Chapter 6
Delay Effects in Visual Tracking Problems for an Optronic Sighting System

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Abstract In this chapter, we study the delay effects in visual tracking problems for an optronic sighting system. We first describe the physical model and then give a simplified version defined by an integrator and a time-delay. We then state the visual tracking problems that are considered. To solve these problems, we first have to study the stabilization problem for the system defined above. Since this problem is a particular case of the general problem of parametrizing all the stabilizing controllers of a stable perturbation of a (infinite-dimensional) stabilizable plant, this problem is studied in its generality. Within the fractional representation approach to synthesis problems, we give an elementary proof for the existence of a general parametrization of all the stabilizing controllers of a stabilizable plant which does not necessarily admit doubly coprime factorizations. Only the knowledge of a (finite-dimensional) stabilizing controller is required. If the plant admits doubly coprime factorizations, then this parametrization yields the Youla-Kučera parametrization. Finally, using the above results, we study the tracking problems and show numerical simulations in which our results are compared with a PID and a H_∞-controller.

6.1 Automatic Visual Tracker

In practice, a gyrostabilized optronic payload orients its line of sight in the space [3, 5]. In this chapter, we shall only consider a simplified version, namely the plane case. A visual tracker is a combination of the following four elements [2, 4]:

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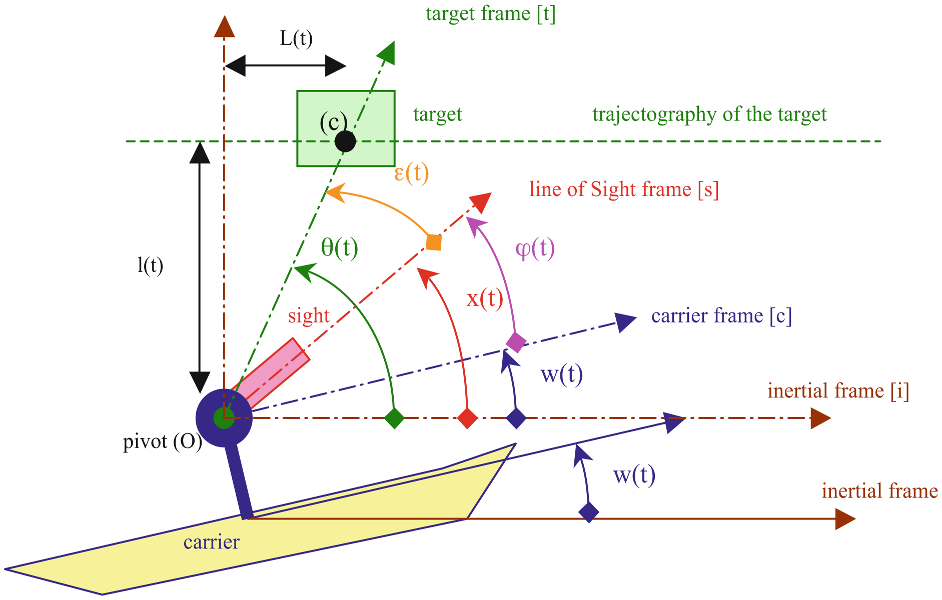

Fig. 6.1 Inertially stabilized boat camera platform

- A gyrostabilized optronic payload is a torque motorized speed controlled platform using the gyrometer speed.
- The optics consists of a digital video camera or a thermal imager set on the platform.
- Using image processing, an automatic image tracker detects the target and returns its coordinates in the frame images with a certain delay.
- A tracker controls the inertial speed of the inertially stabilized imager.

Let $[i]$ be the inertial frame with origin (O). See Fig. 6.1.

To each body, we attach a frame at (O): $[c]$ is the frame of the carrier, $[s]$ is the frame of the line of sight of the video camera and $[t]$ is the frame attached to the *target* located at the point (C). The line of sight can rotate from the carrier $[c]$ thanks to a motorized pivot linkage at (O). Let us now introduce the different angles:

- x is the angle defined by the line of sight of the camera in the inertial frame $[i]$.
- θ is the polar coordinate of the target in the inertial frame $[i]$.
- The angle between $[t]$ and $[s]$ is given by the image obtained by the camera:

$$\varepsilon := \theta - x. \tag{6.1}$$

The angle ε between the target and the line of sight is not directly accessible. To measure it, one can use an image processing device called the *image tracker*. See Fig. 6.2.

The coordinates of the target are determined by means of digit image correlation techniques or *centroid detection* methods. The position of the target can be characterized in terms of a shift of N pixels from the center of the image. At the focal distance of the camera D_f and the size of the optical sensor D, we have $\varepsilon \simeq \tan \varepsilon = \frac{D\,N}{D_f\,N_{\max}}$. The image processing introduces delays and two constraints:

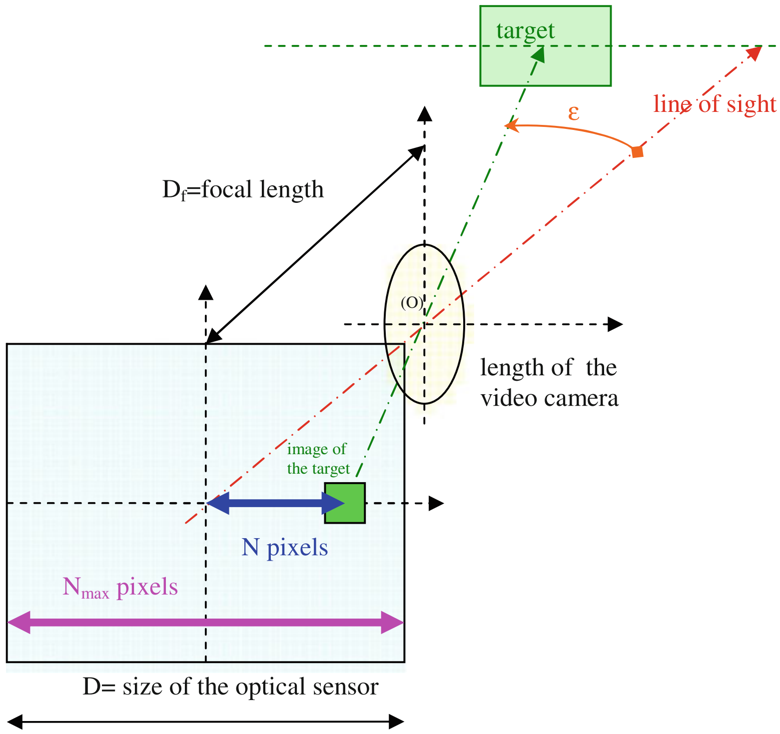

Fig. 6.2 Image tracker

- The image tracker yields a time-delay $T \in [0.04\,\text{s}, 0.2\,\text{s}]$ and a distributed delay $\tau_i \in [1\,\text{ms}, 40\,\text{ms}]$ (which can be neglected), and is corrupted by a noise u_1:

$$\widehat{e}_1 = \widehat{u}_1 - G\widehat{\varepsilon}, \quad G := e^{-Ts}\,\frac{1-e^{-\tau_i s}}{\tau_i\, s} \simeq e^{-Ts}. \tag{6.2}$$

- The image of the target must stay within the image of the video camera, i.e., $|\varepsilon(t)| \leq C_1$ for all t, and the signal processing imposes $|\dot{\varepsilon}(t)| \leq C_2$ for all t.

The movement of the target is unknown. Its cartesian coordinates are $(L,\ l)$ in the inertial frame $[i]$. See Fig. 6.1. The trigonometric relation between the polar coordinates and the cartesian ones yield $\theta = \arctan\,(l/L)$. If we set $\theta_0^{(n)} := \theta^{(n)}(0) \in \mathbb{R}$, where $\theta^{(n)}$ denotes the n^{th} derivative of θ with respect to t, then the following three scenarios are admissible:

- **Scenario 1:** Constant position: $\theta = \theta_0$.
- **Scenario 2:** Constant angular speed: $\theta = {\theta_0}^{(1)}\, t + \theta_0$.
- **Scenario 3:** Constant angular acceleration: $\theta = {\theta_0}^{(2)}\, t^2 + {\theta_0}^{(1)}\, t + \theta_0$.

A gyrometer observes the speed $\dot{x}$ of the line of sight. The gyrostabilized platform is modeled by an inner speed loop used by the outer video tracking loop. The transfer function from the reference speed of the inner loop y_1 to the real sight speed $\dot{x}$ can be written as $s\,\widehat{x} = F\,\widehat{y}_1$, where F is a low-pass filter and $\lim_{s\to 0} F(s) = 1$. For more details, [9]. We consider a controller $\widehat{y}_1 = C'\,\widehat{e}_1$ from the output signal $\widehat{e}_1$ of the

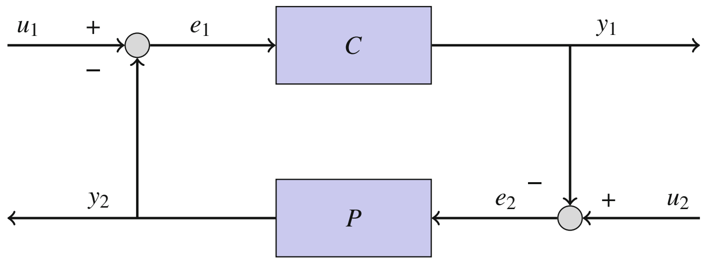

Fig. 6.3 Closed-loop system

tracker to the reference signal $\widehat{y}_1$. If we set $\widehat{u}_2 := \frac{s}{F}\,\widehat{\theta} \approx s\,\widehat{\theta}$, where $\widehat{u}_2$ is the target speed, then (6.1), and (6.2) yield:

$$\begin{cases} \widehat{\varepsilon} = \widehat{\theta} - \widehat{x} = \frac{F}{s}\,(\widehat{u}_2 - \widehat{y}_1), \\ \widehat{e}_1 = \widehat{u}_1 - G\,\widehat{\varepsilon}. \end{cases} \tag{6.3}$$

The main goal is to design a stabilizing controller C' such that $\lim_{t \to +\infty} \varepsilon(t) = 0$ for the above scenarios. To do that, let us first review results on stabilizability.

6.2 Parametrizations of all Stabilizing Controllers

Within the *fractional representation approach* to analysis and synthesis problems [1, 10], the class of systems that are considered are defined by transfer matrices with entries in the quotient field $Q(A) := \{n/d \,|\, 0 \neq d,\ n \in A\}$ of an integral domain A of SISO stable plants. Integral domains commonly considered are the Hardy algebra $H_\infty(\mathbb{C}_+)$ of bounded holomorphic functions in $\mathbb{C}_+ := \{s \in \mathbb{C} \mid \Re(s) > 0\}$, $RH_\infty := \mathbb{R}(s) \cap H_\infty(\mathbb{C}_+)$, the Wiener algebras $\widehat{\mathscr{A}}$ or W_+, the disc algebra $A(\mathbb{D})$, ... For more details, see [1, 10]. Let us recall a few standard definitions [1, 10].

Definition 1 Let A be an integral domain of stable SISO plants and $K := Q(A)$.

- A *fractional representation* of the transfer matrix $P \in K^{q \times r}$ is any representation of the form $P = D^{-1}\,N = \widetilde{N}\,\widetilde{D}^{-1}$, where $D \in A^{q \times q}$, $\det D \neq 0$, $N \in A^{q \times r}$, $\widetilde{N} \in A^{q \times r}$, $\widetilde{D} \in A^{r \times r}$ and $\det \widetilde{D} \neq 0$.
- The plant $P \in K^{q \times r}$ is said to be *(internally) stabilizable* if there exists a *stabilizing controller* $C \in K^{r \times q}$ of P, namely a controller $C \in K^{r \times q}$ such that

$$H(P,\ C) := \begin{pmatrix} I_q & P \\ C & I_r \end{pmatrix}^{-1} \in A^{(q+r) \times (q+r)},$$

where, using Fig. 6.3, $(e_1^T \quad e_2^T)^T = H(P, C)\,(u_1^T \quad u_2^T)^T$ is defined by:

$$H(P,\ C) = \begin{pmatrix} (I_q - P\,C)^{-1} & -(I_q - P\,C)^{-1}\,P \\ -C\,(I_q - P\,C)^{-1} & I_r + C\,(I_q - P\,C)^{-1}\,P \end{pmatrix} = \begin{pmatrix} I_q + P\,(I_r - C\,P)^{-1}\,C & -P\,(I_r - C\,P)^{-1} \\ -(I_r - C\,P)^{-1}\,C & (I_r - C\,P)^{-1} \end{pmatrix}. \tag{6.4}$$

- A transfer matrix $P \in K^{q\times r}$ admits a *left-coprime factorization* if there exist $D \in A^{q\times q}$, $\det D \neq 0$, $N \in A^{q\times r}$, $X \in A^{q\times q}$ and $Y \in A^{r\times q}$ such that $P = D^{-1}\,N$ and $D\,X - N\,Y = I_q$.
- A transfer matrix $P \in K^{q\times r}$ admits a *right-coprime factorization* if there exist $\widetilde{D} \in A^{r\times r}$, $\det \widetilde{D} \neq 0$, $\widetilde{N} \in A^{q\times r}$, $\widetilde{Y} \in A^{r\times q}$ and $\widetilde{X} \in A^{r\times r}$ such that $P = \widetilde{N}\,\widetilde{D}^{-1}$ and $-\widetilde{Y}\,\widetilde{N} + \widetilde{X}\,\widetilde{D} = I_r$.
- A transfer matrix $P \in K^{q\times r}$ admits a *doubly coprime factorization* if P admits a left- and a right-coprime factorizations $P = D^{-1}\,N = \widetilde{N}\,\widetilde{D}^{-1}$ such that:

$$\begin{pmatrix} D & -N \\ -\widetilde{Y} & \widetilde{X} \end{pmatrix} \begin{pmatrix} X & \widetilde{N} \\ Y & \widetilde{D} \end{pmatrix} = I_{q+r}. \tag{6.5}$$

In what follows, we characterize stabilizablity and give a parametrization of all stabilizing controllers obtained in [8]. Contrary to [7, 8], which are based on modern algebraic methods, following [6], we present here elementary proofs of these results.

Proposition 1 ([7]) *Let $P \in K^{q\times r}$ be a plant, $M := (I_q \quad -P) \in K^{q\times(q+r)}$ and $\widetilde{M} := (P^T \quad I_r^T)^T \in K^{(q+r)\times r}$. Then, $P \in K^{q\times r}$ is stabilizable iff one of the following equivalent assertions is satisfied*:

1. *There exists a matrix $L := (S_o^T \quad U^T)^T \in A^{(q+r)\times q}$, $S_o \in A^{q\times q}$, $\det S_o \neq 0$, $U \in A^{r\times q}$, such that*:
   a. $L\,P = \begin{pmatrix} S_o\,P \\ U\,P \end{pmatrix} \in A^{(q+r)\times r}$,
   b. $M\,L = S_o - P\,U = I_q$.

   Then $C = U\,S_o^{-1}$ stabilizes P, $S_o = (I_q - P\,C)^{-1}$ is the output sensitivity transfer matrix and $U = C\,(I_q - P\,C)^{-1}$.
2. *There exists a matrix $\widetilde{L} := (-\widetilde{U} \quad \widetilde{S}_i) \in A^{r\times(q+r)}$, $\widetilde{U} \in A^{r\times q}$, $\widetilde{S}_i \in A^{r\times r}$, $\det \widetilde{S}_i \neq 0$, such that:*
   a. $P\,\widetilde{L} = (-P\,\widetilde{U} \quad P\,\widetilde{S}_i) \in A^{q\times(q+r)}$,
   b. $\widetilde{L}\,\widetilde{M} = -\widetilde{U}\,P + \widetilde{S}_i = I_r$.

   Then $\widetilde{C} = \widetilde{S}_i^{-1}\,\widetilde{U}$ stabilizes P, $\widetilde{S}_i = (I_r - \widetilde{C}\,P)^{-1}$ is the input sensitivity transfer matrix and $\widetilde{U} = (I_r - \widetilde{C}\,P)^{-1}\,\widetilde{C}$

With the above notations, we have $\widetilde{C} = C \Leftrightarrow \widetilde{L}\,L = 0$.

Proof 1. Let C stabilize P. Using (6.4), we get $S_o := (I_q - P\,C)^{-1} \in A^{q\times q}$ and $U := C\,(I_q - P\,C)^{-1} \in A^{r\times q}$. Using again (6.4), $L := (S_o^T \quad U^T)^T \in A^{(q+r)\times q}$ satisfies $L\,P \in A^{(q+r)\times r}$, and thus 1.a. holds. Finally, 1.b. also holds since we have

$S_o - P\,U = (I_q - P\,C)^{-1} - P\,C\,(I_q - P\,C)^{-1} = I_q$. Let us now suppose that 1.a and 1.b hold. Since $\det S_o \neq 0$, $S_o - P\,U = I_q$ yields $I_q - P\,(U\,S_o^{-1}) = S_o^{-1}$, i.e., with the notation $C := U\,S_o^{-1}$, we get $S_o = (I_q - P\,C)^{-1}$. Then, we have $(I_q - P\,C)^{-1} \in A^{q\times q}$ and $U = C\,S_o = C\,(I_q - P\,C)^{-1} \in A^{r\times q}$. Now, using $L\,P \in A^{(q+r)\times r}$, we get $(I_q - P\,C)^{-1}\,P \in A^{q\times r}$ and $C\,(I_q - P\,C)^{-1}\,P \in A^{r\times r}$, and thus $H(P, C) \in A^{(q+r)\times(q+r)}$, i.e., C stabilizes P. 2 can be proved similarly. Finally, we have $\widetilde{C} = C$ iff $-\widetilde{U}\,S_o + \widetilde{S}_i\,U = 0$, i.e., iff $\widetilde{L}\,L = 0$. □

Corollary 1 ([7]) *P is stabilizable iff there exists $U \in A^{r\times q}$ such that:*

$$\begin{cases} T_i := -U\,P \in A^{r\times r}, \\ T_o := -P\,U \in A^{q\times q}, \\ R := (I_q + P\,U)\,P = P\,(I_r + U\,P) \in A^{q\times r}, \\ \det(I_q + P\,U) = \det(I_r + U\,P) \neq 0. \end{cases} \tag{6.6}$$

Then, $C := U\,(P\,U + I_q)^{-1} = (U\,P + I_r)^{-1}\,U$ is a stabilizing controller of P,

$$\mathrm{Feedback}(C, P) := U = C\,(I_q - P\,C)^{-1} = (I_r - C\,P)^{-1}\,C, \tag{6.7}$$

T_i (resp., T_o) is the complementary input (resp. output) sensitivity transfer matrix.

Proof By 1 of Proposition 1, P is stabilizable iff there exists $U \in A^{r\times q}$ such that $S_o = I_q + P\,U \in A^{q\times q}$, $\det S_o \neq 0$, $S_o\,P \in A^{q\times r}$ and $U\,P \in A^{r\times r}$, i.e., iff $-T_o := P\,U \in A^{q\times q}$, $R := (I_q + P\,U)\,P = P\,(I_r + U\,P) \in A^{q\times r}$, $-T_i := U\,P \in A^{r\times r}$ and $\det(I_q + P\,U) = \det(I_r + U\,P) \neq 0$.

If C stabilizes P, then the matrices L and $\widetilde{L}$ defined by

$$\begin{cases} S_o := (I_q - P\,C)^{-1}, \\ U = \widetilde{U} := C\,(I_q - P\,C)^{-1} = (I_r - C\,P)^{-1}\,C, \\ S_i = \widetilde{S}_i := (I_r - C\,P)^{-1}, \end{cases}$$

satisfy 1 and 2 of Proposition 1 and $R = S_o\,P = P\,S_i$, 1.b and 2.b show that

$$\Pi_C := L\,M = \begin{pmatrix} S_o & -R \\ U & T_i \end{pmatrix}, \quad \Pi_P := \widetilde{M}\,\widetilde{L} = \begin{pmatrix} T_o & R \\ -U & S_i \end{pmatrix}$$

are two *complementary projectors* of $A^{(q+r)\times(q+r)}$, i.e., $\Pi_C^2 = \Pi_C$, $\Pi_P^2 = \Pi_P$ and $\Pi_C + \Pi_P = I_{q+r}$. Using (6.4), Fig. 6.3 and $T_i = -C\,R$, we can easily check that:

$$\begin{pmatrix} e_1 \\ y_1 \end{pmatrix} = \Pi_C \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad \begin{pmatrix} y_2 \\ e_2 \end{pmatrix} = \Pi_P \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}.$$

Corollary 2 *Let $\Delta \in A^{q\times r}$. The following assertions are equivalent*:

1. $C \in K^{r\times q}$ *stabilizes* $P \in K^{q\times r}$.
2. $C' := C\,(I_q - \Delta\, C)^{-1} = (I_r - C\,\Delta)^{-1}\, C$ *stabilizes* $P' := P - \Delta$.

Proof By Corollary 1, C stabilizes P iff $U := C\,(I_q - P\,C)^{-1} = (I_r - C\,P)^{-1}\,C$ is such that $U \in A^{r\times q}$ and satisfies (6.6). Now, for any $\Delta \in A^{q\times r}$, $U \in A^{r\times q}$ and (6.6) are equivalent to $U \in A^{r\times q}$ and

$$\begin{cases} U\,(P-\Delta) = U\,P - U\,\Delta \in A^{r\times r}, \\ (P-\Delta)\,U = P\,U - \Delta\,U \in A^{q\times q}, \\ ((P-\Delta)\,U + I_q)\,(P-\Delta) = (P\,U + I_q)\,P - (P\,U+I_q)\,\Delta - \Delta\,(U\,P) + \Delta\,U\,\Delta \in A^{q\times r}, \end{cases}$$

which shows that $P \in K^{q\times r}$ is stabilized by $C \in K^{r\times q}$ iff $P - \Delta$ is stabilized by $C' := U\,((P-\Delta)\,U + I_q)^{-1} = (U\,(P-\Delta) + I_r)^{-1}\,U$. Finally, using $C = U\,(P\,U + I_q)^{-1} = (U\,P + I_r)^{-1}\,U$, we obtain:

$$\begin{aligned} C' &= U\,(P\,U + I_q - \Delta\,U)^{-1} = U\,((I_q - \Delta\,U\,(P\,U+I_q)^{-1})\,(P\,U+I_q))^{-1} \\ &= U\,(P\,U+I_q)^{-1}\,(I_q - \Delta\,U\,(P\,U+I_q)^{-1})^{-1} = C\,(I_q - \Delta\,C)^{-1}, \end{aligned}$$

$$\begin{aligned} C' &= (U\,P + I_r - U\,\Delta)^{-1}\,U = ((U\,P+I_r)\,(I_r - (U\,P+I_r)^{-1}\,U\,\Delta))^{-1}\,U \\ &= (I_r - (U\,P+I_r)^{-1}\,U\,\Delta))^{-1}\,(U\,P+I_r)^{-1}\,U = (I_r - C\,\Delta)^{-1}\,C. \end{aligned}$$ □

The next results gives a parametrization of all the stabilizing controllers of a stabilizable plant. Only the explicit knowledge of a stabilizing controller is assumed.

Theorem 1 ([8]) *Let $C_\star \in K^{r\times q}$ be a stabilizing controller of $P \in K^{q\times r}$ and*:

$$\begin{aligned} &U := C_\star\,(I_q - P\,C_\star)^{-1} = (I_r - C_\star\,P)^{-1}\,C_\star \in A^{r\times q}, \\ &S_i := (I_r - C_\star\,P)^{-1} \in A^{r\times r}, \quad S_o := (I_q - P\,C_\star)^{-1} \in A^{q\times q}. \end{aligned}$$

Then, all stabilizing controllers of P are given by

$$C(\Lambda) = (U + \Lambda)\,(S_o + P\,\Lambda)^{-1} = (S_i + \Lambda\,P)^{-1}\,(U + \Lambda), \tag{6.8}$$

where Λ is any matrix which belongs to the following A-module

$$\Omega = \{\,\Lambda \in A^{r\times q} \mid \Lambda\,P \in A^{r\times r},\ P\,\Lambda \in A^{q\times q},\ P\,\Lambda\,P \in A^{q\times r}\,\}, \tag{6.9}$$

and satisfies:

$$\det(S_o + P\,\Lambda) \neq 0, \quad \det(S_i + \Lambda\,P) \neq 0. \tag{6.10}$$

Proof Let C_1 and $C_2 \in K^{r\times q}$ be two stabilizing controllers of $P \in K^{q\times r}$ and:

$$\begin{aligned} &S_{ik} := (I_r - C_k\,P)^{-1} \in A^{r\times r}, \quad S_{ok} := (I_q - P\,C_k)^{-1} \in A^{q\times q}, \\ &U_k := C_k\,(I_q - P\,C_k)^{-1} = (I_r - C_k\,P)^{-1}\,C_k := \widetilde{U}_k \in A^{r\times q}, \end{aligned} \quad k = 1, 2.$$

We then have $C_k = U_k\, S_{o_k}^{-1} = S_{i_k}^{-1}\, U_k$ for $k = 1, 2$. Moreover, the matrices $L_k := (S_{o_k}^T \quad U_k^T)^T \in A^{(q+r)\times q}$ and $\widetilde{L}_k := (-U_k \quad S_{ik}) \in A^{r\times(q+r)}$ satisfy 1.a, 1.b, 2.a and 2.b of Proposition 1. Using 1.b and 2.b, we get:

$$\begin{cases} S_{o2} - S_{o1} = P\, U_2 + I_q - P\, U_1 - I_q = P\,(U_2 - U_1), \\ S_{i2} - S_{i1} \;= U_2\, P + I_r - U_1\, P - I_r = (U_2 - U_1)\, P. \end{cases}$$

Now, using 1.a and 2.a, we obtain:

$$\begin{cases} (U_2 - U_1)\, P \quad = S_{i2} - S_{i1} \in A^{r\times r}, \\ P\,(U_2 - U_1) \quad = S_{o2} - S_{o1} \in A^{q\times q}, \\ P\,(U_2 - U_1)\, P = (S_{o2} - S_{o1})\, P = P\,(S_{i2} - S_{i1}) \in A^{q\times r}, \end{cases} \Rightarrow\; U_2 - U_1 \in \Omega.$$

Hence, if $\Lambda := U_2 - U_1 \in \Omega$, then we have $U_2 = U_1 + \Lambda$, $S_{o2} = S_{o1} + P\,\Lambda$ and $S_{i2} = S_{i1} + \Lambda\, P$, and if $\det(S_{o1} + P\,\Lambda) \neq 0$ and $\det(S_{i1} + \Lambda\, P) \neq 0$, then we get:

$$C_2 = U_2\, S_{o2}^{-1} = (U_1 + \Lambda)\,(S_{o1} + P\,\Lambda)^{-1},\; C_2 = S_{i2}^{-1}\, U_2 = (S_{i1} + \Lambda\, P)^{-1}(U_1 + \Lambda).$$

If $S_o := S_{o1}$, $U := U_1$ and $S_i := S_{i1}$, then we get $C_2 = C(\Lambda)$, where $C(\Lambda)$ is defined by (6.8) for a certain $\Lambda \in \Omega$ which satisfies (6.10). Finally, let us prove that for every $\Lambda \in \Omega$ which satisfies (6.10), the controller $C(\Lambda)$ defined by (6.8) stabilizes P. Let $L(\Lambda) := ((S_o + P\,\Lambda)^T \quad (U + \Lambda)^T)^T$, $\widetilde{L}(\Lambda) := (-(U + \Lambda) \quad S_i + \Lambda\, P)$, $M := (I_q \quad -P)$ and $\widetilde{M} := (P^T \quad I_r^T)^T$. Since $\Lambda \in \Omega$, we have $U + \Lambda \in A^{r\times q}$, $S_o + P\,\Lambda \in A^{q\times q}$ and

$$L(\Lambda)\, P = \begin{pmatrix} S_o\, P + P\,\Lambda\, P \\ U\, P + \Lambda\, P \end{pmatrix} \in A^{(q+r)\times r}, \quad M\, L(\Lambda) = S_o - P\, U = I_q,$$

and thus $C(\Lambda) = (U + \Lambda)\,(S_o + P\,\Lambda)^{-1}$ stabilizes P by 1 of Proposition 1. Similarly, since $\Lambda \in \Omega$, we get $U + \Lambda \in A^{r\times q}$, $S_i + \Lambda\, P \in A^{r\times r}$ and

$$\begin{cases} P\,\widetilde{L}(\Lambda) = (-(P\, U + P\,\Lambda) \quad P\, S_i + P\,\Lambda\, P) \in A^{q\times(q+r)}, \\ \widetilde{L}(\Lambda)\,\widetilde{M} = -U\, P + S_i = I_r, \end{cases}$$

i.e., $C(\Lambda) = (S_i + \Lambda\, P)^{-1}\,(U + \Lambda)$ stabilizes P by 2 of of Proposition 1. □

Proposition 2 ([8]) *Let $C_\star \in K^{r\times q}$ be a stabilizing controller of $P \in K^{q\times r}$ and:*

$$\begin{cases} L = \begin{pmatrix} (I_q - P\, C_\star)^{-1} \\ C_\star\,(I_q - P\, C_\star)^{-1} \end{pmatrix} \in A^{(q+r)\times q}, \\ \widetilde{L} = (-(I_r - C_\star\, P)^{-1}\, C_\star \quad (I_r - C_\star\, P)^{-1}) \in A^{r\times(q+r)}. \end{cases}$$

Then, the A-module Ω defined by (6.9) satisfies $\Omega = \widetilde{L}\, A^{(q+r)\times(q+r)}\, L$. Hence, the A-module Ω is generated by the $(q+r)^2$ matrices $\widetilde{L}_{\bullet i}\, L_{j\bullet}$, where $\widetilde{L}_{\bullet i}$ is the ith column of $\widetilde{L}$ and $L_{j\bullet}$ the jth row of L, i.e., $\Omega = \left\{\sum_{i,j=1}^{q+r} a_{ij}\, \widetilde{L}_{\bullet i}\, L_{j\bullet} \mid a_{ij} \in A\right\}$.

Proof Let $\Lambda \in \Omega$, i.e., $\Lambda \in A^{r\times q}$ is such that $\Lambda\, P \in A^{r\times r}$, $P\,\Lambda \in A^{q\times q}$ and $P\,\Lambda\, P \in A^{q\times r}$, $M := (I_q - P)$ and $\widetilde{M} := (P^T \quad I_r^T)^T$. In particular, we have $\widetilde{M}\,\Lambda \in A^{(q+r)\times q}$, $\Lambda\, M \in A^{r\times(q+r)}$ and $P\,\Lambda\, M \in A^{q\times(q+r)}$. Now, using 2.b of Proposition 1, i.e., $\widetilde{L}\,\widetilde{M} = I_r$, we get $\Lambda = \widetilde{L}\,(\widetilde{M}\,\Lambda)$. Using 1.b of Proposition 1, i.e., $M\, L = I_q$, we get $\Lambda = (\Lambda\, M)\, L$, and thus $P\,\Lambda = (P\,\Lambda\, M)\, L$. Substituting $\Lambda = (\Lambda\, M)\, L$ and $P\,\Lambda = (P\,\Lambda\, M)\, L$ into $\widetilde{M}\,\Lambda = ((P\,\Lambda)^T \quad \Lambda^T)^T$, we get $\widetilde{M}\,\Lambda = \Theta\, L$, where $\Theta := ((P\,\Lambda\, M)^T (\Lambda\, M)^T)^T \in A^{(q+r)\times(q+r)}$ and, since $\Lambda = \widetilde{L}\,(\widetilde{M}\,\Lambda)$, we finally obtain $\Lambda = \widetilde{L}\,\Theta\, L$, i.e., $\Omega \subseteq \widetilde{L}\, A^{(q+r)\times(q+r)}\, L$.

Now, let $\Lambda \in \widetilde{L}\, A^{(q+r)\times(q+r)}\, L$, i.e., $\Lambda = \widetilde{L}\,\Theta\, L$ for $\Theta \in A^{(q+r)\times(q+r)}$ and L and $\widetilde{L}$ satisfy 1 and 2 of Proposition 1. Then, using 1.a and 2.a of Proposition 1, we obtain $\Lambda \in A^{r\times q}$, $\Lambda\, P = \widetilde{L}\,\Theta\,(L\, P) \in A^{r\times r}$, $P\,\Lambda = (P\,\widetilde{L})\,\Theta\, L \in A^{q\times q}$, $P\,\Lambda\, P = (P\,\widetilde{L})\,\Theta\,(L\, P) \in A^{q\times r}$, i.e., $\Lambda \in \Omega$, and thus $\Omega = \widetilde{L}\, A^{(q+r)\times(q+r)}\, L$.

Finally, $\Theta \in A^{(q+r)\times(q+r)}$ can be written as $\Theta = \sum_{i,j=1}^{q+r} \Theta_{ij}\, E_{ij}$ where $\Theta_{ij} \in A$ and E_{ij} is the matrix defined by 1 in the ith row and the jth column and 0 elsewhere, and thus every $\Lambda \in \Omega$ can be written as $\Lambda = \widetilde{L}\,\Theta\, L = \sum_{i,j=1}^{q+r} \Theta_{ij}\,(\widetilde{L}\, E_{ij}\, L)$. Therefore, $\{\widetilde{L}\, E_{ij}\, L\}_{i,j=1,\ldots,q+r}$ is a family of generators of the A-module Ω and $\widetilde{L}\, E_{ij}\, L$ is the product of the ith column $\widetilde{L}_{\bullet i}$ of $\widetilde{L}$ by the jth row $L_{j\bullet}$ of L. □

Combining Theorem 1 and Proposition 2, we obtain the following result.

Corollary 3 ([8]) *With the notations of Theorem 1, if $C_\star \in K^{r\times q}$ is a stabilizing controller of $P \in K^{q\times r}$, then all the stabilizing controllers of P are of the form (6.8), where Λ is any matrix which belongs to $\Omega = \sum_{i,j=1}^{q+r} A\,(\widetilde{L}_{\bullet i}\, L_{j\bullet})$ and satisfies (6.10).*

In Corollary 3, only the explicit knowledge of a stabilizing controller is assumed. For many classes of infinite-dimensional systems, (PID, finite-dimensional) stabilizing controllers are known which is not the case for doubly coprime factorizations.

Corollary 4 *1. If $P \in K^{q\times r}$ admits a left-coprime factorization $P = D^{-1}\, N$, $D\, X - N\, Y = I_q$, with $(X^T \quad Y^T)^T \in A^{(q+r)\times q}$ and $\det X \neq 0$, then the matrix $L = ((X\, D)^T \quad (Y\, D)^T)^T \in A^{(q+r)\times q}$ satisfies 1.a and 1.b of Proposition 1, and $C = Y\, X^{-1}$ is a stabilizing controller of P.*

2. If $P \in K^{q\times r}$ admits a right-coprime factorization $P = \widetilde{N}\,\widetilde{D}^{-1}$, $-\widetilde{Y}\,\widetilde{N} + \widetilde{X}\,\widetilde{D} = I_r$, with $(-\widetilde{Y} \quad \widetilde{X}) \in A^{r\times(q+r)}$ and $\det \widetilde{X} \neq 0$, then the matrix $\widetilde{L} = (-\widetilde{D}\,\widetilde{Y} \quad \widetilde{D}\,\widetilde{X}) \in A^{r\times(q+r)}$ satisfies 2.a and 2.b of Proposition 1, and $C = \widetilde{X}^{-1}\,\widetilde{Y}$ is a stabilizing controller of P.

Proof Let us prove 1. If $P = D^{-1}\, N$, $D\, X - N\, Y = I_q$, is a left-coprime factorization of P, then $(X\, D)\, P = X\, N \in A^{q\times r}$, $(Y\, D)\, P = Y\, N \in A^{r\times r}$ and $D\, X - N\, Y = I_q \Rightarrow X - P\, Y = D^{-1} \Rightarrow (X\, D) - P\,(Y\, D) = I_q$, i.e., $L = ((X\, D)^T \quad (Y\, D)^T)^T \in A^{(q+r)\times q}$ satisfies 1 of Proposition 1, and thus $C = (Y\, D)\,(X\, D)^{-1} = Y\, X^{-1}$ stabilizes P. 2 can be proved similarly. □

From Corollary 4, the existence of a doubly coprime factorization of P is a sufficient but not a necessary condition for stabilizability.

Proposition 3 ([8]) *If $P \in K^{q\times r}$ admits the doubly coprime factorization (6.5), then the A-module Ω defined by (6.9) satisfies $\Omega = \widetilde{D}\,A^{r\times q}\,D$.*

Proof Let $\Lambda \in \widetilde{D}\,A^{r\times q}\,D$, i.e., $\Lambda = \widetilde{D}\,Q\,D$ for a certain $Q \in A^{r\times q}$. Then, we have $\Lambda = \widetilde{D}\,Q\,D \in A^{r\times q}$, $\Lambda\,P = \widetilde{D}\,Q\,N \in A^{r\times r}$, $P\,\Lambda = \widetilde{N}\,Q\,D \in A^{q\times q}$ and $P\,\Lambda\,P = \widetilde{N}\,Q\,N \in A^{q\times r}$, which shows that $\Lambda \in \Omega$. Conversely, let $\Lambda \in \Omega$ and $Q := \widetilde{D}^{-1}\,\Lambda\,D^{-1} \in K^{r\times q}$. From (6.5), we get the identities $D^{-1} = X - P\,Y$ and $\widetilde{D}^{-1} = \widetilde{X} - \widetilde{Y}\,P$, which yield $Q = \widetilde{D}^{-1}\,\Lambda\,D^{-1} = (\widetilde{X} - \widetilde{Y}\,P)\,\Lambda\,(X - P\,Y) = \widetilde{X}\,\Lambda\,X - \widetilde{X}\,(\Lambda\,P)\,Y - \widetilde{Y}\,(P\,\Lambda)\,X + \widetilde{Y}\,(P\,\Lambda\,P)\,Y \in A^{r\times q}$ since $\Lambda \in \Omega$ and the entries of X, Y, $\widetilde{X}$ and $\widetilde{Y}$ belong to A. Therefore, we get $\Lambda = \widetilde{D}\,Q\,D$ for a certain $Q \in A^{r\times q}$, i.e., $\Lambda \in \widetilde{D}\,A^{r\times q}\,D$, which finally proves that $\Omega = \widetilde{D}\,A^{r\times q}\,D$. □

The next corollary shows that the parametrization (6.8) gives rise to the Youla-Kučera parametrization when the plant P admits a doubly coprime factorization.

Corollary 5 ([8]) *Let $P \in K^{q\times r}$ admit a doubly coprime factorization $P = D^{-1}\,N = \widetilde{N}\,\widetilde{D}^{-1}$, where (6.5) is satisfied. Then, all the stabilizing controllers of P are of the form $C(Q) = (Y + \widetilde{D}\,Q)\,(X + \widetilde{N}\,Q)^{-1} = (\widetilde{X} + Q\,N)^{-1}\,(\widetilde{Y} + Q\,D)$, where Q is any matrix of $A^{r\times q}$ such that $\det(X + \widetilde{N}\,Q) \neq 0$ and $\det(\widetilde{X} + Q\,N) \neq 0$.*

Proof By Proposition 3, we have $\Omega = \widetilde{D}\,A^{r\times q}\,D$. Moreover, by 1 of Corollary 4, $C = (Y\,D)\,(X\,D)^{-1} = Y\,X^{-1}$ is a stabilizing controller of P. Moreover, by 2 of Corollary 4, $\widetilde{C} = (\widetilde{D}\,\widetilde{X})^{-1}\,(\widetilde{D}\,\widetilde{Y}) = \widetilde{X}^{-1}\,\widetilde{Y}$ is a stabilizing controller of P. By (6.5), $-\widetilde{Y}\,X + \widetilde{X}\,Y = 0$, which shows that $\widetilde{C} = C$. Therefore, by Theorem 1 or Corollary 3, we obtain that all the stabilizing controllers of P are of the form

$$\begin{aligned} C^{\circ}(Q) &:= C(\widetilde{D}\,Q\,D) = (Y\,D + \widetilde{D}\,Q\,D)\,(X\,D + P\,\widetilde{D}\,Q\,D)^{-1} \\ &= (Y\,D + \widetilde{D}\,Q\,D)\,(X\,D + \widetilde{N}\,Q\,D)^{-1} = (Y + \widetilde{D}\,Q)\,D\,D^{-1}\,(X + \widetilde{N}\,Q)^{-1} \\ &= (Y + \widetilde{D}\,Q)\,(X + \widetilde{N}\,Q)^{-1}, \end{aligned}$$

$$\begin{aligned} C^{\circ}(Q) &:= C(\widetilde{D}\,Q\,D) = (\widetilde{D}\,\widetilde{X} + \widetilde{D}\,Q\,D\,P)^{-1}\,(\widetilde{D}\,\widetilde{Y} + \widetilde{D}\,Q\,D) \\ &= (\widetilde{D}\,\widetilde{X} + \widetilde{D}\,Q\,N)^{-1}\,(\widetilde{D}\,\widetilde{Y} + \widetilde{D}\,Q\,D) = (\widetilde{X} + Q\,N)^{-1}\,\widetilde{D}^{-1}\,\widetilde{D}\,(\widetilde{Y} + Q\,D) \\ &= (\widetilde{X} + Q\,N)^{-1}\,(\widetilde{Y} + Q\,D), \end{aligned}$$

where $Q \in A^{r\times q}$ is any matrix such that $\det(X + \widetilde{N}\,Q) \neq 0$ and $\det(\widetilde{X} + Q\,N) \neq 0$. □

The following result is a direct consequence of Corollaries 3 and 2.

Theorem 2 *Let $\Delta \in A^{q\times r}$ and $C_\star \in K^{r\times q}$ be a stabilizing controller of the plant $P \in K^{q\times r}$. Then, all the stabilizing controllers C' of $P' := P - \Delta$ are of the form*

$$C'(\Lambda) = C(\Lambda)\,(I_q - \Delta\,C(\Lambda))^{-1} = (I_r - C(\Lambda)\Delta)^{-1}C(\Lambda) = \text{Feedback}\,(C(\Lambda), \Delta)\,,$$

where $C(\Lambda)$ is the parametrization (6.8) of all the stabilizing controllers of P.

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We have the following straightforward consequence of Theorem 2.

Corollary 6 *Let $P \in K^{q \times r}$ admits a doubly coprime factorization $P = D^{-1} N = \widetilde{N}\, \widetilde{D}^{-1}$, where (6.5) is satisfied, and $\Delta \in A^{q \times r}$. Then, $P' := P - \Delta$ admits the doubly coprime factorization $P' = D^{-1}\,(N - D\,\Delta) = (\widetilde{N} - \Delta\,\widetilde{D})\,\widetilde{D}^{-1}$ and:*

$$\begin{pmatrix} D & -(N - D\,\Delta) \\ -\widetilde{Y} & \widetilde{X} - \widetilde{Y}\,\Delta \end{pmatrix} \begin{pmatrix} X - \Delta\,Y & \widetilde{N} - \Delta\,\widetilde{D} \\ Y & \widetilde{D} \end{pmatrix} = I_{q+r}.$$

Hence, if $C(Q) := (Y + \widetilde{D}\,Q)\,(X + \widetilde{N}\,Q)^{-1} = (\widetilde{X} + Q\,N)^{-1}\,(\widetilde{Y} + Q\,D)$ is the Youla-Kučera parametrization of all the stabilizing controllers of P, then Youla-Kučera parametrization $C'(Q)$ of all the stabilizing controllers of P' satisfies:

$$C'(Q) = C(Q)\,(I_q - \Delta\,C(Q))^{-1} = (I_r - C(Q)\,\Delta)^{-1}\,C(Q) = \text{Feedback}\,(C(Q), \Delta)\,. \tag{6.11}$$

Theorem 2 and Corollary 6 are particularly interesting when P is a rational transfer matrix for which different techniques can be used to find a particular finite-dimensional controller or doubly coprime factorization.

Example 1 Let $F \in A := H_\infty(\mathbb{C}_+)$ be such that $F_0 := F(0) \neq 0$, $P := \frac{F}{s}$, $\Delta := F\,\frac{(1-e^{-T\,s})}{s} \in A$, and $P' := P - \Delta = F\,\frac{e^{-T\,s}}{s}$. Clearly, P admits the coprime factorization $P = \frac{N}{D}$, $D\,X - N\,Y = 1$, where $\alpha \in \mathbb{R}_{>0} := \{x \in \mathbb{R} \mid x > 0\}$,

$$N = \frac{F}{s + \alpha}, \quad D = \frac{s}{s + \alpha}, \quad X = 1 + \alpha\,\frac{1 - \frac{F}{F_0}}{s}, \quad Y = -\frac{\alpha}{F_0}.$$

The only point to check is that $X \in A$, i.e., $Z := (X - 1)/\alpha = \frac{1 - \frac{F}{F_0}}{s} \in A$. Clearly, Z is a holomorphic function in $\mathbb{C}_+$, has no poles in the imaginary axis and

$$\left| \frac{1 - \frac{F(i\,\omega)}{F_0}}{i\,\omega} \right| \leq \frac{1 + \left| \frac{F(i\,\omega)}{F_0} \right|}{|\omega|} \leq \frac{1 + \parallel \frac{F}{F_0} \parallel_\infty}{|\omega|},$$

which proves that $Z \in A$. By Corollary 5, the Youla-Kučera parametrization of all the stabilizing controllers of P is then defined by:

$$\forall\, Q \in A: \;\; C(Q) = \frac{Y + D\,Q}{X + N\,Q} = \frac{-\frac{\alpha}{F_0} + \frac{s}{s+\alpha}\,Q}{1 + \alpha\,\frac{1 - \frac{F}{F_0}}{s} + \frac{F}{s+\alpha}\,Q}. \tag{6.12}$$

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By Corollary 6, the Youla-Kučera parametrization of $P' = F\, \frac{e^{-T\,s}}{s}$ is then

$$C'(Q) := \frac{C(Q)}{1 - \Delta\, C(Q)} = \frac{Y + D\, Q}{(X - \Delta\, Y) + (N - \Delta\, D)\, Q} = \frac{-\frac{\alpha}{F_0} + \frac{s}{s+\alpha}\, Q}{1 + \alpha\, \frac{1 - \frac{F}{F_0}\, e^{-T\,s}}{s} + Q\, F\, \frac{e^{-Ts}}{s+\alpha}}, \tag{6.13}$$

for the coprime factorization $P' = \frac{N - D\,\Delta}{D}$, $D\,(X - \Delta\, Y) - (N - D\,\Delta)\, Y = 1$.

Example 2 We can apply Corollary 6 to $P = \frac{1}{2\,e}\, \frac{s+1}{s-1}$ and $\Delta = \frac{1}{2\,e}\, \frac{(s+1) - 2\, e^{1-\sqrt{s}}}{s-1} \in A := H_\infty(\mathbb{C}_+)$ to get the Youla-Kučera parametrization of $P' := P - \Delta = \frac{e^{-\sqrt{s}}}{s-1}$.

6.3 Study of the Tracking Problem and Numerical Simulations

In this section, we study the tracking problems introduced in Sect. 6.1.

Let $F \in A := H_\infty(\mathbb{C}_+)$ be such that $F_0 := F(0) \neq 0$. In many situations, we have $F \in RH_\infty$. Using Fig. 6.4 and (6.3), let us introduce the following two systems:

$$\begin{cases} \widehat{\varepsilon} = P\, \widehat{e}_2, \\ \widehat{y}_2 = G\, \widehat{\varepsilon}, \end{cases} \quad \begin{cases} P := \frac{F}{s}, \\ G := e^{-Ts}, \end{cases} \quad \begin{cases} \widehat{e}_1 = \widehat{u}_1 - \widehat{y}_2, \\ \widehat{e}_2 = \widehat{u}_2 - \widehat{y}_1. \end{cases}$$

We then have $\widehat{y}_2 = P'\, \widehat{e}_2$, where $P' := P\, G = \frac{F\, e^{-Ts}}{s}$ was introduced in Example 1. Considering the controller $\widehat{y}_1 = C'\, \widehat{e}_1$, we then obtain:

$$\begin{pmatrix} \widehat{\varepsilon} \\ \widehat{y}_1 \end{pmatrix} = \frac{1}{1 - P'\, C'} \begin{pmatrix} -P\, C' & P \\ C' & -P'\, C' \end{pmatrix} \begin{pmatrix} \widehat{u}_1 \\ \widehat{u}_2 \end{pmatrix}. \tag{6.14}$$

Lemma 1 *With the above notations, the following assertions are equivalent*:

1. $C'_\star \in Q(A)$ *is such that* $\frac{P\, C'_\star}{1 - P'\, C'_\star}, \frac{P}{1 - P'\, C'_\star}, \frac{C'_\star}{1 - P'\, C'_\star}, \frac{P'\, C'_\star}{1 - P'\, C'_\star} \in A$.
2. $C'_\star \in Q(A)$ *stabilizes* P', *i.e.*, $\frac{1}{1 - P'\, C'_\star}, \frac{C'_\star}{1 - P'\, C'_\star}, \frac{P'}{1 - P'\, C'_\star} \in A$.

If $C(Q)$ is the Youla-Kučera parametrization (6.12) *of $P = \frac{F}{s}$, then the Youla-Kučera parametrization of $P' = P - \Delta$, where $\Delta := F\, \frac{(1 - e^{-T\,s})}{s} \in A$, is given by* (6.13).

Proof Let $C'_\star$ satisfy 1. Then, we have $\frac{1}{1 - P'\, C'_\star} = \frac{P\, C'_\star}{1 - P'\, C'_\star} - 1, \frac{C'_\star}{1 - P'\, C'_\star} \in A$. We then get $\frac{P'}{1 - P'\, C'_\star} = G\, \frac{P}{1 - P'\, C'_\star} \in A$ since $G = e^{-Ts} \in A$, which proves 2. Now, let us suppose that $C'_\star$ stabilizes P', i.e., satisfies 2. We then need to check that $\frac{P}{1 - P'\, C'_\star}, \frac{P\, C'_\star}{1 - P'\, C'_\star} \in A$. Using (6.13), we have $C'_\star = C'(Q)$ for a certain $Q \in A$ and:

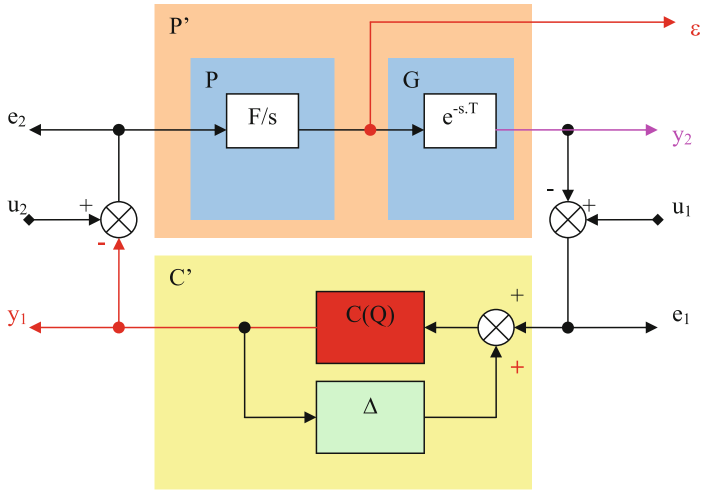

Fig. 6.4 Closed-loop system with the feedback structure of $C'(Q)$

$$\begin{cases} \frac{P}{1-P'\,C'_\star} = \frac{F}{s+\alpha}\left(1+\alpha\,\frac{1-\frac{F}{F_0}\,e^{-Ts}}{s}+Q\,F\,\frac{e^{-Ts}}{s+\alpha}\right)\in A,\\ \frac{P\,C'_\star}{1-P'\,C'_\star} = \frac{F}{s+\alpha}\left(-\frac{\alpha}{F_0}+\frac{s}{s+\alpha}\,Q\right)\in A. \end{cases}$$ □

The sensibility transfer function $S(Q) := (1-P'\,C'(Q))^{-1}$ corresponding to the Youla-Kučera parametrization $C'(Q)$ of $P' = P - \Delta = F\,\frac{e^{-Ts}}{s}$ (see (6.13)) is:

$$S(Q) = \frac{s}{s+\alpha}\left(1+\alpha\,\frac{1-\frac{F}{F_0}e^{-Ts}}{s}+Q\,F\,\frac{e^{-Ts}}{s+\alpha}\right).$$

Let us now investigate the asymptotic tracking of the target. In what follows, we consider the noiseless case, i.e., $\widehat{u}_1 = 0$. Using (6.14), we then get $\widehat{\varepsilon} = \frac{P}{1-P'\,C'}\,\widehat{u}_2$. Therefore, we have to find a stabilizing controller $C'(Q)$ of P' which is such that

$$\lim_{t\to+\infty}\varepsilon(t) = \lim_{s\to 0} s\,\widehat{\varepsilon}(s) = \lim_{s\to 0} s\,S(Q)\,P\,\widehat{u}_2 = 0,$$

where $\widehat{u}_2 = \frac{s}{F}\,\widehat{\theta}$ (see Sect. 6.1). Letting $\gamma = \theta_0^{(n)}$, $m = 1$ for **scenario 2** and $m = 2$ for **scenario 3**, $\widehat{u}_2$ can be decomposed as a sum of terms of the form $\widehat{u}_2 = \frac{\gamma}{s^n\,F}$, where $\gamma \in \mathbb{R}$ and $0 \le n \le m$. Using $s\,P = F$, we get $\lim_{t\to+\infty}\varepsilon(t) = \lim_{s\to 0} s\,S(Q)\,P\,\frac{\gamma}{s^n\,F} = \lim_{s\to 0}\frac{S(Q)\,\gamma}{s^n}$. Hence, if we set

$$E := \frac{(s+\alpha)^2\,S(Q)}{s^n} = \frac{(s+\alpha)\left(1+\alpha\,\frac{1-F/F_0\,e^{-T\,s}}{s}\right)+Q\,F\,e^{-T\,s}}{s^{n-1}},$$

then we have to determine the parameter $Q \in A$ such that $\lim_{s\to 0} E\,\frac{\gamma}{(s+\alpha)^2} = 0$.

Let us consider **scenario 3** which corresponds to $m = 2$. Let us consider power series expansion of Q, F and $e^{-T\,s}$ at $s = 0$, i.e., $Q = q_0 + q_1\,s + O(s^2)$,

$$F = F_0 + F_1\,s + \frac{F_2}{2}\,s^2 + O(s^3), \quad e^{-Ts} = 1 - T\,s + \frac{T^2}{2}\,s^2 + O(s^3).$$

where $F_i := F^{(i)}(0)$. For $n = 2$, we get $E_2 := s\,E = e_0 + e_1\,s + O(s^2)$, where:

$$\begin{cases} e_0 = q_0\,F_0 + \alpha + \alpha^2\,T - \alpha^2\,\frac{F_1}{F_0}, \\ e_1 = 1 + q_1\,F_0 + q_0\,F_1 - \alpha\,\frac{F_1}{F_0} - \frac{\alpha^2\,F_2}{2\,F_0} + \left(\alpha - q_0\,F_0 + \alpha^2\,\frac{F_1}{F_0}\right)\,T - \frac{\alpha^2\,T^2}{2}. \end{cases}$$

Hence, we have $E_2 = O(s^2)$, i.e., $e_0 = 0$ and $e_1 = 0$, iff:

$$\begin{cases} q_0 = -\,\dfrac{\alpha\,(1 + \alpha\,T)}{F_0} + \dfrac{\alpha^2}{F_0^2}\,F_1, \\ q_1 = -\,\dfrac{1 + 2\,\alpha\,T + \frac{\alpha^2\,T^2}{2}}{F_0} + \dfrac{\alpha(2 + \alpha\,T)\,F_1 + \alpha^2\,\frac{F_2}{2}}{F_0^2} - \dfrac{\alpha^2}{F_0^3}\,F_1^2. \end{cases}$$

For the numerical simulations, we take $Q := \frac{N_Q}{D_Q}$, where $N_Q := q_0 + q_2'\,s$, $D_Q := 1 + q_1'\,s$, $q_0,\ q_2' \in \mathbb{R}$, $q_1' \in \mathbb{R}_{\geq 0} = \{x \in \mathbb{R} \mid x \geq 0\}$, so that we get $Q = q_0 + (q_2' - q_0\,q_1')\,s + O(s^2)$, and thus we can choose arbitrarily $q_1' \in \mathbb{R}_{\geq 0}$ and:

$$\begin{cases} q_0 = -\,\dfrac{\alpha\,(1 + \alpha\,T)}{F_0} + \dfrac{\alpha^2}{F_0^2}\,F_1, \\ q_2' = -\,\dfrac{1 + \alpha\,q_1' + (\alpha\,q_1' + 2)\,\alpha\,T + \frac{\alpha^2\,T^2}{2}}{F_0} + \dfrac{((\alpha\,q_1' + 2)\,\alpha + \alpha^2\,T)\,F_1 + \alpha^2\,\frac{F_2}{2}}{F_0^2} - \dfrac{\alpha^2}{F_0^3}\,F_1^2. \end{cases}$$

We then have two degrees of freedom: $\alpha \in \mathbb{R}_{>0}$ and $q_1' \in \mathbb{R}_{\geq 0}$. We can check that we then have $\lim_{s \to 0} s\,S(Q)\,P\,\frac{\gamma}{s^n\,F} = 0$ for $0 \leq n \leq 1$. The form of Q can be used to study **scenario 2**, i.e., $Q = q_0 \in \mathbb{R}$, by considering $q_1' = 0$ and $q_2' = 0$.

For the visual tracking developed in Sect. 6.1, we have:

$$F := \frac{1 + \tau_2\,s}{(1 + \tau_1\,s)\,(1 + \tau_2\,s + \tau_2^2\,s^2)}, \ \tau_1 = \frac{1}{60\,\pi}, \ \tau_2 = \frac{1}{30\,\pi}, \ T = 0.18.$$

In the Matlab simulations, we take $\alpha = 0.95$ and $q_1' = \frac{1}{5\,\pi}$ to get a gain margin of 5.4dB and a phase margin of 42° at 0.84Hz. See Black's diagram of the closed-loop (the blue plot in Fig. 6.5), the step response (the blue plot in Fig. 6.6) and compare with the results obtained with a PID controller (the black plots) and a H_∞-controller (the red plots).

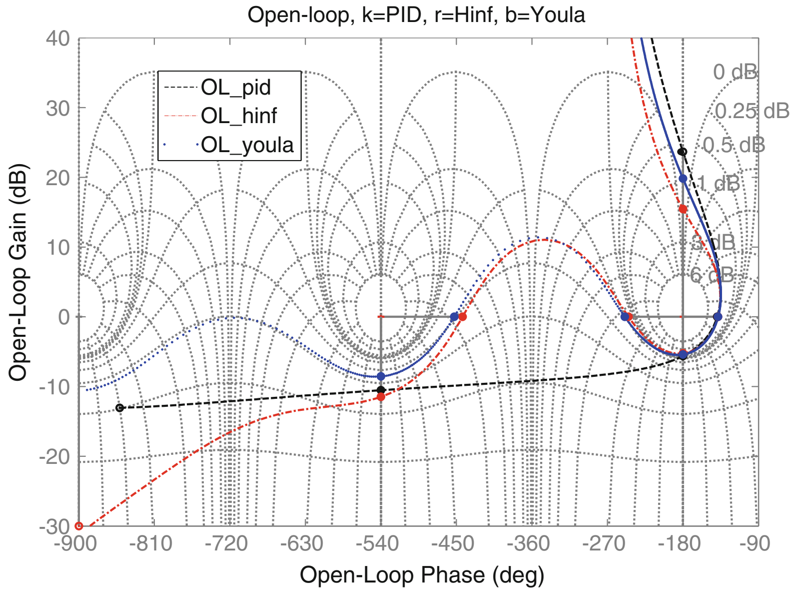

Fig. 6.5 Black's diagram and margins

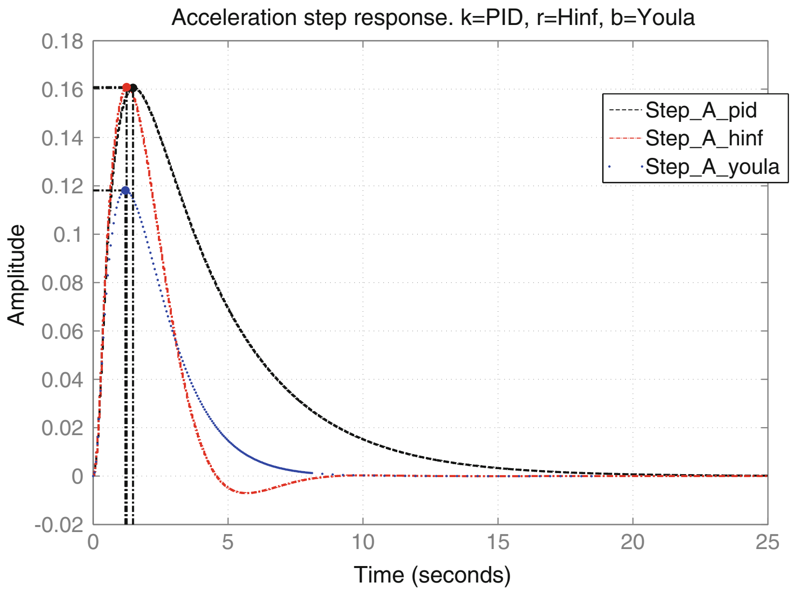

Fig. 6.6 Step response

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