Explicit H_∞ controllers for 1^{st} to 3^{rd} order single-input single-output systems with parameters

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Abstract: The purpose of this paper is to explicitly characterize H_∞ controllers for single-input single-output (SISO) systems of order 1, 2 and 3 in terms of their coefficients considered as unknown parameters. In the SISO case, computing H_∞ controllers requires to find the real positive definite solution of an algebraic Riccati equation. Due to the system parameters, no purely numerical method can be used to find such a solution, and thus parametric H_∞ controllers. Using elimination techniques of zero-dimensional polynomial systems, we first give a parametrization of all the solutions of the algebraic Riccati equation associated with the H_∞ control problem. Since the problem reduces to solving an univariate polynomial of degree less than or equal to 4, closed-form solutions are then obtained for the solutions of the algebraic Riccati equation by means of radicals. Using the concept of discriminant variety, we show that the maximal real root of this polynomial is always defined by the same closed-form expression, which yields the positive definite solution of the algebraic Riccati equation. Finally, we use the above results to explicitly compute the H_∞ criteria γ_{opt} and H_∞ controllers in terms of the system parameters and study them with respect to parameter variations.

Keywords: Robust control theory, parametric control, linear systems, algebraic systems theory, symbolic computation, polynomial methods.

1. INTRODUCTION

H_∞ control theory aims at designing stabilizing controllers which satisfy robustness constraints defined in the frequency domain (e.g., stability margins). These controllers are usually computed numerically for a given system using, e.g., γ-iteration via *Algebraic Riccati Equations* (ARE) or Linear Matrix Inequalities (LMI), see, e.g., Zhou et al. (1996) and the references therein.

An alternative approach to this numerical approach is to solve the problem symbolically for a set of systems depicted by some parameters to obtain a robust controller that depends on the system parameters (Rance et al. (2016); Kanno et al. (2007, 2012)). Once such a closed-form controller is known, only an evaluation is required to obtain a controller for a particular value of the system parameters.

We foresee many practical applications of this symbolic approach which motivate the present paper. For instance, while *designing* a project, the designer is interested in testing if his model can reach the desired specifications and this check has to be done quickly. The knowledge of a parametric controller and of its performance and robustness margins with respect to its parameters can be a very efficient method for a design perspective. Parametric controllers can be used to determine the values of the parameters so that the controller achieves desired performances, or to analytically prove that these specifications cannot be obtained for certain values of the parameters.

The symbolic approach can also be used for the design of *robust adaptive controllers* which adjust themselves while coupling with parameter estimation methods. Since only evaluations of closed-form solutions are required, these controllers could easily be embedded. Finally, the explicit formulas for the robust controllers can easily be used without any knowledge of the H_∞ control theory.

In this paper, based on Rance et al. (2016), we present the explicit forms of H_∞ controllers for SISO systems of order 1 to 3. These controllers are obtained by means of the computation of the positive definite solution of an ARE, which can be reduced to the finding of the maximal real root of a univariate polynomial $\mathcal{P}$. Since this polynomial $\mathcal{P}$ is of order less than or equal to 4 for the class of systems under study, its roots can be found by radicals and the maximal real solution can be explicitly determined over the entire space of the parameters, which yields a closed-form solution for the positive definite solution of the ARE. Then, the computation of the H_∞ criterion γ_{opt} is reduced to the finding of the maximal real root of a characteristic polynomial $\mathcal{H}$ of degree less or equal to 3. Again, we can express this root by radicals since the degrees of $\mathcal{H}$ are less or equal to 3 and we can prove that this maximal real root is defined by the same expression over the entire parameter space. Hence, γ_{opt} has always the same closed-form. We can then explicitly compute H_∞ controllers that have the same expression in the whole space of the parameters for SISO systems of order 1 to 3. Finally, the closed-form solution of γ_{opt} can be used to study its dependence upon the system parameters and can determine them so that some performances and robustness criteria are achieved.

The paper is organized as follows. Section 2 reviews key results of H_∞ loop-shaping theory. In Section 3 we give a parametrization of all the solutions of an ARE which has to be solved for the H_∞ control problem. Section 4 shows how to explicitly compute the positive definite solution of the ARE. Then, we explicitly compute the H_∞ criterion γ_{opt} and the stabilizing (sub)-optimal H_∞ controllers. Finally, the results are illustrated on (two) mass-spring examples.

2. THE STANDARD H_∞-CONTROL PROBLEM

In this paper, we shall consider 1st to 3rd SISO finite-dimensional linear systems (Figure 1) given by $y_1 = G\, e_1$, where the strictly proper transfer function G is defined by

$$G := \frac{c_{n-1}\, s^{n-1} + \ldots + c_1\, s + c_0}{s^n + a_{n-1}\, s^{n-1} + \ldots + a_1\, s + a_0}\,, \tag{1}$$

where n is the order of G which satisfies $1 \le n \le 3$, and $a_i,\, c_i \in \mathbb{R}$ for $i = 0, \ldots, n-1$. We note $a := (a_0,\, \ldots,\, a_{n-1})$ and $c := (c_0,\, \ldots,\, c_{n-1})$ the system parameters of (1).

Let us consider its controllable canonical form defined by the following state-space representation:

$$\dot{x} = A\, x + B\, e_1, \quad y_1 = C\, x, \tag{2}$$

$$A := \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \ldots & 0 & 1 \\ -a_0 & -a_1 & \ldots & -a_{n-2} & -a_{n-1} \end{pmatrix} \in \mathbb{R}^{n \times n},$$
$$B := (0\ \ldots\ 0\ \ 1)^T \in \mathbb{R}^{n \times 1},$$
$$C := (c_0\ \ \ldots\ \ c_{n-1}) \in \mathbb{R}^{1 \times n}. \tag{3}$$

Let K be a rational controller, i.e., an element of the field of rational functions with real coefficients $\mathbb{R}(s)$. Let us also consider the closed-loop system defined in Figure 1. Then, we have:

$$\begin{pmatrix} e_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} S & K\,S \\ G\,S & G\,K\,S \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad S := (1 + G\,K)^{-1}.$$

Let us consider the following standard control problem.

Robust Control Problem (RCP): Given $\gamma > 0$, find a controller K which stabilizes G (i.e., such that the rational transfer functions S, $K\,S$ and $G\,S$ are proper and stable) and is such that:

$$\left\| \begin{pmatrix} S & K\,S \\ G\,S & G\,K\,S \end{pmatrix} \right\|_\infty < \gamma. \tag{4}$$

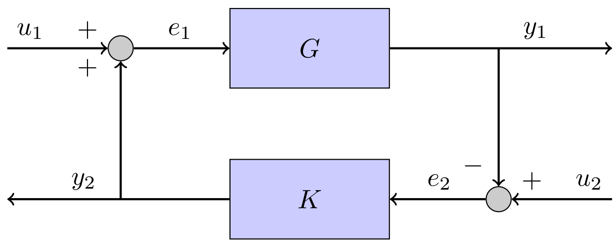

Fig. 1. Control scheme

For more details, the reader is referred to Glover et al. (1989); Zhou et al. (1996); Vinnicombe et al. (2001) and the references therein. The RCP yields a compromise between the performance of the closed-loop system and the robustness with respect to the perturbations u_1 and u_2.

We briefly state a standard result of *H_∞-control theory*.

Theorem 1. (Glover et al., 1989, Cor. 5.1), (Zhou et al., 1996, Ch. 18) Let (A, B, C) be an observable state-space representation (2) of the transfer function G defined by (1). Then, the minimal value of γ of (4) is given by

$$\gamma_{\text{opt}} = \sqrt{1 + \lambda_{\max}(Y X)},$$

where X is the unique real positive definite solution of the following *Algebraic Riccati Equation* (ARE)

$$\mathscr{R} := X\,A + A^T\,X - X\,B\,B^T\,X + C^T\,C = 0, \tag{5}$$

and $Y := Q\,X\,Q$, where $Q = Q^T$ is the Hankel matrix defined by

$$Q^{-1} := P = (P_1\ \cdots\ P_n)\,, \tag{6}$$
$$P_i^{\,T} := C \sum_{j=0}^{n-i} a_{n-j}\, A^{n-i-j}, \quad i = 1, \ldots, n,$$

and $\lambda_{\max}$ is the greatest eigenvalue of $Y\,X$ (which one has only real positive eigenvalues). For $\gamma > \gamma_{\text{opt}}$, a controller K_γ satisfying the RCP is defined by

$$\dot{z} = A_\gamma\, z + B_\gamma\, e_2, \quad y_2 = C_\gamma\, z, \tag{7}$$

with the following notations:

$$\begin{cases} Z_\gamma := \left(I + Y\,X - \gamma^2\, I\right)^{-1}, \\ A_\gamma := A - B\,B^T\,X + \gamma^2\, Z_\gamma\, Y\, C^T\, C, \\ B_\gamma := -\gamma^2\, Z_\gamma\, Y\, C^T, \\ C_\gamma := B^T\, X. \end{cases} \tag{8}$$

Remark 1. Q is related to Kalman's observability matrix $\mathscr{O} := \left(C, C A, \ldots, C A^{n-1}\right)^T$ since, with the notation

$$\Delta_o := \det(\mathscr{O}),$$

it only exists when the system is observable due to:

$$\begin{cases} n = 1, & \det(Q) = \Delta_o^{-1}, \\ n \in \{2, 3\}, & \det(Q) = -\Delta_o^{-1}. \end{cases}$$

In what follows, we shall suppose that (2) is observable.

In this paper, for systems of order 1 to 3 for which the *a_i's and c_j's are unknown parameters and not fixed numerical values*, we focus on the symbolic computation of X and

$\lambda_{\max}(YX)$. In this case, *numerical algorithms* for the computation of the positive definite solutions of ARE cannot be used. Our approach is based on Algorithm. 1 of Rance et al. (2016), which develops a symbolic-numeric method for solving the RCP. Since we work with small order systems, this symbolic-numeric algorithm yields a purely symbolic one for the computation of H_∞ controllers.

3. PARAMETRIZATION OF ALL THE SOLUTIONS OF THE ARE $\mathscr{R} = 0$

For $1 \le n \le 3$, let us introduce the following notations

$$\begin{cases} b_k := 0, & k < 0, \\ b_k := x_{k+1,n} + a_k, & 0 \le k \le n-1, \\ b_k := 1, & k = n, \\ b_k := 0, & k > n, \end{cases} \tag{9}$$

and $b = (b_0, \ldots, b_{n-1})$. According to Proposition 2 of Rance et al. (2016), the entries $x_{i,j}$ of a solution $X = X^T$ of $\mathscr{R} = 0$ can be expressed using a, b and c as shown in Table 1. Moreover the b_k's satisfy the polynomial systems $\mathcal{B}$ defined in Table 2, where the $d_{2\,k}$'s are defined in Table 3.

Table 1: Solutions X of $\mathscr{R} = 0$

n	$x_{i,j} = f(a, b, c)$
1	$x_{1,1} = b_0 - a_0$
2	$x_{1,2} = b_0 - a_0$ $x_{2,2} = b_1 - a_1$ $x_{1,1} = b_0\,b_1 - (a_0\,a_1 + c_0\,c_1)$
3	$x_{1,3} = b_0 - a_0$ $x_{2,3} = b_1 - a_1$ $x_{3,3} = b_2 - a_2$ $x_{1,1} = b_0\,b_1 - a_0\,a_1 - c_0\,c_1$ $x_{1,2} = b_0\,b_2 - a_0\,a_2 - c_0\,c_2$ $x_{2,2} = b_1\,b_2 - b_0 - a_1\,a_2 - c_1\,c_2 + a_0$

Table 2: Polynomial system $\mathcal{B}$ satisfied by the b_k's

n	Polynomial system $\mathcal{B}$
1	$\mathcal{B}_0 := b_0^2 - d_0 = 0$
2	$\mathcal{B}_0 := b_0^2 - d_0 = 0$ $\mathcal{B}_1 := b_1^2 - 2\,b_0 - d_2 = 0$
3	$\mathcal{B}_0 := b_0^2 - d_0 = 0$ $\mathcal{B}_1 := b_1^2 - 2\,b_0\,b_2 - d_2 = 0$ $\mathcal{B}_2 := b_2^2 - 2\,b_1 - d_4 = 0$

Table 3: Definition of the d_{2k}'s

n	Variables d_{2k}
1	$d_0 := a_0^2 + c_0^2$
2	$d_0 := a_0^2 + c_0^2$ $d_2 := \left(a_1^2 + c_1^2\right) - 2\,a_0$
3	$d_0 := a_0^2 + c_0^2$ $d_2 := \left(a_1^2 + c_1^2\right) - 2\,(a_0\,a_2 + c_0\,c_2)$ $d_4 := \left(a_2^2 + c_2^2\right) - 2\,a_1$

Remark 2. Note that $b_0 = \pm\sqrt{d_0}$. Hence, for first order systems, we get all the solutions of $\mathscr{R} = 0$.

As explained in (Rance et al., 2016, Section IV), we can find a rational parametrization of the solutions of $\mathcal{B}$ in the variable b_{n-1}, where b_{n-1} is a root of a univariate polynomial $\mathcal{P}$. For second order systems, since b_0 is known, we have $\mathcal{P} := \mathcal{B}_1$, where $\mathcal{P}$ is a univariate polynomial in b_1. For third order systems, using $\mathcal{B}_2 = 0$, one can find b_1 in terms of b_2, and substituting it into $\mathcal{B}_1 = 0$ to obtain a univariate polynomial $\mathcal{P}$ in b_2. The different parametrizations are given in Table 4. For more details on (rational) parametrizations of polynomial systems, we refer to Rouillier et al. (1999); Cox et al. (2015).

Table 4: Parametrization of all the solutions of $\mathcal{B}$

n	Polynomial system in b
1	$\emptyset$
2	$b_1 = b_1$ $\mathcal{P} := b_1^2 - 2\,b_0 - d_2 = 0$
3	$b_1 = 1/2\,\left(b_2^2 - d_4\right)$ $b_2 = b_2$ $\mathcal{P} := b_2^4 - 2\,d_4\,b_2^2 - 8\,b_0\,b_2 + d_4^2 - 4\,d_2 = 0$

The computation of the solutions of $\mathscr{R} = 0$ is then reduced to the computation of the roots of the univariate polynomial $\mathcal{P}$. Since the degree of $\mathcal{P}$ is less than or equal to 4, Cardano's and Ferrari's closed-form solutions can be used to express the roots of $\mathcal{P}$ by radicals (Tignol (2002)).

4. POSITIVE DEFINITE SOLUTIONS OF $\mathscr{R} = 0$

According to Kanno et al. (2009), Theorem 4 of Rance et al. (2016), the real positive definite solution of $\mathscr{R} = 0$ is obtained by means of the maximal real root σ of $\mathcal{P}$. Furthermore, $X > 0$ solution of $\mathscr{R} = 0$ verifies the following property.

Proposition 1. (Rance et al. (2016), Proposition 5).
$X > 0$ verifies $b_0 = \sqrt{d_0}$, where $d_0 := a_0^2 + c_0^2$.

4.1 First order systems

For first order systems, the ARE $\mathscr{R} = 0$ is a single quadratic polynomial equation, which solutions are known (see Remark 2). According to Proposition 1, we have:

$$X := x_{1,1} = \sigma - a_0 = \sqrt{a_0^2 + c_0^2} - a_0. \tag{10}$$

4.2 Second order systems

The univariate polynomial $\mathcal{P}$ is of degree 2 in b_1, where $b_0 = \sqrt{a_0^2 + c_0^2} \ge |a_0|$. Since $b_0 - a_0 \ge 0$, $\mathcal{P}$ has always 2 real roots which greatest real one is defined by:

$$\sigma := \sqrt{2\,b_0 + d_2} = \sqrt{2\,(b_0 - a_0) + a_1^2 + c_1^2}. \tag{11}$$

4.3 Third order systems

If $p := -2\,d_4$, $q := -8\,b_0$, $r := d_4^2 - 4\,d_2$, then we have:

$$\mathcal{P}(b_2) := b_2^4 + p\,b_2^2 + q\,b_2 + r.$$

To obtain the solutions of $\mathcal{P}$ by radicals, we first introduce the following notations (see Tignol (2002)):

$$\begin{cases} \varepsilon_1 := \pm 1,\ \varepsilon_2 := \pm 1,\ \varepsilon := (\varepsilon_1, \varepsilon_2), \\ p_2 := 4\,d_2 - \dfrac{4}{3}\,d_4^2, \quad q_2 := \dfrac{8}{3}\,d_2\,d_4 - \dfrac{16}{27}\,d_4^3 - 8\,b_0^2, \\ \alpha := \left(\dfrac{1}{2}\left(-27\,q_2 + \sqrt{27\,(4\,p_2^3 + 27\,q_2^2)}\right)\right)^{1/3}, \\ u := \dfrac{1}{3}\left(\alpha - \dfrac{3\,p_2}{\alpha} + 2\,d_4\right), \quad \Delta_2 := 2\,d_4 + \varepsilon_1\,\dfrac{8\,b_0}{\sqrt{2\,u}} - u. \end{cases}$$

Since we can suppose that $q = -8\,b_0 \neq 0$ ($b_0 = 0$ is equivalent to $a_0 = c_0 = 0$, i.e., G is then of order 2, which case has already been studied in Section 4.2), the roots of $\mathcal{P}$ can be expressed as follows:

$$b_2(\varepsilon) = \frac{\sqrt{2}}{2}\left(\varepsilon_1 \sqrt{u} + \varepsilon_2 \sqrt{\Delta_2}\right).$$

To determine which one of the b_2's is the maximal real one, we compute the discriminant variety of $\mathcal{P}$ (Lazard et al. (2007)), i.e., the discriminant of $\mathcal{P}$ in b_2 as $\mathcal{P}$ is monic in b_2. Given an open connected set in the space of parameters which does not encounter the discriminant variety of $\mathcal{P}$, for any values of the parameters in this set, $\mathcal{P}$ has a constant number of real roots. Over the discriminant variety, some roots are crossing, i.e., 2 closed-form solutions can define the same maximal real root. For instance, the `Maple` command `CellDecomposition` applied to $\mathcal{P}$ computes, as in Corvez et al. (2003), a partition of the ambient space made of $\mathcal{P}$ itself and of the cells of maximal dimension of the *Cylindrical Algebraic Decomposition* adapted to $\mathcal{P}$ (see Collins et al. (1976)) where all the polynomials defining $\mathcal{P}$ have all a non null constant sign. In each cell, we can choose a particular value of the parameters and find which root is the greatest real one. Ignoring cells where there are no real solutions (as we assume that (2) is observable and thus that the RCP has always a solution), we show that

$$\sigma = b_2(1,1) = \frac{\sqrt{2}}{2}\left(\sqrt{u} + \sqrt{2\,d_4 + \frac{8\,b_0}{\sqrt{2\,u}} - u}\right)$$

is the maximal real root of $\mathcal{P}$ in each cell, so it is for any values of the parameters.

5. CHARACTERISTIC POLYNOMIAL OF YX

Now that the solution $X > 0$ of $\mathscr{R} = 0$ is determined, we have to compute the eigenvalues of $YX = QXQX$. In Table 5, we give the explicit form of Q defined by (6). Then, in Table 6, the matrix Y is shown. Finally, in Table 7, the characteristic polynomial $\mathcal{H}(\lambda, a, c)$ of YX is computed.

Table 5: Matrices Q

n	Hankel matrix Q
1	$Q := \Delta_o^{-1} = c_0^{-1}$
2	$Q := \frac{1}{\Delta_o}\begin{pmatrix} q_1 & q_2 \\ q_2 & q_3 \end{pmatrix}$ $q_1 := -c_1, \quad q_2 := c_0, \quad q_3 := c_1\,a_0 - a_1\,c_0$
3	$Q := \frac{1}{\Delta_o}\begin{pmatrix} q_1 & q_2 & q_3 \\ q_2 & q_3 & q_4 \\ q_3 & q_4 & q_5 \end{pmatrix}$ $q_1 := a_1\,c_2^2 - (a_2\,c_1 + c_0)\,c_2 + c_1^2$ $q_2 := -a_0\,c_2^2 + a_2\,c_0\,c_2 - c_0\,c_1$ $q_3 := a_0\,c_1\,c_2 - a_1\,c_0\,c_2 + c_0^2$ $q_4 := a_0\,c_0\,c_2 - a_0\,c_1^2 + a_1\,c_0\,c_1 - a_2\,c_0^2$ $q_5 := (a_0\,c_2 - a_2\,c_0)^2 - a_0\,a_1\,c_1\,c_2 + a_0\,a_2\,c_1^2$ $\quad +a_1^2\,c_0\,c_2 - a_1\,a_2\,c_0\,c_1 + a_0\,c_0\,c_1 - a_1\,c_0^2$

Table 6: Matrices Y

n	Matrix Y
1	$Y := \left(\sqrt{a_0^2 + c_0^2} - a_0\right) c_0^{-2}$
2	$Y := \frac{1}{\Delta_o^2}\begin{pmatrix} z_{1,1} & z_{1,2} \\ z_{1,2} & z_{2,2} \end{pmatrix}$ $z_{1,1} := q_1^2\,x_{1,1} + 2\,q_1\,q_2\,x_{1,2} + q_2^2\,x_{2,2}$ $z_{1,2} := (q_2\,x_{1,1} + q_3\,x_{1,2})\,q_1 + (q_2\,x_{1,2} + q_3\,x_{2,2})\,q_2$ $z_{2,2} := q_2^2\,x_{1,1} + 2\,q_2\,q_3\,x_{1,2} + q_3^2\,x_{2,2}$
3	$Y := \frac{1}{\Delta_o^2}\begin{pmatrix} z_{1,1} & z_{1,2} & z_{1,3} \\ z_{1,2} & z_{2,2} & z_{2,3} \\ z_{1,3} & z_{2,3} & z_{3,3} \end{pmatrix}$ $z_{1,1} := q_1^2\,x_{1,1} + q_2^2\,x_{2,2} + q_3^2\,x_{3,3}$ $\quad +2\,(x_{1,2}\,q_1\,q_2 + x_{1,3}\,q_1\,q_3 + x_{2,3}\,q_2\,q_3)$ $z_{1,2} := q_2^2\,x_{1,2} + q_3^2\,x_{2,3} + (q_1\,x_{1,2} + q_4\,x_{3,3})\,q_3$ $\quad + ((x_{1,3} + x_{2,2})\,q_3 + q_1\,x_{1,1} + q_4\,x_{2,3})\,q_2 + q_1\,q_4\,x_{1,3}$ $z_{1,3} := q_3^2\,x_{1,3} + q_1\,(q_4\,x_{1,2} + q_5\,x_{1,3}) + q_2\,(q_4\,x_{2,2} + q_5\,x_{2,3})$ $\quad + (q_1\,x_{1,1} + q_2\,x_{1,2} + q_4\,x_{2,3} + q_5\,x_{3,3})\,q_3$ $z_{2,2} := q_2^2\,x_{1,1} + q_3^2\,x_{2,2} + q_4^2\,x_{3,3}$ $\quad +2\,(q_2\,x_{1,3} + q_3\,x_{2,3})\,q_4 + 2\,q_2\,q_3\,x_{1,2}$ $z_{2,3} := q_3^2\,x_{1,2} + q_4^2\,x_{2,3} + (q_2\,x_{1,2} + q_5\,x_{3,3})\,q_4$ $\quad + ((x_{1,3} + x_{2,2})\,q_4 + q_2\,x_{1,1} + q_5\,x_{2,3})\,q_3 + q_2\,q_5\,x_{1,3}$ $z_{3,3} := q_3^2\,x_{1,1} + q_4^2\,x_{2,2} + q_5^2\,x_{3,3} + 2\,q_3\,q_4\,x_{1,2}$ $\quad +2\,(q_3\,x_{1,3} + q_4\,x_{2,3})\,q_5$

Table 7: Characteristic polynomial $\mathcal{H}$ of YX

n	Polynomial $\mathcal{H}(\lambda, a, c)$
1	$\mathcal{H}(\lambda, a, c) := \lambda - YX = \lambda - \left(\sqrt{a_0^2 + c_0^2} - a_0\right)^2 c_0^{-2}$
2	$\mathcal{H}(\lambda, a, c) := \lambda^2 + \nu_1(a, c)\,\lambda + \nu_0(a, c) = 0$ $\nu_1(a, c) := \beta_1(a, c, b)\,\Delta_o^{-2}$ $\nu_0(a, c) := \beta_0(a, c, b)\,\Delta_o^{-4} = \left(\det(X)\,\Delta_o^{-1}\right)^2$ $\beta_1(a, c, b) := -\left(z_{1,1}\,x_{1,1} + z_{2,2}\,x_{2,2} + 2\,z_{1,2}\,x_{1,2}\right)$ $\beta_0(a, c, b) := (\Delta_o\,\det(X))^2$
3	$\mathcal{H}(\lambda, a, c) := \lambda^3 + \nu_2\,\lambda^2 + \nu_1\,\lambda + \nu_0 = 0$ $\nu_k := \beta_k\,\Delta_o^{2(k-n)},\ 0 \leq k \leq 2$ $\beta_2 := -\,(x_{1,1}\,z_{1,1} + x_{2,2}\,z_{2,2} + x_{3,3}\,z_{3,3}$ $\quad +2\,(x_{1,2}\,z_{1,2} + x_{1,3}\,z_{1,3} + x_{2,3}\,z_{2,3}))$ $\beta_1 := \left(z_{1,2}^2 - z_{1,1}\,z_{2,2}\right) x_{1,2}^2 + \left(z_{1,3}^2 - z_{1,1}\,z_{3,3}\right) x_{1,3}^2$ $\quad + \left(z_{2,3}^2 - z_{2,2}\,z_{3,3}\right) x_{2,3}^2 - x_{1,1}\,x_{2,2}\,z_{1,2}^2$ $\quad -x_{1,1}\,x_{3,3}\,z_{1,3}^2 - x_{2,2}\,x_{3,3}\,z_{2,3}^2 + z_{1,1}\,z_{3,3}\,x_{1,1}\,x_{3,3}$ $\quad +z_{1,1}\,z_{2,2}\,x_{1,1}\,x_{2,2} + z_{2,2}\,z_{3,3}\,x_{2,2}\,x_{3,3}$ $\quad +2\,[x_{1,3}\,(z_{1,2}\,z_{1,3} - z_{1,1}\,z_{2,3})$ $\quad +x_{2,3}\,(z_{1,2}\,z_{2,3} - z_{1,3}\,z_{2,2})$ $\quad +x_{3,3}\,(z_{1,2}\,z_{3,3} - z_{1,3}\,z_{2,3})]\,x_{1,2}$ $\quad +2\,[x_{2,3}\,(z_{1,3}\,z_{2,3} - z_{1,2}\,z_{3,3})$ $\quad +x_{2,2}\,(z_{1,3}\,z_{2,2} - z_{1,2}\,z_{2,3})]\,x_{1,3}$ $\quad +2\,x_{1,1}\,x_{2,3}\,(z_{1,1}\,z_{2,3} - z_{1,2}\,z_{1,3})$ $\beta_0 := -\Delta_o^4\,\det(X)^2$

6. EXPLICIT γ_{OPT} AND H_∞ CONTROLLERS

As $X > 0$ and $Y > 0$, all the roots of $\mathcal{H}$ are real positive. Since $\mathcal{H}$ is of degree less than or equal to 4 in λ, it is possible to compute its roots by means of radicals (Tignol (2002)). We detail below the computation of its maximal real root at each order. Then, given this maximal root, one can deduce $\gamma_{\mathrm{opt}} := \sqrt{1 + \lambda_{\max}}$ and thus (sub)-optimal H_∞ controllers using (7), (8) and Tables 1 and 6.

6.1 First order systems

According to Table 7, γ_{opt} is trivial. We also note that γ_{opt} only involves $\tau := 1/G(0) = a_0/c_0$ and the sign of c_0.

Theorem 2. For $n = 1$, the optimal H_∞ criterion γ_{opt} is:

$$\gamma_{\text{opt}} := \sqrt{1 + y_{1,1}\, x_{1,1}} = \sqrt{1 + \left(\operatorname{sgn}(c_0)\sqrt{1+\tau^2} - \tau\right)^2}.$$

Figure 2 represents $\gamma_{\text{opt}}(\tau)$ depending on $\operatorname{sgn}(c_0)$.

If $\tau = 0$, i.e., $G = c_0/s$, then $\gamma_{\text{opt}} = \sqrt{2}$ for all c_0 (see also Kanno et al. (2007)). Moreover, we have:

$$\lim_{\tau \to +\infty} \gamma_{\text{opt}} = \begin{cases} 1 & c_0 > 0, \\ +\infty & c_0 < 0, \end{cases} \quad \lim_{\tau \to -\infty} \gamma_{\text{opt}} = \begin{cases} +\infty & c_0 > 0, \\ 1 & c_0 < 0. \end{cases}$$

In both cases, γ_{opt} diverges for $a_0 < 0$, i.e., when the pole of G is unstable. On the contrary, γ_{opt} converges to 1 for $a_0 > 0$, i.e., when the pole of G is stable.

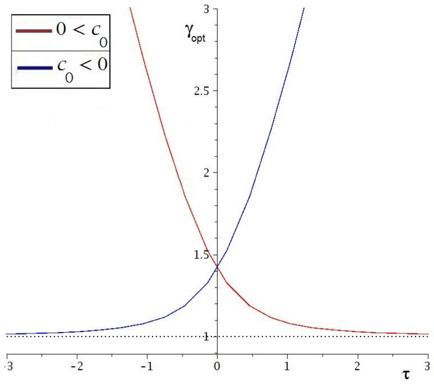

Fig. 2. $\gamma_{\text{opt}} = \Gamma_2(\tau)$ for a first order SISO system

For $n = 1$, as $BB^T = 1$ and $YC^TC = x_{1,1}$, (8) yields

$$\begin{cases} Z_\gamma = \left(\gamma_{\text{opt}}^2 - \gamma^2\right)^{-1}, \\ A_\gamma = -a_0 - |c_0| \sqrt{\gamma_{\text{opt}}^2 - 1}\, \dfrac{\gamma_{\text{opt}}^2 - 2\gamma^2}{\gamma_{\text{opt}}^2 - \gamma^2}, \\ B_\gamma = -\dfrac{\operatorname{sgn}(c_0)\,\gamma^2}{\gamma_{\text{opt}}^2 - \gamma^2} \sqrt{\gamma_{\text{opt}}^2 - 1}, \\ C_\gamma = |c_0| \sqrt{\gamma_{\text{opt}}^2 - 1}, \end{cases}$$

and we get the following (sub)-optimal H_∞ controller:

$$K_\gamma(s) = \frac{-c_0\,\gamma^2\,\left(\gamma_{\text{opt}}^2 - 1\right)}{(s + a_0)(\gamma_{\text{opt}}^2 - \gamma^2) + |c_0|\,(\gamma_{\text{opt}}^2 - 2\,\gamma^2)\,\sqrt{\gamma_{\text{opt}}^2 - 1}}. \tag{12}$$

Remark 3. We have $K_{\gamma_{\text{opt}}} = \operatorname{sgn}(c_0)\sqrt{\gamma_{\text{opt}}^2 - 1}$.

6.2 Second order systems

According to Table 7, the characteristic polynomial $\mathcal{H}$ of YX is of the form

$$\mathcal{H}(\lambda, a, c) = \lambda^2 + \nu_1(a, c)\,\lambda + \nu_0(a, c) = 0,$$

which leads to the following theorem.

Theorem 3. For $n = 2$, the optimal H_∞ criterion γ_{opt} is:

$$\gamma_{\text{opt}} := \sqrt{1 + \frac{-\beta_1 + \sqrt{\beta_1^2 - 4\,\beta_0}}{2\,\Delta_o^2}},$$

where β_1 and β_0 are given in Table 7.

Proof. Since $\mathcal{H}$ is of degree 2, its roots are given by:

$$\begin{cases} \lambda_1 := \dfrac{-\beta_1 + \sqrt{\beta_1^2 - 4\,\beta_0}}{2\,\Delta_o^2} > 0, \\ \lambda_2 := \dfrac{-\beta_1 - \sqrt{\beta_1^2 - 4\,\beta_0}}{2\,\Delta_o^2} > 0. \end{cases}$$

As $X > 0$ and $Y > 0$, both of these roots are real positive, which means $\sqrt{\beta_1^2 - 4\,\beta_0} > 0$, so that the maximal one is:

$$\lambda_{\max} := \lambda_1 = \frac{-\beta_1 + \sqrt{\beta_1^2 - 4\,\beta_0}}{2\,\Delta_o^2}.$$

Then, we deduce $\gamma_{\text{opt}} := \sqrt{1 + \lambda_{\max}}$.

6.3 Third order systems

To find the roots of $\mathcal{H}(\lambda) = \lambda^3 + \nu_2\,\lambda_2 + \nu_1\,\lambda + \nu_0$, as in Tignol (2002), we introduce the following notations :

$$\begin{cases} \mu_1 := -\dfrac{1}{3}\,\nu_2^2 + \nu_1, \\ \mu_0 := \dfrac{2}{27}\,\nu_2^3 + \nu_0 \\ \qquad -\dfrac{1}{3}\,\nu_1\,\nu_2, \end{cases} \begin{cases} \alpha := \left(\dfrac{-27\,\mu_0 + \sqrt{\Delta}}{2}\right)^{\frac{1}{3}}, \\ \Delta = 27\left(4\,\mu_1^3 + 27\,\mu_0^2\right), \\ j := -\dfrac{1}{2} + i\,\dfrac{\sqrt{3}}{2}. \end{cases} \tag{13}$$

Theorem 4. For $n = 3$, the optimal H_∞ criterion γ_{opt} is

$$\gamma_{\text{opt}} := \sqrt{1 + \frac{1}{3}\left(\alpha - \frac{3\,\mu_1}{\alpha} - \nu_2\right)},$$

where α and μ_1 are given in (13) and ν_2 in Table 7.

Proof. The roots of $\mathcal{H}$ are given by (Tignol (2002)):

$$\begin{cases} \lambda_1 = \dfrac{1}{3}\left(\alpha - \dfrac{3\,\mu_1}{\alpha} - \nu_2\right), \\ \lambda_2 = \dfrac{1}{3}\left(\alpha\,j^2 - \dfrac{3\,\mu_1}{\alpha\,j} - \nu_2\right), \\ \lambda_3 = \dfrac{1}{3}\left(j\,\alpha - \dfrac{3\,\mu_1}{\alpha\,j^2} - \nu_2\right). \end{cases}$$

Since $X > 0$ and $Y > 0$, $\mathcal{H}$ is a polynomial of degree 3 with 3 real roots. Using the concept of *discriminant variety* as in Section 4.3, we can show that its maximal root is $\lambda_{\max} = \lambda_1$. Then, we deduce $\gamma_{\text{opt}} := \sqrt{1 + \lambda_{\max}}$.

7. A MASS-SPRING-DAMPER SYSTEM

We consider a mass-spring-damper system (Zhou et al., 1996, §10.2) (Figure 3). A mass m is linked to a motionless support by a spring of stiffness k and a damper of magnitude ξ. Referring to the notations of Figure 1, we study the displacement of m, denoted by y_1, while m is excited by a force e_1.

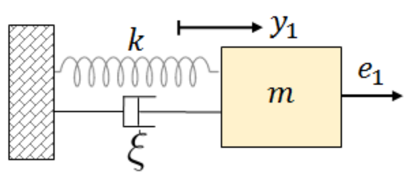

Fig. 3. Mass-spring-damper system

We consider the transfer function G of the plant:

$$G := \frac{y_1}{e_1} = \frac{c_0}{s^2 + a_1\, s + a_0},\; c_0 := \frac{1}{m},\; a_1 := \frac{\xi}{m},\; a_0 := \frac{k}{m}.$$

Since the parameter $c_1 := 0$, the expressions are a bit simpler than in the general case. Also, we can notice that $\gamma_{\rm opt}$ only depends on $\tau^{-1} := G(0) = c_0/a_0 = 1/k$ and $\rho := a_1/\sqrt{a_0} = \xi/\sqrt{k\,m}$. See Figure 4 for the plot of $\gamma_{\rm opt}$ in function of τ and ρ. We denote by Δ_G (resp., Δ_Φ) the *gain* (resp., *phase*) *margin*. Given $\gamma > \gamma_{\rm opt}$, we can compute a parametric H_∞ controller satisfying the RCP which guarantees the following gain and phase margins:

$$\begin{cases} \Delta_{\rm G}(G, K_\gamma) \geq \delta_{\rm G}(\gamma) := \dfrac{1+\gamma^{-1}}{1-\gamma^{-1}}, \\ \Delta_\Phi(G, K_\gamma) \geq \delta_\Phi(\gamma) := 2 \arcsin\left(\gamma^{-1}\right). \end{cases}$$

For more details, see Vinnicombe et al. (2001). For instance, for $m = 1$, $k = 1$, $\xi \in \{0, 1/2, 1\}$, the margins are given in Table 8 (evaluations of symbolic expressions), the Bode diagrams of the optimal controllers in Figure 5, and the Black-Nichols diagrams of the open-loop with the same optimal controllers in Figure 6.

Remark 4. Practically, in order to follow a target reference, the controller K should contain one integrator. This can be obtained by first using the weight $W := \frac{c_c + s}{s}$, then defining the new plant $G' := G\,W$, and finally computing a controller K' stabilizing G' by means of $3^{\rm rd}$ order formulas. Then, as detailed in Vinnicombe et al. (2001), the controller $K = W\,K'$ stabilizes G and satisfies:

$$\|S\|_\infty < \gamma, \quad \|GKS\|_\infty < \gamma.$$

Table 8: $\gamma_{\rm opt}$ for $\xi \in \{0, 1/2, 1\}$

ξ	0	1/2	1
$\gamma_{\rm opt}$	$\simeq 1.80$	$\simeq 1.37$	$\simeq 1.22$
$\delta_\Phi(\gamma_{\rm opt})$ (°)	$\simeq 67.5$	$\simeq 93.9$	$\simeq 110$
$\Delta_\Phi(G, K_{\gamma_{\rm opt}})$ (°)	$\simeq 68.9$	∞	∞
$\delta_{\rm G}(\gamma_{\rm opt})$ (dB)	$\simeq 10.9$	$\simeq 16.1$	$\simeq 20.2$
$\Delta_{\rm G}(G, K_{\gamma_{\rm opt}})$ (dB)	$\simeq 99.1$	$\simeq 96.7$	$\simeq 95.7$

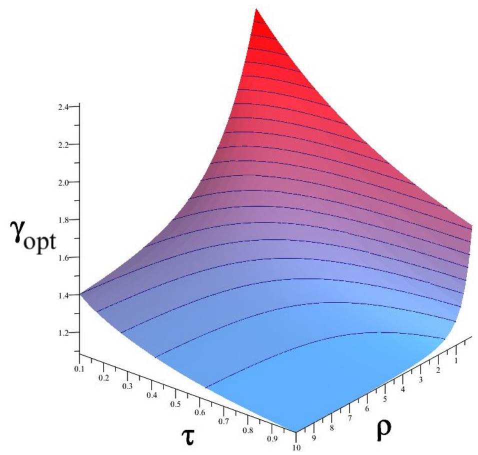

Fig. 4. $\gamma_{\rm opt}$ of a mass-spring-damper system.

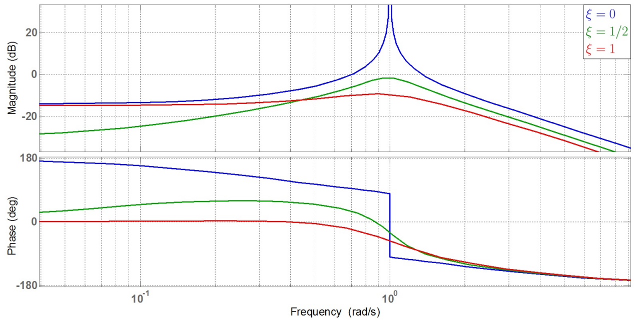

Fig. 5. Bode diagram of $K_{\gamma_{\rm opt}}$ for $m = 1$, $k = 1$

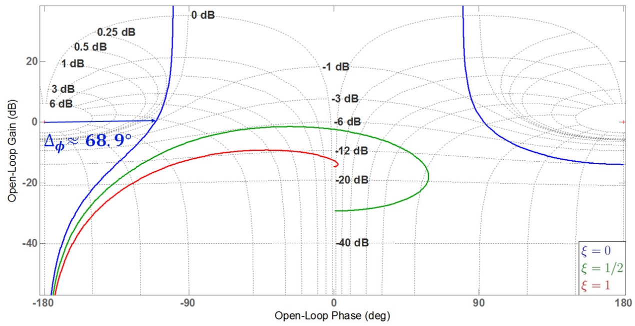

Fig. 6. Black-Nichols diagram of $K_{\gamma_{\rm opt}}\,G$ for $m = 1$, $k = 1$

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