On the Ore Extension Ring of Differential Time-Varying Delay Operators

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Abstract In this work, we propose an algebraic method to study linear differential time-varying delay (DTVD) systems. Our goal is to give an effective construction of the ring of DTVD operators as an Ore extension, thanks to the concept of skew polynomial rings developed by Ore in the 30s. Some algebraic properties of the DTVD operators ring are analyzed, such as its Noetherianity, its homological and Krull dimensions, and the existence of Gröbner bases, all given in terms of the time-varying delay function. The algebraic analysis framework for linear systems theory allows us to study linear DTVD systems and essential properties such as the existence of autonomous elements, controllability, parametrizability, flatness, etc., through methods coming from module theory, homological algebra, and constructive algebra.

1 Introduction

In control theory, differential time-delay systems are a vastly studied subject in the literature, most commonly considering constant or distributed time-delays. Numerous applications have been investigated in this domain. Nevertheless, the class of *differential time-varying delay (DTVD) systems* has been studied mainly from a stability analysis viewpoint, such as in incompressible fluid flows in pipes [17], material or vehicular flows [26], metal-rolling processes [10], communication networks [29]. More details on linear DTVD systems can be found in [2, 8, 13, 16, 19, 28] and in their references.

In the 90s, Oberst, Fliess, and Pommaret introduced the *algebraic analysis method* [1, 11, 12] for linear systems theory, see [7, 20, 22, 30]. In this context, an intrinsic

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study of linear systems can be elaborated based on module theory, homological algebra, and functional analysis. In particular, built-in properties of linear systems can be characterized through module properties providing a representation independent of the system representation itself. Furthermore, *Willems' behavioral approach* [21] can also be realized and developed within this framework [20].

An *effective algebraic analysis method* to linear functional systems was initiated in [3, 6], based on properties of *skew polynomials* [18] and *Ore algebras* [5], computer algebra methods (e.g., *Gröbner* or *Janet bases* [3, 5]) and a constructive approach to both module theory and homological algebra [25]. This effective method allows the study of large classes of linear functional systems (e.g., differential systems, constant time-delay systems, discrete systems). Moreover, this approach yields the development of the OREMODULES and OREMORPHISMS packages [4, 6], allowing a complete study of control concepts such as controllability, observability, parametrizability, flatness, and equivalences.

In this work, our goal is to develop an algebraic analysis method for linear DTVD systems. First, the ring of DTVD operators is shown to be an *Ore extension* $\mathcal{D}$ [18]. Important algebraic and homological properties of $\mathcal{D}$ are then characterized. Since $\mathcal{D}$ is also a *bijective skew Poincaré-Birkhoff-Witt (PBW) extension* [9], Gröbner basis techniques can be applied over $\mathcal{D}$ (using [14]), allowing to test effectively standard module properties and thus system properties. The corresponding Gröbner basis algorithms will be implemented in the future in a computer algebra system.

The outline of the paper is the following. In Sect. 2, we give basic definitions of skew polynomial rings and Ore extensions and some examples. An explicit construction of the ring $\mathcal{D}$ of DTVD operators as an Ore extension is provided in Sect. 3. Algebraic properties of this Ore extension are analyzed in Sect. 4, in particular within a homological framework. In Sect. 5, system properties are characterized in terms of module theory and homological algebra. Finally, in Sect. 6, we discuss some open questions concerning $\mathcal{D}$.

2 Skew Polynomial Rings and Ore Extensions

In the sequel, basic definitions and properties of skew polynomials rings and Ore structures are recalled.

Definition 1 ([18]) Let A be a ring.

(i) We denote by $\mathrm{End}(A)$ the ring of endomorphisms of A, where a map $\sigma : A \longrightarrow A$ is an *endomorphism* of A if it satisfies the following conditions:

$$\forall\, a,\, b \in A, \quad \begin{cases} \sigma(1) = 1, \\ \sigma(a+b) = \sigma(a) + \sigma(b), \\ \sigma(a\, b) = \sigma(a)\, \sigma(b). \end{cases}$$

(ii) For $\sigma \in \mathrm{End}(A)$, a *$\sigma$-derivation* δ of A is a map $\delta : A \longrightarrow A$ satisfying the following conditions:

$$\forall\, a,\ b \in A, \quad \begin{cases} \delta(a+b) = \delta(a) + \delta(b), \\ \delta(a\,b) = \delta(a)\,b + \sigma(a)\,\delta(b). \end{cases}$$

Definition 2 Let A be a ring.

(i) A *derivation* $\delta : A \to A$ is a linear map satisfying Leibniz's rule, that is δ is a id_A-derivation (where id_A is the identity map of A).
(ii) A *differential ring* (A, δ) is a ring equipped with a derivation δ.

Let us introduce the definition of an *Ore extension* which will play a fundamental role in what follows.

Definition 3 ([18]) Let A be a ring, $\sigma \in \mathrm{End}(A)$ and δ be a σ-derivation of A. An *Ore extension* $A[\partial; \sigma, \delta]$ *of* A is the noncommutative polynomial ring in ∂ with coefficients in A, formed by the elements of the form $\sum_{i=0}^{n} a_i\, \partial^i$ with $a_i \in A$, and equipped with a new product given by:

$$\forall\, a \in A, \quad \partial\, a = \sigma(a)\, \partial + \delta(a). \tag{1}$$

The Ore extension $A[\partial; \sigma, \delta]$ of A is also called a *skew polynomial ring* over A.

Example 1 Consider a differential ring (A, δ) such as, e.g., $A = \mathbb{K}[t]$ (resp., $\mathbb{K}(t)$) the commutative ring of polynomials (resp., rational functions) in t with coefficients in a field $\mathbb{K}$ (e.g., $\mathbb{K} = \mathbb{Q}, \mathbb{R}$) or A the ring of smooth/analytic/meromorphic functions in t, and $\delta = \frac{d}{dt}$. The Ore extension $\mathcal{O} = A\left[\partial; \mathrm{id}_A, \frac{d}{dt}\right]$ of A is the ring of differential operators in ∂ with coefficients in A. An element $a \in A$ acts as an operator of A by multiplication $b \in A \mapsto a\,b \in A$ and an element of $\mathcal{O}$ can be written as $\sum_{i=0}^{n} a_i\, \partial^i$, where the concatenation denotes the composition of the operators acting on A:

$$\begin{aligned} a : A &\longrightarrow A \\ b &\longmapsto a\,b, \end{aligned} \qquad \begin{aligned} \partial : A &\longrightarrow A \\ a &\longmapsto \delta(a). \end{aligned}$$

For all $a \in A$, we have

$$(\partial\, a)(b) = \partial(a\,b) = \delta\,(a\,b) = a\,\delta(b) + \delta(a)\,b = (a\,\partial + \delta(a))\,(b),$$

showing that (1) is satisfied for $\sigma = \mathrm{id}_A$.

Example 2 Let $h \in \mathbb{R}_{>0} = \mathbb{R}_{\geq 0} \setminus \{0\}$, A be a ring of real-valued functions equipped with $\sigma \in \mathrm{End}(A)$ defined by $\sigma(a(t)) = a(t-h)$ for all $a \in A$. As in Example 1, given $a \in A$, we can associate an operator $a : A \to A$ defined by the multiplication $b \in A \mapsto a\,b \in A$ of functions. The Ore extension $\mathcal{O} = A\,[S; \sigma, 0]$ of A is the *ring of time-delay (TD) operators with coefficients in* A. An element of $\mathcal{O}$ can be written as $\sum_{i=0}^{n} a_i\, S^i$, where the concatenation means the composition of two operators and:

$$\begin{aligned} S : A &\longrightarrow A \\ a(t) &\longmapsto a(t-h). \end{aligned}$$

Example 3 In a discrete-time version, if A is a ring of real-valued sequences on $\mathbb{Z}$, i.e. $A = \mathbb{R}^{\mathbb{Z}}$, then an element $a \in A$ can be written as $a = (a_i)_{i\in\mathbb{Z}}$. Let $\sigma \in \mathrm{End}(A)$ be defined by forward (resp., backward) shift $\sigma\,(a_i)_{i\in\mathbb{Z}} = (a_{i+1})_{i\in\mathbb{Z}}$ (resp., $\sigma\,(a_i)_{i\in\mathbb{Z}} = (a_{i-1})_{i\in\mathbb{Z}}$). Given $a \in A$, there is an operator $a : A \to A$ defined by the multiplication of functions: $a\,b = (a_i\,b_i)_{i\in\mathbb{Z}}$ for all $b = (b_i)_{i\in\mathbb{Z}} \in A$. Then, the Ore extension $A\,[S;\sigma, 0]$ is the *skew polynomial ring of forward (resp., backward) shift operators*. An element of $\mathscr{O}$ can be written as $\sum_{k=0}^{n} a_k\,S^k$, where the concatenation means the composition of two operators and:

$$\begin{aligned} S : A &\longrightarrow A \\ (a_i)_{i\in\mathbb{Z}} &\longmapsto (a_{i+1})_{i\in\mathbb{Z}}\,. \end{aligned}$$

Skew polynomial rings satisfy the following important *universal property*.

Proposition 1 ([18], 1.2.5) *Let A and E be two rings and $f : A \longrightarrow E$ a ring homomorphism (i.e., $f(a+b) = f(a) + f(b)$, $f(a\,b) = f(a)\,f(b)$, for all $a,\ b \in A$, and $f(1) = 1$). Let $\sigma \in \mathrm{End}(A)$, δ be a σ-derivation of A, and $A[\partial;\sigma,\delta]$ the corresponding Ore extension of A. Moreover, let $e \in E$ be such that:*

$$\forall\, a \in A, \quad e\,f(a) = f(\sigma(a))\,e + f(\delta(a)). \tag{2}$$

Then, there exists a unique ring homomorphism $g : A[\partial;\sigma,\delta] \longrightarrow E$ extending f, satisfying:

$$\begin{cases} g_{|A} = f, \\ g(\partial) = e. \end{cases}$$

In Proposition 1, considering $E = A[\partial;\sigma,\delta]$, $\sigma' \in \mathrm{End}(A)$, and

$$\begin{aligned} f : A &\longrightarrow E \\ a &\longmapsto \sigma'(a), \end{aligned}$$

we obtain the following corollary.

Corollary 1 *Let A be a ring, $\sigma,\ \sigma' \in \mathrm{End}(A)$, and δ a σ-derivation of A. Moreover, let $e \in A[\partial;\sigma,\delta]$ be such that:*

$$\forall\, a \in A, \quad e\,\sigma'(a) = \sigma'(\sigma(a))\,e + \sigma'(\delta(a)). \tag{3}$$

Then, there exists a unique ring homomorphism $\overline{\sigma} \in \mathrm{End}(A[\partial;\sigma,\delta])$ satisfying:

$$\overline{\sigma}_{|A} = \sigma', \quad \overline{\sigma}(\partial) = e.$$

Proposition 1 and Corollary 1 will play an important role in what follows.

In the case of multiple σ-derivations and ring endomorphisms, for instance, for partial differential equations, or differential-difference equations, or differential time-delay (DTD) equations, an iteration of Ore extensions can be constructed as follows. If $\mathscr{O} = A[\partial; \sigma, \delta]$ is an Ore extension of A, $\sigma_2 \in \mathrm{End}(\mathscr{O})$ and δ_2 is a σ_2-derivation of $\mathscr{O}$, then we define the following Ore extension of $\mathscr{O}$:

$$B = \mathscr{O}\,[\partial_2; \sigma_2, \delta_2] = A\,[\partial; \sigma, \delta]\,[\partial_2; \sigma_2, \delta_2]\,.$$

By definition, the following equation holds:

$$\forall\, o \in \mathscr{O}, \quad \partial_2\, o = \sigma_2(o)\,\partial_2 + \delta_2(o).$$

Hence, we have $\partial_2\,\partial = \sigma_2(\partial)\,\partial_2 + \delta_2(\partial)$, therefore usually $\partial_2\,\partial \neq \partial\,\partial_2$ so ∂ and ∂_2 do not commute. Nevertheless, in many cases, the operators might commute, leading to the definition of an *Ore algebra*.

Definition 4 ([5]) If A is a $\mathbb{K}$-algebra, then an Ore extension $A\,[\partial_1; \sigma_1, \delta_1] \cdots [\partial_n; \sigma_n, \delta_n]$ of A is called an *Ore algebra* if the following conditions are satisfied

$$\begin{cases} \sigma_j(A) \subseteq A, \quad \delta_j(A) \subseteq A, \quad 1 \leq j \leq n, \\ \sigma_j(\partial_i) = \partial_i, \quad \delta_j(\partial_i) = 0, \quad 1 \leq i < j \leq n, \end{cases}$$

and the restrictions $\sigma_i|_A$ and $\delta_j\big|_A$ commute two by two, for all $1 \leq i, j \leq n$.

Example 4 Consider the polynomial ring $A = \mathbb{R}\,[x_1, x_2]$. Notice that $\beta_1 := \frac{\partial}{\partial x_1}$ is a derivation of A. The Ore extension $\mathscr{O}$ can be defined, $\mathscr{O} = A\,[\partial_1; \mathrm{id}_A, \beta_1]$ of A consists of the differential operators in ∂_1 with coefficients in A. Now consider the following derivation of $\mathscr{O}$:

$$\begin{aligned} \beta_2 := \frac{\partial}{\partial x_2} &: \mathscr{O} \longrightarrow \mathscr{O} \\ \textstyle\sum_{i=0}^{r} a_i(x_1, x_2)\,\partial_1^i &\longmapsto \textstyle\sum_{i=0}^{r} \frac{\partial a_i(x_1, x_2)}{\partial x_2}\,\partial_1^i. \end{aligned}$$

Then we can construct the Ore extension $B = \mathscr{O}\,[\partial_2; \mathrm{id}_{\mathscr{O}}, \beta_2]$ of $\mathscr{O}$. Since $\beta_2(\partial_1) = 0$, the operators ∂_1 and ∂_2 commute and B is an Ore algebra.

Example 5 Let A be a differential $\mathbb{R}$-algebra of real-valued functions of t with support on $\mathbb{R}_{\geq 0}$ or on $\mathbb{R}$, and $\delta = \frac{d}{dt}$. The Ore extension $\mathscr{O} = A\,[\partial; \mathrm{id}_A, \delta]$ of A is the ring of ordinary differential operators with coefficients in A. Let $h \in \mathbb{R}_{>0}$ and consider $\sigma \in \mathrm{End}(A)$ defined by $\sigma(a(t)) = a(t-h)$ for all $a \in A$. Using (1), we obtain:

$$\forall\, a \in A, \quad \partial\,\sigma(a) = \partial\,a(\cdot - h) = a(\cdot - h)\,\partial + \dot{a}(\cdot - h) = \sigma(a)\,\partial + \sigma(\delta(a)).$$

By Corollary 1, there exists a unique $\overline{\sigma} \in \mathrm{End}(\mathscr{O})$ such that $\overline{\sigma}_{|A} = \sigma$ and $\overline{\sigma}(\partial) = \partial$. To simplify the notations, we write σ for $\overline{\sigma}$. Hence, $\sigma \in \mathrm{End}(\mathscr{O})$ is defined by:

$$\begin{aligned} \sigma : \mathscr{O} &\longrightarrow \mathscr{O} \\ \textstyle\sum_{i=0}^{r} a_i(t)\, \partial^i &\longmapsto \textstyle\sum_{i=0}^{r} a_i(t-h)\, \partial^i. \end{aligned}$$

Then, by Proposition 1, we can construct the Ore extension $B = \mathscr{O}[S; \sigma, 0]$ of $\mathscr{O}$. For $a \in A$, we have

$$(\delta \circ \sigma)\,(a(t)) = \frac{d}{dt}\, a(t-h) = \dot{a}(t-h) = (\sigma \circ \delta)\,(a(t)),$$

which shows that $(\delta \circ \sigma)|_A = (\sigma \circ \delta)|_A$. By construction, it is clear that the remaining conditions for B to be an Ore algebra hold. In particular, we have:

$$S\, \partial = \sigma(\partial)\, S = \partial\, S.$$

Hence, $B = A\,[\partial; \mathrm{id}_A, \delta]\,[S; \sigma, 0]$ is the commutative polynomial ring of differential constant time-delay operators with coefficients in A.

Now, we consider the ring of *differential time-varying delay* (DTVD) operators. This ring is not an Ore algebra, as we now briefly explain.

In the remainder of the paper, we shall suppose that the function $h : \mathbb{R} \to \mathbb{R}$ is smooth or analytic function, and, since we only consider time-delay operators, that:

$$\forall\, t \geq 0, \quad h(t) \geq 0. \tag{4}$$

Remark 1 It is usually assumed that h satisfies $\dot{h}(t) < 1$ for all $t \in \mathbb{R}_{\geq 0}$. That can be interpreted as the fact that the delay cannot be faster, or as fast as, than time itself.

Let us consider a differential ring of real-valued functions $\left(A, \delta = \frac{d}{dt}\right)$ equipped with $\sigma \in \mathrm{End}(A)$ defined by:

$$\forall\, a \in A, \quad \sigma\,(a(t)) = a\,(t - h(t))\,. \tag{5}$$

For instance, we can consider $A = C^{\infty}(\mathbb{R})$. Then, we have:

$$\begin{aligned} (\delta \circ \sigma)(a(t)) &= \delta(a(t-h(t))) = (1 - \dot{h}(t))\, \dot{a}(t - h(t)) \\ &= (1 - \dot{h}(t))\, (\sigma \circ \delta)(a(t)). \end{aligned}$$

That implies:

$$(\delta \circ \sigma)|_A = \left(1 - \dot{h}\right)\, (\sigma \circ \delta)|_A\,. \tag{6}$$

If h is not a constant function (i.e., $\dot{h} \neq 0$), δ and σ do not commute, and it is then not possible to construct an Ore algebra of DTVD operators (see Definition 4).

3 An Ore Extension Construction

In the above Section, we have seen that the ring of DTVD operators cannot give rise to an Ore algebra. Nevertheless, it is possible to construct an Ore *extension* for this operator ring, and that is the goal of the present Section.

Let $\left(A, \delta = \frac{d}{dt}\right)$ be a differential ring of real-valued functions (see Definition 2) equipped with $\sigma \in \text{End}(A)$ defined by (5). Generalizing the construction made in Example 2, we can consider the skew polynomial ring $\mathcal{O} = A[S; \sigma, 0]$ of time-varying delay operators. If we note $e = (1 - \dot{h}(t))\, S \in \mathcal{O}$, then we have:

$$\forall\, a \in A, \quad e\, a = (1 - \dot{h}(t))\, S\, a = (1 - \dot{h}(t))\, a(t - h(t))\, S = \sigma(a)\, e.$$

By Corollary 1, there exists a unique $\sigma' \in \text{End}(\mathcal{O})$ such that:

$$\forall\, a \in A, \quad \sigma'(a) = a, \quad \sigma'(S) = e = (1 - \dot{h}(t))\, S.$$

Let us consider the following map:

$$\begin{aligned} \delta' : A[S; \sigma, 0] &\longrightarrow A[S; \sigma, 0] \\ o = \textstyle\sum_{i=0}^{r} a_i(t)\, S^i &\longmapsto \textstyle\sum_{i=0}^{r} \dot{a}_i(t)\, S^i = \frac{do}{dt}. \end{aligned}$$

In particular, we get $\delta'(a) = \dot{a}$ and $\delta'(S) = 0$. Moreover, we have

$$\begin{cases} \delta'(S\, a) = \delta'(\sigma(a)\, S) = \delta'(a(t - h(t))\, S) = (1 - \dot{h}(t))\, \dot{a}(t - h(t))\, S, \\ \delta'(S)\, a + \sigma'(S)\, \delta'(a) = e\, \delta'(a) = (1 - \dot{h}(t))\, \dot{a}(t - h(t))\, S, \end{cases}$$

for all $a \in A$, which shows that:

$$\forall\, a \in A, \quad \delta'(S\, a) = \delta'(S)\, a + \sigma'(S)\, \delta'(a) = e\, \delta'(a) = \sigma(\delta'(a))\, e.$$

More generally, we can check that δ' is a σ'-derivation of $\mathcal{O}$.
Hence, the Ore extension $\mathcal{D} = A[S; \sigma, 0]\left[\partial; \sigma', \delta'\right]$ of $\mathcal{O}$ can be defined. We have:

$$\forall\, o \in \mathcal{O}, \quad \partial\, o = \sigma'(o)\, \partial + \delta'(o).$$

In particular, we obtain:

$$\begin{cases} \partial\, a(t) = \sigma'(a(t))\, \partial + \delta'(a(t)) = a(t)\, \partial + \dot{a}(t), \\ \partial\, S = \sigma'(S)\, \partial + \delta'(S) = (1 - \dot{h}(t))\, S\, \partial. \end{cases}$$

This last identity encodes the commutation rule of ∂ and S in terms of h (see Sect. 6).

An element of $\mathscr{D}$ is of the form $d = \sum_{i=0}^{r} o_i \, \partial^i = \sum_{i=0}^{r} \sum_{j=0}^{s} a_{ij} \, S^j \, \partial^i$ with $a_{ij} \in A$. In the remaining of the paper, this ring $\mathscr{D}$ will be called a *ring of differential time-varying delay (DTVD) operators over* A.

Remark 2 Under the hypothesis that the function h satisfies the condition

$$\forall \, t, \quad \dot{h}(t) \neq 1, \tag{7}$$

a second construction of $\mathscr{D}$ can be given. Note that (7) is satisfied if the hypothesis of Remark 1 is assumed. Let $\mathscr{O}' = A[\partial; \mathrm{id}_A, \delta]$ be the Ore extension of A defined in Example 1, i.e., the ring of OD operators with coefficients in A. Moreover, let $\sigma \in \mathrm{End}(A)$ be defined by (5) and note $e = (1 - \dot{h})^{-1} \, \partial \in \mathscr{O}'$. Then, we have

$$\begin{aligned} e \, \sigma(a) &= \frac{1}{1 - \dot{h}(t)} \, \partial \, a(t - h(t)) = a(t - h(t)) \, \frac{1}{1 - \dot{h}(t)} \, \partial + \dot{a}(t - h(t)) \\ &= a(t - h(t)) \, e + \dot{a}(t - h(t)) = \sigma(a) \, e + \sigma(\dot{a}), \end{aligned}$$

for all $a \in A$. By Corollary 1, there exists a unique endomorphism of $\mathscr{O}'$, simply denoted by σ, which satisfies $\sigma(\partial) = e = (1 - \dot{h})^{-1} \, \partial$, i.e. $\sigma \in \mathrm{End}(\mathscr{O}')$ is given by:

$$\begin{aligned} \sigma : \mathscr{O}' &\longrightarrow \mathscr{O}' \\ \textstyle\sum_{i=0}^{r} a_i(t) \, \partial^i &\longmapsto \textstyle\sum_{i=0}^{r} a_i(t - h(t)) \left(\frac{1}{1 - \dot{h}} \, \partial \right)^i. \end{aligned} \tag{8}$$

If $h(t) = h \in \mathbb{R}_{\geq 0}$, then we note that $\sigma(\partial) = \partial$ and we find again Example 5.
Let us interpret (8). First denote $x(t) := t - h(t)$. Then, we have

$$\frac{d}{dt} = \frac{dx}{dt} \, \frac{d}{dx} = \left(1 - \dot{h}(t)\right) \, \frac{d}{dx},$$

and from that, it results:

$$\frac{d}{dx} = \frac{1}{1 - \dot{h}(t)} \, \frac{d}{dt}.$$

Set $\partial_t := \partial = \frac{d}{dt}$ and $\partial_x := \frac{d}{dx}$. Then, the map σ satisfies:

$$\sigma(a(t)) = a(x), \quad \sigma(\partial_t) = \partial_x.$$

Hence, σ is a *change of time-scale*, from time t to time $x(t) := t - h(t)$, and:

$$\sigma \left(\sum_{i=0}^{r} a_i(t) \, \partial_t^i \right) = \sum_{i=0}^{r} a_i(x) \, \partial_x^i. \tag{9}$$

Now, thanks to $\sigma \in \mathrm{End}(\mathscr{O}')$, an Ore extension of $\mathscr{O}'$ can be constructed by:

$$\mathscr{D}' = \mathscr{O}'\,[S;\sigma,0] = A\left[\partial;\,\mathrm{id}_A,\,\frac{d}{dt}\right][S;\sigma,0]\,.$$

An element $d' \in \mathscr{D}'$ can be written as $d' = \sum_{j=0}^{s} o'_j\, S^j = \sum_{j=0}^{s}\sum_{i=0}^{r} a_{ij}\,\partial^i\, S^j$ with $a_{ij} \in A$. By definition of an Ore extension, we have $S\,o = \sigma(o)\,S$ for all $o \in \mathscr{O}'$. In particular, setting $o = \partial$, we get $S\,\partial = (1-\dot{h})^{-1}\,\partial\,S$, i.e. $\partial\,S = (1-\dot{h})\,S\,\partial$ (see (6)).

We can check that $\mathscr{D}' = \mathscr{D}$, which gives a second construction of the ring of DTVD operators over A.

4 Algebraic Properties of the Ore Extension *D*

In this section, general properties of the ring $\mathscr{D}$ of DTVD operators defined in Sect. 3 are studied. For general concepts and structural definitions, we refer to [3, 18, 25]. Let us start with standard results on Ore extensions.

Theorem 1 ([18]) *Let A be a noncommutative ring and $D = A[\partial;\sigma,\delta]$ a skew polynomial ring.*

1. *If A is a domain*[1] *and σ is injective, then D is a domain.*
2. *If A is a left Ore domain*[2] *and σ is injective, then D is a left Ore domain.*
3. *If A is a left (right) Noetherian ring*[3] *and σ is bijective, then D is a left (right) Noetherian ring. Moreover, if A is a domain, then D is a left (right) Ore domain.*

A very useful corollary of Theorem 1 can then be proven:

Corollary 2 *Let $\mathscr{D} = A\left[\partial;\,\mathrm{id}_A,\,\frac{d}{dt}\right][S;\sigma,0]$ be the ring of DTVD operators defined in Sect. 3, where $\dot{h} \neq 1$. Then, we have:*

1. *If A is a domain, then so is $\mathscr{D}$.*
2. *If A is a left Ore domain, then so is $\mathscr{D}$.*
3. *If A is a left (right) Noetherian ring and if $\sigma|_A$ is an automorphism of A, then $\mathscr{D}$ is a left (right) Noetherian ring. Moreover, $\sigma|_A$ is an automorphism of A if and only if the function ℓ defined by*

$$\begin{aligned} \ell : \mathbb{R}_{>0} &\longrightarrow \mathbb{R} \\ t &\longmapsto t - h(t), \end{aligned} \tag{10}$$

admits an inverse $\ell^{-1} \in A$ such that $a \circ \ell^{-1} \in A$, for all $a \in A$. Finally, if A is a domain, then $\mathscr{D}$ is a left (right) Ore domain.

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[1] A ring A is a *domain* if A has no non-zero zero divisors.

[2] A ring A is a *left Ore domain* if A is a domain such that for $a_1,\, a_2 \in A \setminus \{0\}$, the intersection of the left ideals $A\,a_1 \cap A\,a_2 \neq 0$.

[3] A ring A is a *left (right) Noetherian ring* if every left (right) ideal of A is finitely generated.

Proof Since id_A and σ (as defined in (9), Sect. 3) are injective endomorphisms of A, then (1) and (2) are direct consequences of (1) and (2) of Theorem 1.

To prove (3), consider first that σ is an automorphism of A, so for every $a \in A$, there exists a unique $b \in A$ such that $a(t) = \sigma(b(t)) = b(\ell(t))$. In particular, for $t \mapsto t \in A$, we obtain $t = (b \circ \ell)(t)$. That is $b \circ \ell = \mathrm{id}_{\mathbb{R}_{>0}}$, so ℓ is injective and thus bijective on its image, and $b = \ell^{-1} \in A$. In addition, given $a \in A$, then $\sigma^{-1}(a) \in A$, where $\sigma^{-1}(a(t)) = a(\sigma^{-1}(t)) = a(\ell^{-1}(t))$ and that implies $a \circ \ell^{-1} \in A$. Conversely, if (10) admits an inverse $\ell^{-1} \in A$ and $a \circ \ell^{-1} \in A$ for all $a \in A$, then we can define $\sigma' \in \mathrm{End}(A)$ by:

$$\begin{aligned} \sigma' : A &\longrightarrow A \\ a(t) &\longmapsto a(\ell^{-1}(t)). \end{aligned}$$

We can check that $\sigma' \circ \sigma = \mathrm{id}_A = \sigma \circ \sigma'$, i.e., $\sigma = {\sigma'}^{-1}$ is an automorphism of A.

Next, we show that the automorphism σ of A extends to an automorphism of the ring $\mathcal{O} = A\left[\partial; \mathrm{id}_A, \frac{d}{dt}\right]$ of differential operators. For that, we consider

$$\begin{aligned} \sigma' : \mathcal{O} &\longrightarrow \mathcal{O} \\ a(t) &\longmapsto a(y) \\ \partial_t &\longmapsto \partial_y, \end{aligned}$$

where $y := \ell^{-1}$, hence $\partial_y := \frac{1}{\dot{y}(t)}\,\partial_t$. Similarly as it was proven for (8), it results that $\sigma' \in \mathrm{End}\,(\mathcal{O})$:

$$\begin{aligned} \sigma' : \mathcal{O} &\longrightarrow \mathcal{O} \\ \sum_{i=0}^{r} a_i(t)\,\partial_t^i &\longmapsto \sum_{i=0}^{r} a(y)\,\partial_y^i, \end{aligned}$$

Using $y \circ \ell = \mathrm{id} = \ell \circ y$, we have:

$$\begin{aligned} \ell(y(t)) = t \quad &\Rightarrow \quad \dot{\ell}(y(t))\,\dot{y}(t) = 1, \\ y(\ell(t)) = t \quad &\Rightarrow \quad \dot{y}(\ell(t))\,\dot{\ell}(t) = 1. \end{aligned}$$

Since $\dot{\ell} = 1 - \dot{h} \in A$, then $\frac{1}{\dot{y}} = \dot{\ell}(y) \in A$. Furthermore,

$$\begin{aligned} (\sigma' \circ \sigma)\left(\textstyle\sum_{i=0}^{r} a_i(t)\,\partial_t^i\right) = \sigma'\left(\textstyle\sum_{i=0}^{r} a_i(\ell(t))\,\left(\frac{1}{\dot{\ell}(t)}\,\partial_t\right)^i\right) &= \textstyle\sum_{i=0}^{r} a_i(t)\,\left(\frac{1}{\dot{\ell}(y(t))\,\dot{y}(t)}\,\partial_t\right)^i \\ &= \textstyle\sum_{i=0}^{r} a_i(t)\,\partial_t^i, \end{aligned}$$

$$(\sigma \circ \sigma')\left(\sum_{i=0}^{r} a_i(t)\, \partial_t^i\right) = \sigma\left(\sum_{i=0}^{r} a_i(y(t))\, \left(\frac{1}{\dot{y}(t)}\, \partial_t\right)^i\right) = \sum_{i=0}^{r} a_i(t)\, \left(\frac{1}{\dot{y}(\ell(t))\, \dot{\ell}(t)}\, \partial_t\right)^i$$
$$= \sum_{i=0}^{r} a_i(t)\, \partial_t^i$$

and it follows that σ is an automorphism of $\mathscr{O}$ and $\sigma' = \sigma^{-1}$. Finally, the results are consequences of (3) of Theorem 1. ■

An example of a function h as in (10) is given by $t \mapsto q\, t + \alpha \in A$ with $\alpha \in \mathbb{R}_{>0}$ and $0 < q < 1$.

Remark 3 In the proof of Corollary 2, we used the second construction of the ring of DTVD operators given in Remark 2. But the first construction can also be used. Note that the condition $\dot{h} \neq 1$ is also required so that $\sigma'(S) = (1 - \dot{h})\, S$ is injective.

Example 6 Consider two functions $h_1 : t \mapsto \frac{1}{1+t^2}$ and $h_2 : t \mapsto 1 - h_1(t)$. Then, we have $h_1, h_2 : \mathbb{R}_{>0} \to [0,\ 1[$. Moreover, the equations $1 - \dot{h}_1(t) = 1 + \frac{2\,t}{(1+t^2)^2} = 0$ and $1 - \dot{h}_2(t) = 1 - \frac{2\,t}{(1+t^2)^2} = 0$ have no real (positive) solutions. In addition, the functions $\ell_i : t \in \mathbb{R}_{>0} \longmapsto t - h_i(t) \in [0,\ 1[$ are bijective for $i = 1, 2$. Given $x \geq 0$, the equation $\ell_1(t) = t - h_1(t) = x$ admits a unique real positive solution defined by:

$$\ell_1^{-1}(x) := \frac{x}{3} + \frac{\alpha_1^{\frac{1}{3}}}{6} - \frac{1}{3\,\alpha_1^{\frac{1}{3}}}\,(3 - x^2),$$

where $\alpha_1 = 8\,x^3 + 72\,x + 108 + 12\,\sqrt{3\,(4\,x^4 + 4\,x^3 + 8\,x^2 + 36\,x + 31)}$. In a similar way, given $x \geq 0$, the equation $\ell_2(t) = t - h_2(t) = x$ admits a unique real solution defined by:

$$\ell_2^{-1}(x) = \frac{1}{3}\,(x + 1) + \frac{1}{6}\,\alpha_2^{\frac{1}{3}} + \frac{(x^2 + 2\,x - 2)}{3\,\alpha_2^{\frac{1}{3}}},$$

where $\alpha_2 = 8\,x^3 + 24\,x^2 + 96\,x + 188 + \sqrt{3\,(4\,x^4 + 20\,x^3 + 44\,x^2 + 80\,x + 83)}$. Hence, for $i = 1, 2$, if A_i is a differential field such that $a \circ \ell_i \in A$ and $a \circ \ell_i^{-1} \in A_i$ for all $a \in A_i$, then using Corollary 2, $\mathscr{D}_i = A_i\left[\partial; \mathrm{id}_{A_i}, \frac{d}{dt}\right][S_i; \sigma_{h_i}, 0]$ are Noetherian domains.

In what follows, for a given ring $\mathbf{A}$, we denote by $\mathbf{A}^{p \times q}$ the $\mathbf{A}$-module of $p \times q$-matrices with entries in $\mathbf{A}$. The following Theorem provides a helpful result on the effective study of linear systems over the ring $\mathscr{D}$ of DTVD operators.

Theorem 2 *Every left ideal of the Ore extension $\mathscr{D} = A\left[\partial; \mathrm{id}_A, \frac{d}{dt}\right][S; \sigma, 0]$, where A is a field and σ an automorphism, admits a Gröbner basis for an admissible monomial order [15] which can be computed by means of Buchberger's algorithm*

[15]. More generally, every left $\mathcal{D}$-submodule of $\mathcal{D}^{1\times p}$ admits a Gröbner basis for $p \in \mathbb{Z}_{\geq 0}$.

Proof If A is a field, then the Ore extension $\mathcal{D} = A\left[\partial; \mathrm{id}_A, \frac{d}{dt}\right][S; \sigma, 0]$ of DTVD operators is a bijective skew PBW extension (see [9] for the definitions). The authors in [14] proved that Gröbner techniques [15] hold over a bijective skew PBW extension, so the Theorem follows. ■

4.1 Algebraic Analysis Framework

This subsection contains some generalities on the algebraic analysis framework for linear systems theory. Here, D will denote an arbitrary ring.

Definition 5 Let D be a ring (of functional operators), $R \in D^{q\times p}$ and $\mathcal{F}$ a left D-module. A *linear functional system* or a *behavior* is defined by the Abelian group:

$$\ker_{\mathcal{F}}(R.) := \{\eta \in \mathcal{F}^{p\times 1} \mid R\,\eta = 0\}.$$

Let us consider the left D-submodule $D^{1\times q}\,R = \{\mu\,R \mid \mu \in D^{1\times q}\}$ of $D^{1\times p}$ consisting of all the left D-linear combinations of the columns of R. Let us denote by $M = D^{1\times p}/(D^{1\times q}\,R)$ the factor left D-module. The Abelian group of D-homomorphisms (i.e., left D-linear maps) from M to $\mathcal{F}$ [25] is denoted by $\hom_D(M, \mathcal{F})$. An important note concerns the D-module isomorphism,[4] following a standard homological algebra result:

$$\ker_{\mathcal{F}}(R.) \cong \hom_D(M, \mathcal{F}). \tag{11}$$

More details can be found in [3, 20, 23, 30]. A remarkable consequence of (11) is that a linear system can be studied by means of M (that encodes the system equations) and by $\mathcal{F}$ (that is the functional space where the solutions are sought).[5] More importantly, built-in properties of the linear system $\ker_{\mathcal{F}}(R.)$ such as controllability, observability, and flatness can be characterized by module properties of M.

Next, we state basic module properties and definitions [25].

Definition 6 Let D be a domain and M a left D-module. The *dual module* $M^\star$ is the right D-module defined by $M^\star := \hom_D(M, D)$.

There is a canonical homomorphism between M and its bidual module $M^{\star\star}$:

$$\begin{aligned} \varepsilon : M &\longrightarrow M^{\star\star} = \hom_D(\hom_D(M, D), D) \\ m &\longmapsto \varepsilon(m), \ \text{ with } \varepsilon(m)(f) := f(m), \ \forall f \in M^\star. \end{aligned} \tag{12}$$

[4] i.e., a bijective homomorphism of left D-modules

[5] The space $\mathcal{F}$ is also called the *signal space* in the behavioral approach [20].

Torsion modules play an important role in the theory of D-modules.

Definition 7 The *torsion submodule* $t(M)$ of a D-module M is defined by:

$$t(M) = \{m \in M \mid \exists\, d \in D \setminus \{0\} :\ d\, m = 0\}.$$

We recall that a left D-module M is *finitely generated* if $M = \sum_{i=1}^{r} D\, m_i$ with $m_i \in M$ and $r \in \mathbb{Z}_{\geq = 0}$.

Definition 8 Let D be a domain and M a *finitely generated* left D-module.

1. M is *free* if there exists $r \in \mathbb{Z}_{\geq 0}$ such that $M \cong D^{1\times r}$.
2. M is *stably free* if there exist $r,\ s \in \mathbb{Z}_{\geq 0}$ such that $M \oplus D^{1\times s} \cong D^{1\times r}$.
3. M is *projective* if there exist $r \in \mathbb{Z}_{\geq 0}$ and a left D-module N such that:

$$M \oplus N \cong D^{1\times r}.$$

4. M is *reflexive* if the canonical homomorphism $\varepsilon : M \longrightarrow M^{\star\star}$ defined by (12) is an isomorphism.
5. M is *torsion-free* if $t(M) = 0$.
6. M is *torsion* if $t(M) = M$.

It is clear that a free module is stably free, and a stably free module is projective. If M is a finitely generated free module, then $M \cong M^{\star\star}$, so M is a reflexive module.

If M is a reflexive module, then M is torsion-free. To prove that, note that:

$$\ker(\varepsilon) = \left\{m \in M \mid \forall\, f \in M^{\star} :\ f(m) = 0\right\}.$$

If $m \in t(M)$, then there exists $d \in D \setminus \{0\}$ such that $d\, m = 0$, hence $d\, f(m) = f(d\, m) = f(0) = 0$ for all $f \in M^{\star}$, and so $f(m) = 0$, since D is a domain. That shows $t(M) \subseteq \ker(\varepsilon)$, and consequently, M is torsion-free whenever M is reflexive.

There is a strong relation between basic ideas in control theory, such as recognizing when a system is *parametrizable* or *flat* [3, 7, 23, 24], and the equivalence of the above Definition 8. Recall that if D is a *principal ideal domain*[6] (e.g., $D = \mathbb{K}[\partial; \mathrm{id}, 0]$ with $\mathbb{K}$ a field), then a torsion-free module is free (see for instance [25]).

Homological invariants concepts (defined in [18, 25]) of the ring of DTVD operators are characterized in the sequel. These invariants play an important role in the algebraic analysis approach [3, 23]. We start by recalling basic definitions of homological algebra [25].

Definition 9 Let $(M_i)_{i\in\mathbb{Z}}$ be a sequence of left (resp., right) D-modules. For $i \in \mathbb{Z}$, let $\delta_i \in \hom_D(M_i, M_{i-1})$.

[6] In a principal ideal domain D, every left (resp. right) ideal of D is generated by a single element of D, that is, of the form $D\, d$ (resp., $d\, D$) for a certain $d \in D$ [25].

1. The sequence $(M_i, \delta_i)_{i\in\mathbb{Z}}$ forms a *complex* if $\delta_i \circ \delta_{i+1} = 0$, that is, if we have $\operatorname{im}(\delta_{i+1}) \subseteq \ker(\delta_i)$ for all $i \in \mathbb{Z}$. This complex is denoted by:

$$M_\bullet : \ \ldots \xrightarrow{\delta_{i+2}} M_{i+1} \xrightarrow{\delta_{i+1}} M_i \xrightarrow{\delta_i} M_{i-1} \xrightarrow{\delta_{i-1}} \ldots$$

2. The *defect of exactness* of the complex $M_\bullet$ at M_i is the left (resp., right) D-module defined by:
$$H_i(M_\bullet) = \ker(\delta_i)\,/\operatorname{im}(\delta_{i+1})\,.$$

3. The complex $M_\bullet$ is said to be *exact at* M_i if we have $H_i(M_\bullet) = 0$, i.e., if we have $\ker(\delta_i) = \operatorname{im}(\delta_{i+1})$, and is *exact* if:

$$H_i(M_\bullet) = 0, \ \forall\, i \in \mathbb{Z}.$$

We also say that this sequence is an *exact sequence*.
4. A *projective* (resp., *free*) *resolution* of a left D-module M is an exact sequence of the form

$$P_\bullet : \ \ldots \xrightarrow{\delta_3} P_2 \xrightarrow{\delta_2} P_1 \xrightarrow{\delta_1} P_0 \xrightarrow{\delta_1} M \longrightarrow 0,$$

where the P_i's are projective (resp., free) left D-modules. If $P_i = 0$ for $i \geq m+1$, then the *length* of the resolution $P_\bullet$ is said to be m. Similar definitions hold for right D-modules.
5. The *left projective dimension* of a left D-module M, denoted by $\operatorname{lpd}_D(M)$, is the length of the shortest projective resolution of M (and similarly for the *right projective dimension* $\operatorname{rpd}_D(M)$ of a right D-module M).
6. The *left* (resp., *right*) *global dimension* of D, denoted by $\operatorname{lgd}(D)$ (resp., $\operatorname{rgd}(D)$), is the supremum of $\operatorname{lpd}_D(M)$ (resp., $\operatorname{rpd}_D(M)$) over all the left (resp., right) D-modules M.

Theorem 3 (Auslander's theorem, Corollary 8.28 of [25]) *If A is a* Noetherian ring *(i.e., both left and right Noetherian ring), then:*

$$\operatorname{lgd}(A) = \operatorname{rgd}(A)\,.$$

We simply denote it by $\operatorname{gd}(A)$.

Definition 10 A ring A is called a *left* (resp., *right*) *regular ring* if every finitely generated left (resp., right) A-module has finite projective dimension.

Theorem 4 *([18]) Let A be a ring with finite left (resp., right) global dimension* $\operatorname{lgd}(A)$ *(resp.,* $\operatorname{rgd}(A)$*) and $\sigma \in$* $\operatorname{End}(A)$. *Then, we have:*

1. $\operatorname{lgd}(A) \leq \operatorname{lgd}(A[\partial; \sigma, \delta]) \leq \operatorname{lgd}(A) + 1$
(resp., $\operatorname{rgd}(A) \leq \operatorname{rgd}(A[\partial; \sigma, \delta]) \leq \operatorname{rgd}(A) + 1$*).*

2. *If* $\delta = 0$, *then* $\mathrm{lgd}\,(A[\partial; \sigma, \delta]) = \mathrm{lgd}\,(A) + 1$ *(resp.,* $\mathrm{rgd}\,(A[\partial; \sigma, \delta]) = \mathrm{rgd}\,(A) + 1$*).*
3. *If A is a semisimple Artinian ring (e.g., A is a field), then* $\mathrm{lgd}\,(A[\partial; \sigma, \delta]) = 1$ *(resp.,* $\mathrm{rgd}\,(A[\partial; \sigma, \delta]) = 1$*).*

4.2 Some Properties of the Ring of DTVD Operators

Using the results recalled in (4.1), we can prove the following results:

Proposition 2 *Let* $\mathscr{D} = A\left[\partial; \mathrm{id}_A, \frac{d}{dt}\right][S; \sigma, 0]$ *be the ring of DTVD operators defined in Sect. 3. Assume that the function* $\ell : t \longmapsto t - h(t)$ *is bijective.*[7] *If A has finite left (resp., right) global dimension* $\mathrm{lgd}\,(A)$ *(resp.,* $\mathrm{rgd}\,(A)$*), then:*

$$\mathrm{lgd}\,(A) + 1 \leq \mathrm{lgd}\,(\mathscr{D}) \leq \mathrm{lgd}\,(A) + 2$$

(resp., $\mathrm{rgd}\,(A) + 1 \leq \mathrm{rgd}\,(\mathscr{D}) \leq \mathrm{rgd}\,(A) + 2$*). In particular, $\mathscr{D}$ is a left (resp., right) regular ring. If A is a semisimple Artinian ring,*[8] *then* $\mathrm{lgd}\,(\mathscr{D}) = 2$ *(resp.,* $\mathrm{rgd}\,(\mathscr{D}) = 2$*). Finally, if A is a Noetherian ring, then* $\mathrm{gd}\,(\mathscr{D}) = \mathrm{lgd}\,(\mathscr{D}) = \mathrm{rgd}\,(\mathscr{D})$.

This Proposition is a consequence of Theorem 4.

Example 7 Considering Example 6, if $\mathscr{D} = \mathscr{D}_1$ or $\mathscr{D} = \mathscr{D}_2$, then we have $\mathrm{gd}\,(\mathscr{D}) = 2$.

Definition 11 A ring A is said to be *projective stably free* if every finitely generated projective left/right A-module is stably free.

Theorem 5 ([27]) *If A is a left regular Noetherian ring and projective stably free (e.g., A is a field), then so is the ring* $\mathscr{D} = A\left[\partial; \mathrm{id}_A, \frac{d}{dt}\right][S; \sigma, 0]$.

Proof It is a consequence of Corollary 12.3.3 of [18]. ■

Corollary 3 *Let* $\mathscr{D} = A\left[\partial; \mathrm{id}_A, \frac{d}{dt}\right][S; \sigma, 0]$ *be the ring of DTVD operators defined in Sect. 3. Assume that the function* $\ell : t \longmapsto t - h(t)$ *is bijective. Let* $R \in \mathscr{D}^{q \times p}$ *and* $M = \mathscr{D}^{1 \times p} / (\mathscr{D}^{1 \times q}\, R)$ *be the left $\mathscr{D}$-module finitely presented by R. Then, M admits a free resolution of length less than or equal to* $\mathrm{lgd}\,(\mathscr{D}) + 1$.

Proof This can be proved as in Proposition 8 of [3]. ■

The reader may refer to [3, 23] and to the OREMODULES package [4] for an explicit way to compute free resolutions of finitely generated left D-modules based on Gröbner basis techniques.

We denote by $\mathrm{lKdim}\,(A)$ the *left Krull dimension* of the ring A. For more details, see [18] and the references therein.

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[7] e.g., $h(t) = q\,t + h$ with $h \in \mathbb{R}_{\geq 0}$ and $0 < q < 1$.

[8] e.g., A is a field.

Theorem 6 (Proposition 6.5.4 of [18]) *If A is a left Noetherian ring, σ an automorphism of A and δ a σ-derivation, then:*

1. $\mathrm{lKdim}\,(A) \leq \mathrm{lKdim}\,(A[\partial; \sigma, \delta]) \leq \mathrm{lKdim}\,(A) + 1$.
2. $\mathrm{lKdim}\,(A[S; \sigma, 0]) = \mathrm{lKdim}\,(A)$.
3. *If A is a left Artinian ring, then* $\mathrm{lKdim}\,(A[S; \sigma, 0]) = 1$.

Corollary 4 *Let $\mathscr{D} = A\left[\partial; \mathrm{id}_A, \frac{d}{dt}\right][S; \sigma, 0]$ be the ring of DTVD operators defined in Sect. 3, where the function $\ell : t \longmapsto t - h(t)$ is bijective, then:*

$$\mathrm{lKdim}\,(A) \leq \mathrm{lKdim}\,(\mathscr{D}) \leq \mathrm{lKdim}\,(A) + 1.$$

If D is a left Noetherian domain, then we can define its *total quotient field/field of fractions* $K := \{d^{-1}\, n \mid 0 \neq d,\, n \in D\}$ and the *rank* of a left D-module M is defined by $\mathrm{rank}_D(M) := \dim_K(K \otimes_D M)$, where $K \otimes_D M$ is the left K-vector space obtained by extending the scalars of M from D to K [25].

Theorem 7 (Theorems 11.1.14 and 11.1.17 of [18]) *If D is a left Noetherian domain, then any stably free left D-module M with* $\mathrm{rank}_D(M) \geq \mathrm{lKdim}(D) + 1$ *is free.*

5 Algebraic Analysis Approach

Let D be a Noetherian domain. In this Section, we briefly introduce the concept of *extension modules* [25]. Let $M = D^{1\times p}/(D^{1\times q}\, R)$ be the left D-module *finitely presented* by the system matrix $R \in D^{q\times p}$, which is associated with the linear system:

$$\ker_{\mathscr{F}}(R.) = \{\eta \in \mathscr{F}^p \mid R\,\eta = 0\}.$$

Then, we have the following exact sequence

$$0 \longrightarrow \ker_D(.R) \xrightarrow{\;i\;} D^{1\times q} \xrightarrow{\;.R\;} D^{1\times p} \xrightarrow{\;\pi\;} M \longrightarrow 0\,,$$

where $.R \in \hom_D(D^{1\times q}, D^{1\times p})$ is defined by $(.R)(\mu) = \mu\, R$ for all $\mu \in D^{1\times q}$, $\ker_D(.R) = \{\mu \in D^{1\times q} \mid \mu\, R = 0\}$ its kernel, i is the canonical injection and $\pi \in \hom_D(D^{1\times p}, M)$ is the canonical projection which sends $\lambda \in D^{1\times p}$ onto its *residue class* $\pi(\lambda) \in M$ (note that $\pi(\lambda) = \pi(\lambda')$ iff $\lambda - \lambda' \in D^{1\times q}\, R$). Since D is a left Noetherian ring, then the finitely generated left D-module $D^{1\times q}$ is Noetherian, and thus, $\ker_D(.R)$ is a finitely generated left D-module. Therefore, there exists a finite set of generators $\{\mu_j\}_{j=1,\ldots,r}$ of $\ker_D(.R)$ and we have $\ker_D(.R) = \mathrm{im}_D(.R_2) = D^{1\times r}\, R_2$ for $R_2 = (\mu_1^T\ \ldots\ \mu_r^T)^T \in D^{r\times q}$. Thus, we obtain the following long exact sequence of left D-modules:

$$0 \longrightarrow \ker_D(.R_2) \xrightarrow{\;i_2\;} D^{1\times r} \xrightarrow{\;.R_2\;} D^{1\times q} \xrightarrow{\;.R\;} D^{1\times p} \xrightarrow{\;\pi\;} M \longrightarrow 0.$$

Repeating the same reasoning as above for $\ker_D(.R_2)$ and so on, we finally obtain a *free resolution* of M

$$\ldots \xrightarrow{.R_3} D^{1\times p_2} \xrightarrow{.R_2} D^{1\times p_1} \xrightarrow{.R_1} D^{1\times p_0} \xrightarrow{\pi} M \longrightarrow 0,$$

where $p_0 = p$, $p_1 = q$, $p_2 = r$ and $R_1 = R$. Applying the *contravariant left exact functor* $\hom_D(\,\cdot\,, \mathscr{F})$ [25] to the *truncated free resolution* of M defined by

$$\ldots \xrightarrow{.R_3} D^{1\times p_2} \xrightarrow{.R_2} D^{1\times p_1} \xrightarrow{.R_1} D^{1\times p_0} \longrightarrow 0, \tag{13}$$

i.e., dualizing (13), we get the following complex of Abelian groups

$$\mathscr{F}^{\bullet} : \ \ldots \xleftarrow{R_3.} \mathscr{F}^{p_2} \xleftarrow{R_2.} \mathscr{F}^{p_1} \xleftarrow{R_1.} \mathscr{F}^{p_0} \longleftarrow 0,$$

where $(R_i.)(\eta) = R_i\,\eta$ for all $\eta \in \mathscr{F}^{p_{i-1}}$ for $i \geq 1$. The defects of exactness of $\mathscr{F}^{\bullet}$ are characterizes by the *cohomology* groups of $\mathscr{F}^{\bullet}$ defined by:

$$\begin{cases} H^0(\mathscr{F}^{\bullet}) = \ker_{\mathscr{F}}(R_1.), \\ H^i(\mathscr{F}^{\bullet}) = \ker_{\mathscr{F}}(R_{i+1}.)/\mathrm{im}_{\mathscr{F}}(R_i.), \quad i \geq 1. \end{cases}$$

Using (11), we obtain $H^0(\mathscr{F}^{\bullet}) \cong \hom_D(M, \mathscr{F})$. More generally, an important result of homological algebra proves that the cohomologies $H^i(\mathscr{F}^{\bullet})$ do not depend on the choice of the free resolution of M, i.e., up to isomorphism, they depend only on M and $\mathscr{F}$ [25]. They are then denoted by

$$\begin{cases} \mathrm{ext}^i_D(M, \mathscr{F}) = \hom_D(M, \mathscr{F}), \\ \mathrm{ext}^i_D(M, \mathscr{F}) = H^i(\mathscr{F}^{\bullet}), \quad i \geq 1, \end{cases}$$

and are called *extension Abelian groups*. In the case where $\mathscr{F} = D$, we can prove that the $\mathrm{ext}^i_D(M, D)$ inherit a right D-module structure. Similarly, if M is a right D-module, then the $\mathrm{ext}^i_D(M, D)$ inherit left D-module structures. For an implementation of the computation of the $\mathrm{ext}^i_D(M, D)$'s for certain Ore algebras, see the OREMODULES package [4].

Due to the homological nature of the main results of [3] (see also [23]), they can directly be applied to the ring of DTVD operators.

Theorem 8 *Let A be a regular Noetherian ring and $\mathscr{D} = A\left[\partial; \mathrm{id}_A, \frac{d}{dt}\right][S; \sigma, 0]$ the ring of DTVD operators with coefficients in A defined in Sect. 3. Assume that the function $\ell : t \longmapsto t - h(t)$ is bijective. Moreover, let $R \in \mathscr{D}^{q\times p}$, $M = \mathscr{D}^{1\times p}/(\mathscr{D}^{1\times q}\, R)$ and $N = \mathscr{D}^{q\times 1}/(R\, \mathscr{D}^{p\times 1})$ the so-called* Auslander transpose *of M. Then, we have:*

1. *$t(M) \cong \mathrm{ext}^1_{\mathscr{D}}(N, \mathscr{D})$.*
2. *M is torsion-free if and only if $\mathrm{ext}^1_{\mathscr{D}}(N, \mathscr{D}) = 0$.*

3. *M is reflexive if and only if* $\mathrm{ext}^1_{\mathscr{D}}(N, \mathscr{D}) = \mathrm{ext}^2_{\mathscr{D}}(N, \mathscr{D}) = 0$.
4. *M is projective if and only if* $\mathrm{ext}^i_{\mathscr{D}}(N, \mathscr{D}) = 0$ *for* $i = 1, \ldots, \mathrm{gd}\,(\mathscr{D})$.
5. *If A is a projective free ring, then M is projective left $\mathscr{D}$-module if and only if M is stably free left $\mathscr{D}$-module. Moreover, M is free when:*

$$\mathrm{rank}_{\mathscr{D}}(M) \geq \mathrm{lKdim}(\mathscr{D}) + 1.$$

Proof Assertions 1 and 2 are direct consequences of Theorem 5 of [3]. The assertion 3 (resp., 4) is a consequence of Theorem 6 (resp., Theorem 7) of [3]. Finally, let us prove 5. A stably free left $\mathscr{D}$-module is well-known to be projective. The converse is proved in Corollary 12.3.3 of [18]. ■

Within the algebraic analysis framework, the concept of an *injective cogenerator signal space* $\mathscr{F}$ plays a similar role as an algebraic *closed field* in algebraic geometry (think about the solutions of $x^2 + 1 = 0$ in $\mathbb{R}$). A non-trivial module $M = D^{1\times p}/(D^{1\times q}\, R)$ then defines a non-zero linear system/behavior $\ker_{\mathscr{F}}(R.)$. Moreover, a complete duality exists between linear systems/behaviors and finitely presented left modules [3, 20, 23, 30].

Definition 12 ([25])

1. A left D-module $\mathscr{F}$ is *cogenerator* if for every left D-module M and $m \in M \setminus \{0\}$, there exists $f \in \hom_D(M, \mathscr{F})$ such that $f(m) \neq 0$.
2. A left D-module $\mathscr{F}$ is *injective* if for all left D-modules M, $\mathrm{ext}^i_D(M, \mathscr{F}) = 0$ for all $i \geq 1$.

For a given ring D, it can be shown that an injective cogenerator left D-module $\mathscr{F}$ always exists (see, e.g., [25]).

According to [3, 7, 20, 22, 23, 30], we can state the following general definitions.

Definition 13 Let D be a Noetherian domain, $R \in D^{q\times p}$, $\mathscr{F}$ an injective cogenerator left D-module and $\ker_{\mathscr{F}}(R.)$ the linear system defined by R.

1. An *observable* $\psi(\eta)$ of $\ker_{\mathscr{F}}(R.)$ is a left D-linear combination of the system variables, i.e., $\psi(\eta) = \sum_{i=1}^p d_i\, \eta_i$, where $d_i \in D$ and $\eta = (\eta_1 \ \ldots \ \eta_p)^T \in \ker_{\mathscr{F}}(R.)$.
2. An observable $\psi(\eta)$ is called *autonomous* if it satisfies a left D-linear relation by itself, i.e., $d\, \psi(\eta) = 0$ for some $d \in D \setminus \{0\}$. It is called *free* it is not autonomous.
3. The linear system is said to be *controllable* if every observable is free.
4. The linear system is said to be *parametrizable* if there exists $Q \in D^{p\times m}$ such that

$$\ker_{\mathscr{F}}(R.) = \mathrm{im}_{\mathscr{F}}(Q.) = Q\, \mathscr{F}^m,$$

   i.e., if for every $\eta \in \ker_{\mathscr{F}}(R.)$, there exists $\xi \in \mathscr{F}^m$ such that $\eta = Q\, \xi$. Then, Q is called a *parametrization* and ξ a *potential*.
5. The linear system is said to be *flat* if there exists a parametrization $Q \in D^{p\times m}$ which admits a left inverse $T \in D^{m\times p}$, i.e. $T\, Q = I_p$. In other words, a flat system is a parametrizable system such that every component ξ_i of a potential ξ is an observable of the system. The potential ξ is then called a *flat output*.

The next Theorem explicitly characterizes the above definitions in terms of properties of modules.

Theorem 9 ([3, 7]) *Let $M = D^{1\times p}/(D^{1\times q}\, R)$ be the finitely presented left D-module associated with the linear system $\ker_{\mathscr{F}}(R.) = \{\eta \in \mathscr{F}^p \mid R\,\eta = 0\}$. With the hypotheses of Definition 13, we have:*

1. *The observables of the linear system are in one-to-one correspondence with the elements of M.*
2. *The autonomous elements of the linear system are in one-to-one correspondence with the torsion elements of M. The linear system is controllable if and only if M is torsion-free.*
3. *The linear system is parametrizable if and only if there exists a matrix $Q \in D^{p\times m}$ such that $M \cong D^{1\times p}\, Q$, i.e. M is a torsion-free left D-module. Then, the matrix Q is a parametrization, i.e. $\ker_{\mathscr{F}}(R.) = Q\,\mathscr{F}^m$. A parametrization Q can be computed by checking that $\mathrm{ext}^1_D(N, D) = 0$, where $N = D^{q\times 1}/(R\, D^{p\times 1})$.*
4. *The linear system is flat if and only if M is a free left D-module. Then, the bases of M are in one-to-one correspondence with the flat outputs of the linear system.*

Combining Theorems 8 and 9, the system properties listed in Definition 13 can be explicitly characterized in terms of module properties and in terms of the vanishing of the extension modules $\mathrm{ext}^i_D(N, D)$. Hence, the future implementation of Gröbner basis techniques for the ring D (see Theorem 2) in the OreModules package [4] will give us an effective way to check the system properties given in Definition 13.

Example 8 Let us consider the system $\dot{x}(t) = u(t - h(t))$ with $\ell : t \longmapsto t - h(t)$ bijective. Let $\mathscr{D}$ be a ring of DTVD operators, $R = (\partial \quad -S) \in \mathscr{D}^{1\times 2}$ the system matrix, $M = \mathscr{D}^{1\times 2}/(\mathscr{D}\, R)$ the left $\mathscr{D}$-module finitely presented by R and $N = \mathscr{D}/(R\,\mathscr{D}^{2\times 1})$ the Auslander transpose of M. Using the identity

$$\partial\, S = (1 - \dot{h})\, S\, \partial = S\, \left(\sigma^{-1}\left(1 - \dot{h}\right)\, \partial\right),$$

if we note

$$Q = \left(S \quad \sigma^{-1}\left(1 - \dot{h}\right)\, \partial\right)^T \in \mathscr{D}^{2\times 1},$$

then we have $\mathrm{im}_{\mathscr{D}}(Q.) = \ker_{\mathscr{D}}(R.)$. Thus, we obtain the free resolution:

$$0 \longleftarrow N \xleftarrow{\kappa} \mathscr{D} \xleftarrow{R.} \mathscr{D}^{2\times 1} \xleftarrow{Q.} \mathscr{D} \longleftarrow 0.$$

Dualizing this free resolution, we get the following complex

$$0 \longrightarrow \mathscr{D} \xrightarrow{.R} \mathscr{D}^{1\times 2} \xrightarrow{.Q} \mathscr{D} \longrightarrow 0,$$

which yields:

$$\begin{cases} \mathrm{ext}^1_{\mathscr{D}}(N, \mathscr{D}) = \ker_{\mathscr{D}}(.Q)/\mathrm{im}_{\mathscr{D}}(.R) \cong t(M), \\ \mathrm{ext}^2_{\mathscr{D}}(N, \mathscr{D}) = \mathscr{D}/(\mathscr{D}^{1\times 2}\, Q) = \mathscr{D}/\,(S,\ \partial) \neq 0. \end{cases}$$

We obtain that M is torsion-free but it is not a projective left $\mathscr{D}$-module since $\ker_{\mathscr{D}}(.Q) = \mathrm{im}_{\mathscr{D}}(.R)$ and $\mathrm{gd}\,(\mathscr{D}) = 2$.

If $\mathscr{F}$ is an injective left $\mathscr{D}$-module, then applying the contravariant exact functor $\hom_{\mathscr{D}}(\,\cdot\,, \mathscr{F})$ to the above complex results in the exact sequence of Abelian groups:

$$0 \longleftarrow \mathscr{F} \xleftarrow{R.} \mathscr{F}^2 \xleftarrow{Q.} \mathscr{F} \longleftarrow \ker_{\mathscr{F}}(Q.) \longleftarrow 0.$$

We get $\ker_{\mathscr{F}}(R.) = \mathrm{im}_{\mathscr{F}}(Q.)$, showing that Q is a parametrization of $\ker_{\mathscr{F}}(R.)$, so:

$$\begin{cases} x(t) = S\,\xi(t) = \xi(t - h(t)), \\ u(t) = \sigma^{-1}\left(1 - \dot{h}\right)\partial\,\xi(t) = \left(1 - \dot{h}(\ell^{-1}(t))\right)\dot{\xi}(t). \end{cases}$$

Since σ is an automorphism of $A\left[\partial;\, \mathrm{id}_A,\, \frac{d}{dt}\right]$, then we can define *the skew Laurent polynomial ring* $E = A\left[\partial;\, \mathrm{id}_A,\, \frac{d}{dt}\right][S;\, \sigma, 0][S^{-1};\, \sigma^{-1}, 0]$ [18]. Hence, $E \otimes_{\mathscr{D}} M$ is a free left E-module of rank 1 and x is a basis of $E \otimes_{\mathscr{D}} M$, i.e., the DTVD system is a *S-flat* system [7].

6 Conclusion

In this work, an algebraic analysis framework for the ring of differential time-varying delay operators was proposed via its realization as an Ore extension $\mathscr{D}$. An explicit construction of this ring was given, providing a novel algebraic approach for the study of linear differential varying time-delay systems. Homological algebraic properties of $\mathscr{D}$ were studied, and its global and Krull dimensions were analyzed.

Directions for future research include the possibility of $\mathscr{D}$ being an Auslander regular ring or a Cohen Macaulay ring [18]. Another important point would be to study the existence of an *involution* of $\mathscr{D}$ so that Gröbner bases can be calculated for right $\mathscr{D}$-modules (see [3] for more details). As a significant issue for the constructive aspects, Gröbner basis techniques for $\mathscr{D}$ will then be implemented.

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