INSTITUT NATIONAL DE RECHERCHE EN INFORMATIQUE ET EN AUTOMATIQUE

On the Baer extension problem for multidimensional linear systems

Alban Quadrat — Daniel Robertz

N° 6307

September 2007

Thème SYM

Rapport de recherche

ISSN 0249-6399 ISRN INRIA/RR--6307--FR+ENG

On the Baer extension problem for multidimensional linear systems

Alban Quadrat* , Daniel Robertz†

Thème SYM — Systèmes symboliques
Projet APICS

Rapport de recherche n° 6307 — September 2007 — 43 pages

Abstract: Within an algebraic analysis approach, the purpose of the paper is to constructively solve the following problem: given two fixed multidimensional linear systems S_1 and S_2, parametrize the multidimensional linear systems S which contain S_1 as a subsystem and satisfy that S/S_1 is isomorphic to S_2. In order to study this problem, we use Baer's classical interpretation of the extension functor and give an explicit characterization and parametrization of the equivalence classes of multidimensional linear systems S solving this problem. We then use these results to parametrize the equivalence classes of multidimensional linear systems S which admit a fixed parametrizable subsystem S_1 and satisfy that S/S_1 is isomorphic to a fixed autonomous system S_2. We illustrate the main results by means of explicit examples of differential time-delay systems.

Key-words: Multidimensional linear systems, Baer's interpretation of the extension functor, constructive algebra, module theory, homological algebra, behavioural approach, differential time-delay systems, parametrizations, autonomous elements, controllability.

6307

* INRIA Sophia Antipolis, APICS project, 2004 Route des Lucioles BP 93, 06902 Sophia Antipolis Cedex, France, `Alban.Quadrat@sophia.inria.fr`

† Lehrstuhl B für Mathematik RWTH - Aachen, Templergraben 64, 52056 Aachen, Germany, `daniel@momo.math.rwth-aachen.de`

Sur le problème d'extention de Baer pour les systèmes linéaires multidimensionnels

Résumé : A l'aide de l'analyse algébrique, le but de ce papier est de résoudre de manière constructive le problème suivant: étant donnés deux systèmes linéaires multidimensionnels S_1 et S_2, paramétrer les systèmes linéaires multidimensionnels S qui contiennent S_1 comme sous-système et tels que S/S_1 soient isomorphes à S_2. Pour étudier ce problème, nous utilisons l'interprétation de Baer du foncteur extension et nous donnons une caractérisation et une paramétrisation explicite des classes d'équivalence des systèmes linéaires multidimensionnels S satisfaisant à ce problème. Nous utilisons alors ces résultats pour paramétrer les classes d'équivalence des systèmes linéaires multidimensionnels S qui admettent un sous-système paramétrisable donné S_1 et sont tels que S/S_1 soient isomorphes à un système autonome donné S_2. Nous illustrons les résultats importants par des exemples explicites de systèmes d'équations différentielles à retard.

Mots-clés : Systèmes linéaires multidimensionnels, interprétation de Baer du foncteur extension, algèbre constructive, théorie des modules, algèbre homologique, approche comportementale, systèmes différentiels à retard, paramétrisations, éléments autonomes, contrôlabilité.

1 Introduction

A well-known result due to R. E. Kalman states that every time-invariant 1-D linear system defined by a state-space representation decomposes into the direct sum of its controllable (i.e., parametrizable) and autonomous subsystems ([13]). See [21] for a behavioural generalization of this result to time-invariant polynomial linear systems. Within a module-theoretic approach, R. E. Kalman's result has been extended in [11] to the case of linear systems of ordinary differential equations with varying coefficients belonging to a differential field (e.g., $\mathbb{Q}(t)$). The same result also holds, for instance, when the coefficients of the system belong to the polynomial ring $k[t]$ (k is a field of characteristic 0), the ring $\mathbb{C}[\![t]\!]$ of formal power series or the ring $\mathbb{C}\{t\}$ of locally convergent power series. However, it is well-known that this result does not generally admit a generalization for multidimensional linear systems and, in particular, for differential time-delay systems.

In the recent works [25, 26], we have constructively characterized when a multidimensional linear system decomposed into a direct sum of its parametrizable subsystem and the system formed by its autonomous elements. The corresponding algorithms have been implemented in the library OreModules ([5, 6]) and illustrated on many explicit examples. Finally, we have applied this result to the so-called *Monge problem* which questions the existence of parametrizations of the solutions of multidimensional linear systems, to optimal control and variational problems ([25, 26]).

Discussing about [25] in a private communication with the first author, S. Shankar (Chennai Mathematical Institute) proposed the challenging problem of classifying all the multidimensional linear systems S whose parametrizable subsystems are exactly isomorphic to a given parametrizable system S_p and such that S/S_p are isomorphic to a given autonomous system S_a, i.e., $S/S_p \cong S_a$. In particular, S_a and S_p can be chosen as the parametrizable subsystem and the system formed by the autonomous elements of a given multidimensional linear system Σ. Solving this last problem would allow us to parametrize all multidimensional linear systems which have the same parametrizable subsystem and autonomous system as Σ. Moreover, S. Shankar pointed out that this parametrization could be obtained if we could compute the abelian group $\mathrm{ext}^1_D(P, N)$, where P and N are two finitely presented left D-modules defining respectively the systems S_p and S_a, and interpret the elements of this abelian group in terms of multidimensional linear systems using Baer's classical interpretation ([1, 15, 27]).

The purpose of the paper is to constructively answer the problem proposed by S. Shankar. We focus here on the algorithmic issues of this problem and we refer to [3, 29] and future publications for further development and applications of this problem to multidimensional systems theory.

In this paper, we study the computation of the abelian group $\mathrm{ext}^1_D(M, N)$, where M and N are two finitely presented left D-modules. If D is a commutative polynomial ring, we then give an explicit algorithm which computes the D-module $\mathrm{ext}^1_D(M, N)$. This algorithm is implemented in the library *homalg* ([2]). If D is a non-commutative polynomial ring over which Gröbner bases can be constructed, then we show how to compute elements of this abelian group and the corresponding algorithm is implemented in the OreModules package Morphisms ([7]).

We recall Baer's interpretation of the elements of the abelian group $\mathrm{ext}^1_D(M, N)$ in terms of equivalence classes of *extensions* of N by M, namely, in terms of equivalence classes of exact sequences of the form:

$$0 \longrightarrow N \xrightarrow{\alpha} E \xrightarrow{\beta} M \longrightarrow 0.$$

From the previous exact sequence, it is clear that the left D-module $N \cong \alpha(N)$ is contained in E and $E/\alpha(N) \cong M$. Using the duality existing between the system-theroretic and behavioural-theoretic approaches (see [18, 20, 24, 30] and the references therein), whenever the signal space $\mathcal{F}$ in which we seek the system solutions is an *injective left D-module* ([14, 27]), we then obtain that the multidimensional linear system S defined by E admits as a subsystem the system S_1 defined by the left D-module M and satisfies that S/S_1 is isomorphic to the system S_2 defined by the left D-module

N. Contrary to the homological algebra literature, we make constructive Baer's interpretation giving, for instance, an explicit formula for the family of left D-modules defining the equivalence classes of extensions of N by M in terms of matrices characterizing the elements of the abelian group $\mathrm{ext}^1_D(M, N)$ and the matrices defining the finitely presented left D-modules M and N.

We illustrate the previous results in the particular situation where $N = t(P)$ is the torsion left D-submodule of a given finitely presented left D-module P and $M = P/t(P)$. Up to a certain equivalence relation that we shall define in Section 2, this result allows us to classify all the possible presentations of the left D-modules E satisfying that $t(E) \cong t(P)$ and $E/t(E) \cong P/t(P)$. If we denote by Σ the system corresponding to the finitely presented left D-module P, then the previous result shows that, up to this equivalence relation, we have a parametrization of all the multidimensional linear systems which have the same parametrizable subsystem and autonomous system as Σ.

Finally, we illustrate the main results on explicit differential time-delay systems considered in the classical literature and particularly in [10, 17, 19]. The different results have been implemented in the library *homalg* ([2]) and in the package Morphisms of the library OreModules ([8, 7]).

2 A module-theoretic approach to linear systems theory

Let D be a non-commutative ring, $R \in D^{q\times p}$ a $q \times p$ matrix with entries in D and $\mathcal{F}$ a left D-module, namely, an abelian group $\mathcal{F}$ satisfying:

$$\forall\, a_1, a_2 \in D,\ \forall\, f_1, f_2 \in \mathcal{F}:\ a_1\, f_1 + a_2\, f_2 \in \mathcal{F}.$$

Then, a *linear system* or *behaviour* is the abelian group defined by:

$$\ker_{\mathcal{F}}(R.) = \{\eta = (\eta_1\ \dots\ \eta_p)^T \in \mathcal{F}^p \mid R\,\eta = 0\}.$$

In what follows, we shall denote by $D^{1\times p}$ the left D-module of row vectors of length p with entries in D. Let us consider the following D-*morphism*, namely, the D-linear map defined by:

$$\begin{array}{ccc} D^{1\times q} & \xrightarrow{.R} & D^{1\times p} \\ \lambda = (\lambda_1\ \dots\ \lambda_q) & \longmapsto & \lambda\, R. \end{array}$$

The image $\mathrm{im}_D(.R) = D^{1\times q}\, R$ of the D-morphism $.R$ is formed by the left D-linear combinations of the rows of R. The cokernel of the D-morphism $.R$ is then defined by:

$$\mathrm{coker}_D\,(.R) = D^{1\times p}/(D^{1\times q}\, R).$$

Let us denote by $M = D^{1\times p}/(D^{1\times q}\, R)$ and $\{e_i\}_{1\le i\le p}$ (resp., $\{f_j\}_{1\le j\le q}$) the canonical basis of the left D-module $D^{1\times p}$ (resp., $D^{1\times q}$), namely, e_i is the row vector formed by 1 at the i^{th} position and 0 elsewhere. Let us denote by $\pi : D^{1\times p} \longrightarrow M$ the canonical D-morphism sending any element $\lambda \in D^{1\times p}$ onto its residue class $\pi(\lambda) \in M$ and, for $i = 1, \dots, p$, $y_i = \pi(e_i)$. Then, we have:

$$\forall\, j = 1, \dots, q, \quad f_j\, R = (R_{j1}, \dots, R_{jp}) = \sum_{i=1}^{p} R_{ji}\, e_i \in (D^{1\times q}\, R).$$

Therefore, we get:

$$\forall\, j = 1, \dots, q, \quad \pi(f_j\, R) = \sum_{i=1}^{p} R_{ji}\, \pi(e_i) = \sum_{i=1}^{p} R_{ji}\, y_i = 0. \tag{1}$$

Hence, the left D-module $M = D^{1\times p}/(D^{1\times q}\, R)$, *finitely presented by* R, is defined by the generators $\{y_i\}_{1\le i\le p}$ which satisfy the relations (1) and their left D-linear combinations.

The left D-module $M = D^{1\times p}/(D^{1\times q}\, R)$ plays a central role as it was first shown by B. Malgrange. Let us state an important result ([16]).

Theorem 1. *Let $R \in D^{q \times p}$ be a matrix with entries in D, $M = D^{1 \times p}/(D^{1 \times q} R)$ the left D-module finitely presented by R and $\mathcal{F}$ a left D-module. Then, the morphism ψ of abelian groups*

$$\begin{array}{rcl} \psi : \ker_{\mathcal{F}}(R.) & \longrightarrow & \hom_D(M, \mathcal{F}), \\ \eta = (\eta_1 \, \ldots \, \eta_p)^T & \longmapsto & \psi(\eta), \end{array}$$

where $\psi(\eta)$ is defined by

$$\forall \, i = 1, \ldots, p, \quad \psi(\eta)(\pi(e_i)) = \eta_i,$$

is an isomorphism.

Theorem 1 states that we have the following isomorphism of abelian groups

$$\ker_{\mathcal{F}}(R.) \cong \hom_D(M, \mathcal{F}),$$

a fact showing that a linear system $\ker_{\mathcal{F}}(R.)$ only depends on the two left D-modules:

1. The finitely presented left D-module $M = D^{1 \times p}/(D^{1 \times q} R)$.
2. The functional space or the signal space $\mathcal{F}$.

If D is a commutative ring, then we note that $\hom_D(M, \mathcal{F})$ and $\ker_{\mathcal{F}}(R.)$ are D-modules.

Example 1. Let us consider the model of the move of a fluid in a one-dimensional tank ([10])

$$\left\{ \begin{array}{l} y_1(t - 2\,h) + y_2(t) - 2\,\dot{y}_3(t - h) = 0, \\ y_1(t) + y_2(t - 2\,h) - 2\,\dot{y}_3(t - h) = 0, \end{array} \right. \tag{2}$$

where h is a strictly positive real number.

Let $D = \mathbb{Q}\,[\partial, \delta]$ be the commutative polynomial ring of differential time-delay operators, namely,

$$\partial\, f(t) = \dot{f}(t), \quad \delta\, f(t) = f(t - h),$$

and consider the system matrix of (2), namely,

$$R = \begin{pmatrix} \delta^2 & 1 & -2\,\partial\,\delta \\ 1 & \delta^2 & -2\,\partial\,\delta \end{pmatrix} \in D^{2 \times 3}, \tag{3}$$

and the finitely presented D-module defined by $M = D^{1 \times 3}/(D^{1 \times 2} R)$.

If we consider the D-module $\mathcal{F} = C^{\infty}(\mathbb{R})$ of smooth real-valued functions, Theorem 1 then shows that the differential time-delay linear system (2) corresponds to:

$$\ker_{\mathcal{F}}(R.) = \{y = (y_1 \; y_2 \; y_3)^T \in \mathcal{F}^3 \mid R\,y = 0\} \cong \hom_D(M, \mathcal{F}).$$

Example 2. Let us denote by $\partial_i = \partial/\partial x_i$ the partial derivative with respect to the independent variable x_i and let us consider the commutative polynomial ring $D = \mathbb{Q}[\partial_1, \partial_2, \partial_3]$ of differential operators with rational constant coefficients, the matrix $R = (\partial_1 \;\; \partial_2 \;\; \partial_3) \in D^{1 \times 3}$ and the finitely presented D-module $M = D^{1 \times 3}/(D\,R)$.

If $\mathcal{F} = C^{\infty}(\mathbb{R})$ is the D-module of smooth real-valued functions on $\mathbb{R}$, then we obtain that

$$\ker_{\mathcal{F}}(R.) = \{y = (y_1 \; y_2 \; y_3)^T \in \mathcal{F}^3 \mid R\,y = \partial_1\,y_1 + \partial_2\,y_2 + \partial_3\,y_3 = 0\}$$

is the linear system defining the divergence operator in $\mathbb{R}^3$. By Theorem 1, we obtain that the D-module $\ker_{\mathcal{F}}(R.)$ is isomorphic to the D-module $\hom_D(M, \mathcal{F})$.

We introduce a few concepts of homological algebra. See [15, 27] for more details.

Definition 1. 1. A *complex* is a sequence of left D-modules P_i and D-morphisms

$$d_i : P_i \longrightarrow P_{i-1}, \quad i \in \mathbb{Z},$$

satisfying $d_{i-1} \circ d_i = 0$, i.e., $\operatorname{im} d_i \subseteq \ker d_{i-1}$, for all $i \in \mathbb{Z}$. We denote the complex by:

$$P : \ldots \xrightarrow{d_{i+1}} P_{i+1} \xrightarrow{d_i} P_i \xrightarrow{d_{i-1}} P_{i-1} \xrightarrow{d_{i-2}} \ldots$$

2. The *defects of exactness* of the complex P are defined by:

$$\forall\, i \in \mathbb{Z}, \quad H_i(P) = \ker d_{i-1}/\operatorname{im} d_i.$$

3. The complex P is said to be *exact* at P_i if $H_i(P) = 0$, i.e., if $\ker d_{i-1} = \operatorname{im} d_i$. By extension, P is said to be *exact* if $H_i(P) = 0$ for all $i \in \mathbb{Z}$.

Example 3. The complex of left D-modules

$$0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$$

is exact iff f is injective, namely, $\ker f = 0$, g is surjective, namely, $\operatorname{im} g = M''$, and $\ker g = \operatorname{im} f$.

A *finite free resolution* of a left D-module M is an exact sequence of the form

$$\ldots \xrightarrow{.R_3} D^{1\times p_2} \xrightarrow{.R_2} D^{1\times p_1} \xrightarrow{.R_1} D^{1\times p_0} \xrightarrow{\pi} M \longrightarrow 0, \tag{4}$$

where $R_i \in D^{p_i \times p_{i-1}}$, $i \geq 1$, and the D-morphism $.R_i$ is defined by

$$\forall\, \lambda \in D^{1\times p_i},\ (.R_i)(\lambda) = \lambda\, R_i \in D^{1\times p_{i-1}},$$

and π denotes the canonical projection onto the left D-module $M = D^{1\times p_0}/(D^{1\times p_1} R_1)$.

Over the commutative polynomial ring $D = k[x_1, \ldots, x_n]$, where k is a *computable field* (e.g., $k = \mathbb{Q}$, $\mathbb{F}_p$), every D-module admits a finite free resolution which can be explicitly computed by means of Gröbner or Janet bases. Extensions of this result exist for some classes of non-commutative polynomial rings such as the ring of differential operators with polynomial or rational coefficients, the so-called *Weyl algebras*, or some classes of *Ore algebras*. For more details, we refer the reader to [5, 6] and the references therein.

Let $\mathcal{F}$ be a left D-module. A classical result of homological algebra proves that the defect of exactness at position $i \geq 1$ of the following complex

$$\ldots \xleftarrow{R_3.} \mathcal{F}^{p_2} \xleftarrow{R_2.} \mathcal{F}^{p_1} \xleftarrow{R_1.} \mathcal{F}^{p_0} \longleftarrow 0 \tag{5}$$

– where $R_i. : \mathcal{F}^{p_{i-1}} \longrightarrow \mathcal{F}^{p_i}$ is defined by $(R_i.)(\zeta) = R_i\, \zeta \in \mathcal{F}^{p_i}$, for all $\zeta \in \mathcal{F}^{p_{i-1}}$ – namely,

$$\operatorname{ext}^i_D(M, \mathcal{F}) = \ker_{\mathcal{F}}(R_{i+1}.)/(R_i\, \mathcal{F}^{p_{i-1}}),\ i \geq 1,$$

only depends on M and $\mathcal{F}$ and not on the choice of the finite free resolution (and, more generally, on a *projective resolutions* of M), i.e., on the choice of the matrices R_i, $i \geq 1$. See [15, 27] for more details. Moreover, we can prove that we have:

$$\operatorname{ext}^0_D(M, \mathcal{F}) = \ker_{\mathcal{F}}(R_1.) \cong \hom_D(M, \mathcal{F}).$$

Example 4. Let us consider the commutative polynomial ring $D = \mathbb{Q}[\partial, \delta]$ of differential time-delay operators, the matrix $R_1 = (1-\delta \quad \partial)^T$ and the D-module finitely presented by the matrix R_1:

$$M = D/(D^{1\times 2}\, R_1) = D/\left(D\,(1-\delta) + D\,\partial\right).$$

Using the fact that the greatest common divisor of $1-\delta$ and ∂ is 1 and denoting by $R_2 = (\partial \quad \delta - 1)$, we can easily check that we have the finite free resolution of M:

$$0 \longrightarrow D \xrightarrow{.R_2} D^{1\times 2} \xrightarrow{.R_1} D \xrightarrow{\pi} M \longrightarrow 0.$$

Let us consider the D-module $\mathcal{F} = C^\infty(\mathbb{R})$. Then, we have $\mathrm{ext}^1_D(M, \mathcal{F}) = \ker_{\mathcal{F}}(R_2.)/(R_1\,\mathcal{F})$. The D-module $\mathrm{ext}^1_D(M, \mathcal{F})$ is non-trivial because, if we denote by c_1 and c_2 two different real constants, then $\zeta = (c_1 \quad c_2)^T \in \mathcal{F}^2$ satisfies:

$$R_2\,\zeta = \partial\, c_1 + (\delta - 1)\, c_2 = 0.$$

However, ζ does not belong to the D-module $R_1\,\mathcal{F}$ as from the second equation of the following system

$$\begin{cases} \xi(t) - \xi(t-1) = c_1, \\ \dot{\xi}(t) = c_2, \end{cases} \tag{6}$$

we obtain $\xi(t) = c_2\, t + c_3$, where $c_3 \in \mathbb{R}$ is a constant, and substituting this result into the first equation of (6), we then get $\xi(t) - \xi(t-1) - c_1 = c_2 - c_1 = 0$, which contradicts the fact that $c_1 \neq c_2$. Therefore, we obtain that $\mathrm{ext}^1_D(M, \mathcal{F}) \neq 0$.

We say that the complex (5) is obtained by *applying the contravariant left exact functor* $\hom_D(\,\cdot\,, \mathcal{F})$ to the *truncated finite free resolution* of M, namely, the complex defined by:

$$M^\bullet : \ldots \xrightarrow{.R_3} D^{1\times p_2} \xrightarrow{.R_2} D^{1\times p_1} \xrightarrow{.R_1} D^{1\times p_0} \longrightarrow 0.$$

See [27] for more details.

Example 5. Let us consider again Example 2, namely, the ring $D = \mathbb{Q}[\partial_1, \partial_2, \partial_3]$ of differential operators with rational constant coefficients, the matrix $R = (\partial_1 \;\, \partial_2 \;\, \partial_3)$ defining the divergence operator in $\mathbb{R}^3$ and the D-module $M = D^{1\times 3}/(D\,R)$.

Applying the contravariant left exact functor $\hom_D(\,\cdot\,, D)$ to the truncated free resolution of M

$$0 \longrightarrow D \xrightarrow{.R} D^{1\times 3} \longrightarrow 0,$$

we obtain the complex $0 \longleftarrow D \xleftarrow{R.} D^3 \longleftarrow 0$. Hence, we get $\mathrm{ext}^1_D(M, D) = D/(R\,D^3)$. We note that $1 \in D$ but $1 \notin (R\,D^3) = \partial_1\,D + \partial_2\,D + \partial_3\,D$. Therefore, the residue class $\overline{1}$ of 1 in $\mathrm{ext}^1_D(M, D) = D/(D\,\partial_1 + D\,\partial_2 + D\,\partial_3)$ is non-zero, a fact showing that $\mathrm{ext}^1_D(M, D) \neq 0$.

To finish, let us introduce the concepts of *injective* and *cogenerator* left D-modules ([14, 27]).

Definition 2.
1. A left D-module $\mathcal{F}$ is said to be *injective* if, for every left D-module M and for all $i \geq 1$, we have $\mathrm{ext}^i_D(M, \mathcal{F}) = 0$.
2. A left D-module $\mathcal{F}$ is said to be *cogenerator* if $\hom_D(M, \mathcal{F}) = 0$ implies that $M = 0$.

If $\mathcal{F}$ is a cogenerator left D-module and $M = D^{1\times p}/(D^{1\times q}\,R)$ is a non-zero finitely presented left D-module, then we have $\hom_D(M, \mathcal{F}) \neq 0$, which, by Theorem 1, shows that $\ker_{\mathcal{F}}(R.) \neq 0$, i.e., the linear system $\ker_{\mathcal{F}}(R.)$ admits a non-trivial solution in $\mathcal{F}$.

Example 6. Let $D = \mathbb{R}[\partial_1, \ldots, \partial_n]$ be the commutative polynomial ring of differential operators in $\partial_i = \partial/\partial x_i$ with real constant coefficients. If Ω is a convex open subset of $\mathbb{R}^n$, then it was shown by B. Malgrange that the D-module $C^\infty(\Omega)$ (resp. $\mathcal{D}'(\Omega)$, $\mathcal{S}'(\Omega)$) of smooth functions (resp., distributions, temperate distributions) is an injective cogenerator ([18, 20, 30]).

Example 4 shows that the $D = \mathbb{Q}[\partial, \delta]$-module $\mathcal{F} = C^\infty(\mathbb{R})$ is not injective. Moreover, if we consider the non-trivial D-module $M = D/(D\,\delta)$, then the D-module $\hom_D(M, \mathcal{F})$ is isomorphic to

$$\ker_{\mathcal{F}}(\delta.) = \{\eta \in \mathcal{F} \mid \forall\, t \in \mathbb{R},\ \eta(t-1) = 0\} = 0,$$

which shows that $\mathcal{F}$ is not a cogenerator D-module. However, we have the important proposition.

Proposition 1. *([27]) For every non-commutative ring D, there exists an injective cogenerator left D-module $\mathcal{F}$.*

To finish this section, we recall the following fundamental result.

Proposition 2. *([27]) Let $0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$ be a short exact sequence of left D-modules and N a left D-module. Then, there are natural connecting morphisms δ^i such that we have the following exact sequence of abelian groups*

$$\begin{array}{c} 0 \longrightarrow \hom_D(M'', N) \xrightarrow{g^\star} \hom_D(M, N) \xrightarrow{f^\star} \hom_D(M', N) \xrightarrow{\delta^1} \mathrm{ext}^1_D(M'', N) \longrightarrow \mathrm{ext}^1_D(M, N) \\ \longrightarrow \mathrm{ext}^1_D(M', N) \xrightarrow{\delta^2} \mathrm{ext}^2_D(M'', N) \longrightarrow \ldots, \end{array}$$

where, for all $h \in \hom_D(M'', N)$, $g^\star(h) = h \circ g$, and similarly for $f^\star$.

We refer the reader to [15, 27] for information concerning module theory and homological algebra.

3 Baer extensions

Let us introduce the concept of *extension*. For more details, see [15, 27].

Definition 3. 1. Let M and N be two left D-modules. An *extension of N by M* is an exact sequence of left D-modules of the form:

$$\xi : 0 \longrightarrow N \xrightarrow{f} E \xrightarrow{g} M \longrightarrow 0.$$

2. Two extensions of N by M,

$$\xi : 0 \longrightarrow N \xrightarrow{f} E \xrightarrow{g} M \longrightarrow 0, \quad \xi' : 0 \longrightarrow N \xrightarrow{f'} E' \xrightarrow{g'} M \longrightarrow 0,$$

are said to be *equivalent*, denoted by $\xi \sim \xi'$, if there exists a D-morphism $\phi : E \longrightarrow E'$ such that we have the commutative exact diagram

$$\begin{array}{ccccccc} 0 \longrightarrow & N & \xrightarrow{f} & E & \xrightarrow{g} & M & \longrightarrow 0 \\ & \| & & \downarrow \phi & & \| & \\ 0 \longrightarrow & N & \xrightarrow{f'} & E' & \xrightarrow{g'} & M & \longrightarrow 0, \end{array}$$

i.e., such that the identities $f' = \phi \circ f$ and $g = g' \circ \phi$ hold.

3. We denote by $[\xi]$ the *equivalence class* of the extension ξ for the equivalence relation defined by $\sim$. The set of all equivalence classes of extensions of N by M is denoted by $\mathrm{e}_D(M, N)$.

4. A short exact sequence $0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$ is said to *split* if we have $M \cong M' \oplus M''$, where $\oplus$ denotes the direct sum.

Using the *snake lemma* ([27]), we can check that ϕ defined in 2 of Definition 3 is an isomorphism.

The condition (4) of Definition 3 is equivalent to the existence of a D-morphism $h : M'' \longrightarrow M$ such that $g \circ h = \mathrm{id}_{M''}$ or, equivalently, to the existence of a D-morphism $k : M \longrightarrow M'$ satisfying $k \circ f = \mathrm{id}_{M'}$. See [27] for more details.

Example 7. Let us consider the following split exact sequence:

$$\xi : 0 \longrightarrow M' \underset{\overset{k}{\longleftarrow}}{\overset{f}{\longrightarrow}} M \underset{\overset{h}{\longleftarrow}}{\overset{g}{\longrightarrow}} M'' \longrightarrow 0.$$

We can easily check that we have the following commutative exact diagram

$$\begin{array}{ccccccccc} 0 \longrightarrow & M' & \xrightarrow{f} & M & \xrightarrow{g} & M'' & \longrightarrow 0 \\ & \| & & \downarrow (k,\, g) & & \| & \\ 0 \longrightarrow & M' & \xrightarrow{i_1} & M' \oplus M'' & \xrightarrow{p_2} & M'' & \longrightarrow 0, \end{array}$$

with the following notations:

$$\begin{array}{cccccccc} \xi' : 0 \longrightarrow & M' & \xrightarrow{i_1} & M' \oplus M'' & \xrightarrow{p_2} & M'' & \longrightarrow 0. \\ & m' & \longmapsto & (m', 0) & & & \\ & & & (m', m'') & \longmapsto & m'' & \end{array}$$

We obtain that the extension ξ is equivalent to the extension ξ', i.e., we have $[\xi] = [\xi']$ in $\mathrm{e}_D(M', M'')$.

Let us introduce the classical concepts of *pushout* and *pullback* ([15, 27]).

Definition 4. 1. Let us consider two morphisms of left D-modules $f : A \longrightarrow B$ and $g : A \longrightarrow C$. If we denote by E the cokernel of the D-morphism defined by

$$\begin{array}{rcl} (f, -g) : A & \longrightarrow & B \oplus C, \\ a & \longmapsto & (f(a), -g(a)) \end{array}$$

$\sigma : B \oplus C \longrightarrow E$ the canonical projection onto E, $\gamma : B \longrightarrow E$ and $\alpha : C \longrightarrow E$ the D-morphisms defined by $\gamma(b) = \sigma((b, 0))$ and $\alpha(c) = \sigma((0, c))$, then we get the following commutative diagram:

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow g & & \downarrow \gamma \\ C & \xrightarrow{\alpha} & E. \end{array}$$

This commutative diagram is a *pushout square* and this construction is the *pushout* of f and g.

2. Let $f : B \longrightarrow A$ and $g : C \longrightarrow A$ be two morphisms of left D-modules. If we denote by $E = \{(b, c) \in B \oplus C \mid f(b) = g(c)\}$ and define the following D-morphisms

$$\begin{array}{rclcrcl} \alpha : E & \longrightarrow & C, & \quad & \beta : E & \longrightarrow & B, \\ (b, c) & \longmapsto & c & & (b, c) & \longmapsto & b \end{array}$$

then we get the following commutative diagram:

$$\begin{array}{ccc} E & \xrightarrow{\alpha} & C \\ \downarrow \beta & & \downarrow g \\ B & \xrightarrow{f} & A. \end{array}$$

This commutative diagram is a *pullback square* and this construction is the *pullback* of f and g.

The following lemma is standard in homological algebra (see, e.g., [15, 27]).

Lemma 1. *1. Let $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{f'} A' \longrightarrow 0$ be an exact sequence of left D-modules and $g : A \longrightarrow C$ a D-morphism. Then, we have the following commutative exact diagram*

$$\begin{array}{ccccccc} 0 \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{f'} & A' & \longrightarrow 0 \\ & \downarrow g & & \downarrow \gamma & & \| & \\ 0 \longrightarrow & C & \xrightarrow{\alpha} & E & \xrightarrow{\beta} & A' & \longrightarrow 0, \end{array}$$

in which the first square is the pushout of f and g and, with the notations of 1 of Definition 4, $\beta : E \longrightarrow A'$ is defined by:

$$\forall\, (b, c) \in B \oplus C, \quad \beta(\sigma((b, c))) = f'(b).$$

If ξ denotes the first horizontal exact sequence of the previous commutative exact diagram, then we shall denote by $g_\star(\xi)$ the second one.

2. Let $0 \longrightarrow A' \xrightarrow{f'} B \xrightarrow{f} A \longrightarrow 0$ be an exact sequence of left D-modules and $g : C \longrightarrow A$ a D-morphism. Then, we have the following commutative exact diagram

$$\begin{array}{ccccccc} 0 \longrightarrow & A' & \xrightarrow{\alpha'} & E & \xrightarrow{\alpha} & C & \longrightarrow 0 \\ & \| & & \downarrow \beta & & \downarrow g & \\ 0 \longrightarrow & A' & \xrightarrow{f'} & B & \xrightarrow{f} & A & \longrightarrow 0, \end{array}$$

in which the last square is the pullback of f and g and, with the notations of 2 of Definition 4, the D-morphism $\alpha' : A' \longrightarrow E$ is defined by:

$$\forall\, a' \in A', \quad \alpha'(a') = (f'(a'), 0).$$

If ξ denotes the second horizontal exact sequence of the previous commutative exact diagram, then we shall denote by $g^\star(\xi)$ the first horizontal one.

If we consider two extensions of N by M,

$$\xi : 0 \longrightarrow N \xrightarrow{f} E \xrightarrow{g} M \longrightarrow 0, \quad \xi' : 0 \longrightarrow N \xrightarrow{f'} E' \xrightarrow{g'} M \longrightarrow 0, \tag{7}$$

then we shall denote by $\xi \oplus \xi'$ the following exact sequence

$$0 \longrightarrow N \oplus N \xrightarrow{f \oplus f'} E \oplus E' \xrightarrow{g \oplus g'} M \oplus M \longrightarrow 0,$$

where, for all $(n, n') \in N \oplus N$, $(f \oplus f')((n, n')) = (f(n), f'(n'))$ and similarly for $g \oplus g'$.

Let us introduce the following D-morphisms:

$$\begin{array}{rcl} \nabla : N \oplus N & \longrightarrow & N, \\ (n_1, n_2) & \longmapsto & n_1 + n_2, \end{array} \qquad \begin{array}{rcl} \Delta : M & \longrightarrow & M \oplus M, \\ m & \longmapsto & (m, m). \end{array} \tag{8}$$

Using the notations of Lemma 1, the *Baer sum* of the two extensions ξ and ξ' is the extension defined as follows:

$$\xi + \xi' = \Delta^\star(\nabla_\star(\xi \oplus \xi')) = \nabla_\star(\Delta^\star(\xi \oplus \xi')). \tag{9}$$

For instance, using Lemma 1, $\nabla_\star(\xi \oplus \xi')$ is an exact sequence of the form

$$0 \longrightarrow N \longrightarrow F \longrightarrow M \oplus M \longrightarrow 0,$$

and $\Delta^\star(\nabla_\star(\xi \oplus \xi'))$ is then an exact sequence of the form:

$$0 \longrightarrow N \longrightarrow G \longrightarrow M \longrightarrow 0.$$

The set $\mathrm{e}_D(M, N)$ equipped with this sum forms an abelian group: the equivalence class of the split exact sequence

$$0 \longrightarrow N \longrightarrow N \oplus M \longrightarrow M \longrightarrow 0$$

defines the zero element of $\mathrm{e}_D(M, N)$ and the inverse of $[\xi]$ is defined by the equivalence class of the following equivalent two extensions (see, e.g., [27]):

$$0 \longrightarrow N \xrightarrow{-f} E \xrightarrow{g} M \longrightarrow 0, \quad 0 \longrightarrow N \xrightarrow{f} E \xrightarrow{-g} M \longrightarrow 0.$$

A more tractable characterization of the sum of two extensions of N by M can be found in the classical book of Cartan and Eilenberg ([4]): let (7) be two extensions of N by M and let us define the following two D-morphisms:

$$\begin{array}{rcl} -f \oplus f' : N & \longrightarrow & E \oplus E' \\ n & \longmapsto & (-f(n), f'(n')) \end{array} \qquad \begin{array}{rcl} (g, -g') : E \oplus E' & \longrightarrow & M \\ (e, e') & \longmapsto & g(e) - g'(e'). \end{array}$$

Then, the sum $\xi + \xi'$ is defined by the left D-module $E'' = \ker(g, -g')/\mathrm{im}\,(-f \oplus f')$.

If we denote by $\gamma : \ker(g, -g') \longrightarrow E''$ the canonical projection onto E'', then we have the following short exact sequence of left D-modules:

$$\begin{array}{ccccccc} 0 \longrightarrow & N & \xrightarrow{f''} & E'' & \xrightarrow{g''} & M & \longrightarrow 0. \\ & n & \longmapsto & \gamma((f(n), 0)) = \gamma((0, f'(n))) & & & \\ & & & \gamma((e, e')) & \longmapsto & g(e) = g'(e') & \end{array}$$

4 An important isomorphism

The following result due to R. Baer explains the etymology of the extension functor ([1]).

Theorem 2 ([15, 27])**.** *Let M and $\mathcal{F}$ be two left D-modules. Then, the abelian groups $\mathrm{ext}^1_D(M, \mathcal{F})$ and $\mathrm{e}_D(M, \mathcal{F})$ are isomorphic.*

The purpose of this section is to prove Theorem 2 in the case of a finitely generated left D-module M over a left noetherian domain D ([14, 27]). We do not claim any novelty in this proof apart from the fact that the classical proofs are turned as constructive as possible, a fact which will play an important role in Section 5.

Let us consider a finite free resolution of the left D-module M:

$$\ldots \xrightarrow{.R_3} D^{1\times r} \xrightarrow{.R_2} D^{1\times q} \xrightarrow{.R} D^{1\times p} \xrightarrow{\pi} M \longrightarrow 0. \tag{10}$$

Applying the contravariant left exact functor $\hom_D(\,\cdot\,, \mathcal{F})$ to the corresponding truncated free resolution of M, namely,

$$M^\bullet : \ldots \xrightarrow{.R_3} D^{1\times r} \xrightarrow{.R_2} D^{1\times q} \xrightarrow{.R} D^{1\times p} \longrightarrow 0, \tag{11}$$

we then get the complex $\hom_D(M^\bullet, \mathcal{F})$:

$$\ldots \xleftarrow{R_3.} \mathcal{F}^r \xleftarrow{R_2.} \mathcal{F}^q \xleftarrow{R.} \mathcal{F}^p \longleftarrow 0. \tag{12}$$

We obtain:

$$\mathrm{ext}^1_D(M, \mathcal{F}) = \ker_{\mathcal{F}}(R_2.)/(R\,\mathcal{F}^p).$$

Hence, $\bar{\zeta} \in \mathrm{ext}^1_D(M, \mathcal{F})$ is represented by $\zeta \in \ker_{\mathcal{F}}(R_2.)$, i.e., $\zeta \in \mathcal{F}^q$ satisfies $R_2\, \zeta = 0$.

Let us denote by $M_2 = D^{1\times q}/(D^{1\times r}\, R_2)$ and $\kappa : D^{1\times q} \longrightarrow M_2$ the canonical projection onto M_2. For all $\mu \in (D^{1\times r}\, R_2)$, there exists $\nu \in D^{1\times r}$ such that $\mu = \nu\, R_2$ and we then get $\mu\, R = \nu\, R_2\, R = 0$ as we have $R_2\, R = 0$. Hence, the restriction of the D-morphism $.R$ to the left D-submodule $D^{1\times r}\, R_2$ of $D^{1\times q}$ is the zero morphism. Therefore, we get the following exact sequence

$$\xi : 0 \longrightarrow M_2 \xrightarrow{d_R} D^{1\times p} \xrightarrow{\pi} M \longrightarrow 0, \tag{13}$$

where, for all $\mu \in D^{1\times q}$, $d_R(\kappa(\mu)) = \mu\, R$.

By Theorem 1, we have an isomorphism $\psi : \ker_{\mathcal{F}}(R_2.) \longrightarrow \hom_D(M_2, \mathcal{F})$ and if $\{f_i\}_{1\leq i\leq q}$ denotes the standard basis of $D^{1\times q}$, then $\psi(\zeta) : M_2 \longrightarrow \mathcal{F}$ is defined by:

$$\forall\, i = 1, \ldots, q, \quad \psi(\zeta)(\kappa(f_i)) = \zeta_i.$$

Pushing out the D-morphisms d_R and $\psi(\zeta)$

$$\begin{array}{ccccccc} 0 \longrightarrow & M_2 & \xrightarrow{d_R} & D^{1\times p} & \xrightarrow{\pi} & M & \longrightarrow 0, \\ & \downarrow \psi(\zeta) & & & & & \\ & \mathcal{F} & & & & & \end{array}$$

we then obtain the commutative exact diagram (see 1 of Lemma 1)

$$\begin{array}{ccccccc} 0 \longrightarrow & M_2 & \xrightarrow{d_R} & D^{1\times p} & \xrightarrow{\pi} & M & \longrightarrow 0 \\ & \downarrow \psi(\zeta) & & \downarrow \gamma & & \| & \\ 0 \longrightarrow & \mathcal{F} & \xrightarrow{\alpha} & E & \xrightarrow{\beta} & M & \longrightarrow 0, \end{array}$$

where E denotes the cokernel of the D-morphism $\varphi : M_2 \longrightarrow D^{1\times p} \oplus \mathcal{F}$ defined by:

$$\forall\, \mu \in D^{1\times q}, \quad \varphi(\kappa(\mu)) = (d_R(\kappa(\mu)), -\psi(\zeta)(\kappa(\mu))) = (\mu\, R, -\mu\, \zeta).$$

If we denote by $\sigma : D^{1\times p} \oplus \mathcal{F} \longrightarrow E$ the canonical projection onto E, then α, β and γ are defined by:

$$\begin{array}{ll} \forall\, f \in \mathcal{F}, & \alpha(f) = \sigma((0, f)), \\ \forall\, \lambda \in D^{1\times p}, & \gamma(\lambda) = \sigma((\lambda, 0)), \\ \forall\, \lambda \in D^{1\times p},\ \forall\, f \in \mathcal{F}, & \beta(\sigma((\lambda, f))) = \pi(\lambda). \end{array} \tag{14}$$

The last horizontal exact sequence of the previous commutative exact diagram is $\psi(\zeta)_\star(\xi)$.

Using the fact that d_R is injective, we obtain that φ is injective and we get the exact sequence:

$$0 \longrightarrow M_2 \xrightarrow{\varphi} D^{1\times p} \oplus \mathcal{F} \xrightarrow{\sigma} E \longrightarrow 0.$$

Combining the previous short exact sequence with the following long exact sequence

$$\ldots \xrightarrow{.R_3} D^{1\times r} \xrightarrow{.R_2} D^{1\times q} \xrightarrow{\kappa} M_2 \longrightarrow 0,$$

we then obtain the following long exact sequence:

$$\begin{array}{cccccccc} \ldots \xrightarrow{.R_3} D^{1\times r} \xrightarrow{.R_2} & D^{1\times q} & \xrightarrow{.(R \quad -\zeta)} & D^{1\times p} \oplus \mathcal{F} & \xrightarrow{\sigma} & E \longrightarrow 0. \\ & \mu & \longmapsto & \mu\,(R \quad -\zeta) & & \end{array} \tag{15}$$

To sum up, for all $\zeta \in \ker_{\mathcal{F}}(R_2.)$, we obtain that the left D-module E defined by (15) yields the following extension of $\mathcal{F}$ by M

$$\psi(\zeta)_\star(\xi) : 0 \longrightarrow \mathcal{F} \xrightarrow{\alpha} E \xrightarrow{\beta} M \longrightarrow 0, \tag{16}$$

where α and β are defined by (14).

Let us prove that the previous construction only depends on the residue class $\overline{\zeta}$ of the element $\zeta \in \ker_{\mathcal{F}}(R_2.)$ in $\mathrm{ext}^1_D(M, \mathcal{F})$. If $\zeta' \in \ker_{\mathcal{F}}(R_2.)$ is another pre-image of $\overline{\zeta} \in \mathrm{ext}^1_D(M, \mathcal{F})$, then there exists $\xi \in \mathcal{F}^p$ such that $\zeta' = \zeta + R\,\xi$. Applying the previous construction to ζ', we get the following extension

$$\psi(\zeta')_\star(\xi) : 0 \longrightarrow \mathcal{F} \xrightarrow{\alpha'} E' \xrightarrow{\beta'} M \longrightarrow 0, \tag{17}$$

where α', β' and γ' are similarly defined as in (14) and $E' = (D^{1\times p} \oplus \mathcal{F})/(D^{1\times q}\,(R \quad -\zeta - R\,\xi))$.

Now, we can check that we have

$$(R \quad -\zeta - R\,\xi) \begin{pmatrix} I_p & \xi \\ 0 & \mathrm{id}_{\mathcal{F}} \end{pmatrix} = (R \quad -\zeta),$$

which induces an isomorphism $\phi : E' \longrightarrow E$ defined by:

$$\forall\, \lambda \in D^{1\times p}, \quad \forall\, f \in \mathcal{F}, \quad \phi(\sigma'((\lambda, f))) = \sigma\left((\lambda, f) \begin{pmatrix} I_p & \xi \\ 0 & \mathrm{id}_{\mathcal{F}} \end{pmatrix}\right) = \sigma((\lambda, f + \lambda\,\xi)),$$

where $\sigma' : D^{1\times p} \oplus \mathcal{F} \longrightarrow E'$ denotes the canonical projection. For all $f \in \mathcal{F}$ and $\lambda \in D^{1\times p}$, we have

$$\begin{cases} (\phi \circ \alpha')(f) = \phi(\sigma'((0, f))) = \sigma((0, f)) = \alpha(f), \\ (\beta \circ \phi)(\sigma'((\lambda, f))) = \beta(\sigma((\lambda, f + \lambda\,\xi))) = \pi(\lambda) = \beta'(\sigma'((\lambda, f))), \end{cases}$$

which proves that $\alpha = \phi \circ \alpha'$ and $\beta' = \beta \circ \phi$ and the extensions (16) and (17) of $\mathcal{F}$ by M are then equivalent, i.e., $[\psi(\zeta)_\star(\xi)] = [\psi(\zeta')_\star(\xi)]$, where ξ is the exact sequence defined by (13).

Hence, with the previous notations, we obtain the following result.

Proposition 3. *For all $\overline{\zeta} \in \mathrm{ext}^1_D(M, \mathcal{F})$, the left D-module E defined by (15) satisfies (16), where α and β are defined by (14), and defines an equivalence class $[\psi(\zeta)_\star(\xi)]$ of extensions of $\mathcal{F}$ by M, i.e., an element of $\mathrm{e}_D(M, \mathcal{F})$, where ξ is the exact sequence defined by (13).*

By Proposition 3, we obtain the following well-defined map:

$$\begin{array}{rcl} \Pi : \mathrm{ext}^1_D(M, \mathcal{F}) & \longrightarrow & \mathrm{e}_D(M, \mathcal{F}), \\ \overline{\zeta} & \longmapsto & [\psi(\zeta)_\star(\xi)]\,. \end{array}$$

It is not totally straightforward to prove that Π is a morphism of abelian groups ([15, 27]). In Theorem 4, we shall detail the proof of this result for a particular but important class of left D-modules $\mathcal{F}$ in mathematical systems theory. As the lines of the two proofs are quite similar, we let the reader adapt the proof of Theorem 4 to show that Π is a morphism of abelian groups.

Remark 1. We note that if we apply the contravariant left exact functor $\hom_D(\,\cdot\,, \mathcal{F})$ to the exact sequence (15), we obtain the exact sequence

$$\mathcal{F}^q \xleftarrow{(R \quad -\zeta).} \mathcal{F}^p \oplus \mathrm{end}_D(\mathcal{F}) \xleftarrow{\sigma^\star} \hom_D(E, \mathcal{F}) \longleftarrow 0,$$

where $\mathrm{end}_D(\mathcal{F})$ denotes the non-commutative ring of D-endomorphisms of $\mathcal{F}$, which shows that:

$$\hom_D(E, \mathcal{F}) \cong \{(\eta,\, \omega) \in \mathcal{F}^p \oplus \mathrm{end}_D(\mathcal{F}) \mid R\,\eta = (\omega(\zeta_1), \ldots, \omega(\zeta_q))^T\}.$$

Taking $\omega = 0$, we get $\hom_D(M, \mathcal{F}) \subseteq \hom_D(E, \mathcal{F})$ by Theorem 1.

This last result can also be proved by applying the contravariant left exact functor $\hom_D(\,\cdot\,, \mathcal{F})$ to the exact sequence (16) to get the exact sequence:

$$\mathrm{end}_D(\mathcal{F}) \xleftarrow{\alpha^\star} \hom_D(E, \mathcal{F}) \xleftarrow{\beta^\star} \hom_D(M, \mathcal{F}) \longleftarrow 0.$$

Example 8. We consider again Example 4. There, we proved that, for $c_1 \neq c_2$, the residue class $\overline{\zeta}$ of $\zeta = (c_1,\, c_2)^T \in \mathbb{R}^2$ in $\mathrm{ext}^1_D(M, \mathcal{F})$ is non-zero. By Proposition 3, a non-trivial extension of $\mathcal{F}$ by N is then defined by:

$$E = (D \oplus \mathcal{F}) / \left(D^{1\times 2} \begin{pmatrix} 1-\delta & -c_1 \\ \partial & -c_2 \end{pmatrix} \right).$$

Using Remark 1, we obtain that:

$$\hom_D(E, \mathcal{F}) = \left\{ (\xi,\, \omega) \in \mathcal{F} \oplus \mathrm{end}_D(\mathcal{F}) \mid \xi(t) - \xi(t-1) = \omega(c_1),\ \dot{\xi}(t) = \omega(c_2) \right\}.$$

Using the fact that $c_i \in \mathbb{R}$ and $\omega \in \mathrm{end}_D(\mathcal{F})$, we get:

$$\partial\, c_i = 0 \ \Rightarrow\ \omega\,(\partial\, c_i) = 0 \ \Rightarrow\ \partial\, \omega(c_i) = 0 \ \Rightarrow\ \omega(c_i) \in \mathbb{R}.$$

Hence, by integration, we obtain that the system

$$\begin{cases} \xi(t) - \xi(t-1) = \omega(c_1), \\ \dot{\xi}(t) = \omega(c_2), \end{cases}$$

admits a solution iff we have $\omega(c_1) = \omega(c_2)$, i.e., $\omega(c_1 - c_2) = 0$ (see Example 4). For instance, if we take $\omega = \partial$, then we get that $(a, \partial) \in \hom_D(E, \mathcal{F})$, where a denotes any real constant. Similarly, for any real constant a, we have $(a, 1-\delta) \in \hom_D(E, \mathcal{F})$. However, we note that we cannot take $\omega = \mathrm{id}_{\mathcal{F}}$ as we have shown that (6) does not admit any solution in $\mathcal{F}$.

Example 9. We consider again Example 5. There, we proved that the residue class $\overline{1}$ of 1 in $\mathrm{ext}^1_D(M, D)$ is non-zero. Therefore, by Proposition 3, a non-trivial extension is defined by:

$$0 \longrightarrow D \xrightarrow{.(\partial_1\ \partial_2\ \partial_3\ -1)} D^{1\times 4} \xrightarrow{\sigma} E \longrightarrow 0. \tag{18}$$

We can easily check that E is a free D-module of rank 3, i.e., $E \cong D^{1\times 3}$. In particular, if we denote by $\{e_i\}_{1\leq i\leq 4}$ the standard basis of $D^{1\times 4}$, then $\{\sigma(e_1), \sigma(e_2), \sigma(e_3)\}$ forms a basis of E.

Moreover, we have the short exact sequence

$$0 \longrightarrow D \xrightarrow{\alpha} E \xrightarrow{\beta} M \longrightarrow 0,$$

where the D-morphism α is defined by $\alpha(1) = \sigma(e_4) = \partial_1\, \sigma(e_1) + \partial_2\, \sigma(e_2) + \partial_3\, \sigma(e_3)$. We obtain that the extension (18) of D by M is nothing else than the following finite free resolution of M:

$$0 \longrightarrow D \xrightarrow{.R} D^{1\times 3} \xrightarrow{\pi} M \longrightarrow 0. \tag{19}$$

This is a particular example of a result due to J.-P. Serre (Proposition 2 of [28]). We refer to [3] for results in multidimensional systems theory using Serre's theorem. Moreover, we can check that the matrix R does not admit a right-inverse over D as 1 does not belong to the ideal of D defined by ∂_1, ∂_2 and ∂_3. Hence, the previous extension does not split, a fact which is coherent with the fact that $\overline{1}$ is not equal to $\overline{0}$ in $\mathrm{ext}^1_D(M, D) = D/(D\,\partial_1 + D\,\partial_2 + D\,\partial_3)$. Therefore, there exist only two equivalence classes of extensions of D by M, namely, the split one defined by $0 \longrightarrow D \longrightarrow M \oplus D \longrightarrow D \longrightarrow 0$ and the one defined by (19).

Now, let us prove that any extension of $\mathcal{F}$ by M, namely, any exact sequence of the form

$$\xi : 0 \longrightarrow \mathcal{F} \xrightarrow{\alpha} E \xrightarrow{\beta} M \longrightarrow 0, \tag{20}$$

defines an element $\overline{\zeta} \in \mathrm{ext}^1_D(M, \mathcal{F}) = \ker_{\mathcal{F}}(R_2.)/(R\,\mathcal{F}^p)$. We point out that α and β are not defined by (14) anymore but are general D-morphisms of left D-modules.

We use the same notations as in Section 2. Let us consider a finite free resolution (10) of M and the following diagram:

$$\begin{array}{ccccccccc} D^{1\times r} & \xrightarrow{.R_2} & D^{1\times q} & \xrightarrow{.R} & D^{1\times p} & \xrightarrow{\pi} & M & \longrightarrow 0 \\ & & & & & & \| & \\ 0 & \longrightarrow & \mathcal{F} & \xrightarrow{\alpha} & E & \xrightarrow{\beta} & M & \longrightarrow 0. \end{array} \tag{21}$$

Let us consider the canonical projection $\pi : D^{1\times p} \longrightarrow M$ and define $\pi(e_i) = y_i$, for $i = 1, \ldots, p$. Using the fact that β is a surjective D-morphism, for $i = 1, \ldots, p$, there exists $a_i \in E$ such that $y_i = \beta(a_i)$. If we define the D-morphism $\gamma : D^{1\times p} \longrightarrow E$ by $\gamma(e_i) = a_i$, $i = 1, \ldots, p$, we then get that $\pi = \beta \circ \gamma$.

We now have $\beta \circ \gamma \circ (.R) = \pi \circ (.R) = 0$. Hence, for $j = 1, \ldots, q$, we get $\gamma(f_j\,R) \in \ker \beta = \mathrm{im}\,\alpha$ and using the fact that α is an injective D-morphism, there exists a unique element $\zeta_j \in \mathcal{F}$ satisfying $\gamma(f_j\,R) = \alpha(\zeta_j)$. If we define the D-morphism of left D-modules

$$\begin{array}{rcl} \psi : D^{1\times q} & \longrightarrow & \mathcal{F} \\ f_j & \longmapsto & \psi(f_j) = \zeta_j, \quad j = 1, \ldots, q, \end{array}$$

then we get $\gamma \circ (.R) = \alpha \circ \psi$.

We have $\alpha \circ \psi \circ (.R_2) = \gamma \circ (.R) \circ (.R_2) = 0$ and if we denote by $\{g_k\}_{1\leq k\leq r}$ the canonical basis of $D^{1\times r}$, then, for $k = 1, \ldots, r$, we have $\alpha(\psi(g_k\,R_2)) = 0$. The fact that α is injective implies that $\psi(g_k\,R_2) = 0$, for $k = 1, \ldots, r$. Expanding $g_k\,R_2$ with respect to the standard basis $\{f_j\}_{1\leq j\leq q}$ of $D^{1\times q}$, we obtain $g_k\,R_2 = \sum_{j=1}^{q} (R_2)_{kj}\, f_j$, which implies that

$$\forall\, k = 1, \ldots, r, \quad \psi(g_k\,R_2) = \sum_{j=1}^{q} (R_2)_{kj}\, \psi(f_j) = \sum_{j=1}^{q} (R_2)_{kj}\, \zeta_j = 0,$$

i.e., if we denote by $\zeta = (\zeta_1, \ldots, \zeta_q)^T \in \mathcal{F}^q$, we then get $R_2\,\zeta = 0$. Hence, ζ defines an element $\overline{\zeta} \in \mathrm{ext}^1_D(M, \mathcal{F}) = \ker_{\mathcal{F}}(R_2.)/(R\,\mathcal{F}^p)$ and we have the following commutative exact diagram:

$$\begin{array}{ccccccccc} D^{1\times r} & \xrightarrow{.R_2} & D^{1\times q} & \xrightarrow{.R} & D^{1\times p} & \xrightarrow{\pi} & M & \longrightarrow 0 \\ \downarrow & & \downarrow \psi & & \downarrow \gamma & & \| & \\ 0 & \longrightarrow & \mathcal{F} & \xrightarrow{\alpha} & E & \xrightarrow{\beta} & M & \longrightarrow 0. \end{array} \tag{22}$$

Let us prove that $\overline{\zeta}$ only depends on the extension (16) and not on the choice of the pre-images a_i of y_i. For $i = 1, \ldots, p$, let us consider different pre-images $a'_i \in E$ of y_i. Then, for $i = 1, \ldots, p$, we have $\beta(a'_i) = \beta(a_i)$, i.e., $\beta(a'_i - a_i) = 0$, and thus, there exists $\xi_i \in \mathcal{F}$ such that $a'_i - a_i = \alpha(\xi_i)$. Let us define $s : D^{1\times p} \longrightarrow \mathcal{F}$ by $s(e_i) = \xi_i$, for $i = 1, \ldots, p$, and $\gamma' : D^{1\times p} \longrightarrow E$ defined by $\gamma'(e_i) = a'_i = \gamma(e_i) + (\alpha \circ s)(e_i)$, for $i = 1, \ldots, p$. Therefore, we get $\gamma' = \gamma + \alpha \circ s$ and, using the fact that $\gamma(f_j\,R) = \alpha(\zeta_j)$, we get:

$$\begin{array}{rl} \forall\, j = 1, \ldots, q, \quad \gamma'(f_j\,R) & = \gamma(f_j\,R) + (\alpha \circ s)(f_j\,R) \\ & = \alpha(\zeta_j) + (\alpha \circ s)\left(\sum_{i=1}^{p} R_{ji}\, e_i\right) \\ & = \alpha\left(\zeta_j + \sum_{i=1}^{p} R_{ji}\, \xi_i\right). \end{array}$$

Hence, the D-morphism defined by

$$\begin{array}{rcl} \psi' : D^{1\times q} & \longrightarrow & \mathcal{F} \\ f_j & \longmapsto & \zeta_j + \sum_{i=1}^{p} R_{ji}\,\xi_i, \quad j = 1, \dots, q, \end{array}$$

satisfies $\psi' = \psi + s \circ (.R)$ and $\gamma' \circ (.R) = \alpha \circ \psi'$. Then, we get

$$\zeta' = (\psi'(f_1), \dots, \psi'(f_q))^T = \zeta + R\,\xi,$$

which shows $\overline{\zeta'} = \overline{\zeta} \in \mathrm{ext}^1_D(M, \mathcal{F}) = \ker_{\mathcal{F}}(R_2.)/(R\,\mathcal{F}^p)$, and thus, the previous construction does not depend on the choice of the pre-images a_i of y_i, for $i = 1, \dots, p$.

Finally, let us consider an extension $\xi' : 0 \longrightarrow \mathcal{F} \xrightarrow{\alpha'} E' \xrightarrow{\beta'} M \longrightarrow 0$ of $\mathcal{F}$ by M which belongs to the same equivalence class as the extension ξ defined by (20). Combining the commutative exact diagram (22) with the commutative exact diagram

$$\begin{array}{ccccccccc} 0 \longrightarrow & \mathcal{F} & \xrightarrow{\alpha} & E & \xrightarrow{\beta} & M & \longrightarrow 0 \\ & \| & & \downarrow \phi & & \| & \\ 0 \longrightarrow & \mathcal{F} & \xrightarrow{\alpha'} & E' & \xrightarrow{\beta'} & M & \longrightarrow 0, \end{array}$$

expressing the equivalence between ξ and ξ', we obtain the commutative exact diagram

$$\begin{array}{ccccccccc} D^{1\times r} & \xrightarrow{.R_2} & D^{1\times q} & \xrightarrow{.R} & D^{1\times p} & \xrightarrow{\pi} & M & \longrightarrow 0 \\ \downarrow & & \downarrow \psi & & \downarrow \phi\circ\gamma & & \| & \\ 0 & \longrightarrow & \mathcal{F} & \xrightarrow{\alpha'} & E' & \xrightarrow{\beta'} & M & \longrightarrow 0, \end{array}$$

which proves that $\overline{\zeta} \in \mathrm{ext}^1_D(M, \mathcal{F})$ only depends on the equivalence class of the extension ξ of $\mathcal{F}$ by M defined by (20), i.e., $[\xi] \in \mathrm{e}_D(M, \mathcal{F})$.

Hence, with the previous notations, we obtain the following result.

Proposition 4. *An equivalence class of extensions $[\xi]$ of $\mathcal{F}$ by M defines an element $\overline{\zeta} \in \mathrm{ext}^1_D(M, \mathcal{F})$.*

By Proposition 4, we obtain the following well-defined map:

$$\begin{array}{rcl} \Gamma : \mathrm{e}_D(M, \mathcal{F}) & \longrightarrow & \mathrm{ext}^1_D(M, \mathcal{F}) \\ [\xi] & \longmapsto & \overline{\zeta}. \end{array}$$

It is not totally obvious to prove that Γ is a morphism of abelian groups ([15, 27]).

Example 10. Let us consider the commutative polynomial ring $D = \mathbb{Q}\,[\partial, \delta]$ of differential time-delay operators and the following matrix of functional operators

$$R = \begin{pmatrix} \partial & -\partial\,\delta & -1 \\ 2\,\partial\,\delta & -\partial\,(1+\delta^2) & 0 \end{pmatrix} \in D^{2\times 3} \tag{23}$$

which describes the torsion of a flexible rod with a force applied on one end:

$$\left\{ \begin{array}{l} \dot{y}_1(t) - \dot{y}_2(t-h) - y_3(t) = 0, \\ 2\,\dot{y}_1(t-h) - \dot{y}_2(t) - \dot{y}_2(t-2\,h) = 0. \end{array} \right. \tag{24}$$

See [17] for more details. Moreover, let us consider the following matrix

$$R' = \begin{pmatrix} -2\,\delta & 1+\delta^2 & 0 \\ -\partial & \partial\,\delta & 1 \\ \partial\,\delta & -\partial & \delta \end{pmatrix} \in D^{3\times 3}, \tag{25}$$

which corresponds to the parametrizable (controllable) subsystem of (24). We can easily check that:

$$D^{1\times 2}\, R \subsetneq D^{1\times 3}\, R'.$$

Hence, if we denote by $M' = (D^{1\times 3}\, R')/(D^{1\times 2}\, R)$ and $M'' = D^{1\times 3}/(D^{1\times 3}\, R')$, we then have the following exact sequence

$$\xi : 0 \longrightarrow M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \longrightarrow 0,$$

where α is the canonical injection and β a projection satisfying the relation $\pi' = \beta \circ \pi$, where $\pi : D^{1\times 3} \longrightarrow M$ and $\pi' : D^{1\times 3} \longrightarrow M'$ are respectively the canonical projections onto M and M'. Let us show that the extension ξ of M' by M'' defines an element $\overline{\zeta} \in \operatorname{ext}^1_D(M'', M')$. M'' admits the following finite free resolution

$$0 \longrightarrow D \xrightarrow{.R'_2} D^{1\times 3} \xrightarrow{.R'} D^{1\times 3} \xrightarrow{\pi'} M'' \longrightarrow 0,$$

where $R'_2 = (\partial \quad -\delta \quad 1) \in D^{1\times 3}$. Proceeding similarly as in the proof of Proposition 4, we obtain the following commutative exact diagram

$$\begin{array}{ccccccccc}
0 \longrightarrow & D & \xrightarrow{.R'_2} & D^{1\times 3} & \xrightarrow{.R'} & D^{1\times 3} & \xrightarrow{\pi'} & M'' & \longrightarrow 0 \\
& \downarrow & & \downarrow \psi & & \downarrow \pi & & \| & \\
& 0 & \longrightarrow & M' & \xrightarrow{\alpha} & M & \xrightarrow{\beta} & M'' & \longrightarrow 0,
\end{array}$$

where $\psi : D^{1\times 3} \longrightarrow M'$ is defined by

$$\begin{cases} \psi(f_1) = -2\,\delta\, y_1 + (1+\delta^2)\, y_2, \\ \psi(f_2) = 0, \\ \psi(f_3) = 0, \end{cases}$$

and $\{f_j\}_{1\leq j\leq 3}$ denotes the standard basis of $D^{1\times 3}$ and $\{y_i = \pi(f_i)\}_{1\leq i\leq 3}$ is a set of generators of M. Finally, we obtain

$$(\psi \circ (.R'_2))(1) = \psi(R'_2) = \partial\, \psi(f_1) - \delta\, \psi(f_2) + \psi(f_3) = \partial\,(-2\,\delta\, y_1 + (1+\delta^2)\, y_2) = 0,$$

which shows that the element of M'^3 defined by

$$\zeta = (\psi(f_1), \psi(f_2), \psi(f_3))^T = (-2\,\delta\, y_1 + (1+\delta^2)\, y_2,\, 0,\, 0)^T$$

satisfies $R'_2\, \zeta = 0$, and thus, defines a non-zero element $\overline{\zeta} \in \operatorname{ext}^1_D(M'', M') = \ker_{M'}(R'_2.)/(R'\, M'^3)$.

Let us prove that we have the following identities:

$$\Gamma \circ \Pi = \mathrm{id}_{\operatorname{ext}^1_D(M,\mathcal{F})}, \quad \Pi \circ \Gamma = \mathrm{id}_{\mathrm{e}_D(M,\mathcal{F})}.$$

Let us first prove that $\Gamma \circ \Pi = \mathrm{id}_{\operatorname{ext}^1_D(M,\mathcal{F})}$. Let us consider the equivalence class $\overline{\zeta} \in \operatorname{ext}^1_D(M, \mathcal{F})$ and $\Pi(\overline{\zeta}) = [\psi(\zeta)_\star(\xi)]$ of the extension (16), where α, β and γ are defined by (14). Let us compute $\Gamma([\psi(\zeta)_\star(\xi)])$. Considering (21) and proceeding similarly as it was done after Diagram (21), we obtain that γ can be chosen as follows:

$$\begin{array}{rcl} \gamma : D^{1\times p} & \longrightarrow & E \\ e_i & \longmapsto & \sigma((e_i, 0)), \quad i = 1, \ldots, p. \end{array}$$

Then, for $j = 1, \ldots, q$, we have $\gamma(f_j\, R) = \sigma((f_j\, R \quad 0)) \in \operatorname{im} \alpha$. Using the fact that, for $j = 1, \ldots, q$, $f_j\,(R \quad -\zeta) \in (D^{1\times q}\,(R \quad -\zeta))$, we obtain that:

$$\sigma((f_j\,(R \quad -\zeta))) = 0 \;\Rightarrow\; \sigma((f_j\, R,\, 0)) = \sigma((0,\, f_j\, \zeta)) = \sigma((0,\, \zeta_j)).$$

Therefore, for $j = 1, \ldots, q$, we get $\gamma(f_j\, R) = \sigma((0, \zeta_j)) = \alpha(\zeta_j)$, i.e., we can take $\psi : D^{1\times q} \longrightarrow \mathcal{F}$ defined by $\psi(f_j) = \zeta_j$, for $j = 1, \ldots, q$, which proves that:

$$(\Gamma \circ \Pi)(\overline{\zeta}) = \Gamma([\psi(\zeta)_\star(\xi)]) = \overline{\zeta}.$$

Conversely, let us consider a class of equivalence $[\xi]$ of extensions of the form (20). Repeating the same procedure as the one explained after Example 9, we obtain the commutative exact diagram (22) and $\Gamma([\xi]) = \overline{\zeta} \in \mathrm{ext}^1_D(M, \mathcal{F})$, where $\overline{\zeta}$ denotes the residue class of the element $\zeta = (\psi(f_1), \ldots, \psi(f_q))^T \in \mathcal{F}^q$ satisfying $R_2\, \zeta = 0$. Defining the left D-module $M_2 = D^{1\times q}/(D^{1\times r}\, R_2)$ and denoting by $\kappa : D^{1\times q} \longrightarrow M_2$ the standard projection, we get the commutative exact diagram

$$\begin{array}{ccccccccc} 0 \longrightarrow & M_2 & \xrightarrow{d_R} & D^{1\times p} & \xrightarrow{\pi} & M & \longrightarrow 0 \\ & \downarrow \overline{\psi} & & \downarrow \gamma & & \| & \\ 0 \longrightarrow & \mathcal{F} & \xrightarrow{\alpha} & E & \xrightarrow{\beta} & M & \longrightarrow 0, \end{array}$$

where $\overline{\psi} : M_2 \longrightarrow \mathcal{F}$ is defined by:

$$\forall\, \mu \in D^{1\times q}, \quad \overline{\psi}(\kappa(\mu)) = \psi(\mu) = \mu\, \zeta.$$

Let us prove E is the pushout of the D-morphisms $d_R : M_2 \longrightarrow D^{1\times p}$ and $\overline{\psi} : M_2 \longrightarrow \mathcal{F}$.

Let us consider $e \in E$ and using the fact that $\beta(e) \in M$, there exists $\lambda \in D^{1\times p}$ such that $\beta(e) = \pi(\lambda)$. If we denote by $e' = \gamma(\lambda) \in E$, using the fact that $\beta \circ \gamma = \pi$, we obtain that:

$$\beta(e) - \beta(e') = \beta(e) - (\beta \circ \gamma)(\lambda) = \beta(e) - \pi(\lambda) = 0.$$

Hence, we get $\beta(e - e') = 0$, i.e., $e - e' \in \ker \beta = \mathrm{im}\, \alpha$. Using the fact that α is an injective D-morphism, there exists a unique $f \in \mathcal{F}$ such that $e = e' + \alpha(f) = \gamma(\lambda) + \alpha(f)$. Therefore, we obtain the surjective D-morphism ε defined by:

$$\begin{array}{rcl} \varepsilon : D^{1\times p} \oplus \mathcal{F} & \longrightarrow & E \\ (\lambda, f) & \longmapsto & \gamma(\lambda) + \alpha(f). \end{array}$$

Let us compute $\ker \varepsilon = \{(\lambda, f) \in D^{1\times p} \oplus \mathcal{F} \mid \gamma(\lambda) = -\alpha(f)\}$. If $(\lambda, f) \in \ker \varepsilon$, then we have

$$\beta(\alpha(f)) = 0 \Rightarrow \beta(\gamma(\lambda)) = \pi(\lambda) = 0,$$

and thus, there exists $z \in M_2$ such that $\lambda = d_R(z)$. Using the relation $\gamma \circ d_R = \alpha \circ \overline{\psi}$, we obtain:

$$\gamma(\lambda) = \gamma(d_R(z)) = \alpha(\overline{\psi}(z)).$$

Using the fact that $\gamma(\lambda) = -\alpha(f)$, we then obtain $\alpha(\overline{\psi}(z) + f) = 0$ and, using the fact that α is injective D-morphism, we get $f = -\overline{\psi}(z)$, which shows that:

$$\ker \varepsilon = \{(d_R(z), -\overline{\psi}(z)) \in D^{1\times p} \oplus \mathcal{F} \mid z \in M_2\}.$$

Hence, E is the pushout of the D-morphisms $d_R : M_2 \longrightarrow D^{1\times p}$ and $\overline{\psi} : M_2 \longrightarrow \mathcal{F}$, which finally proves that $(\Pi \circ \Gamma)([\xi]) = \Pi(\overline{\zeta}) = [\xi]$ and Theorem 2.

For more details on Baer's extensions, we refer the reader to [15, 27].

The following result will play an important role in what follows ([15, 27]).

Proposition 5. *Every extension of $\mathcal{F}$ by M splits iff $\mathrm{ext}^1_D(M, \mathcal{F}) = 0$, i.e., iff $\mathrm{e}_D(M, \mathcal{F}) = 0$.*

Proof. Applying the contravariant left exact functor $\hom_D(\,\cdot\,,\mathcal{F})$ to the short exact sequence (20) and using $\mathrm{ext}^1_D(M,\mathcal{F})=0$, we obtain the following long exact sequence:

$$0 \longleftarrow \mathrm{end}_D(\mathcal{F}) \xleftarrow{\alpha^\star} \hom_D(E,\mathcal{F}) \xleftarrow{\beta^\star} \hom_D(M,\mathcal{F}) \longleftarrow 0. \tag{26}$$

Using the fact that $\alpha^\star$ is surjective, there exists $\chi \in \hom_D(E,\mathcal{F})$ such that $\mathrm{id}_{\mathcal{F}} = \alpha^\star(\chi) = \chi \circ \alpha$, which shows that (15) splits (see the comment after Definition 3).

Conversely, a standard result of homological algebra states that the functor $\hom_D(\,\cdot\,,\mathcal{F})$ transforms split exact sequences of left D-modules into split exact sequences of abelian groups (see, e.g., [27]). Hence, if (16) is a split exact sequence, applying the functor $\hom_D(\,\cdot\,,\mathcal{F})$ to (15), we obtain the split exact sequence (26), and thus, $\mathrm{ext}^1_D(M,\mathcal{F})=0$, which proves the result.

Finally, the result follows from Theorem 2. □

If $\mathcal{F}$ is an injective left D-module, by 1 of Definition 2, we then have $\mathrm{ext}^1_D(M,\mathcal{F})=0$ which, by Proposition 5, shows that every extension of $\mathcal{F}$ by M splits.

5 Computing extensions of finitely presented left D-modules

The purpose of this section is to use the results of Section 4 in order to constructively characterize the first extension module of a finitely presented left D-module $M = D^{1\times p}/(D^{1\times q}\,R)$ with value in a finitely presented left D-module $N = D^{1\times s}/(D^{1\times t}\,S)$.

In order to do that, we shall first use a finite free resolution of M to compute $\mathrm{ext}^1_D(M,N)$ and then use Theorem 2 and the morphism Π to construct elements of $\mathrm{e}_D(M,N)$.

We shall need the following results to compute elements of $\mathrm{ext}^1_D(M,N)$. For more details, see [8].

Lemma 2. *Let M and M' be two left D-modules respectively defined by the finite presentations:*

$$D^{1\times q} \xrightarrow{.R} D^{1\times p} \xrightarrow{\pi} M \longrightarrow 0,$$
$$D^{1\times q'} \xrightarrow{.R'} D^{1\times p'} \xrightarrow{\pi'} M' \longrightarrow 0.$$

Let $f : M \longrightarrow M'$ be a D-morphism defined by the matrices $P \in D^{p\times p'}$ and $Q \in D^{q\times q'}$ satisfying $R\,P = Q\,R'$, i.e., f is defined by the following commutative exact diagram

$$\begin{array}{ccccccc} D^{1\times q} & \xrightarrow{.R} & D^{1\times p} & \xrightarrow{\pi} & M & \longrightarrow 0 \\ \downarrow .Q & & \downarrow .P & & \downarrow f & \\ D^{1\times q'} & \xrightarrow{.R'} & D^{1\times p'} & \xrightarrow{\pi'} & M' & \longrightarrow 0, \end{array}$$

where, for all $\lambda \in D^{1\times p}$, $f(\pi(\lambda)) = \pi'(\lambda\,P)$.

Then, the kernel, image and cokernel of the D-morphism f are defined by:

1. *If we denote by $S \in D^{r\times p}$ a matrix satisfying*

$$\ker_D\left(.\begin{pmatrix} P \\ R' \end{pmatrix}\right) = D^{1\times r}\,(S \;\; -T),$$

then we have:

$$\ker f = \{\pi(\lambda) \mid \lambda \in D^{1\times p} : \exists\, \mu \in D^{1\times q'},\ \lambda\,P = \mu\,R'\} = (D^{1\times r}\,S)/(D^{1\times q}\,R).$$

Moreover, if we denote by $S_2 \in D^{s\times r}$ *(resp.,* $L \in D^{q\times r}$*) a matrix satisfying* $\ker_D(.S) = D^{1\times s}\, S_2$ *(resp.,* $R = L\, S$*), we then have:*

$$\ker f = (D^{1\times r}\, S)/(D^{1\times q}\, R) \cong D^{1\times r} \Big/ \left(D^{1\times (q+s)} \begin{pmatrix} L \\ S_2 \end{pmatrix} \right).$$

2. $\operatorname{im} f = \left(D^{1\times (p+q')} \begin{pmatrix} P \\ R' \end{pmatrix} \right) / (D^{1\times q'}\, R')$.

3. $\operatorname{coker} f = D^{1\times p'} \Big/ \left(D^{1\times (p+q')} \begin{pmatrix} P \\ R' \end{pmatrix} \right)$.

Let us show how we can compute elements of the abelian group $\operatorname{ext}^1_D(M, N)$. We first consider a finite free resolution (10) of the left D-module M. Applying the contravariant left exact functor $\hom_D(\,\cdot\,, N)$ to the truncated free resolution (11), we get the following complex of abelian groups:

$$\ldots \xleftarrow{R_3.} N^r \xleftarrow{R_2.} N^q \xleftarrow{R.} N^p \longleftarrow 0.$$

Applying the covariant right exact functor $D^m \otimes_D \cdot$ to the following finite free resolution of the left D-module N

$$D^{1\times t} \xrightarrow{.S} D^{1\times s} \xrightarrow{\delta} N \longrightarrow 0, \tag{27}$$

and using the fact that D^m is a free, and thus, a *flat* right D-module, we obtain the exact sequence:

$$D^{m\times t} \xrightarrow{.S} D^{m\times s} \xrightarrow{\mathrm{id}_m \otimes \delta} N^m \longrightarrow 0.$$

See, e.g., [4, 15, 27] for more details. Then, we can easily check that we have the commutative diagram:

$$\begin{array}{ccccc}
0 & & 0 & & 0 \\
\uparrow & & \uparrow & & \uparrow \\
N^r & \xleftarrow{R_2.} & N^q & \xleftarrow{R.} & N^p \\
\uparrow \mathrm{id}_r \otimes \delta & & \uparrow \mathrm{id}_q \otimes \delta & & \uparrow \mathrm{id}_p \otimes \delta \\
D^{r\times s} & \xleftarrow{R_2.} & D^{q\times s} & \xleftarrow{R.} & D^{p\times s} \\
\uparrow .S & & \uparrow .S & & \uparrow .S \\
D^{r\times t} & \xleftarrow{R_2.} & D^{q\times t} & \xleftarrow{R.} & D^{p\times t}.
\end{array}$$

Using Lemma 2, we obtain the following lemma which characterizes the abelian group $\operatorname{ext}^1_D(M, N)$.

Lemma 3. *With the previous notations, we have the following abelian groups:*

$$\ker_N(R_2.) \triangleq \{\zeta \in N^q \mid R_2\, \zeta = 0\} = \{(\mathrm{id}_q \otimes \delta)(A) \mid A \in D^{q\times s} :\ \exists\, B \in D^{r\times t},\ R_2\, A = B\, S\}. \tag{28}$$

$$\operatorname{im}_N(R.) \triangleq R\, N^p = (R\, D^{p\times s} + D^{q\times t}\, S)/(D^{q\times t}\, S). \tag{29}$$

If we define the abelian group $\Omega = \{A \in D^{q\times s} \mid \exists\, B \in D^{r\times t},\ R_2\, A = B\, S\}$*, then the abelian group*

$$\operatorname{ext}^1_D(M, N) = \ker_N(R_2.)/\operatorname{im}_N(R.)$$

satisfies:

$$\operatorname{ext}^1_D(M, N) \cong \Omega/(R\, D^{p\times s} + D^{q\times t}\, S). \tag{30}$$

If $\ker_D(.R) = 0$, i.e., $R_2 = 0$, then we note that $\Omega = D^{q \times s}$.

From Lemma 3, we obtain that $\mathrm{ext}_D^1(M, N) = 0$ iff, for every matrix $A \in D^{q \times s}$ satisfying the relation $R_2\, A = B\, S$ for a certain $B \in D^{r \times t}$, there exist $U \in D^{p \times s}$ and $V \in D^{q \times t}$ such that:

$$A = R\, U + V\, S.$$

Remark 2. If D is a commutative polynomial ring with coefficients in a computable field k (e.g., $k = \mathbb{Q}$, $\mathbb{F}_p$) and $U \in D^{a \times b}$, $V \in D^{b \times c}$ and $W \in D^{c \times d}$ are three matrices, using the standard relation

$$U\, V\, W = \mathrm{row}(V)\, (U^T \otimes W),$$

where $\mathrm{row}(V)$ denotes the row vector formed by stacking the rows of V the ones after the others and $\otimes$ the *Kronecker product*, namely, $A \otimes B = (a_{ij}\, B)$, we obtain that the relation $R_2\, A = B\, S$ is equivalent to:

$$\mathrm{row}(A)\, (R_2^T \otimes I_s) = \mathrm{row}(B)\, (I_r \otimes S) \;\Leftrightarrow\; (\mathrm{row}(A) \;\; -\mathrm{row}(B)) \begin{pmatrix} R_2^T \otimes I_s \\ I_r \otimes S \end{pmatrix} = 0.$$

Moreover, an element $R\, X + Y\, S \in (R\, D^{p \times s} + D^{q \times t}\, S)$ is equivalently defined by:

$$(\mathrm{row}(X) \quad \mathrm{row}(Y)) \begin{pmatrix} R^T \otimes I_s \\ I_q \otimes S \end{pmatrix}.$$

Hence, using Gröbner bases computation, we can explicitly describe the D-module defined by

$$\ker_D \left(. \begin{pmatrix} R_2^T \otimes I_s \\ I_r \otimes S \end{pmatrix} \right) \Big/ \left(D^{1 \times (p\, s + q\, t)} \begin{pmatrix} R^T \otimes I_s \\ I_q \otimes S \end{pmatrix} \right)$$

by means of generators and relations. Hence, we can compute the D-module $\mathrm{ext}_D^1(M, N)$. See [2, 8] for more details. We refer to the package *homalg* ([2]) for an implementation of the previous algorithm.

If D is a non-commutative ring, then $\mathrm{ext}_D^1(M, N)$ is an abelian group and not a left D-module. It is generally a k-vector space, where k denotes the field of constants of D. If M and N are two finite-dimensional k-vector spaces or two holonomic left modules over the Weyl algebras, then we can compute a basis of the finite-dimensional k-vector space $\mathrm{ext}_D^1(M, N)$. However, $\mathrm{ext}_D^1(M, N)$ is generally an infinite-dimensional k-vector space. If D is a non-commutative polynomial ring over which Gröbner bases exist (e.g., the Weyl algebras, some classes of Ore algebras [5]), then we can compute the matrices $A \in D^{q \times s}$ with a fixed order in the functional operators and a fixed degree (resp., fixed degrees) in the polynomial (resp., rational) coefficients which satisfy $R_2\, A \in D^{r \times t}\, S$. See [8, 25] for more details.

Example 11. Let us consider the commutative polynomial ring $D = \mathbb{Q}\,[\partial, \delta]$ of differential time-delay operators and the following two matrices:

$$R = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 + \delta^2 & -2\, \partial\, \delta \end{pmatrix}, \quad S = \begin{pmatrix} \delta^2 & 1 \\ 1 & 1 \end{pmatrix}. \tag{31}$$

Let us consider the two D-modules defined by $M = D^{1 \times 3}/(D^{1 \times 2}\, R)$ and $N = D^{1 \times 2}/(D^{1 \times 2}\, S)$. We can check that $\ker_D(.R) = 0$, i.e., $R_2 = 0$, a fact implying that $\Omega = D^{2 \times 2}$ and we get:

$$\mathrm{ext}_D^1(M, N) \cong D^{2 \times 2}/(R\, D^{3 \times 2} + D^{2 \times 2}\, S).$$

Using the Kronecker product, we obtain that:

$$\mathrm{ext}_D^1(M, N) \cong D^{1 \times 4} \Big/ \left(D^{1 \times 10} \begin{pmatrix} R^T \otimes I_2 \\ I_2 \otimes S \end{pmatrix} \right).$$

We can check that the last matrix admits a left-inverse over D, a fact showing that $\mathrm{ext}^1_D(M,N)=0$. In particular, the matrices defined by

$$X=-\frac{1}{2}\begin{pmatrix}0 & 1\\ 1 & 0\\ 0 & 0\end{pmatrix},\quad Y=\frac{1}{2}\begin{pmatrix}0 & 1\\ 1 & 1\end{pmatrix}$$

satisfy the relation $R\,X+Y\,S=I_2$.

Example 12. Let us consider the commutative polynomial ring $D=\mathbb{Q}(\alpha)\,[\partial,\delta]$ of differential time-delay operators, where $\alpha\in\mathbb{R}$, and the following two matrices of functional operators:

$$R=\begin{pmatrix}1 & 1 & 0\\ 0 & 1+\delta^2 & -\alpha\,\partial\,\delta\end{pmatrix},\quad S=\begin{pmatrix}\partial & -\partial\\ \partial\delta^2 & -\partial\end{pmatrix}. \tag{32}$$

Let us consider the D-modules defined by $M=D^{1\times 3}/(D^{1\times 2}\,R)$ and $N=D^{1\times 2}/(D^{1\times 2}\,S)$. As in the previous example, we can easily check that $R_2=0$, a fact implying that $\Omega=D^{2\times 2}$ and:

$$\mathrm{ext}^1_D(M,N)\cong D^{2\times 2}/(R\,D^{3\times 2}+D^{2\times 2}\,S).$$

Using the Kronecker product, we obtain that:

$$\mathrm{ext}^1_D(M,N)\cong D^{1\times 4}/\left(D^{1\times 10}\begin{pmatrix}R^T\otimes I_2\\ I_2\otimes S\end{pmatrix}\right).$$

Let us denote by L the matrix appearing in the right-hand side of the previous isomorphism, $P=D^{1\times 4}/(D^{1\times 10}\,L)$ the D-module finitely presented by L and $\epsilon: D^{1\times 4}\longrightarrow P$ the canonical projection onto P. Denoting by $v_i=\epsilon(g_i)$ the residue class of the i^{th} vector of the standard basis $\{g_i\}_{1\le i\le 4}$ of $D^{1\times 4}$ in P, we obtain that the generators $\{v_i\}_{1\le i\le 4}$ satisfy the relations:

$$\begin{cases}v_1=0,\\ v_2=0,\\ (1+\delta^2)\,v_i=0,\quad i=3,\,4,\\ \partial\,v_i=0,\quad i=3,\,4.\end{cases}$$

Hence, the D-module $P=D^{1\times 4}/(D^{1\times 10}\,L)$ is generated by the elements $v_3=\epsilon((0,0,1,0))$ and $v_4=\epsilon((0,0,0,1))$. Transforming back the row vectors g_3 and g_4 into 2×2 matrices, we obtain that the D-module $\mathrm{ext}^1_D(M,N)$ is generated by the residue classes $\overline{A_1}$ and $\overline{A_2}$ of the two matrices

$$A_1=\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix},\quad A_2=\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}, \tag{33}$$

in the D-module $D^{2\times 2}/(R\,D^{3\times 2}+D^{2\times 2}\,S)$. They satisfy the following relations:

$$(1+\delta^2)\,\overline{A_i}=0,\quad \partial\,\overline{A_i}=0,\quad i=1,2.$$

By Proposition 3, we know that an extension of N by M can be defined by (16), where the left D-module E is defined by (15). Using the fact that N is a left D-module presented by the matrix S, we can precisely characterize the left D-module E.

Using (27), we obtain the exact sequence

$$D^{1\times t}\xrightarrow{.(0,\,S)}D^{1\times p}\oplus D^{1\times s}\xrightarrow{\mathrm{id}_p\oplus\delta}D^{1\times p}\oplus N\longrightarrow 0,$$

and thus, we get the following commutative exact diagram

$$\begin{array}{ccccc}
0 & & 0 & & \\
\uparrow & & \uparrow & & \\
D^{1\times q} & \xrightarrow{.(R \quad -\zeta)} & D^{1\times p} \oplus N & \xrightarrow{\sigma} & E \longrightarrow 0, \\
\| & & \uparrow \mathrm{id}_p \oplus \delta & & \\
D^{1\times q} & \xrightarrow{.(R \quad -A)} & D^{1\times p} \oplus D^{1\times s} & & \\
\uparrow & & \uparrow .(0,\ S) & & \\
0 & & D^{1\times t} & &
\end{array}$$

where $\zeta = (\mathrm{id}_q \otimes \delta)(A) \in N^q$ satisfies $R_2\, \zeta = 0$ and $A \in D^{q\times s}$ is any matrix satisfying $R_2\, A \in D^{r\times t}\, S$. By 3 of Lemma 2, we then obtain:

$$E = \operatorname{coker} \sigma = (D^{1\times p} \oplus D^{1\times s})/(D^{1\times q}\, (R \quad -A) + D^{1\times t}\, (0 \quad S)).$$

We can now state our first main result.

Theorem 3. *Let $R \in D^{q\times p}$ and $S \in D^{t\times s}$ be two matrices with entries in D and $M = D^{1\times p}/(D^{1\times q}\, R)$ and $N = D^{1\times s}/(D^{1\times t}\, S)$ two finitely presented left D-modules. Moreover, let us denote by $R_2 \in D^{r\times q}$ a matrix satisfying $\ker_D(.R) = D^{1\times r}\, R_2$. Then, any extension of N by M*

$$0 \longrightarrow N \xrightarrow{\alpha} E \xrightarrow{\beta} M \longrightarrow 0 \tag{34}$$

is defined by the left D-module E finitely presented by

$$D^{1\times(q+t)} \xrightarrow{.Q} D^{1\times(p+s)} \xrightarrow{\varrho} E \longrightarrow 0,$$

where the matrix $Q \in D^{(q+t)\times(p+s)}$ is defined by

$$Q = \begin{pmatrix} R & -T \\ 0 & S \end{pmatrix},$$

and T is an element of the abelian group $\Omega = \{A \in D^{q\times s} \mid \exists\, B \in D^{r\times t} : R_2\, A = B\, S\}$.

Finally, the equivalence classes of extensions of N by M only depend on the residue class of $A \in \Omega$ in the abelian group:

$$\mathrm{ext}^1_D(M, N) \cong \Omega/(R\, D^{p\times s} + D^{q\times t}\, S).$$

Remark 3. Using Theorem 3, we easily obtain the following commutative exact diagram:

$$\begin{array}{ccccccccc}
0 \longrightarrow & D^{1\times t} & \xrightarrow{.(0 \quad I_t)} & D^{1\times(q+t)} & \xrightarrow{.(I_q \quad 0)^T} & D^{1\times q} & \longrightarrow 0 \\
& \downarrow .S & & \downarrow .Q & & \downarrow .R & \\
0 \longrightarrow & D^{1\times s} & \xrightarrow{.(0 \quad I_s)} & D^{1\times(p+s)} & \xrightarrow{.(I_p \quad 0)^T} & D^{1\times p} & \longrightarrow 0 \\
& \downarrow \delta & & \downarrow \varrho & & \downarrow \pi & \\
0 \longrightarrow & N & \xrightarrow{\alpha} & E & \xrightarrow{\beta} & M & \longrightarrow 0. \\
& \downarrow & & \downarrow & & \downarrow & \\
& 0 & & 0 & & 0 &
\end{array}$$

In particular, for all $\lambda_2 \in D^{1\times s}$, we obtain that the D-morphism α is defined by

$$\begin{array}{ccc}
N & \xrightarrow{\alpha} & E \\
\delta(\lambda_2) & \longmapsto & \varrho(\lambda_2\, (0 \quad I_s)) = \varrho((0, \lambda_2)),
\end{array}$$

and, for all $\lambda_1 \in D^{1\times p}$, $\lambda_2 \in D^{1\times s}$, the D-morphism β is defined by:

$$\begin{array}{ccc}
E & \xrightarrow{\beta} & M \\
\varrho((\lambda_1, \lambda_2)) & \longmapsto & \pi((\lambda_1, \lambda_2)\, (I_p \quad 0)^T) = \pi(\lambda_1).
\end{array}$$

Remark 4. Using (28) and (29), we obtain that a matrix $T \in D^{q \times s}$ of the form

$$T = R\,U + V\,S,$$

where $U \in D^{p \times s}$ and $V \in D^{q \times t}$, satisfies that the residue class of $(\mathrm{id}_q \otimes \delta)(T)$ in $\ker_N(R_2.)/(R\,N^p)$ is zero, i.e., defines the zero element in $\mathrm{ext}^1_D(M, N)$. By Proposition 5, the corresponding extension is then equivalent to the split one. This result can also be easily checked as the matrix defined by

$$Q = \begin{pmatrix} R & -R\,U - V\,S \\ 0 & S \end{pmatrix}$$

satisfies the relation

$$W\,Q = \begin{pmatrix} R & 0 \\ 0 & S \end{pmatrix} Z,$$

with the notations:

$$W = \begin{pmatrix} I_q & V \\ 0 & I_t \end{pmatrix}, \quad Z = \begin{pmatrix} I_p & -U \\ 0 & I_s \end{pmatrix}.$$

Using the fact that the matrices Z and W are invertible over D, we obtain that the D-morphism $\phi : M \oplus N \longrightarrow E$ defined by

$$\forall\, \lambda_1 \in D^{1 \times p}, \quad \forall\, \lambda_2 \in D^{1 \times s}, \quad \phi((\pi(\lambda_1), \delta(\lambda_2))) = \varrho((\lambda_1, \lambda_2)\,Z) = \varrho((\lambda_1, -\lambda_1\,U + \lambda_2)),$$

is an isomorphism, which shows that $E \cong M \oplus N$.

Let us consider the following split exact sequence:

$$\begin{array}{ccccccc} 0 \longrightarrow & N & \xrightarrow{i_2} & M \oplus N & \xrightarrow{p_1} & M & \longrightarrow 0. \\ & \delta(\lambda_2) & \longmapsto & (0, \delta(\lambda_2)) & & & \\ & & & (\pi(\lambda_1), \delta(\lambda_2)) & \longmapsto & \pi(\lambda_1) & \end{array}$$

Using the commutative exact diagram shown in Remark 3, we obtain that $\alpha \circ \delta = \varrho \circ (.(0 \quad I_s))$, and thus, for all $\lambda_2 \in D^{1 \times s}$, we have

$$(\phi \circ i_2)(\delta(\lambda_2)) = \phi((0, \delta(\lambda_2))) = \varrho((0, \lambda_2)) = \alpha(\delta(\lambda_2)),$$

which proves that $\phi \circ i_2 = \alpha$. Finally, using the relation $\beta \circ \varrho = \pi \circ (.(I_p \quad 0)^T)$ obtained from the commutative exact diagram given in Remark 3, for all $\lambda_1 \in D^{1 \times p}$, and $\lambda_2 \in D^{1 \times s}$, we get

$$(\beta \circ \phi)((\pi(\lambda_1), \delta(\lambda_2))) = \beta(\varrho((\lambda_1, -\lambda_1\,U + \lambda_2))) = \pi(\lambda_1) = p_1((\pi(\lambda_1), \delta(\lambda_2))),$$

which proves $\beta \circ \phi = p_1$ and (34) is equivalent to the previous split exact sequence, i.e., belongs to the trivial equivalence class of extensions of N by M.

Example 13. If we apply Theorem 3 to the D-modules M and N defined in Example 11 and use Remark 4, we obtain that the only equivalence class of extensions of N by M is the trivial one (split one) defined by the D-module E

$$E = D^{1 \times 5} / \left(D^{1 \times 4} \begin{pmatrix} R & 0 \\ 0 & S \end{pmatrix} \right), \tag{35}$$

where R and S are given by (31).

Example 14. Let us consider again Example 12. By Theorem 3, we obtain that there exist two non-trivial equivalence classes of extensions of N by M respectively defined by the D-modules

$$E_1 = D^{1\times 5} \Big/ \left(D^{1\times 4} \begin{pmatrix} R & -A_1 \\ 0 & S \end{pmatrix} \right), \tag{36}$$

$$E_2 = D^{1\times 5} \Big/ \left(D^{1\times 4} \begin{pmatrix} R & -A_2 \\ 0 & S \end{pmatrix} \right), \tag{37}$$

where the matrices R and S are given by (32) and the matrices A_1 and A_2 by (33). Finally, the trivial extension of N by M (split extension) is defined by the D-module:

$$E_0 = D^{1\times 5} \Big/ \left(D^{1\times 4} \begin{pmatrix} R & 0 \\ 0 & S \end{pmatrix} \right). \tag{38}$$

Applying the contravariant left exact functor $\hom_D(\,\cdot\,, \mathcal{F})$ to the commutative exact diagram defined in Remark 3, we obtain the following commutative exact diagram of abelian groups:

$$\begin{array}{ccccccccc}
0 \longleftarrow & \mathcal{F}^t & \xleftarrow{(0 \quad I_t).} & \mathcal{F}^{q+t} & \xleftarrow{(I_t \quad 0)^T.} & \mathcal{F}^q & \longleftarrow 0 \\
& \uparrow S. & & \uparrow Q. & & \uparrow R. & \\
0 \longleftarrow & \mathcal{F}^s & \xleftarrow{(0 \quad I_s).} & \mathcal{F}^{p+s} & \xleftarrow{(I_p \quad 0)^T.} & \mathcal{F}^p & \longleftarrow 0 \\
& \uparrow \delta^\star & & \uparrow \varrho^\star & & \uparrow \pi^\star & \\
& \ker_{\mathcal{F}}(S.) & \xleftarrow{\alpha^\star} & \ker_{\mathcal{F}}(Q.) & \xleftarrow{\beta^\star} & \ker_{\mathcal{F}}(R.) & \longleftarrow 0. \\
& \uparrow & & \uparrow & & \uparrow & \\
& 0 & & 0 & & 0 &
\end{array}$$

Using the previous commutative exact diagram, we obtain the following corollary of Theorem 3.

Corollary 1. *Using the notations of Theorem 3 and denoting by $\mathcal{F}$ a left D-module, we obtain:*

1. *We have the following exact sequence*

$$\ker_{\mathcal{F}}(S.) \xleftarrow{\alpha^\star} \ker_{\mathcal{F}}(Q.) \xleftarrow{\beta^\star} \ker_{\mathcal{F}}(R.) \longleftarrow 0,$$

where the D-morphism $\beta^\star$ is defined by

$$\forall\, \eta \in \ker_{\mathcal{F}}(R.),\ \beta^\star(\eta) = \begin{pmatrix} I_p \\ 0 \end{pmatrix} \eta = \begin{pmatrix} \eta \\ 0 \end{pmatrix},$$

and the D-morphism $\alpha^\star$ is defined by:

$$\forall\, \zeta = \begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix} \in \ker_{\mathcal{F}}(Q.), \quad \zeta_1 \in \mathcal{F}^p, \quad \zeta_2 \in \mathcal{F}^s, \quad \alpha^\star(\zeta) = (0 \quad I_s)\,\zeta = \zeta_2.$$

2. *If $\mathcal{F}$ is an injective left D-module, then we have the following exact sequence:*

$$0 \longleftarrow \ker_{\mathcal{F}}(S.) \xleftarrow{\alpha^\star} \ker_{\mathcal{F}}(Q.) \xleftarrow{\beta^\star} \ker_{\mathcal{F}}(R.) \longleftarrow 0. \tag{39}$$

3. *If $\mathcal{F}$ is an injective cogenerator left D-module, then (39) is exact iff (34) is exact.*

Remark 5. Using the fact that $T \in \Omega$, i.e., there exists $L \in D^{r \times t}$ satisfying $R_2\, T = L\, S$, we have:

$$\left\{ \begin{array}{l} R\,\zeta_1 - T\,\zeta_2 = 0, \\ S\,\zeta_2 = 0, \end{array} \right. \quad \Rightarrow \left\{ \begin{array}{l} R_2\,R\,\zeta_1 - R_2\,T\,\zeta_2 = 0, \\ S\,\zeta_2 = 0, \end{array} \right. \quad \Rightarrow \left\{ \begin{array}{l} L\,S\,\zeta_2 = 0, \\ S\,\zeta_2 = 0, \end{array} \right. \quad \Rightarrow S\,\zeta_2 = 0.$$

Hence, eliminating ζ_1 from the system $Q\,(\zeta_1^T\ \zeta_2^T)^T = 0$, we exactly obtain that $\zeta_2 \in \mathcal{F}^s$ satisfies $S\,\zeta_2$. This last result explains why T must belong to Ω as, otherwise, ζ_2 satisfies a subsystem of $S\,\zeta_2 = 0$.

In what follows, we shall need the following technical result proved in [8].

Lemma 4. *Let $R \in D^{q \times p}$ and $R' \in D^{q' \times p}$ be two matrices satisfying $(D^{1 \times q}\, R) \subseteq (D^{1 \times q'}\, R')$. If we denote by $R'' \in D^{q \times q'}$ (resp., $R'_2 \in D^{r' \times q'}$) satisfying $R = R''\, R'$ (resp., $\ker_D(.R') = D^{1 \times r'}\, R'_2$), then we have the following isomorphism:*

$$(D^{1 \times q'}\, R')/(D^{1 \times q}\, R) \cong D^{1 \times q'} \Big/ \left(D^{1 \times (q+r')} \left(\begin{array}{c} R'' \\ R'_2 \end{array} \right) \right).$$

Finally, the next theorem gives an explicit description of the finitely presented left D-module defining the sum of two extensions. Let us state the second main result of this paper.

Theorem 4. *Let $R \in D^{q \times p}$ and $S \in D^{t \times s}$ be two matrices with entries in D and $M = D^{1 \times p}/(D^{1 \times q}\, R)$ and $N = D^{1 \times s}/(D^{1 \times t}\, S)$ two finitely presented left D-modules. Let us denote by $R_2 \in D^{r \times q}$ a matrix satisfying $\ker_D(.R) = D^{1 \times r}\, R_2$ and let us consider two extensions ξ_1 and ξ_2 of N by M,*

$$\xi_i : 0 \longrightarrow N \xrightarrow{\alpha_i} E_i \xrightarrow{\beta_i} M \longrightarrow 0, \quad i = 1, 2,$$

where the left D-module E_i is finitely presented by

$$D^{1 \times (q+t)} \xrightarrow{.Q_i} D^{1 \times (p+s)} \xrightarrow{\varrho_i} E_i \longrightarrow 0, \tag{40}$$

and the matrix $Q_i \in D^{(q+t) \times (p+s)}$ is given by

$$Q_i = \left(\begin{array}{cc} R & -T_i \\ 0 & S \end{array} \right), \tag{41}$$

where T_i is an element of $\Omega = \{A \in D^{q \times s} \mid \exists\, B \in D^{r \times t} : R_2\, A = B\, S\}$.

Then, the extension $\xi_3 = \xi_1 + \xi_2$ of N by M is defined by (40) with $i = 3$, where the left D-module E_3 is finitely presented by

$$E_3 = D^{1 \times (p+s)}/(D^{1 \times (q+t)}\, Q_3),$$

and the matrix Q_3 is defined by (41) with:

$$T_3 = T_1 + T_2.$$

Proof. To prove the result, we can use the definition of $\xi_3 = \xi_1 + \xi_2$ in terms of $\xi_3 = \nabla_\star(\Delta^\star(\xi_1 \oplus \xi_2))$, where ∇ and Δ are defined by (8). However, its leads to lengthly involved computations. In order to avoid that, we use the more tractable characterization of the sum of extensions ξ_1 and ξ_2 due to Cartan and Eilenberg ([4]) and presented at the end of Section 2: the extension ξ_3 can be defined by the left D-module $E_3 = \ker(\beta_1, -\beta_2)/\mathrm{im}\,(-\alpha_1 \oplus \alpha_2)$. Using Lemma 2, we shall explicitly compute E_3.

The left D-module $E_1 \oplus E_2$ is clearly defined by the following finite presentation:

$$D^{1 \times (q+t)} \oplus D^{1 \times (q+t)} \xrightarrow{.(Q_1 \oplus Q_2)} D^{1 \times (p+s)} \oplus D^{1 \times (p+s)} \xrightarrow{\varrho_1 \oplus \varrho_2} E_1 \oplus E_2 \longrightarrow 0.$$

We can easily check that the D-morphism $(\beta_1, -\beta_2) : E_1 \oplus E_2 \longrightarrow M$ induces the following *morphism of complexes* ([27]), i.e., the following commutative exact diagram

$$\begin{array}{ccccccc} D^{1\times(q+t)} \oplus D^{1\times(q+t)} & \xrightarrow{.(Q_1 \oplus Q_2)} & D^{1\times(p+s)} \oplus D^{1\times(p+s)} & \xrightarrow{\varrho_1 \oplus \varrho_2} & E_1 \oplus E_2 & \longrightarrow 0 \\ \downarrow V & & \downarrow U & & \downarrow (\beta_1, -\beta_2) & \\ D^{1\times q} & \xrightarrow{.R} & D^{1\times p} & \xrightarrow{\pi} & M & \longrightarrow 0, \end{array}$$

with the notations:

$$U = \begin{pmatrix} I_p \\ 0 \\ -I_p \\ 0 \end{pmatrix} \in D^{(p+s+p+s)\times p}, \quad V = \begin{pmatrix} I_q \\ 0 \\ -I_q \\ 0 \end{pmatrix} \in D^{(q+t+q+t)\times q}.$$

Let us determine the left D-module $\ker(\beta_1, -\beta_2)$. By 1 of Lemma 2, we first need to compute $\ker_D(.(U^T \;\; R^T)^T)$. Let us consider $\mu = (\mu_1, \ldots, \mu_5) \in \ker_D(.(U^T \;\; R^T)^T)$, i.e., $\mu_1 - \mu_3 + \mu_5 R = 0$. Hence, we get

$$\mu = \mu_2\,(0 \quad I_s \quad 0 \quad 0 \quad 0) + \mu_3\,(I_p \quad 0 \quad I_p \quad 0 \quad 0) + \mu_4\,(0 \quad 0 \quad 0 \quad I_s \quad 0) - \mu_5\,(R \quad 0 \quad 0 \quad 0 \quad -I_q),$$

which shows that

$$\ker_D\left(.\begin{pmatrix} U \\ R \end{pmatrix}\right) = D^{1\times(s+p+s+q)}\,(P \quad -P'), \quad P = \begin{pmatrix} 0 & I_s & 0 & 0 \\ I_p & 0 & I_p & 0 \\ 0 & 0 & 0 & I_s \\ R & 0 & 0 & 0 \end{pmatrix}, \quad P' = \begin{pmatrix} 0 \\ 0 \\ 0 \\ I_q \end{pmatrix}. \tag{42}$$

By 1 of Lemma 2, we then obtain:

$$\ker(\beta_1, -\beta_2) = (D^{1\times(s+p+s+q)}\,P)/(D^{1\times(q+t+q+t)}\,(Q_1 \oplus Q_2)).$$

Let us characterize the left D-module $\mathrm{im}\,(-\alpha_1 \oplus \alpha_2)$. The D-morphism $-\alpha_1 \oplus \alpha_2 : N \longrightarrow E_1 \oplus E_2$ induces the following morphism of complexes, i.e., the following commutative exact diagram

$$\begin{array}{ccccccc} D^{1\times t} & \xrightarrow{.S} & D^{1\times s} & \xrightarrow{\delta} & N & \longrightarrow 0 \\ \downarrow .Y & & \downarrow .X & & \downarrow -\alpha_1 \oplus \alpha_2 & \\ D^{1\times(q+t)} \oplus D^{1\times(q+t)} & \xrightarrow{.(Q_1 \oplus Q_2)} & D^{1\times(p+s)} \oplus D^{1\times(p+s)} & \xrightarrow{\varrho_1 \oplus \varrho_2} & E_1 \oplus E_2 & \longrightarrow 0, \end{array}$$

with the notations:

$$X = (0 \quad -I_s \quad 0 \quad I_s), \quad Y = (0 \quad -I_t \quad 0 \quad I_t).$$

Using 2 of Lemma 2, we obtain:

$$\mathrm{im}\,(-\alpha_1 \oplus \alpha_2) = (D^{1\times(s+q+t+q+t)}\,(X^T \;\; (Q_1 \oplus Q_2)^T)^T/(D^{1\times(q+t+q+t)}\,(Q_1 \oplus Q_2)).$$

Using the classical *third isomorphism theorem* (see, e.g., [27]), we then get:

$$E_3 = \ker(\beta_1, -\beta_2)/\mathrm{im}\,(-\alpha_1 \oplus \alpha_2) \cong E_4 = (D^{1\times(s+p+s+q)}\,P)/(D^{1\times(s+q+t+q+t)}\,(X^T \;\; (Q_1 \oplus Q_2)^T)^T.$$

We denote by ϕ the previous isomorphism between E_3 and E_4.

Using Lemma 4, let us find a finite presentation of the left D-module E_4. We can easily check that we have the following factorization

$$\left(\begin{array}{cccc} 0 & -I_s & 0 & I_s \\ R & -T_1 & 0 & 0 \\ 0 & S & 0 & 0 \\ 0 & 0 & R & -T_2 \\ 0 & 0 & 0 & S \end{array}\right) = \left(\begin{array}{cccc} -I_s & 0 & I_s & 0 \\ -T_1 & 0 & 0 & -I_q \\ S & 0 & 0 & 0 \\ 0 & R & -T_2 & I_q \\ 0 & 0 & S & 0 \end{array}\right) \left(\begin{array}{cccc} 0 & I_s & 0 & 0 \\ I_p & 0 & I_p & 0 \\ 0 & 0 & 0 & I_s \\ -R & 0 & 0 & 0 \end{array}\right),$$

i.e., $(X^T \, (Q_1 \oplus Q_2)^T)^T = F\, P$, where P denotes the matrix defined by (42) and F is the first matrix in the previous right hand side. Moreover, an element $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4) \in \ker_D(.P)$ satisfies

$$\left\{\begin{array}{l} \lambda_2 - \lambda_4\, R = 0, \\ \lambda_1 = 0, \\ \lambda_2 = 0, \\ \lambda_3 = 0, \end{array}\right.$$

i.e., $\lambda = (0, 0, 0, \lambda_4)$, where $\lambda_4 \in \ker_D(.R)$. Hence, if we denote by $R_2 \in D^{r \times q}$ a matrix satisfying that $\ker_D(.R) = D^{1 \times r}\, R_2$ and $G = (0 \quad 0 \quad 0 \quad R_2)$, then we get $\ker_D(.P) = D^{1 \times r}\, G$. Using Lemma 4, we then obtain:

$$E_4 \cong D^{1 \times (s+p+s+q)} / \left(D^{1 \times (s+q+t+q+t+r)} \left(\begin{array}{c} F \\ G \end{array}\right)\right).$$

The finitely presented left D-module E_4 is defined by the following relations between its generators y:

$$\left(\begin{array}{c} F \\ G \end{array}\right) y = 0 \quad \Leftrightarrow \quad \left\{\begin{array}{l} -y_1 + y_3 = 0, \\ -T_1\, y_1 - y_4 = 0, \\ S\, y_1 = 0, \\ R\, y_2 - T_2\, y_3 + y_4 = 0, \\ S\, y_3 = 0, \\ R_2\, y_4 = 0, \end{array}\right. \quad \Leftrightarrow \quad \left\{\begin{array}{l} y_1 = y_3, \\ y_4 = -T_1\, y_3, \\ S\, y_3 = 0, \\ R\, y_2 - T_1\, y_3 - T_2\, y_3 = 0, \\ R_2\, T_1\, y_3 = 0. \end{array}\right.$$

Using the fact that $T_1 \in \Omega$, we get that there exists $L_1 \in D^{r \times t}$ such that $R_2\, T_1 = L_1\, S$, which shows that $R_2\, T_1\, y_3 = L_1\, S\, y_3 = 0$ is a direct consequence of the equation $S\, y_3 = 0$. Hence, we get that the finitely presented left D-module E_4 can be generated by the components of the vectors y_2 and y_3 which satisfy the following relations:

$$\left\{\begin{array}{l} R\, y_2 - (T_1 + T_2)\, y_3 = 0, \\ S\, y_3 = 0. \end{array}\right.$$

We obtain that $E_4 \cong E_5 = D^{1 \times (p+s)} / (D^{1 \times (q+t)}\, Q_3)$, where Q_3 is defined by (41) with $T_3 = T_1 + T_2$. Let us denote by ψ the previous isomorphism between E_4 and E_5.

We can easily check that we have the following commutative exact diagram

$$\begin{array}{ccccccccc} 0 \longrightarrow & N & \xrightarrow{\alpha_3} & E_3 & \xrightarrow{\beta_3} & M & \longrightarrow 0 \\ & \| & & \downarrow \psi \circ \phi & & \| & \\ 0 \longrightarrow & N & \xrightarrow{\psi \circ \phi \circ \alpha_3} & E_5 & \xrightarrow{\beta_3 \circ \phi^{-1} \circ \psi^{-1}} & M & \longrightarrow 0, \end{array}$$

which finally proves that the extensions of N by M defined by E_3 and E_5 belong to the same equivalence class in $\mathrm{e}_D(N, M)$.

□

6 Applications to multidimensional systems theory

Let $R \in D^{q \times p}$ be a matrix with entries in a left noetherian domain D. We know that D is then a *left Ore domain*, namely, a domain satisfying that, for all $a,\, b \in D \setminus \{0\}$, there exist $c, d \in D \setminus \{0\}$ such that $c\, a = d\, b$ ([9, 14]). If $M = D^{1 \times p}/(D^{1 \times q}\, R)$ denotes a left D-module finitely presented by R, then

$$t(M) = \{m \in M \mid \exists\, 0 \neq a \in D :\ a\, m = 0\}$$

is a left D-submodule of M and we have the following canonical short exact sequence ([9, 14, 27]):

$$0 \longrightarrow t(M) \xrightarrow{\iota} M \xrightarrow{\tau} M/t(M) \longrightarrow 0. \tag{43}$$

An element of $t(M)$ is called a *torsion element* of M and M is said to be *torsion-free* if $t(M) = 0$ and *torsion* if $t(M) = M$ (see, e.g., [14, 27]).

Results obtained in [5, 23] show that there exists a matrix $R' \in D^{q' \times p}$ satisfying:

$$\begin{cases} t(M) = (D^{1 \times q'}\, R')/(D^{1 \times q}\, R), \\ M/t(M) = D^{1 \times p}/(D^{1 \times q'}\, R'). \end{cases}$$

In Section 6.1, using Baer's interpretation of the extension functor, we shall give another proof of this result. We refer to [5, 6] for more details on a constructive algorithm which computes the matrix R', on its implementations in the library OreModules and applications in control theory and mathematical physics.

In the control theory and mathematical physics literatures, it has been shown in the past years that the concepts of parametrizability and controllability are related to the one of torsion-free module. In particular, the elements of the torsion submodule correspond to *constrained observables* or *autonomous elements*. For more details, we refer to [5, 12, 20, 22, 30] and the references therein.

More precisely, if $\mathcal{F}$ is an injective left D-module and $M = D^{1 \times p}/(D^{1 \times q}\, R)$ a finitely presented left D-module associated with the system $\ker_{\mathcal{F}}(R.) \cong \hom_D(M, \mathcal{F})$, applying the contravariant exact functor $\hom_D(\,\cdot\,, \mathcal{F})$ to the exact sequence (43), we then get the exact sequence of abelian groups:

$$0 \longleftarrow \hom_D(t(M), \mathcal{F}) \xleftarrow{\iota^\star} \hom_D(M, \mathcal{F}) \xleftarrow{\tau^\star} \hom_D(M/t(M), \mathcal{F}) \longleftarrow 0.$$

The system $\ker_{\mathcal{F}}(R'.) \cong \hom_D(M/t(M), \mathcal{F})$ corresponds to the *parametrizable subsystem* of the system $\ker_{\mathcal{F}}(R.) \cong \hom_D(M, \mathcal{F})$. In some contexts in control theory, it is also called the *controllable subsystem* of $\ker_{\mathcal{F}}(R.)$. For more details, see [12, 20, 24, 30] and the references therein. We shall see that there always exists a matrix $Q \in D^{p \times m}$ such that $\ker_{\mathcal{F}}(R'.) = Q\, \mathcal{F}^m$, i.e., any solution $\eta \in \mathcal{F}^p$ of the system $R'\, \eta = 0$ has the form $\eta = Q\, \xi$ for a certain $\xi \in \mathcal{F}^m$. This fact explains the terminology.

Moreover, if we denote by $R'' \in D^{q \times q'}$ (resp., $R'_2 \in D^{r' \times q'}$) a matrix satisfying $R = R''\, R'$ (resp., $\ker_D(.R') = D^{1 \times r'}\, R'_2$), then we shall recall in Proposition 6 that we have the following isomorphism:

$$t(M) \cong D^{1 \times q'} \Big/ \left(D^{1 \times (q + r')} \begin{pmatrix} R'' \\ R'_2 \end{pmatrix} \right).$$

The *autonomous system* defined by $\ker_{\mathcal{F}}((R''^T\ R_2'^T)^T.) \cong \hom_D(t(M), \mathcal{F})$ then satisfies:

$$\ker_{\mathcal{F}}((R''^T\ R_2'^T)^T.) \cong \ker_{\mathcal{F}}(R.)/\tau^\star(\ker_{\mathcal{F}}(R'.)).$$

This last system will be called the *autonomous quotient* of the system $\ker_{\mathcal{F}}(R.)$.

The purpose of Section 6.2 is to parametrize all the equivalence classes of multidimensional linear systems which have a fixed parametrizable subsystem and a fixed autonomous system.

6.1 An extension characterization of torsion submodules

In what follows, we shall assume that D is a left and right noetherian domain. In particular, it implies that D is a left and a right Ore domain and the existence of a skew field of fractions K of D ([9, 14]).

Let us consider a matrix $R \in D^{q\times p}$, the finitely presented left D-module $M = D^{1\times p}/(D^{1\times q}\, R)$ and the following finite presentation of M:

$$D^{1\times q} \xrightarrow{.R} D^{1\times p} \xrightarrow{\pi} M \longrightarrow 0.$$

Let us define the finitely presented right D-module $N = D^q/(R\, D^p)$ called the *transposed module* of M ([5]). If we denote by $M^\star$ the right D-module $\hom_D(M, D)$, we then have the following exact sequence of right D-modules obtained by applying the contravariant left exact functor $\hom_D(\,\cdot\,, D)$ to the previous exact sequence:

$$0 \longleftarrow N \xleftarrow{\kappa} D^q \xleftarrow{R.} D^p \xleftarrow{\pi^\star} M^\star \longleftarrow 0. \tag{44}$$

If we denote by $R_2 \in D^{p\times m}$ a matrix satisfying $\ker_D(R.) = R_2\, D^m$ and by $R' \in D^{q'\times p}$ a matrix such that $\ker_D(.R_2) = D^{1\times q'}\, R'$, then we obtain that:

$$\mathrm{ext}^1_D(N, D) = \ker_D(.R_2)/(D^{1\times q}\, R) = (D^{1\times q'}\, R')/(D^{1\times q}\, R).$$

Hence, every $\zeta \in \mathrm{ext}^1_D(N, D)$ has the form $\zeta = \pi(\nu\, R')$, where $\nu \in D^{1\times q'}$ and $\pi : D^{1\times p} \longrightarrow M$ denotes the canonical projection onto M. Let us denote by

$$Q = \begin{pmatrix} R \\ -\nu\, R' \end{pmatrix} \in D^{(q+1)\times p},$$

and let us define the right D-module $E = D^{q+1}/(Q\, D^p)$. Then, a version of the results developed in Section 4 for right D-modules proves that we get the extension of the right D-modules D by N:

$$0 \longrightarrow D \xrightarrow{\alpha} E \xrightarrow{\beta} N \longrightarrow 0. \tag{45}$$

A version for right D-modules of the exact sequence (15) gives the following finite free resolution of the right D-module E:

$$0 \longleftarrow E \xleftarrow{\sigma} D^{q+1} \xleftarrow{Q.} D^p \xleftarrow{R_2.} D^m \longleftarrow \ldots$$

In particular, we get $\ker_D(Q.) = R_2\, D^m = \ker_D(R.) \cong M^\star$.

The left D-module $P = D^{1\times p}/(D^{1\times (q+1)}\, Q)$ admits the following finite presentation:

$$D^{1\times (q+1)} \xrightarrow{.Q} D^{1\times p} \xrightarrow{\epsilon} P \longrightarrow 0.$$

Applying the contravariant left exact functor $\hom_D(\,\cdot\,, D)$ to the previous exact sequence, we obtain the following exact sequence

$$0 \longleftarrow E \xleftarrow{\sigma} D^{q+1} \xleftarrow{Q.} D^p \xleftarrow{\epsilon^\star} P^\star \longleftarrow 0, \tag{46}$$

which proves that $\ker_D(Q.) \cong P^\star$ and the following isomorphism:

$$M^\star \cong P^\star. \tag{47}$$

Now, using the inclusion $D^{1\times q}\,R \subseteq D^{1\times (q+1)}\,Q$, we get the commutative exact diagram:

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ 0 \longrightarrow & & D^{1\times q}\,R & \longrightarrow & D^{1\times (q+1)}\,Q & & \\ & & \downarrow & & \downarrow & & \\ 0 \longrightarrow & & D^{1\times p} & \longrightarrow & D^{1\times p} & \longrightarrow 0 & \\ & & \downarrow & & \downarrow & & \\ & & M & \longrightarrow & P & \longrightarrow 0. & \\ & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & \end{array}$$

The snake lemma in homological algebra ([27]) then shows that we have the short exact sequence:

$$0 \longrightarrow (D^{1\times (q+1)}\,Q)/(D^{1\times q}\,R) \longrightarrow M \longrightarrow P \longrightarrow 0. \tag{48}$$

We recall that D admits a skew field of fractions (see, e.g., [14]):

$$K = \{a^{-1}\,b = c\,d^{-1} \mid 0 \neq a,\, 0 \neq d,\, b,\, c \in D\}.$$

The *rank* of a finitely generated left D-module M is then defined by

$$\mathrm{rank}_D(M) = \dim_K(K \otimes_D M),$$

where $\dim_K$ stands for the dimension of the *left division ring* $K \otimes_D M$ ([14]). A similar definition holds for right D-modules. We have the following lemma.

Lemma 5. *Let D be a left and right noetherian domain and M a finitely generated left D-module. Then, we have:*

$$\mathrm{rank}_D(M^\star) = \mathrm{rank}_D(M).$$

Proof. We first have $\mathrm{rank}_D(M^\star) = \dim_K(M^\star \otimes_D K) = \dim_K(\hom_D(M, D) \otimes_D K)$. Now, using the fact that K is a *flat* left D-module ([14, 27]), D is a left noetherian domain and M is a finitely generated left D-module, we have:

$$\hom_D(M, D) \otimes_D K \cong \hom_K(K \otimes_D M, K).$$

See, e.g., Theorem 3.84 of [27]. Hence, if we set $l = \mathrm{rank}_D(M)$, then we get

$$\hom_K(K \otimes_D M, K) \cong \hom_K(K^{1\times l}, K) \cong K^l,$$

which shows that $\mathrm{rank}_D(M^\star) = l$, i.e., $\mathrm{rank}_D(M^\star) = \mathrm{rank}_D(M)$. □

Applying Lemma 5 to $M^\star \cong P^\star$, we obtain:

$$\mathrm{rank}_D(M) = \mathrm{rank}_D(M^\star) = \mathrm{rank}_D(P^\star) = \mathrm{rank}_D(P).$$

Finally, using the short exact sequence (48) and the classical property of the rank (Euler-Poincaré characteristic) ([27]), we then get

$$\mathrm{rank}_D(M) = \mathrm{rank}_D(P) + \mathrm{rank}_D((D^{1\times (q+1)}\,Q)/(D^{1\times q}\,R)),$$

which proves that $\mathrm{rank}_D((D^{1\times (q+1)}\,Q)/(D^{1\times q}\,R)) = 0$, i.e., the left D-module $(D^{1\times (q+1)}\,Q)/(D^{1\times q}\,R)$ is torsion. Hence, we obtain that $\zeta = \pi(\nu\,R') \in \mathrm{ext}^1_D(N, D)$ is a torsion element of M.

Conversely, let us consider a torsion element $\zeta = \pi(\theta)$ of $M = D^{1\times p}/(D^{1\times q}\, R)$, where $\theta \in D^{1\times p}$. In particular, there exist $0 \neq a \in D$ and $\mu \in D^{1\times q}$ such that $a\,\theta = \mu\, R$. Let us define:

$$Q = \begin{pmatrix} R \\ \theta \end{pmatrix} \in D^{(q+1)\times p}.$$

Clearly, we have $\ker_D(Q.) \subseteq \ker_D(R.)$. Let us consider $\lambda \in \ker_D(R.)$, i.e., $\lambda \in D^p$ satisfying $R\,\lambda = 0$. Post-multiplying the expression $a\,\theta = \mu\, R$ by λ, we get $a\,(\theta\,\lambda) = \mu\,(R\,\lambda) = 0$. Using the fact that $\theta\,\lambda \in D$, $0 \neq a \in D$ and D is an integral domain, we then get $\theta\,\lambda = 0$, i.e., $\lambda \in \ker_D(Q.)$, which proves $\ker_D(Q.) = \ker_D(R.)$. Let us define the right D-modules N and E by (44) respectively (46). If we denote by $R_2 \in D^{p\times m}$ a matrix satisfying $\ker_D(R.) = R_2\, D^m$, using the fact that, for all $\lambda \in \ker_D(R.)$, we have $\theta\,\lambda = 0$, we then get $\theta\, R_2 = 0$ and:

$$\pi(\theta) \in \ker_D(.R_2)/(D^{1\times q}\, R) = \mathrm{ext}^1_D(N, D).$$

Let us consider the following commutative exact diagram

$$\begin{array}{cccccccc}
 & & 0 & & 0 & & 0 & \\
 & & \downarrow & & \downarrow & & \downarrow & \\
 & & \Gamma & & D & & \ker\beta & \\
 & & \downarrow & & \downarrow & & \downarrow & \\
0 \longrightarrow & & Q\,D^p & \longrightarrow & D^{q+1} & \xrightarrow{\sigma} & E & \longrightarrow 0 \\
 & & \downarrow {\scriptstyle (I_q \quad 0).} & & \downarrow {\scriptstyle (I_q \quad 0).} & & \downarrow {\scriptstyle \beta} & \\
0 \longrightarrow & & R\,D^p & \longrightarrow & D^q & \xrightarrow{\kappa} & N & \longrightarrow 0, \\
 & & \downarrow & & \downarrow & & \downarrow & \\
 & & 0 & & 0 & & \mathrm{coker}\,\beta & \\
 & & & & & & \downarrow & \\
 & & & & & & 0 &
\end{array} \tag{49}$$

where, for all $\lambda \in D^{q+1}$, $\beta(\sigma(\lambda)) = \kappa((I_q\ 0)\,\lambda)$, and Γ denotes the kernel of the following D-morphism:

$$\begin{array}{ccc}
Q\,D^p & \xrightarrow{(I_q \quad 0).} & R\,D^p \\
\begin{pmatrix} R \\ \theta \end{pmatrix} \lambda & \longmapsto & R\,\lambda.
\end{array}$$

Using the fact that $\ker_D(R.) = \ker_D(Q.)$, we get that $\Gamma = 0$. Applying the snake lemma to the commutative exact diagram (49), we obtain that $\ker\beta \cong D$ and $\mathrm{coker}\,\beta = 0$, and we get the exact sequence (45), i.e., an extension of D by N defining an element of $\mathrm{e}_D(N, D)$.

We obtain the following theorem which was obtained in [5, 24] by means of a different proof.

Theorem 5. *Let D be a left and right noetherian domain, $R \in D^{q\times p}$ a matrix, $M = D^{1\times p}/(D^{1\times q}\, R)$ the left D-module presented by R and the right D-module $N = D^q/(R\,D^p)$. If we denote by $R_2 \in D^{p\times m}$ a matrix satisfying $\ker_D(R.) = R_2\, D^m$ and by $R' \in D^{q'\times p}$ a matrix such that $\ker_D(.R_2) = D^{1\times q'}\, R'$, then, we have the following isomorphism of left D-modules:*

$$t(M) = \{m \in M \mid \exists\, 0 \neq a \in P:\ a\, m = 0\} \cong \mathrm{ext}^1_D(N, D) = (D^{1\times q'}\, R')/(D^{1\times q}\, R) \cong \mathrm{e}_D(N, D).$$

Using the fact that $M = D^{1\times p}/(D^{1\times q}\, R)$, $t(M) = (D^{1\times q'}\, R')/(D^{1\times q}\, R)$ and the short exact sequence (43), the third isomorphism theorem ([27]) then gives that $M/t(M) = D^{1\times p}/(D^{1\times q'}\, R')$.

Moreover, if $\mathcal{F}$ is an injective left D-module, applying the contravariant exact functor $\hom_D(\,\cdot\,, \mathcal{F})$ to the exact sequence of left D-modules

$$D^{1\times q'} \xrightarrow{.R'} D^{1\times p} \xrightarrow{.R_2} D^{1\times m},$$

we then get the following exact sequence of abelian groups:

$$\mathcal{F}^{q'} \xleftarrow{R'.} \mathcal{F}^p \xleftarrow{R_2.} \mathcal{F}^m.$$

It follows that we have $\ker_{\mathcal{F}}(R'.) = R_2\,\mathcal{F}^m$, meaning that every solution $\zeta \in \mathcal{F}^p$ of the system $R'\,\zeta = 0$ defined by the left D-module $M/t(M)$ has the form $\zeta = R_2\,\xi$ for a certain $\xi \in \mathcal{F}^m$.

Finally, we refer to [5, 23] for a constructive algorithm which computes $\mathrm{ext}^1_D(N, D)$ and $t(M)$ and its implementation in the library OreModules ([6]). See also [5, 6] for explict examples.

6.2 Parametrization of multidimensional systems having fixed parametrizable subsystem and autonomous system

If M and N are respectively a torsion-free and a torsion left D-module defined by means of two finite presentations, using Theorem 3, we can parametrize all the equivalence classes of extensions of N by M. If $\mathcal{F}$ is an injective left D-module, by duality, we then obtain all the equivalence classes of systems admitting $\hom_D(M, \mathcal{F})$ as a parametrizable subsystem and $\hom_D(N, \mathcal{F})$ as autonomous quotient. In what follows, we are going to detail these computations. But, we can first note that if we consider the left D-module $P = M \oplus N$, we then have $t(P) \cong N$ and $P/t(P) \cong M$, and thus, the previous problem is reduced to the case where we only consider the extensions of $t(P)$ by $P/t(P)$ for a finitely presented left D-module P.

Let $L \in D^{m\times l}$ be a matrix with entries in a left and right noetherian domain D and let us consider the finitely presented left D-module $P = D^{1\times l}/(D^{1\times m}\,L)$. As we have seen in Section 6.1, computing the left D-module $\mathrm{ext}^1_D(N, D)$, where $N = D^m/(L\,D^l)$, gives us a matrix $L' \in D^{m'\times l}$ satisfying:

$$\begin{cases} t(P) = (D^{1\times m'}\,L')/(D^{1\times m}\,L), \\ P/t(P) = D^{1\times l}/(D^{1\times m'}\,L'). \end{cases}$$

We denote by $\epsilon : D^{1\times m} \longrightarrow P$ (resp., $\epsilon' : D^{1\times m} \longrightarrow P/t(P)$) the canonical projection onto P (resp., $P/t(P)$). In particular, we can easily check that we have the relation $\epsilon' = \tau \circ \epsilon$, where τ denotes the canonical projection $M \longrightarrow M/t(M)$ (see (43)).

Applying Lemma 4 to the left D-module $t(P) = (D^{1\times m'}\,L')/(D^{1\times m}\,L)$, we obtain the following proposition (see [25] for a system-theoretic interpretation).

Proposition 6. *With the previous notations, if we denote by $L'_2 \in D^{n'\times m'}$ (resp., $L'' \in D^{m\times m'}$) the matrix satisfying $\ker_D(.L') = D^{1\times n'}\,L'_2$ (resp., $L = L''\,L'$), then the D-morphism ϖ defined by*

$$\begin{array}{rcl} N = D^{1\times m'} \Big/ \left(D^{1\times(m+n')} \begin{pmatrix} L'' \\ L'_2 \end{pmatrix} \right) & \xrightarrow{\varpi} & t(P) \\ \delta(\nu) & \longmapsto & \epsilon(\nu\,L'), \end{array}$$

where $\delta : D^{1\times m'} \longrightarrow N$ denotes the canonical projection onto N and $\nu \in D^{1\times m'}$, is an isomorphism, i.e., $t(P) \cong N$.

From Proposition 6, we get the following finite presentation of the left D-module $t(P)$

$$D^{1\times(m+n')} \xrightarrow{.\begin{pmatrix} L'' \\ L'_2 \end{pmatrix}} D^{1\times m'} \xrightarrow{\varpi\circ\delta} t(P) \longrightarrow 0,$$

where, for all $\nu \in D^{1\times m'}$, we have $(\varpi \circ \delta)(\nu) = \epsilon(\nu\,L')$.

We can now use this finite presentation and the results developed in Section 5 to compute elements of the abelian group $\mathrm{ext}^1_D(P/t(P), t(P))$.

We obtain the following corollary of Theorem 3.

Corollary 2. *With the previous notations, an extension of* $t(P)$ *by* $P/t(P)$ *is defined by the left* D*-module* E *finitely presented by*

$$D^{1\times(m'+m+n')} \xrightarrow{.Q} D^{1\times(l+m')} \xrightarrow{\varrho} E \longrightarrow 0,$$

where the matrix Q *has the form*

$$Q = \begin{pmatrix} L' & -T \\ 0 & L'' \\ 0 & L'_2 \end{pmatrix} \in D^{(m'+m+n')\times(l+m')}, \tag{50}$$

and T *is any element of the abelian group* Ω*:*

$$\Omega = \left\{ A \in D^{m'\times m'} \mid \exists\, B \in D^{n'\times(m+n')} :\; L'_2\, A = B \begin{pmatrix} L'' \\ L'_2 \end{pmatrix} \right\}. \tag{51}$$

Moreover, the equivalence classes of the extensions of $t(P)$ *by* $P/t(P)$ *are in 1-1 correspondence with the residue class* $(\mathrm{id}_{m'} \otimes (\varpi \circ \delta))(T)$*,* $T \in \Omega$*, in the abelian group:*

$$\Omega / \left(L'\, D^{l\times m'} + D^{m'\times(m+n')} \begin{pmatrix} L'' \\ L'_2 \end{pmatrix} \right) \cong \mathrm{ext}^1_D(P/t(P), t(P)).$$

Remark 6. If $\mathcal{F}$ denotes a left D-module and $\ker_{\mathcal{F}}(L.) \cong \hom_D(P, \mathcal{F})$, then Corollary 2 gives a parametrization of all equivalence classes of linear systems $\ker_{\mathcal{F}}(Q.) \cong \hom_D(E, \mathcal{F})$ which admits $\ker_{\mathcal{F}}(L'.)$ as a parametrizable subsystem and $\ker_{\mathcal{F}}((L''^T\; L'^T_2)^T.)$ as an autonomous quotient.

Let us illustrate Corollary 2 and the previous remark on two explicit examples.

Example 15. We consider again the differential time-delay system (2) defined in Example 1, the commutative polynomial ring $D = \mathbb{Q}\,[\partial, \delta]$ of differential time-delay operators, the system matrix

$$L = \begin{pmatrix} \delta^2 & 1 & -2\,\partial\,\delta \\ 1 & \delta^2 & -2\,\partial\,\delta \end{pmatrix} \in D^{2\times 3}, \tag{52}$$

and the D-module $P = D^{1\times 3}/(D^{1\times 2}\, L)$.

Using the algorithms developed in [5, 24] and implemented in the library OreModules ([6]), we obtain that

$$\begin{cases} t(P) \cong (D^{1\times 2}\, L')/(D^{1\times 2}\, L), \\ P/t(P) = D^{1\times 3}/(D^{1\times 2}\, L'), \end{cases}$$

where the matrix $L' = R \in D^{2\times 3}$ is defined by (31). Moreover, we have $\ker_D(.L') = 0$ and $L = L''\, L'$, where $L'' = S \in D^{2\times 2}$ is defined by (31). Hence, by Proposition 6, we get $t(P) \cong D^{1\times 2}/(D^{1\times 2}\, L'')$.

In Example 11, we proved that $\mathrm{ext}^1_D(M, N) = 0$ with the notations:

$$M = D^{1\times 3}/(D^{1\times 2}\, R), \quad N = D^{1\times 2}/(D^{1\times 2}\, S).$$

Hence, we get $\mathrm{ext}^1_D(P/t(P), t(P)) = 0$ and, by Proposition 5, we obtain that:

$$P \cong t(P) \oplus (P/t(P)).$$

Then, the D-module E defined by (35) generates the unique and trivial equivalence class of extensions of $t(P)$ by $P/t(P)$. If $\mathcal{F}$ denotes an injective cogenerator D-module, then 2 of Corollary 1 shows that (2) defines the unique and trivial equivalence class of systems having the same parametrizable subsystems and autonomous quotients.

Example 16. Let us consider the following differential time-delay system

$$\left\{ \begin{array}{l} \dot{y}_1(t) - \dot{y}_2(t-2\,h) + \alpha\,\ddot{y}_3(t-h) = 0, \\ \dot{y}_1(t-2\,h) - \dot{y}_2(t) + \alpha\,\ddot{y}_3(t-h) = 0, \end{array} \right. \tag{53}$$

where $\alpha \in \mathbb{R}$ is a constant parameter and h is a strictly positive real number. This system corresponds to a model of a one-dimensional tank containing a fluid subjected to a horizontal move. See [19] for more details.

Let us consider the commutative polynomial ring $D = \mathbb{Q}(\alpha)\,[\partial, \delta]$ of differential time-delay operators, the system matrix

$$L = \left(\begin{array}{ccc} \partial & -\partial\,\delta^2 & \alpha\,\partial^2\,\delta \\ \partial\,\delta^2 & -\partial & \alpha\,\partial^2\,\delta \end{array} \right) \in D^{2\times 3},$$

the D-module $P = D^{1\times 3}/(D^{1\times 2}\,L)$ and $\epsilon : D^{1\times 3} \longrightarrow P$ the canonical projection onto P.

Using a constructive algorithm developed in [5, 24] and implemented in [6], we obtain that

$$\left\{ \begin{array}{l} t(P) \cong (D^{1\times 2}\,L')/(D^{1\times 2}\,L), \\ P/t(P) = D^{1\times 3}/(D^{1\times 2}\,L'), \end{array} \right.$$

where the matrix $L' = R \in D^{2\times 3}$ is defined by (32), i.e., corresponds to the following system:

$$\left\{ \begin{array}{l} y_1(t) + y_2(t) = 0, \\ y_2(t) + y_2(t-2\,h) - \alpha\,\dot{y}_3(t-h) = 0. \end{array} \right. \tag{54}$$

Moreover, we have $\ker_D(.L') = 0$ and $L = L''\,L'$, where $L'' = S \in D^{2\times 2}$ is defined by (32). Hence, by Proposition 6, we obtain $t(P) \cong D^{1\times 2}/(D^{1\times 2}\,L'')$.

The equivalence classes of extensions of $t(P)$ by $P/t(P)$, i.e., the equivalence classes of exact sequences of the form

$$0 \longrightarrow t(P) \xrightarrow{\alpha} E \xrightarrow{\beta} P/t(P) \longrightarrow 0,$$

are in 1-1 correspondence with the elements of the D-module $\mathrm{ext}^1_D(P/t(P), t(P))$. Using Examples 12 and 14, we obtain that the three equivalence classes of extensions are generated by the D-modules E_0, E_1 and E_2 respectively defined by (38), (36) and (37). Hence, the three different equivalence classes of extensions of $t(P)$ by $P/t(P)$ are respectively defined by the following differential time-delay systems:

$$\left\{ \begin{array}{l} z_1(t) + z_2(t) = 0, \\ z_2(t) + z_2(t-2\,h) - \alpha\,\dot{z}_3(t-h) = 0, \\ \dot{z}_4(t) - \dot{z}_5(t) = 0, \\ \dot{z}_4(t-2\,h) - \dot{z}_5(t) = 0, \end{array} \right. \quad \left\{ \begin{array}{l} z_1(t) + z_2(t) = 0, \\ z_2(t) + z_2(t-2\,h) - \alpha\,\dot{z}_3(t-h) - z_4(t) = 0, \\ \dot{z}_4(t) - \dot{z}_5(t) = 0, \\ \dot{z}_4(t-2\,h) - \dot{z}_5(t) = 0, \end{array} \right.$$

$$\left\{ \begin{array}{l} z_1(t) + z_2(t) = 0, \\ z_2(t) + z_2(t-2\,h) - \alpha\,\dot{z}_3(t-h) - z_5(t) = 0, \\ \dot{z}_4(t) - \dot{z}_5(t) = 0, \\ \dot{z}_4(t-2\,h) - \dot{z}_5(t) = 0. \end{array} \right. \tag{55}$$

If $\mathcal{F}$ denotes an injective cogenerator D-module, by 2 of Corollary 1, then we obtain three non-equivalent linear differential time-delay systems which admit the same parametrizable subsystems and autonomous quotients.

Using the results of [8], we can check that the matrices defined by

$$X = \begin{pmatrix} 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & -\partial & 1 & 0 \\ 0 & -\partial & 0 & 1 \end{pmatrix},$$

$$W = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1+\delta^2 & -\alpha\,\partial\,\delta \end{pmatrix}, \quad Z = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix},$$

satisfy the relations

$$L\,X = Y \begin{pmatrix} R & -A_2 \\ 0 & S \end{pmatrix}, \quad \begin{pmatrix} R & -A_2 \\ 0 & S \end{pmatrix} W = Z\,L,$$

where the matrix A_2 is defined by (33), and thus, define the following D-morphisms:

$$\begin{array}{rcl} f : P & \longrightarrow & E_2 \\ \epsilon(\lambda) & \longmapsto & \varrho_2(\lambda\,X), \end{array} \qquad \begin{array}{rcl} g : E_2 & \longrightarrow & P, \\ \varrho_2(\mu) & \longmapsto & \epsilon(\mu\,W). \end{array}$$

Moreover, we can easily check that $g \circ f = \mathrm{id}_P$ and $f \circ g = \mathrm{id}_{E_2}$, which proves that $E_2 \cong P$.

Let us prove that the extensions defined by P and E_2 belong to the same equivalence class, i.e., we have the following commutative exact diagram

$$\begin{array}{ccccccc} 0 \longrightarrow & t(P) & \xrightarrow{\alpha} & E_2 & \xrightarrow{\beta} & P/t(P) & \longrightarrow 0 \\ & \| & & \downarrow g & & \| & \\ 0 \longrightarrow & t(P) & \xrightarrow{\iota} & P & \xrightarrow{\tau} & P/t(P) & \longrightarrow 0, \end{array}$$

where the D-morphisms α and β are defined in Remark 3, namely, for all $\mu \in D^{1\times 2}$, $\lambda_1 \in D^{1\times 3}$ and $\lambda_2 \in D^{1\times 2}$, we have

$$\begin{array}{rcl} t(P) & \xrightarrow{\alpha} & E_2 \\ (\varpi \circ \delta)(\mu) & \longmapsto & \varrho_2(\mu\,(0 \quad I_2)) = \varrho_2((0, \mu)), \end{array}$$

$$\begin{array}{rcl} E_2 & \xrightarrow{\beta} & P/t(P) \\ \varrho_2((\lambda_1, \lambda_2)) & \longmapsto & \epsilon'((\lambda_1, \lambda_2)\,(I_3 \quad 0)^T) = \epsilon'(\lambda_1), \end{array}$$

where $\epsilon' : D^{1\times 3} \longrightarrow P/t(P)$ denotes the canonical projection onto $P/t(P)$.

Let us denote by $\{e_i\}_{1\leq i\leq 2}$ (resp., $\{f_j\}_{1\leq j\leq 5}$, $\{h_i\}_{1\leq i\leq 3}$) the standard basis of $D^{1\times 2}$ (resp., $D^{1\times 5}$, $D^{1\times 3}$). If we denote by $\{y_k\}_{1\leq k\leq 3}$ the set of generators of P, i.e., $y_k = \epsilon(h_k)$, $k = 1, 2, 3$, then we have

$$\left\{ \begin{array}{l} (g \circ \alpha)((\varpi \circ \delta)(e_1)) = g(\varrho_2(f_4)) = y_1 + y_2 = \epsilon(e_1\,L') = \iota((\varpi \circ \delta)(e_1)), \\ (g \circ \alpha)((\varpi \circ \delta)(e_2)) = g(\varrho_2(f_5)) = (1 + \delta^2)\,y_2 - \alpha\,\partial\,\delta\,y_3 = \epsilon(e_2\,L') = \iota((\varpi \circ \delta)(e_2)), \end{array} \right.$$

which proves that $g \circ \alpha = \iota$. Moreover, we get

$$\forall\, j = 1, \ldots, 5, \quad (\tau \circ g)(\varrho_2(f_j)) = \tau(\varrho_2(f_j\,W)) = \epsilon'(f_j\,W),$$

and, more precisely, using (54), we obtain

$$\left\{ \begin{array}{l} (\tau \circ g)(\varrho_2(f_1)) = -y_2 = y_1 = \beta(\varrho_2(f_1)), \\ (\tau \circ g)(\varrho_2(f_2)) = y_2 = \beta(\varrho_2(f_2)), \\ (\tau \circ g)(\varrho_2(f_3)) = y_3 = \beta(\varrho_2(f_3)), \\ (\tau \circ g)(\varrho_2(f_4)) = y_1 + y_2 = 0 = \beta(\varrho_2(f_4)), \\ (\tau \circ g)(\varrho_2(f_5)) = (1 + \delta^2)\,y_2 - \alpha\,\partial\,\delta\,y_3 = 0 = \beta(\varrho_2(f_5)), \end{array} \right.$$

proving $\beta = \tau \circ g$.

Let us consider any D-module $\mathcal{F}$ (e.g., $\mathcal{F} = C^{\infty}(\mathbb{R})$). The fact that $E_2 \cong P$ proves that (53) and (55) are equivalent systems and the invertible transformation mapping $\mathcal{F}$-trajectories of (53) to $\mathcal{F}$-trajectories of (55) is defined by the matrix W, namely,

$$\begin{cases} z_1(t) = -y_2(t), \\ z_2(t) = y_2(t), \\ z_3(t) = y_3(t), \\ z_4(t) = y_1(t) + y_2(t), \\ z_5(t) = y_2(t) + y_2(t - 2\,h) - \alpha\,\dot{y}_3(t - h), \end{cases}$$

and the inverse morphism sending $\mathcal{F}$-trajectories of (55) to $\mathcal{F}$-trajectories of (53) is defined by the matrix X, namely:

$$\begin{cases} y_1(t) = -z_2(t) + z_4(t), \\ y_2(t) = z_2(t), \\ y_3(t) = z_3(t). \end{cases}$$

We note that $T = I_{m'}$ belongs to the abelian group Ω defined by (51) as we have the relation $L_2'\,I_{m'} = 0\,L'' + I_{n'}\,L_2'$. This remark leads to the following interesting result.

Lemma 6. *Using the previous notations where Q defined by (50) with $T = I_{m'} \in \Omega$, we have the following results:*

1. *The D-morphism $f : P \longrightarrow E = D^{1\times(l+m')}/(D^{1\times(m'+m+n')}\,Q)$ defined by*

$$\forall\, \lambda \in D^{1\times l}, \quad f(\epsilon(\lambda)) = \varrho(\lambda\,U),$$

where the matrices $U = (I_l \quad 0) \in D^{l\times(l+m')}$ and $V = (L'' \quad I_m \quad 0) \in D^{m\times(m'+m+n')}$ satisfy that $L\,U = V\,Q$, is an isomorphism, i.e., $E \cong P$.

2. *The extensions $0 \longrightarrow t(P) \xrightarrow{\iota} P \xrightarrow{\tau} P/t(P) \longrightarrow 0$ and $0 \longrightarrow t(P) \xrightarrow{\alpha} E \xrightarrow{\beta} P/t(P) \longrightarrow 0$ belong to the same equivalence class in the abelian group $\mathrm{e}_D(P/t(P), t(P))$.*

Proof. 1. Let us consider the matrix $T = I_{m'} \in \Omega$. The relation $L\,U = V\,Q$ clearly holds, which proves the existence of the D-morphism $f : P \longrightarrow E = D^{1\times(l+m')}/(D^{1\times(m'+m+n')}\,Q)$ defined by $f(\epsilon(\lambda)) = \varrho(\lambda\,U)$, for all $\lambda \in D^{1\times l}$ (see [8] for more details). Moreover, we can easily check that the matrix $(U^T \; Q^T)^T$ admits a left-inverse, which, by 3 of Lemma 2, proves that $\operatorname{coker} f = 0$, i.e., f is surjective. Finally, let us consider $(\lambda_1, \lambda_2, \lambda_3, \lambda_4) \in \ker_D(.(U^T \; Q^T)^T)$. Then, we have

$$\begin{cases} \lambda_1 + \lambda_2\,L' = 0, \\ -\lambda_2 + \lambda_3\,L'' + \lambda_4\,L_2' = 0, \end{cases} \Leftrightarrow \begin{cases} \lambda_1 = -\lambda_2\,L', \\ \lambda_2 = \lambda_3\,L'' + \lambda_4\,L_2', \end{cases} \Leftrightarrow \begin{cases} \lambda_1 = -\lambda_3\,L, \\ \lambda_2 = \lambda_3\,L'' + \lambda_4\,L_2', \end{cases}$$

which proves

$$\ker_D(.(U^T \; Q^T)^T) = D^{1\times(m+n')} \begin{pmatrix} -L & L'' & I_m & 0 \\ 0 & L_2' & 0 & I_{n'} \end{pmatrix},$$

and, using 1 of Lemma 2, $\ker f = (D^{1\times(m+n')}\,(-L^T \quad 0^T)^T)/(D^{1\times m}\,L) = 0$, which proves 1.

2. Let us prove that we have the following commutative exact diagram

$$\begin{array}{ccccccccc} 0 \longrightarrow & t(P) & \xrightarrow{\iota} & P & \xrightarrow{\tau} & P/t(P) & \longrightarrow 0 \\ & \| & & \downarrow f & & \| & \\ 0 \longrightarrow & t(P) & \xrightarrow{\alpha} & E & \xrightarrow{\beta} & P/t(P) & \longrightarrow 0, \end{array}$$

where $\alpha : t(P) \longrightarrow E$ and $\beta : E \longrightarrow P/t(P)$ are defined by:

$$\forall\, \nu \in D^{1\times m'}, \quad \alpha((\varpi \circ \delta)(\nu)) = \varrho((0 \quad \nu)), \quad \forall\, \lambda_1 \in D^{1\times l}, \quad \forall\, \lambda_2 \in D^{1\times m'}, \quad \beta(\varrho((\lambda_1, \lambda_2))) = \epsilon'(\lambda_1).$$

Let us first prove that $\alpha = f \circ \iota$. Using the notations of Proposition 6, we get:

$$\forall\, \nu \in D^{1\times m'}, \quad (f \circ \iota)((\varpi \circ \delta)(\nu)) = f(\epsilon(\nu\, L')) = \varrho((\nu\, L' \quad 0)).$$

Using the definition of E in terms of generators and relations, we have

$$\varrho((\nu\, L' \quad 0)) = \varrho((0 \quad \nu)) = \alpha((\varpi \circ \delta)(\nu)),$$

which proves the result.

Secondly, the identity $\tau = \beta \circ f$ holds as we have

$$\forall\, \lambda \in D^{1\times m}, \quad (\beta \circ f)(\epsilon(\lambda)) = \beta(\varrho((\lambda \quad 0))) = \epsilon'(\lambda) = \tau(\epsilon(\lambda)),$$

which proves 2. □

Remark 7. We point out that the matrix Q defined by (50) with $T = I_{m'}$ was used in [25, 26] in order to parametrize the $\mathcal{F}$-solutions of the system $\ker_{\mathcal{F}}(L.)$ in terms of the $\mathcal{F}$-solutions of $\ker_{\mathcal{F}}(L'.)$ and $\ker_{\mathcal{F}}((L''^T \; L_2'^T)^T.)$. In particular, we first need to solve the following homogeneous system

$$\begin{cases} L''\,\theta = 0, \\ L_2'\,\theta = 0, \end{cases} \tag{56}$$

corresponding to $\hom_D(t(P), \mathcal{F})$ and then solve the inhomogeneous system $L'\,\eta = \theta$. In order to solve the latter, we need to know a particular solution $\eta^\star \in \mathcal{F}^l$ of $L'\,\eta^\star = \theta$ and the general solution of the homogeneous system $L'\,\eta = 0$ associated with the system $\hom_D(P/t(P), \mathcal{F})$. As the subsystem $\hom_D(P/t(P), \mathcal{F})$ of $\hom_D(P, \mathcal{F})$ is parametrizable, using the result developed in Section 6.1 (see also [5, 6]), we can compute a matrix $Q' \in D^{l\times k'}$ satisfying $\ker_{\mathcal{F}}(L'.) = Q'\,\mathcal{F}^{k'}$ whenever $\mathcal{F}$ is an injective left D-module. Then, the solution of $L\,\eta = 0$ has the form $\eta = \eta^\star + Q'\,\xi$, for any $\xi \in \mathcal{F}^{k'}$. We refer to [26] for applications of this result to variational problems and optimal control.

Example 17. Let us consider again Example 16. We recall that the extensions of $t(P)$ by $P/t(P)$ are defined by means of D-modules of the form $E = D^{1\times 5}/(D^{1\times 4}\,Q)$, where Q is defined by:

$$Q = \begin{pmatrix} L & -T \\ 0 & L'' \end{pmatrix} \in D^{4\times 5}, \quad T \in D^{2\times 2}.$$

Moreover, the equivalence classes of extensions are defined by the elements of the D-module:

$$\Theta = D^{2\times 2}/(L'\,D^{3\times 2} + D^{2\times 2}\,L'') \cong \operatorname{ext}^1_D(P/t(P), t(P)).$$

In other words, the matrices T and $T' = T + L'\,U + V\,L''$, where $U \in D^{3\times 2}$ and $V \in D^{2\times 2}$, define the same equivalence class of extensions of $t(P)$ by $P/t(P)$. See Example 12 for more details.

Using the results obtained in Example 12, we find that the following two matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

belong to the same equivalence class in Θ, a fact implying that the following matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

belong to the same equivalence class in Θ. By 2 of Lemma 6, we obtain that P is isomorphic to the D-modules of the form E defined by $T = I_2$ or A_2. We find again the result obtained in Example 16.

If $\mathrm{e}_D(P/t(P), t(P)) = 0$, then we obtain that every extension of $t(P)$ by $P/t(P)$ is equivalent to the following split exact sequence:

$$0 \longrightarrow t(P) \xrightarrow{i_1} t(P) \oplus (P/t(P)) \xrightarrow{p_2} P/t(P) \longrightarrow 0. \tag{57}$$

In particular, we get $P \cong t(P) \oplus (P/t(P))$, showing that the short exact sequence (43) splits.

We give a necessary and sufficient condition for the short exact sequence (43) to split.

Proposition 7. *Let $L \in D^{m \times l}$ and $L' \in D^{m' \times l}$ be two matrices defining the left D-modules:*

$$P = D^{1\times l}/(D^{1\times m}\, L), \quad P/t(P) = D^{1\times l}/(D^{1\times m'}\, L').$$

Then, the short exact sequence (43) splits iff there exist two matrices $X \in D^{l\times m'}$ and $Y \in D^{m'\times m}$ satisfying:

$$L' - L'\, X\, L' = Y\, L. \tag{58}$$

Proof. Let us denote by $L_2' \in D^{n'\times m'}$ (resp., $L'' \in D^{m\times m'}$) a matrix satisfying $\ker_D(.L') = D^{1\times n'}\, L_2'$ (resp., $L = L''\, L'$). Using Corollary 2 and 2 of Lemma 6, we obtain that the extensions of $t(P)$ by $P/t(P)$ defined by $T = I_{m'}$ and $0 \longrightarrow t(P) \longrightarrow P \longrightarrow P/t(P) \longrightarrow 0$ belong to the same equivalence class in $\mathrm{e}_D(P/t(P), t(P))$. Using the group isomorphism $\mathrm{e}_D(P/t(P), t(P)) \cong \mathrm{ext}^1_D(P/t(P), t(P))$ proved in Theorem 2, the previous short exact sequence splits iff $T = I_{m'}$ and 0 belong to the same residue class in

$$\Omega/\left(L'\, D^{l\times m'} + D^{m'\times(m+n')} \begin{pmatrix} L'' \\ L_2' \end{pmatrix}\right) \cong \mathrm{ext}^1_D(P/t(P), t(P)),$$

i.e., iff there exist three matrices $X \in D^{l\times m'}$, $Y \in D^{m'\times m'}$ and $Z \in D^{m'\times n'}$ satisfying:

$$I_{m'} = L'\, X + Y\, L'' + Z\, L_2'. \tag{59}$$

Let us prove that (58) is equivalent to (59). Post-multiplying (59) by the matrix L' and using the fact that we have $L = L''\, L'$ and $L_2'\, L' = 0$, we then obtain (58). Conversely, from (58), we get $(I_{m'} - L'\, X - Y\, L'')\, L' = 0$, which proves that the rows of the matrix $I_{m'} - L'\, X - Y\, L''$ belong to $\ker_D(.L') = D^{1\times n'}\, L_2'$, and thus, there exists $Z \in D^{m'\times n'}$ such that $I_{m'} - L'\, X - Y\, L'' = Z\, L_2'$. □

For a different proof of Proposition 7, see [25].

Remark 8. We consider again Remark 7. As it was shown in [25], Proposition 7 can be used to easily compute a particular solution $\eta^\star \in \mathcal{F}^l$ of the inhomogeneous system $L'\, \eta = \theta$, where $\theta \in \mathcal{F}^{q'}$ is a general solution of the system (56). Indeed, using (59), we obtain that $\theta = L'\, X\, \theta + Y\, L''\, \theta + Z\, L_2'\, \theta = L'\, (X\, \theta)$. Hence, $\eta^\star = X\, \theta \in \mathcal{F}^l$ is a particular solution of the inhomogeneous system $L'\, \eta = \theta$. If $\mathcal{F}$ is an injective left D-module, using the results developed in Remark 7, then we obtain the elements of $\ker_{\mathcal{F}}(L.)$ has the form $\eta = X\, \theta + Q'\, \xi$ for a certain $\xi \in \mathcal{F}^{k'}$. For more details and applications, see [26].

Remark 9. If D is a commutative polynomial ring, using Kronecker products, we then obtain that:

$$(59) \;\Leftrightarrow\; \mathrm{row}(I_{m'}) = (\mathrm{row}(X) \quad \mathrm{row}(Y) \quad \mathrm{row}(Z)) \begin{pmatrix} L'^T \otimes I_{m'} \\ I_{m'} \otimes L'' \\ I_{m'} \otimes L_2' \end{pmatrix}.$$

Hence, using a Gröbner/Janet basis computation, the existence of the matrices X, Y and Z satisfying (59) is reduced to checking whether or not $\mathrm{row}(I_{m'})$ belongs to the Gröbner/Janet basis of the D-module generated by the rows of the last matrix for an elimination order ([25]).

Example 18. Let us consider again Examples 1 and 15. Using Remark 9 or an algorithm developed in [25] and implemented in OreModules ([6]), we obtain that the matrices L and $L' = R$ respectively defined by (52) and (31) satisfy the relation (58), where:

$$X = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 0 \end{pmatrix}, \quad Y = \frac{1}{2} \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}.$$

If $\mathcal{F}$ denotes an injective D-module, using Remark 8, we then can parametrize the system $\ker_{\mathcal{F}}(L.)$. Let us compute a parametrization of $\ker_{\mathcal{F}}(L.)$. In Example 15, we prove that $L'_2 = 0$ and $L'' = S$, where the matrix S is defined by (31). The system $\ker_D(L''.) \cong \hom_D(t(P), \mathcal{F})$ is defined by

$$\begin{cases} \delta^2\,\theta_1 + \theta_2 = 0, \\ \theta_1 + \theta_2 = 0, \end{cases} \quad \Leftrightarrow \quad \begin{cases} \theta_2 = -\theta_1, \\ \delta^2\,\theta_1 - \theta_1 = 0, \end{cases}$$

which proves that θ_1 is a $2\,h$-periodic function of $\mathcal{F}$. Moreover, using the results developed in Section 6.1 (see also [5, 6]), we obtain that $\ker_D(.Q') = D^{1\times 2}\,L'$, where Q' is the matrix defined by:

$$Q' = \begin{pmatrix} 2\,\delta\,\partial & 2\,\delta\,\partial & 1+\delta^2 \end{pmatrix}^T.$$

Then, Remark 8 shows that (2) is parametrized by

$$\begin{cases} y_1(t) = \frac{1}{2}\,\theta_1(t) + 2\,\dot{\xi}(t-h), \\ y_2(t) = -\frac{1}{2}\,\theta_1(t) + 2\,\dot{\xi}(t-h), \\ y_3(t) = \xi(t) + \xi(t-2\,h), \end{cases}$$

where ξ is an arbitrary function of $\mathcal{F}$ and θ_1 an arbitrary $2\,h$-periodic function of $\mathcal{F}$.

If $P/t(P)$ is a *projective* left D-module, namely, a left D-module such that there exists a left D-module N satisfying $(P/t(P)) \oplus N \cong D^{1\times l}$, for a certain $l \in \mathbb{Z}_+ = \{0, 1, 2, \ldots\}$, then we know that (43) splits ([27]). This result is coherent with the fact that $\mathrm{ext}^1_D(P, N) = 0$ whenever P is a projective left D-module (see, e.g., [27]). Moreover, we can prove that the left D-module $P/t(P) = D^{1\times l}/(D^{1\times m'}\,L')$ is projective iff the matrix L' admits a generalized inverse over D, namely, iff there exists a matrix $X \in D^{l\times m'}$ satisfying:

$$L'\,X\,L' = L'.$$

See [5] for more details. A constructive algorithm computing generalized inverses was developed and implemented in OreModules ([6]).

Example 19. We consider again Example 10. The $D = \mathbb{Q}\,[\partial, \delta]$-module $P = D^{1\times 3}/(D^{1\times 2}\,L)$, where the matrix L is defined by (23), satisfies that $P/t(P) = D^{1\times 3}/(D^{1\times 3}\,L')$, where the matrix L' is defined by (25). Moreover, we have $t(P) = (D^{1\times 3}\,L')/(D^{1\times 2}\,L)$ and the matrix defined by

$$Q' = \begin{pmatrix} 1+\delta^2 & 2\,\delta & (1-\delta^2)\,\partial \end{pmatrix}^T$$

satisfies that $\ker_D(.Q') = D^{1\times 3}\,L'$. See [5] for more details. Using an algorithm developed in [5], we can check that L' admits a generalized inverse X over D defined by:

$$X = \begin{pmatrix} \frac{1}{2}\,\delta & 0 & 0 \\ 1 & 0 & 0 \\ -\frac{1}{2}\,\partial\,\delta & 1 & 0 \end{pmatrix} \in D^{3\times 3}.$$

Hence, the D-module $P/t(P)$ is projective, which proves that $\mathrm{e}_D(P/t(P), t(P)) = 0$. Let $\mathcal{F}$ be an injective D-module. Using Remark 8, let us parametrize $\ker_{\mathcal{F}}(L.)$. We can check that we have $L = L'' L'$ and $\ker_D(.L') = D\, L'_2$, with the notations:

$$L'' = \begin{pmatrix} 0 & -1 & 0 \\ 0 & -\delta & 1 \end{pmatrix}, \quad L'_2 = (\partial \quad -\delta \quad 1)\,.$$

Hence, $\hom_D(t(P), \mathcal{F})$ is defined by the following system of functional equations:

$$\begin{cases} -\theta_2 = 0, \\ -\delta\,\theta_2 + \theta_3 = 0, \\ \partial\,\theta_1 - \delta\,\theta_2 + \theta_3 = 0, \end{cases} \Leftrightarrow \begin{cases} \partial\,\theta_1 = 0, \\ \theta_2 = 0, \\ \theta_3 = 0. \end{cases} \Leftrightarrow \begin{cases} \theta_1 = C \in \mathbb{R}, \\ \theta_2 = 0, \\ \theta_3 = 0. \end{cases}$$

Then, Remark 8 shows that (24) admits the following parametrization

$$\begin{cases} y_1(t) = \frac{1}{2}\,C + \xi(t) + \xi(t - 2\,h), \\ y_2(t) = C + 2\,\xi(t - h), \\ y_3(t) = \dot{\xi}(t) - \dot{\xi}(t - 2\,h), \end{cases}$$

where C is an arbitrary constant and ξ an arbitrary function of $\mathcal{F}$.

For more examples, see [26]. We also refer the reader to [8, 7] for related results.

Finally, if D is a *left hereditary ring* ([14, 27]) (e.g., $D = k[t]\,[\partial]$, k a field of characteristic 0) or a left principal ideal domain (e.g., $k(t)\,[\partial]$, k a field), then we know that every torsion-free left D-module is projective and, in particular, the left D-module $P/t(P)$ for any left D-module P. Hence, we find again Kalman's result described in the introduction as well as its different generalizations.

7 Conclusion

In this paper, we have shown how to constructively study Baer's interpretation of the elements of the abelian group $\mathrm{ext}^1_D(M, N)$, where M and N are two finitely presented left D-modules, in terms of equivalence classes of extensions of N by M. This interpretation, combined with constructive algorithms for the computation of elements of this abelian group, allowed us to parametrize, up to an equivalence relation, all the finitely presented left D-modules E which contain N as a left D-submodule and whose quotient by N is isomorphic to M. When the signal space $\mathcal{F}$ was an injective left D-module, the duality between the system-theoretic and the behaviour-theoretic approaches allowed us to solve the problem raised by S. Shankar and described in the introduction of the paper. We have illustrated these results on explicit examples and have shown how to parametrize all the systems having a given parametrizable subsystem and admitting a given system of autonomous elements.

As it was shown in Section 6.1 and more generally in [23], the vanishing of the right D-modules $\mathrm{ext}^i_D(P, D)$, $i \geq 1$, played an important role in the classification of the properties of multidimensional linear systems and in the existence of parametrizations for these systems (i.e., *image representations* in the behavioural approach ([20, 30])). This result has motivated the development of new constructive algorithms for the computation of the right D-modules $\mathrm{ext}^i_D(P, D)$, $i \geq 1$, whenever P is a left D-module presented by a matrix with entries in a non-commutative polynomial ring D of functional operators ([5, 23]). These constructive algorithms, based on Gröbner or Janet bases, were implemented in the library OreModules ([5, 6]) and illustrated on many explicit examples. The problem studied in this paper can be seen as a generalization of this work as we now compute the abelian group $\mathrm{ext}^1_D(P, N)$ and show some of its applications. As it was shown by J.-P. Serre in [28], Baer's interpretation of the extension functor also plays an important role in the reduction of linear multidimensional systems, that is to say, in the search for equivalent systems having a minimal number of equations. See [3] for more details on a system interpretation of Serre's theorem and its applications in mathematical systems theory.

We refer to [29] for results on a behavioural interpretation of the Baer extension problem and its applications in the synthesis of behaviours. We point out that the results obtained in the paper can also be applied to different short exact sequences than (43) as, for instance:

$$0 \longrightarrow M/(M_1 \cap M_2) \longrightarrow (M/M_1) \oplus (M/M_2) \longrightarrow M/(M_1 + M_2) \longrightarrow 0.$$

We shall study these extensions in the future. Finally, in the literature of homological algebra ([31, 15, 27]), Baer's interpretation of $\mathrm{ext}^1_D(M, N)$ was extended by N. Yoneda for $\mathrm{ext}^i_D(M, N)$, $i \geq 1$. The system-theoretic interpretation of the abelian groups $\mathrm{ext}^i_D(M, N)$, $i \geq 2$, will be studied in a future publication.

Acknowledgement

The authors are very grateful to Prof. S. Shankar (Chennai Mathematical Institute) for motivating discussions on the use of Baer's interpretation of the extension functor in mathematical systems theory.

References

[1] R. Baer. Erweiterung von Gruppen und ihren Isomorphismen. *Mathematische Zeitschrift*, 38(1):375–416, 1934.

[2] M. Barakat and D. Robertz. Computing invariants of multidimensional linear systems on an abstract homological level. In *Proc. of the 17th International Symposium on Mathematical Theory of Networks and Systems, Kyoto (Japan)*, 2006, `http://wwwb.math.rwth-aachen.de/homalg/`.

[3] M. S. Boudellioua and A. Quadrat. Reduction of multidimensional linear systems: A module-theoretic approach. *In preparation*, 2007.

[4] H. Cartan and S. Eilenberg. *Homological Algebra*. Princeton University Press, 1956.

[5] F. Chyzak, A. Quadrat, and D. Robertz. Effective algorithms for parametrizing linear control systems over Ore algebras. *Applicable Algebra in Engineering, Communications and Computing*, 16(5):319–376, 2005.

[6] F. Chyzak, A. Quadrat, and D. Robertz. OreModules: A symbolic package for the study of multidimensional linear systems. In J. Chiasson and J.-J. Loiseau, editors, *Applications of Time-Delay Systems*, LNCIS 352, pages 233–264. Springer, 2007, `http://wwwb.math.rwth-aachen.de/OreModules/`.

[7] T. Cluzeau and A. Quadrat. Morphisms: A homological algebra package for factoring and decomposition linear functional systems. *In preparation*, 2007.

[8] T. Cluzeau and A. Quadrat. Factoring and decomposing a class of linear functional systems. *Linear Algebra and Its Applications*, to appear, 2007, `https://hal.inria.fr/inria-00083224`.

[9] P. M. Cohn. *Free Rings and Their Relations*. London Mathematical Society. Academic Press, Second edition, 1985.

[10] F. Dubois, N. Petit, and P. Rouchon. Motion planning and nonlinear simulations for a tank containing a fluid. In *Proc. of the 5th European Control Conf. 1999, Karlsruhe (Germany)*, 1999.

[11] M. Fliess. Some structural properties of generalized linear systems. *Systems & Control Letters*, 15:391–396, 1990.

[12] M. Fliess and H. Mounier. Controllability and observability of linear delay systems: an algebraic approach. *ESAIM Control, Optimization and Calculus of Variations*, 3:301–314, 1998.

[13] R. E. Kalman, P. L. Falb, and M. A. Arbib. *Topics in Mathematical System Theory.* McGraw-Hill, 1969.

[14] T. Y. Lam. *Lectures on Modules and Rings*, volume 189 of *Graduate Texts in Mathematics.* Springer, 1999.

[15] S. MacLane. *Homology.* Springer Verlag, 1995.

[16] B. Malgrange. Systèmes à coefficients constants. *Séminaire Bourbaki*, 246:79–89, 1962-63.

[17] H. Mounier, J. Rudolph, M. Petitot, and M. Fliess. A flexible rod as a linear delay system. In *Proc. of the 3rd European Control Conference, Rome (Italy)*, 1995.

[18] U. Oberst. Multidimensional constant linear systems. *Acta Applicandæ Mathematicæ*, 20:1–175, 1990.

[19] N. Petit and P. Rouchon. Dynamics and solutions to some control problems for water-tank systems. *IEEE Trans. Autom. Contr.*, 47(4):594–609, 2002.

[20] H. Pillai and S. Shankar. A behavioral approach to control of distributed systems. *SIAM Journal on Control and Optimization*, 37:388–408, 1999.

[21] J. W. Polderman and J. C. Willems. *Introduction to Mathematical Systems Theory: A Behavioral Approach*, volume 26 of *Texts in Applied Mathematics.* Springer-Verlag, 1998.

[22] J.-F. Pommaret. *Partial Differential Control Theory*, volume 530 of *Mathematics and Its Applications.* Kluwer, 2001.

[23] J.-F. Pommaret and A. Quadrat. Algebraic analysis of linear multidimensional control systems. *IMA Journal of Control and Information*, 16(3):275–297, 1999.

[24] J.-F. Pommaret and A. Quadrat. A functorial approach to the behaviour of multidimensional control systems. *Applied Mathematics and Computer Science*, 13:7–13, 2003.

[25] A. Quadrat and D. Robertz. Parametrizing all solutions of uncontrollable multidimensional linear systems. In 16^{th} *IFAC World Congress, Prague (Czech Republic)*, 2005.

[26] A. Quadrat and D. Robertz. On the Monge problem and multidimensional optimal control. In 17^{th} *International Symposium on Mathematical Theory of Networks and Systems, Kyoto (Japan)*, 2006.

[27] J. J. Rotman. *An Introduction to Homological Algebra.* Academic Press, 1979.

[28] J.-P. Serre. Sur les modules projectifs. *Séminaire Dubreil-Pisot*, 2:23–34, 1960/61. In Jean-Pierre Serre, *Oeuvres, Collected Papers*, Vol. II 1960-1971, 23-34.

[29] S. Shankar. Synthesis of behaviours. *In preparation*, 2007.

[30] J. Wood. Modules and behaviours in nD systems theory. *Multidimensional Systems and Signal Processing*, 11:11–48, 2000.

[31] N. Yoneda. On Ext and exact sequences. *J. Fac. Sci. Univ. Tokyo Sect. I*, 8:507576, 1960.

Unité de recherche INRIA Sophia Antipolis
2004, route des Lucioles - BP 93 - 06902 Sophia Antipolis Cedex (France)

Unité de recherche INRIA Futurs : Parc Club Orsay Université - ZAC des Vignes
4, rue Jacques Monod - 91893 ORSAY Cedex (France)
Unité de recherche INRIA Lorraine : LORIA, Technopôle de Nancy-Brabois - Campus scientifique
615, rue du Jardin Botanique - BP 101 - 54602 Villers-lès-Nancy Cedex (France)
Unité de recherche INRIA Rennes : IRISA, Campus universitaire de Beaulieu - 35042 Rennes Cedex (France)
Unité de recherche INRIA Rhône-Alpes : 655, avenue de l'Europe - 38334 Montbonnot Saint-Ismier (France)
Unité de recherche INRIA Rocquencourt : Domaine de Voluceau - Rocquencourt - BP 105 - 78153 Le Chesnay Cedex (France)

Éditeur
INRIA - Domaine de Voluceau - Rocquencourt, BP 105 - 78153 Le Chesnay Cedex (France)
http://www.inria.fr
ISSN 0249-6399