Classical orthogonal polynomials

Here, we recall the definition of some classical orthogonal polynomials. In particular, properties of Jacobi and Hermite polynomials are provided. For more details, we refer to [1].

1) Jacobi polynomials:

Jacobi polynomials can be defined by Rodrigues' formula:

$$P_n^{(\alpha,\beta)}(z) = \frac{(-1)^n}{2^n n!}(1-z)^{-\alpha}(1+z)^{-\beta}. \frac{d^n}{dz^n}\left((1-z)^{\alpha}(1+z)^{\beta}\left(1-z^2\right)^n\right).$$

2) Hermite polynomials: The definition of Hermite polynomials is given by:

$$H_n(t) = (-1)^n \, \mathrm{e}^{t^2} \, \frac{d^n}{dt^n} \, \mathrm{e}^{-t^2}.$$

Hermite polynomials of even degree are even functions and those of odd degree are odd functions. Thus we can write

$$H_n(t) = 2^n t^n + h_n(t),$$

where $h_n(t)$ is a polynomial with non-zero coefficients for all even powers of t smaller than n if n is even and for all odd powers if n is odd. Hermite polynomials are orthogonal with respect to the scalar product defined by the weight function $w(t) = \mathrm{e}^{-t^2}$:

$$\langle H_n(t), H_m(t)\rangle = \int_{-\infty}^{\infty} H_m(\tau) H_n(\tau) w(\tau)\, d\tau = 0, \quad m \neq n.$$

B. The Weyl Algebra: basic notions

Definition 3: Let $\mathbb{K}$ be a field of characteristic zero. Let $k \in \mathbb{N} \setminus \{0\}$. The *Weyl algebra* $\mathsf{A}_k = \mathsf{A}_k(\mathbb{K})$ is the free $\mathbb{K}$-algebra generated by $p_1, q_1, \ldots, p_k, q_k$ satisfying the relations

$$1 \leq i, j \leq k, \quad [p_i, q_j] = \delta_{ij}, \quad [p_i, p_j] = [q_i, q_j] = 0,$$

where $[\cdot,\cdot]$ is the *commutator* defined by $[u,v] := uv - vu$ for all $u, v \in \mathsf{A}_k(\mathbb{K})$ and δ_{ij} is the *Kronecker function*, *i.e.* $\delta_{ij} = 1$, if $i = j$ and 0, if $i \neq j$.

A useful realization of the Weyl algebra A_k is to consider it as the $\mathbb{K}$-algebra of polynomial differential operators on $\mathbb{K}[s_1, \ldots, s_k]$ such that $p_i := \partial_{s_i} = \frac{\partial}{\partial s_i}$ is the derivative with respect to s_i and $q_i := s_i$ is interpreted as the multiplication operator $p(s_1, \ldots, s_k) \longmapsto s_i\, p(s_1, \ldots, s_k)$, for $1 \leq i \leq k$.

As a consequence, we can write:

$$\mathsf{A}_k = \mathbb{K}[q_1, \ldots, q_k][p_1, \ldots, p_k] = \mathbb{K}[s_1, \ldots, s_k]\left[\partial_{s_1}, \ldots, \partial_{s_k}\right].$$

Remark 6: The same notation is used for the variable s_i and for the operator "multiplication by s_i".

A closely related algebra to $\mathsf{A}_k(\mathbb{K})$ is defined as the differential operators on $\mathbb{K}[s_1, \ldots, s_k]$ with coefficients in the rational functions field $\mathbb{K}(s_1, \ldots, s_k)$. We denote it by $\mathsf{B}_k(\mathbb{K})$, or B_k for short. We can write:

$$\mathsf{B}_k := \mathbb{K}(q_1, \ldots, q_k)[p_1, \ldots, p_k] = \mathbb{K}(s_1, \ldots, s_k)\left[\partial_{s_1}, \ldots, \partial_{s_k}\right].$$

Proposition 7: A basis for A_k is given by $\left\{q^I p^J \mid I, J \in \mathbb{N}^k\right\}$ where $q^I := q_1^{i_1} \ldots q_k^{i_k}$ and $p^J := p_1^{j_1} \ldots p_k^{j_k}$ if $I = (i_1, \ldots, i_k)$ and $J = (j_1, \ldots, j_k)$. So an operator $F \in \mathsf{A}_k$ can be written in a *canonical form*,

$$F = \sum_{I,J} \lambda_{IJ} q^I p^J \quad \text{with} \quad \lambda_{IJ} \in \mathbb{K}.$$

Example 2: We need later the fallowing useful identity:

$$p^n q^m = q^m p^n + \sum_{k=1}^{n} \binom{n}{i}\binom{m}{i} i!\, q^{m-i} p^{n-i}$$

An element $F \in \mathsf{B}_k$ can be similarly written as

$$F = \sum_I \lambda_I g_I(s) p^I, \quad \text{where} \quad g_I(s) \in \mathbb{K}(s_1, \ldots, s_k).$$

The *order* of an element $F \in \mathsf{B}_k$, $F = \sum_I \lambda_I g_I(s) p^I$ is defined as $\mathrm{ord}(F) := \max\{[I] \mid g_I(s) \neq 0\}$. The same definition holds for the Weyl algebra A_k since $\mathsf{A}_k \subset \mathsf{B}_k$. Some properties of A_k and B_k are given by the following propositions:

Proposition 8: A_k and B_k are simple and Noetherian.

Furthermore, A_k is neither a principal right domain, nor a principal left domain, while this is true for B_k:

Proposition 9: B_1 admits a left division algorithm, that is, if $F, G \in \mathsf{B}_1$, then there exists $Q, R \in \mathsf{B}_1$ such that $F = QG + R$ and $\mathrm{ord}(R) < \mathrm{ord}(G)$. So B_1 is a principal left domain.

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