An algebraic analysis approach to mathematical system theory

NETCA Workshop:

Verification and Theorem Proving for Continuous Systems

Oxford, 26/08/05

Alban Quadrat

INRIA Sophia Antipolis,
CAFE Project,
2004 route des lucioles, BP 93,
06902 Sophia Antipolis cedex,
France.

`Alban.Quadrat@sophia.inria.fr`

`www-sop.inria.fr/cafe/Alban.Quadrat/index.html`

Synthesis problems

Transfer functions

- **Finite-dimensional system:**

$$\dot{x}(t) = x(t) + u(t),\ x(0) = 0 \Rightarrow \hat{x}(s) = \frac{1}{(s-1)}\,\hat{u}(s).$$

- **Differential time-delay system:**

$$\begin{cases} \dot{x}(t) = x(t) + u(t),\ x(0) = 0, \\ y(t) = \begin{cases} 0, & 0 \leq t \leq 1, \\ x(t-1), & t \geq 1, \end{cases} \end{cases}$$

$$\Rightarrow \hat{y}(s) = \frac{e^{-s}}{(s-1)}\,\hat{u}(s).$$

- **System of partial differential equations:**

$$\begin{cases} \frac{\partial^2 z}{\partial t^2}(x,t) - \frac{\partial^2 z}{\partial x^2}(x,t) = 0, \\ \frac{\partial z}{\partial x}(0,t) = 0,\ \frac{\partial z}{\partial x}(1,t) = u(t), \\ y(t) = \frac{\partial z}{\partial t}(1,t), \end{cases}$$

$$\Rightarrow \hat{y}(s) = \frac{(1+e^{-2\,s})}{(1-e^{-2\,s})}\,\hat{u}(s).$$

- The **poles of the transfer functions** ($1, 1, k\,\pi\,i, k \in \mathbb{Z}$) **belong to** $\overline{\mathbb{C}_+} = \{s \in \mathbb{C} \mid \mathrm{Re}(s) \geq 0\} \Rightarrow$ **unstability.**

A module approach of synthesis problems

1. An integral domain A of **SISO stable systems** is chosen (e.g., $A = RH_\infty$, $H_\infty(\mathbb{C}_+)$, $\widehat{\mathcal{A}}\ldots$).

2. **The plant is defined by a transfer matrix**:

$$P \in K^{q\times r},\ K = Q(A) = \{n/d \mid 0 \neq d,\, n \in A\}\,.$$

3. **We write** P as:

$$P = D^{-1}\,N = \tilde{N}\,\tilde{D}^{-1}, \quad \begin{cases} (D \quad -N) \in A^{q\times p}, \\ (\tilde{N}^T \quad \tilde{D}^T)^T \in A^{p\times r}. \end{cases}$$

(e.g., $D = d\,I_q,\ \ N = d\,P,\ \ \tilde{D} = d\,I_r,\ \ \tilde{N} = \ d\,P$).

3. $y = P\,u \Leftrightarrow \begin{cases} (D \quad -N)\begin{pmatrix} y \\ u \end{pmatrix} = 0, \\ \begin{pmatrix} y \\ u \end{pmatrix} = \begin{pmatrix} \tilde{N} \\ \tilde{D} \end{pmatrix} z. \end{cases} \quad (\star)$

4. **Synthesis problems are reformulated in terms of the properties of** $(\star)$.

• **Linear algebra over rings is module theory**

$\Rightarrow$ a module approach to synthesis problems.

Stable algebras A of SISO systems

1. $$RH_\infty = \left\{\frac{n(s)}{d(s)} \in \mathbb{R}(s) \mid \deg n(s) \leq \deg d(s),\ d(s) = 0 \Rightarrow \mathrm{Re}(s) < 0 \right\}$$

$$h_1 = \frac{1}{s-1} = \frac{\left(\frac{1}{s+1}\right)}{\left(\frac{s-1}{s+1}\right)}, \quad \frac{1}{s+1},\ \frac{s-1}{s+1} \in RH_\infty$$

$$\Rightarrow h_1 \in Q(RH_\infty) = \mathbb{R}(s).$$

2. $$\mathcal{A} = \{ f(t) + \sum_{i=0}^{+\infty} a_i\, \delta_{t-t_i} \mid f \in L_1(\mathbb{R}_+),\ (a_i)_{i \geq 0} \in l_1(\mathbb{Z}_+),\ 0 = t_0 \leq t_1 \leq t_2 ...\}$$

and $\widehat{\mathcal{A}} = \{\widehat{g} \mid g \in \mathcal{A}\}$ the **Wiener algebras**.

$$h_2 = \frac{e^{-s}}{s-1} = \frac{\left(\frac{e^{-s}}{s+1}\right)}{\left(\frac{s-1}{s+1}\right)}, \quad \frac{e^{-s}}{s+1},\ \frac{s-1}{s+1} \in \widehat{\mathcal{A}} \Rightarrow h_2 \in Q(\widehat{\mathcal{A}}).$$

3. $\mathbb{C}_+ = \{s \in \mathbb{C} \mid \mathrm{Re}\, s > 0\}$. The **Hardy algebra**

$$H_\infty(\mathbb{C}_+) = \{ \text{holomorphic functions } f \text{ in } \mathbb{C}_+ \mid \| f \|_\infty = \sup_{s \in \mathbb{C}_+} | f(s) | < +\infty\}.$$

$$h_3 = \frac{(1+e^{-2\,s})}{(1-e^{-2\,s})}, 1 + e^{-2\,s}, 1 - e^{-2\,s} \in H_\infty(\mathbb{C}_+)$$

$$\Rightarrow h_3 \in Q(H_\infty(\mathbb{C}_+)).$$

Example

- We consider the transfer matrix:

$$P = \begin{pmatrix} \frac{e^{-s}}{s-1} \\ \frac{e^{-s}}{(s-1)^2} \end{pmatrix}.$$

- Let us consider $A = H_\infty(\mathbb{C}_+)$ and $K = Q(A)$.

- We have:

$$\begin{cases} y_1 = \frac{e^{-s}}{(s-1)}\, u, \\ y_2 = \frac{e^{-s}}{(s-1)^2}\, u \end{cases} \Rightarrow \begin{cases} \frac{(s-1)}{(s+1)}\, y_1 - \frac{e^{-s}}{(s+1)}\, u = 0, \\ \left(\frac{s-1}{s+1}\right)^2\, y_2 - \frac{e^{-s}}{(s+1)^2}\, u = 0, \end{cases}$$

$$\Rightarrow R \begin{pmatrix} y \\ u \end{pmatrix} = 0,$$

$$\text{with } R = \begin{pmatrix} \underbrace{\begin{matrix} \frac{s-1}{s+1} & 0 \\ 0 & \left(\frac{s-1}{s+1}\right)^2 \end{matrix}}_{D} & \underbrace{\begin{matrix} -\frac{e^{-s}}{s+1} \\ -\frac{e^{-s}}{(s+1)^2} \end{matrix}}_{-N} \end{pmatrix} \in A^{2\times 3}.$$

- We have:

$$P = D^{-1}\, N \in K^2.$$

- **Properties of P can be studied by means of the matrix R with entries in the Banach algebra A.**

Doubly coprime factorizations

$$\begin{cases} P = D^{-1} N = \tilde{N}\,\tilde{D}^{-1} \in K^{q\times r}, \\ R = (D \quad -N) \in A^{q\times p}, \\ \tilde{R} = (\tilde{N}^T \quad \tilde{D}^T)^T \in A^{p\times r}. \end{cases}$$

• **Definition**: P admits a **doubly coprime factorization** if there exist

$$\begin{cases} R' = (D' \quad -N') \in A^{q\times p}, \\ \tilde{R}' = (\tilde{N}'^T \quad \tilde{D}'^T)^T \in A^{p\times r}, \\ S = (X^T \quad Y^T)^T \in A^{p\times q}, \\ \tilde{S} = (-\tilde{Y} \quad \tilde{X}) \in A^{r\times p}, \end{cases} \quad \text{such that:}$$

$$\begin{pmatrix} R' \\ \tilde{S} \end{pmatrix} \begin{pmatrix} S & \tilde{R}' \end{pmatrix} = \begin{pmatrix} I_q & 0 \\ 0 & I_r \end{pmatrix} = I_p.$$

• **Theorem**: Let us define the A-modules:

$$M = A^{1\times p}/(A^{1\times q}\,R), \quad N = A^{1\times p}/\left(A^{1\times r}\,\tilde{R}^T\right).$$

Then, we have the following equivalences:

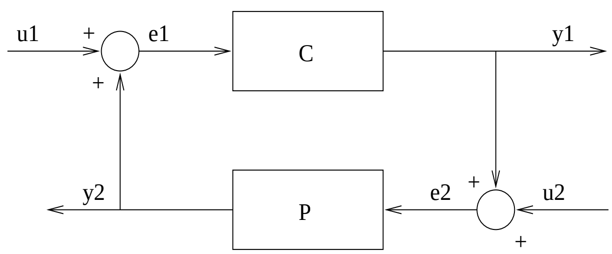
1. P admits a **doubly coprime factorization**.

2. **The A-modules $M/t(M)$ and $N/t(N)$ are free** of rank respectively q and r.

3. **The A-modules $A^{1\times p}\,R^T$ and $A^{1\times p}\,\tilde{R}$ are free** of rank respectively q and r.

Internal stabilizability

- Let A be an **integral domain of SISO stable plants**.
- $K = \{n/d \,|\, 0 \neq d,\, n \in A\}$ field of fractions of A.
- $P \in K^{q \times r}$ **a plant**.
- $C \in K^{r \times q}$ **a controller**.
- The **closed-loop system** is defined by:



$u_1,\, u_2$: external inputs, e_1, e_2: internal inputs, $y_1,\, y_2$: outputs.

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} I_q & -P \\ -C & I_r \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}, \quad \begin{cases} y_1 = e_2 - u_2, \\ y_2 = e_1 - u_1. \end{cases}$$

- **<u>Definition</u>**: C **internally stabilizes** P if the transfer matrix $T = \begin{pmatrix} I_r & -P \\ -C & I_r \end{pmatrix}^{-1}$ **satisfies**:

$$T = \begin{pmatrix} (I_q - P\,C)^{-1} & (I - P_q\,C)^{-1}\,P \\ C(I_q - P\,C)^{-1} & I_r + C\,(I_q - P\,C)^{-1}\,P \end{pmatrix} \in A^{(q+r)\times(q+r)}.$$

- **Internal stability** $\Leftrightarrow \begin{cases} L_2 - L_2 \text{ stability if } A = H_\infty(\mathbb{C}_+), \\ L_\infty - L_\infty \text{ stability if } A = \widehat{\mathcal{A}}. \end{cases}$

Internal stabilizability

$$\begin{cases} P = D^{-1}\, N = \tilde{N}\, \tilde{D}^{-1} \in K^{q \times r}, \\ R = (D \quad -N) \in A^{q \times p}, \\ \tilde{R} = (\tilde{N}^T \quad \tilde{D}^T)^T \in A^{r \times p}. \end{cases}$$

• **Theorem**: Let us define the A-modules:

$$M = A^{1 \times p}/(A^{1 \times q}\, R), \quad N = A^{1 \times p}/\left(A^{1 \times r}\, \tilde{R}^T\right).$$

Then, we have the following equivalences:

1. P is **internally stabilizable**.

2. The A-modules $M/t(M)$ and $N/t(N)$ **are projective** of rank respectively q and r.

3. The A-modules $A^{1 \times p}\, R^T$ and $A^{1 \times p}\, \tilde{R}$ **are projective** of rank respectively q and r.

• **Corollary**: $P = D^{-1}\, N$ is **internally stabilizable** iff $\exists\, S = (X^T \quad Y^T)^T \in K^{p \times q}$ such that:

1. $S\, R = \begin{pmatrix} X\, D & -X\, N \\ Y\, D & -Y\, N \end{pmatrix} \in A^{p \times p}$,

2. $R\, S = D\, X - N\, Y = I_q$.

The controller $C = Y\, X^{-1}$ **internally stabilizes** P.

Example

- We consider the transfer matrix ($A = H_\infty(\mathbb{C}_+)$):

$$P = \begin{pmatrix} \frac{e^{-s}}{(s-1)} \\ \frac{e^{-s}}{(s-1)^2} \end{pmatrix} \in K^2, \quad K = Q(A).$$

- We have $P = D^{-1}\,N$ where R is defined by:

$$R = \begin{pmatrix} \frac{s-1}{s+1} & 0 & -\frac{e^{-s}}{s+1} \\ 0 & \left(\frac{s-1}{s+1}\right)^2 & -\frac{e^{-s}}{(s+1)^2} \end{pmatrix} \in A^{2\times 3}.$$

- The matrix $S = (X^T \quad Y^T)^T \in K^{3\times 2}$ defined by

$$S = \begin{pmatrix} b\left(\frac{s-1}{s+1}\right)^2 + \frac{2}{s-1} & 2\,(b-1)\,\frac{(s-1)}{(s+1)} \\ b\,\frac{(s-1)}{(s+1)^2} - \frac{1}{s-1} & \frac{2\,b}{s+1} + \frac{s+1}{s-1} \\ -a\,\frac{(s-1)}{(s+1)^2} & -\frac{2\,a}{s+1} \end{pmatrix},$$

$$a = \frac{4\,e\,(5\,s-3)}{(s+1)}, \quad b = \frac{(s+1)^3 - 4\,(5\,s-3)\,e^{-(s-1)}}{(s+1)\,(s-1)^2} \in A,$$

satisfies $S\,R \in A^{3\times 3}$ and $R\,S = D\,X - N\,Y = I_2$.

$\Rightarrow P$ is **internally stabilized** by $C = Y\,X^{-1}$, i.e.:

$$C = \frac{-4\,(5\,s-3)\,e\,(s-1)^2}{(s+1)\,((s+1)^3 - 4\,(5\,s-3)\,e^{-(s-1)})}\,(1 \quad 2).$$