# A GENERALIZATION OF THE YOULA-KUČERA PARAMETRIZATION FOR MIMO STABILIZABLE SYSTEMS

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Abstract: The purpose of this paper is to give new necessary and sufficient conditions for internal stabilizability and existence of left/right/doubly coprime factorizations for linear MIMO systems. In particular, we generalize the Youla-Kučera parametrization of all stabilizing controllers for every internally stabilizable MIMO plant which does not necessarily admit doubly coprime factorizations.

Keywords: Youla-Kučera parametrization of all stabilizing controllers, internal stabilizability, left/right/doubly coprime factorizations, lattices.

## 1. INTRODUCTION

For finite-dimensional linear systems (i.e. ordinary differential equations), it is well known that a transfer matrix is internally stabilizable if and only if it admits a doubly coprime factorization (Vidyasagar, 1985). However, this result is generally not true for infinite-dimensional linear systems (e.g. differential time-delay systems, partial differential equations) (Quadrat, 2003; Vidyasagar, 1985) or for multidimensional linear systems (Mori, 2002; Sule, 1994).

The Youla-Kučera parametrization of all stabilizing controllers was developed for transfer matrices which admit doubly coprime factorizations (Vidyasagar, 1985). The fact that this parametrization is affine in a matrix of free parameters highly simplifies the research of all optimal stabilizing controllers. Indeed, this parametrization allows us to transform this non-linear optimal problem with constraints into a free affine, and thus, convex optimal problem.

The purpose of this paper is to give new general necessary and sufficient conditions for the existence of left/right/doubly coprime factorizations and internal stabilizability. In order to do that, we shall introduce the concept of *lattices on vector spaces* (Bourbaki, 1989) into the fractional representation approach to analysis and synthesis problems (Vidyasagar, 1985). Using these results, we shall exhibit the general parametrization of all the stabilizing controllers for a stabilizable plant which does not necessarily admit doubly coprime factorizations. In particular, if the transfer matrix admits a doubly coprime factorization, then the previous parametrization becomes the Youla-Kučera one. These results generalize for MIMO systems the results obtained in (Quadrat, 2003).

## 2. FRACTIONAL REPRESENTATION APPROACH

Let us recall the fractional representation approach to synthesis problems (Vidyasagar, 1985). Let us consider a *commutative integral domain $A$ of (proper) stable SISO plants.*

*Example 1.* For instance, we have the following examples of integral domains of stable systems

$$
\begin{aligned}
RH_\infty &= \{\, n/d \,|\, n,\, d \in \mathbb{R}[s],\, \deg n \leq \deg d \\
&\qquad d(s^\star) = 0 \Rightarrow \operatorname{Re} s^\star < 0\},\\
H_\infty(\mathbb{C}_+) &= \{\, f \in \mathcal{H}(\mathbb{C}_+) \,|\\
&\qquad \parallel f \parallel_\infty = \sup_{s \in \mathbb{C}_+} |f(s)| < +\infty\},\\
\mathcal{A} &= \{\, f(t) + \sum_{i=0}^{\infty} a_i\, \delta(t - t_i) \mid f \in L_1(\mathbb{R}_+),\\
&\qquad (a_i)_{i \geq 0} \in l_1(\mathbb{Z}_+),\, 0 = t_0 \leq t_1 \leq \ldots\},
\end{aligned}
$$

where $\mathbb{C}_+ = \{s \in \mathbb{C} \,|\, \operatorname{Re} s > 0\}$, $\mathcal{H}(\mathbb{C}_+)$ is the ring of holomorphic functions in $\mathbb{C}_+$ (Quadrat, 2003; Vidyasagar, 1985).

A transfer function $p$ belongs to $RH_\infty$ (resp. $H_\infty(\mathbb{C}_+)$, $\hat{\mathcal{A}} = \{\mathcal{L}(f) \,|\, f \in \mathcal{A}\}$, where $\mathcal{L}(f)$ denotes the Laplace transform) iff $p$ is the transfer function of an exponentially stable (resp. $L_2(\mathbb{R}_+)$-stable, $L_\infty(\mathbb{R}_+)$-stable) linear time-invariant finite-dimensional (resp. infinite-dimensional) system.

Let us define the *quotient field* of $A$, namely:

$$K = Q(A) = \{n/d \,|\, 0 \neq d,\, n \in A\}.$$

$K = Q(A)$ corresponds to the class of $A$-stable and $A$-unstable SISO plants. For instance, we have $p = e^{-s}/(s-1) \notin H_\infty(\mathbb{C}_+)$ because $p$ has an unstable pole in $\mathbb{C}_+$. But, $p \in Q(H_\infty(\mathbb{C}_+))$ because we have $p = n/d$, where:

$$n = e^{-s}/(s+1),\; d = (s-1)/(s+1) \in H_\infty(\mathbb{C}_+).$$

More generally, we can consider the class of MIMO plants defined by transfer matrices with entries in $K = Q(A)$. If we have $P \in K^{q \times (p-q)}$, then we can always write $P$ as $P = D^{-1}\, N = \tilde{N}\, \tilde{D}^{-1}$, where:

$$\begin{cases} R = (D \,:\, -N) \in A^{q \times p},\\ \tilde{R} = (\tilde{N}^T \,:\, \tilde{D}^T)^T \in A^{p \times (p-q)}. \end{cases}$$

*Definition 1.* (Vidyasagar, 1985) Let $A$ be an integral domain of stable SISO plants and $K = Q(A)$ its quotient field.

- A transfer matrix $P \in K^{q \times (p-q)}$ is *internally stabilizable* if there exists a *stabilizing controller* $C \in K^{(p-q) \times q}$ of $P$, namely a controller $C \in K^{(p-q) \times q}$ such that all the entries of the following matrix belong to $A$:

$$H(P,\, C) = \begin{pmatrix} I_q & -P \\ -C & I_{p-q} \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} (I_q - P\,C)^{-1} & (I_q - P\,C)^{-1}\,P \\ C\,(I_q - P\,C)^{-1} & I_{p-q} + C\,(I_q - P\,C)^{-1}\,P \end{pmatrix} \quad (1)$$

$$= \begin{pmatrix} I_q + P\,(I_{p-q} - C\,P)^{-1}\,C & P\,(I_{p-q} - C\,P)^{-1} \\ (I_{p-q} - C\,P)^{-1}\,C & (I_{p-q} - C\,P)^{-1} \end{pmatrix} \quad (2)$$

- A transfer matrix $P \in K^{q \times (p-q)}$ admits a *left-coprime factorization* if there exist

$$\begin{cases} R = (D \,:\, -N) \in A^{q \times p},\\ S = (X^T \,:\, Y^T)^T \in A^{p \times q}, \end{cases}$$

such that $\det D \neq 0$, $P = D^{-1}\, N$ and:

$$R\,S = D\,X - N\,Y = I_q.$$

- A transfer matrix $P \in K^{q \times (p-q)}$ admits a *right-coprime factorization* if there exist

$$\begin{cases} \tilde{R} = (\tilde{N}^T \,:\, \tilde{D}^T)^T \in A^{p \times (p-q)},\\ \tilde{S} = (-\tilde{Y} \,:\, \tilde{X}) \in A^{(p-q) \times p}, \end{cases}$$

such that $\det \tilde{D} \neq 0$, $P = \tilde{N}\, \tilde{D}^{-1}$ and:

$$\tilde{S}\,\tilde{R} = -\tilde{Y}\,\tilde{N} + \tilde{X}\,\tilde{D} = I_{p-q}.$$

- A transfer matrix $P \in K^{q \times (p-q)}$ admits a *doubly coprime factorization* iff $P$ admits a left and right-coprime factorization.

## 3. COPRIME FACTORIZATIONS

*Definition 2.* (Bourbaki, 1989) Let $V$ be a finite-dimensional $K = Q(A)$-vector space. An $A$-submodule $M$ of $V$ is called *a lattice on* $V$ if there exist free $A$-submodules $L_1$ and $L_2$ of $V$ such that:

$$\begin{cases} L_1 \subseteq M \subseteq L_2,\\ \operatorname{rk}_A(L_1) = \dim_K(V). \end{cases}$$

*Proposition 1.* (Bourbaki, 1989) An $A$-submodule $M$ of $V$ is a lattice on $V$ iff the $K$-vector space

$$K\,M \triangleq \{k\,m \,|\, k \in K,\, m \in M\} = V$$

and $M$ is contained in a finitely generated $A$-submodule of $V$.

*Example 2.* If $P \in K^{q \times (p-q)}$, then the $A$-module $(I_q \;:\; -P)\,A^p$ is a lattice on the $K$-vector space $K^q$. Similarly, the $A$-module $A^{1 \times p} \begin{pmatrix} P \\ I_{p-q} \end{pmatrix}$ is a lattice on the $K$-vector space $K^{1 \times (p-q)}$.

*Definition 3.* (Bourbaki, 1989) Let $V$ and $W$ be two finite-dimensional $K$-vector spaces and $M$ (resp. $N$) a lattice on $V$ (resp. $W$). Then, we denote by $N : M$ the $A$-submodule of $\hom_K(V, W)$ formed by the $K$-linear maps $f : V \to W$ which satisfy $f(M) \subseteq N$.

*Proposition 2.* (Bourbaki, 1989) Let $V$ and $W$ be two finite-dimensional $K$-vector spaces and $M$ (resp. $N$) a lattice on $V$ (resp. $W$). Then, we have:

(1) $N : M$ is a lattice on

$$\hom_K(V, W) = \{\, f : V \to W \,|\; f \text{ is a } K-\text{linear map}\},$$

(2) the canonical map $N : M \to \hom_A(M, N)$, which maps $f \in N : M$ into $f_{|\,M}$, is bijective.

*Example 3.* If $P \in K^{q\times(p-q)}$, then we have:

$$\begin{aligned} & A : (I_q \,:\, -P)\, A^p \\ &= \{f \in \hom_K(K^q, K) \,|\, f((I_q \,:\, -P)\, A^p) \subseteq A\} \\ &= \{\lambda \in K^{1\times q} \,|\, \lambda\, (I_q \,:\, -P)\, A^p \subseteq A\} \\ &= \{\lambda \in K^{1\times q} \,|\, \lambda \in A^{1\times q},\ \lambda\, P \in A^{1\times(p-q)}\} \\ &= \{\lambda \in A^{1\times q} \,|\, \lambda\, P \in A^{1\times(p-q)}\}. \end{aligned}$$

*Theorem 1.* (1) $P \in K^{q\times(p-q)}$ admits a left-coprime factorization iff there exists a square matrix $D \in A^{q\times q}$ such that $\det D \neq 0$ and

$$(I_q \,:\, -P)\, A^p = D^{-1}\, A^q, \qquad (3)$$

i.e. $(I_q \,:\, -P)\, A^p$ is a free $A$-module of rank $q$.

Then, $P = D^{-1}\, N$ ($N = D\, P \in A^{q\times(p-q)}$) is a left-coprime factorization of $P$.

(2) $P \in K^{q\times(p-q)}$ admits a right-coprime factorization iff there exists a square matrix $\tilde{D} \in A^{(p-q)\times(p-q)}$ such that $\det \tilde{D} \neq 0$ and

$$A^{1\times p} \begin{pmatrix} P \\ I_{p-q} \end{pmatrix} = A^{1\times(p-q)}\, \tilde{D}^{-1}, \qquad (4)$$

is a free $A$-module of rank $p-q$.

Then, $P = \tilde{N}\, \tilde{D}^{-1}$ ($\tilde{N} = P\, \tilde{D} \in A^{q\times(p-q)}$) is a right-coprime factorization of $P$.

*Proof.* 1. $\Rightarrow$ If $P$ admits a left-coprime factorization $P = D^{-1}\, N$, with $D\, X - N\, Y = I_q$, then we have $(I_q \,:\, -D)\, A^p = (D^{-1}\, (D \,:\, -N))\, A^p = D^{-1}\, (D \,:\, -N)\, A^p$ and $(D \,:\, -N)\, A^p = A^q$, because $(D \,:\, -N)\, A^p \subseteq A^q$ and $\forall\, \mu \in A^q$, we have $\mu = (D \,:\, -N)\, \lambda \in (D \,:\, -N)\, A^p$, where:

$$\lambda = \begin{pmatrix} X \\ Y \end{pmatrix} \mu \in A^p.$$

Therefore, we have $(I_q \,:\, -D)\, A^p = D^{-1}\, A^q$, and thus, $(I_q \,:\, -D)\, A^p \cong A^q$ is a free $A$-module of rank $q$ because we have $D^{-1}\, A^q \cong A^q$.

1. $\Leftarrow$ Let us denote by $\{e_i\}_{1\leq i\leq p}$ (resp. $\{f_j\}_{1\leq j\leq q}$) the canonical basis of $A^p$ (resp. $A^q$), namely $e_i$ (resp. $f_i$) is the vector defined by 1 in the $i$th position and 0 elsewhere. Let us also denote

$$P = (P_1 : \ldots : P_{p-q}),\ P_i \in K^q,$$

and $D^{-1} = (D_1^{-1} : \ldots : D_q^{-1})$, where $D_i^{-1} \in A^q$.

Now, if there exists $D \in A^{q\times q}$ such that $\det D \neq 0$ and $(I_q \,:\, -P)\, A^p = D^{-1}\, A^q$, then:

$$\begin{cases} f_i = (I_q \,:\, -P)\, e_i \in (I_q \,:\, -P)\, A^p = D^{-1}\, A^q \\ \Rightarrow \exists\, \lambda_i \in A^q :\ f_i = D^{-1}\, \lambda_i,\ \ 1 \leq i \leq q, \\ \\ -P_j = (I_q \,:\, -P)\, e_{q+j} \in (I_q \,:\, -P)\, A^p = D^{-1}\, A^q \\ \Rightarrow \exists\, \mu_i \in A^q :\ P_j = D^{-1}\, \mu_i,\ \ 1 \leq j \leq p-q, \\ \\ D_k^{-1} = D^{-1}\, f_k \in D^{-1}\, A^q = (I_q \,:\, -P)\, A^p \\ \Rightarrow \exists\, \nu_k \in A^p :\ D_k^{-1} = (I_q \,:\, -P)\, \nu_k,\ \ 1 \leq k \leq q, \end{cases}$$

$$\Rightarrow \begin{cases} \exists\, D' = (\lambda_1 : \ldots : \lambda_q) \in A^{q\times q} :\ I_q = D^{-1}\, D' \\ \Rightarrow D' = D^{-1}, \\ \exists\, N = (\mu_1 : \ldots : \mu_{p-q}) \in A^{q\times(p-q)} : \\ P = D^{-1}\, N, \\ \exists\, S = (\nu_1 : \ldots : \nu_q) = (X^T \,:\, Y^T)^T \in A^{p\times q} : \\ (I_q \,:\, -P)\, S = D^{-1} \Rightarrow D\, X - N\, Y = I_q, \end{cases}$$

which shows that $P$ admits a left-coprime factorization $P = D^{-1}\, N$, $D\, X - N\, Y = I_q$.

2 can be proved similarly.

## 4. INTERNAL STABILIZABILITY

*Theorem 2.* A plant, defined by a transfer matrix $P \in K^{q\times(p-q)}$, is internally stabilizable iff one of the following equivalent assertions is satisfied:

(1) There exists $S = (U^T \,:\, V^T)^T \in A^{p\times q}$ which satisfies $\det U \neq 0$ and:

(a) $S\, P = \begin{pmatrix} U\, P \\ V\, P \end{pmatrix} \in A^{p\times(p-q)}$,

(b) $(I_q \,:\, -P)\, S = U - P\, V = I_q$.

Then, the controller $C = V\, U^{-1}$ internally stabilizes the plant $P$.

(2) There exists $T = (-X \,:\, Y) \in A^{(p-q)\times p}$ which satisfies $\det Y \neq 0$ and:

(a) $P\, T = (-P\, X \,:\, P\, Y) \in A^{q\times p}$,

(b) $T \begin{pmatrix} P \\ I_{p-q} \end{pmatrix} = -X\, P + Y = I_{p-q}$.

Then, the controller $C = Y^{-1}\, X$ internally stabilizes the plant $P$.

(3) There exist $S = (U^T \,:\, V^T)^T \in K^{p\times q}$ and $T = (-X \,:\, Y) \in K^{(p-q)\times p}$ which satisfy $\det U \neq 0$, $\det Y \neq 0$ and:

$$\begin{pmatrix} I_q & -P \\ -X & Y \end{pmatrix} \begin{pmatrix} U & P \\ V & I_{p-q} \end{pmatrix} = \begin{pmatrix} U & P \\ V & I_{p-q} \end{pmatrix} \begin{pmatrix} I_q & -P \\ -X & Y \end{pmatrix} = I_p, \qquad (5)$$

$$\begin{pmatrix} U \\ V \end{pmatrix} (I_q \,:\, -P) \in A^{p\times p}, \qquad (6)$$

$$\begin{pmatrix} P \\ I_{p-q} \end{pmatrix} (-X \,:\, Y) \in A^{p\times p}. \qquad (7)$$

Then, the controller $C = V\, U^{-1} = Y^{-1}\, X$ internally stabilizes the plant $P$.

*Proof.* 1 $\Rightarrow$ Let us suppose that $P \in K^{q\times(p-q)}$ is internally stabilizable, i.e. there exists a controller $C \in K^{(p-q)\times q}$ such that we have:

$$\begin{cases} A_1 = (I_q - P\, C)^{-1} \in A^{q\times q}, \\ A_2 = (I_q - P\, C)^{-1}\, P \in A^{q\times(p-q)}, \\ A_3 = C\, (I_q - P\, C)^{-1} \in A^{(p-q)\times q}, \\ A_4 = I_{p-q} + C\, (I_q - P\, C)^{-1}\, P \in A^{(p-q)\times(p-q)}. \end{cases} \qquad (8)$$

From (8), we obtain $C = A_3\,A_1^{-1}$. If we define $S = (A_1^T : A_3^T)^T \in A^{p\times q}$, then we have:

$$\begin{cases} S\,(I_q : -P) = \begin{pmatrix} A_1 & -A_1\,P \\ A_3 & -A_3\,P \end{pmatrix} \\ \qquad = \begin{pmatrix} A_1 & -A_2 \\ A_3 & I_{p-q} - A_4 \end{pmatrix} \in A^{p\times p}, \\ (I_q : -P)\,S = A_1 - P\,A_3 \\ \qquad = (I_q - P\,C)^{-1} - P\,C\,(I_q - P\,C)^{-1} \\ \qquad = I_q. \end{cases}$$

1 $\Leftarrow$ Let us suppose that there exists a matrix $S = (U^T : V^T)^T \in A^{p\times q}$ satisfying $\det U \neq 0$, and 1.a and 1.b. If we define $C = V\,U^{-1} \in K^{(p-q)\times q}$, then, using point 1.b, we obtain:

$$I_q - P\,C = (U - P\,V)\,U^{-1} = U^{-1} \Rightarrow (I_q - P\,C)^{-1} = U \in A^{q\times q}.$$

Hence, using point 1.a and (1), we obtain

$$H(P,\,C) = \begin{pmatrix} (I_q - P\,C)^{-1} & (I_q - P\,C)^{-1}\,P \\ C\,(I_q - P\,C)^{-1} & I_{p-q} + C\,(I_q - P\,C)^{-1}\,P \end{pmatrix} = \begin{pmatrix} U & U\,P \\ V & I_{p-q} + V\,P \end{pmatrix} \in A^{p\times p},$$

i.e. $C = V\,U^{-1}$ internally stabilizes the plant $P$.

2 can be proved similarly.

3 $\Rightarrow$ Let us suppose that $P \in K^{q\times(p-q)}$ is internally stabilized by $C \in K^{(p-q)\times q}$. Following the proofs of 1 $\Rightarrow$ and 2 $\Rightarrow$, we obtain that $S = (A_1^T : A_3^T)^T$ (resp. $T = (-B_3 : B_4)$, where $B_3 = (I_{p-q} - C\,P)^{-1}\,C$ and $B_4 = (I_{p-q} - C\,P)^{-1}$) satisfies (6) (resp. (7)) and

$$\begin{pmatrix} I_q & -P \\ -B_3 & B_4 \end{pmatrix} \begin{pmatrix} A_1 & P \\ A_3 & I_{p-q} \end{pmatrix} = \begin{pmatrix} I_q & 0 \\ -B_3\,A_1 + B_4\,A_3 & I_{p-q} \end{pmatrix} = I_p, \tag{9}$$

because we have:

$$\begin{aligned} B_3\,A_1 &= ((I_{p-q} - C\,P)^{-1}\,C)\,(I_q - P\,C)^{-1} \\ &= (I_{p-q} - C\,P)^{-1}\,(C\,(I_q - P\,C)^{-1}) \\ &= B_4\,A_3. \end{aligned}$$

Moreover, from (9), we obtain

$$\begin{pmatrix} I_q & -P \\ -B_3 & B_4 \end{pmatrix}^{-1} = \begin{pmatrix} A_1 & P \\ A_3 & I_{p-q} \end{pmatrix} \Rightarrow \begin{pmatrix} A_1 & P \\ A_3 & I_{p-q} \end{pmatrix} \begin{pmatrix} I_q & -P \\ -B_3 & B_4 \end{pmatrix} = I_p,$$

which proves (5).

3 $\Leftarrow$ Let us suppose that there exist

$$\begin{cases} S = (U^T : V^T)^T \in K^{p\times q}, \\ T = (-X : Y) \in K^{(p-q)\times p}, \end{cases}$$

which satisfy (5), (6) and (7). Hence, $S$ (resp. $T$) satisfies 1.$a$ and 1.$b$ (resp. 2.$a$ and 2.$b$), and thus, by point 1 (resp. point 2), $C_1 = V\,U^{-1}$ (resp. $C_2 = Y^{-1}\,X$) is a stabilizing controller of $P$. From (5), we have $X\,U = Y\,V$, and thus, $C_1 = V\,U^{-1} = Y^{-1}\,X = C_2$ is a stabilizing controller of $P$.

*Definition 4.* (1) $0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$ is *a short exact sequence* of $A$-modules if the $A$-linear maps $f$ and $g$ satisfy that $f$ is injective, $g$ is surjective and $\ker g = \operatorname{im} f$.

(2) (Bourbaki, 1989) A short exact sequence is a *split exact sequence* if one of the following equivalent assertions is satisfied:

- there exists $h : M'' \to M$, $A$-linear map, which satisfies $g \circ h = id_{M''}$,
- there exists $k : M \to M'$, $A$-linear map, which satisfies $k \circ f = id_{M'}$,
- there exist two $A$-linear maps

$$\begin{cases} \phi = \begin{pmatrix} g \\ k \end{pmatrix} : M \to M'' \oplus M', \\ \psi = (h : f) : M'' \oplus M' \to M \end{cases}$$

which satisfy:

$$\begin{cases} \phi \circ \psi = \begin{pmatrix} g \\ k \end{pmatrix} (h : f) = id_{M''} \oplus id_{M'}, \\ \psi \circ \phi = (h : f) \begin{pmatrix} g \\ k \end{pmatrix} = id_M. \end{cases}$$

In this case, we say that $M$ is *isomorphic* to $M'' \oplus M'$, denoted by $M \cong M'' \oplus M'$.

We shall denote a split exact sequence by:

$$0 \longleftarrow M'' \underset{\xrightarrow{h}}{\xleftarrow{g}} M \underset{\xrightarrow{k}}{\xleftarrow{f}} M' \longleftarrow 0. \tag{10}$$

(3) (Bourbaki, 1989) An $A$-module $M$ is *projective* if there exist an $A$-module $P$ and $r \in \mathbb{Z}_+$ such that we have $M \oplus P \cong A^r$, i.e. $M$ is a *summand* of *a finite free $A$-module* (namely, a finite product of $A$).

*Corollary 1.* A plant $P \in K^{q\times(p-q)}$ is internally stabilizable iff one of the following equivalent assertions is satisfied:

(1) The $A$-module $(I_q : -P)\,A^p$ is projective.

(2) The $A$-module $A^{1\times p} \begin{pmatrix} P \\ I_{p-q} \end{pmatrix}$ is projective.

*Proof.* 1. Let us define the following $A$-linear map:

$$\begin{aligned} g : A^p &\longrightarrow (I_q : -P)\,A^p, \\ \lambda &\longrightarrow (I_q : -P)\,\lambda. \end{aligned}$$

The $A$-linear map $g$ is surjective and:

$$\ker g = \{\lambda = (\lambda_1^T : \lambda_2^T)^T \in A^p \mid \lambda_1 = P\,\lambda_2\}$$

$$= \{((P\,\lambda_2)^T : \lambda_2^T)^T \in A^p \mid \lambda_2 \in A^{p-q} : P\lambda_2 \in A^q\}$$

$$= \begin{pmatrix} P \\ I_{p-q} \end{pmatrix} \{\lambda_2 \in A^{p-q} \mid P\lambda_2 \in A^q\}$$

$$= \begin{pmatrix} P \\ I_{p-q} \end{pmatrix} A : \left(A^{1\times p} \begin{pmatrix} P \\ I_{p-q} \end{pmatrix}\right).$$

Therefore, we have the following exact sequence

$$0 \longleftarrow (I_q\,:\,-P)\,A^p \overset{g}{\longleftarrow} A^p \overset{f}{\longleftarrow} A : \left(A^{1\times p}\begin{pmatrix} P \\ I_{p-q}\end{pmatrix}\right) \longleftarrow 0, \tag{11}$$

where $f$ is defined by $f(\lambda) = (P^T\,:\,I_{p-q}^T)^T\,\lambda$.

Now, from 1 of Theorem 2, we know that $P$ is internally stabilizable iff there exists a matrix $S = (U^T\,:\,V^T)^T \in A^{p\times q}$ such that

$$\begin{cases} S\,(I_q\,:\,-P) \in A^{p\times p}, \\ (I_q\,:\,-P)\,S = I_q, \end{cases}$$

i.e. iff there exists an $A$-linear map

$$h : (I_q\,:\,-P)\,A^p \longrightarrow A^p, \quad \mu \longrightarrow S\,\mu, \tag{12}$$

satisfying $g\circ h = id_{(I_q:\,-P)\,A^p}$, i.e. iff (11) is a split exact sequence (see 2 of Definition 4). However, if the exact sequence (11) splits, then we have

$$A^p \cong (I_q\,:\,-P)\,A^p \oplus \left(A : \left(A^{1\times p}\begin{pmatrix} P \\ I_{p-q}\end{pmatrix}\right)\right),$$

which shows that $(I_q\,:\,-P)\,A^p$ is a projective $A$-module. Conversely, if $(I_q\,:\,-P)\,A^p$ is a projective $A$-module, then (11) is a split exact sequence because (11) ends by a projective $A$-module (Bourbaki, 1989), and thus, $P$ is internally stabilizable iff $(I_q\,:\,-P)\,A^p$ is projective.

2 can be proved similarly.

*Corollary 2.* (1) If $P \in K^{q\times(p-q)}$ admits a left-coprime factorization $P = D^{-1}\,N$, where

$$D\,X - N\,Y = I_q, \quad \det X \neq 0,$$

and $(X^T\,:\,Y^T)^T \in A^{p\times q}$, then the matrix $S = ((X\,D)^T\,:\,(Y\,D)^T)^T \in A^{p\times q}$ satisfies 1.$a$ and 1.$b$ of Theorem 2 and $C = Y\,X^{-1}$ is a stabilizing controller of $P$.

(2) If $P \in K^{q\times(p-q)}$ admits a right-coprime factorization $P = \tilde{N}\,\tilde{D}^{-1}$, where

$$-\tilde{Y}\,\tilde{N} + \tilde{X}\,\tilde{D} = I_{p-q}, \quad \det \tilde{X} \neq 0,$$

and $(-\tilde{Y}\,:\,\tilde{X}) \in A^{(p-q)\times p}$, then the matrix $T = (-\tilde{D}\,\tilde{Y}\,:\,\tilde{D}\,\tilde{X}) \in A^{(p-q)\times p}$ satisfies 2.$a$ and 2.$b$ of Theorem 2 and $C = \tilde{X}^{-1}\,\tilde{Y}$ is a stabilizing controller of $P$.

## 5. A GENERALIZATION OF THE YOULA-KUČERA PARAMETRIZATION

*Lemma 1.* Let us consider the split exact sequence (10). Then, we have:

(1) All the $A$-linear maps $\overline{h} : M'' \longrightarrow M$ satisfying $g\circ\overline{h} = id_{M''}$ are of the form $\overline{h} = h + f\circ l$, where $l$ is any element of $\hom_A(M'', M')$, namely any $A$-linear map from $M''$ to $M'$.

(2) All the $A$-linear maps $\overline{k} : M \longrightarrow M'$ satisfying $\overline{k}\circ f = id_{M'}$ are of the form $\overline{k} = k + l\circ g$, where $l$ is any element of $\hom_A(M'', M')$, namely any $A$-linear map from $M''$ to $M'$.

(3) For every $l \in \hom_A(M'', M')$, we have:

$$\begin{cases} \begin{pmatrix} g \\ k - l\circ g \end{pmatrix} (h + f\circ l\,:\,f) = id_{M''} \oplus id_{M'}, \\ (h + f\circ l\,:\,f) \begin{pmatrix} g \\ k - l\circ g \end{pmatrix} = id_M. \end{cases}$$

*Theorem 3.* If $P \in K^{q\times(p-q)}$ is internally stabilizable and $S = (U^T\,:\,V^T)^T \in A^{p\times q}$ (resp. $T = (-X\,:\,Y) \in A^{(p-q)\times p}$) is a matrix satisfying $\det U \neq 0$ (resp. $\det Y \neq 0$) and 1.$a$ and 1.$b$ (resp. 2.$a$ and 2.$b$) of Theorem 2. Then, all stabilizing controllers of $P$ are given by

$$C(Q) = (V + Q)\,(U + P\,Q)^{-1} = (Y + Q\,P)^{-1}\,(X + Q), \tag{14}$$

where $Q$ is every matrix which belongs to

$$\Omega \triangleq \left(A : \left(A^{1\times p}\begin{pmatrix} P \\ I_{p-q}\end{pmatrix}\right)\right) : ((I_q\,:\,-P)\,A^p) \tag{15}$$

$$= \{L \in A^{(p-q)\times q} \mid L\,P \in A^{(p-q)\times(p-q)}, \; P\,L \in A^{q\times q}, \; P\,L\,P \in A^{q\times(p-q)}\}, \tag{16}$$

and satisfies $\begin{cases} \det(U + P\,Q) \neq 0, \\ \det(Y + Q\,P) \neq 0. \end{cases}$

*Proof.* Using the fact that $P$ is internally stabilizable, then, by 3 of Theorem 2, there exist

$$\begin{cases} S = (U^T\,:\,V^T)^T \in A^{p\times q}, \\ T = (-X\,:\,Y) \in A^{(p-q)\times p}, \end{cases}$$

which satisfy (5), (6) and (7). Then, the $A$-linear map $h : (I_q\,:\,-P)\,A^p \longrightarrow A^p$, defined by (12), satisfies $f\circ h = id_{(I_q:\,-P)\,A^p}$, and thus, (11) becomes the split exact sequence defined by (13), where $k : A^p \rightarrow A : \left(A^{1\times p}\begin{pmatrix} P \\ I_{p-q}\end{pmatrix}\right)$ is defined by $k(\lambda) = T\,\lambda$, $\forall \lambda \in A^p$. By Lemma 1, we obtain

$$\begin{cases} \begin{pmatrix} g \\ k - l\circ g \end{pmatrix} (h + f\circ l\,:\,f) \\ \qquad = id_{(I_q:\,-P)\,A^p} \oplus id_{A:(A^{1\times p}\,(P^T:\,I_{p-q}^T)^T)}, \\ (h + f\circ l\,:\,f) \begin{pmatrix} g \\ k - l\circ g \end{pmatrix} = id_{A^p}, \end{cases}$$

where $l$ belongs to (see 2 of Proposition 2):

$$\hom_A\left((I_q\,:\,-P)\,A^p,\; A : \left(A^{1\times p}\begin{pmatrix} P \\ I_{p-q}\end{pmatrix}\right)\right) \cong \left(A : \left(A^{1\times p}\begin{pmatrix} P \\ I_{p-q}\end{pmatrix}\right)\right) : ((I_q\,:\,-P)\,A^p).$$

Therefore, every right inverse of $g$ has the form $h + f\circ l$, whereas every left-inverse of $f$ has the form $k - l\circ g$, where $l \in \Omega$. Hence, we have

$$0 \longleftarrow (I_q \,:\, -P)\, A^p \underset{\xrightarrow{h}}{\overset{g}{\longleftarrow}} A^p \underset{\xrightarrow{k}}{\overset{f}{\longleftarrow}} A : \left(A^{1\times p}\begin{pmatrix} P \\ I_{p-q}\end{pmatrix}\right) \longleftarrow 0, \tag{13}$$

$$\begin{aligned} (I_q \,:\, -P)\, A^p &\overset{h+f\circ l}{\longrightarrow} A^p, \\ \nu &\longrightarrow \begin{pmatrix} U + P\,Q \\ V + Q \end{pmatrix} \nu, \\ A^p &\overset{k-l\circ g}{\longrightarrow} A : \left(A^{1\times p}\begin{pmatrix} P \\ I_{p-q}\end{pmatrix}\right), \\ \mu &\longrightarrow (-(X+Q) \,:\, Y + Q\,P)\,\mu, \end{aligned}$$

for every $Q \in \Omega$, and thus, by 3 of Theorem 2, we obtain that every controller of $P$ has the form (14), where $Q \in \Omega$ is such that $\det(U + P\,Q) \neq 0$ and $\det(Y + Q\,P) \neq 0$.

Using the fact that $(I_q \,:\, -P)\, A^p$ is a lattice on $K^q$ and $A : (A^{1\times p}\,(P^T \,:\, I_{p-q}^T)^T))$ is a lattice on $K^{p-q}$, we obtain that:

$$\begin{aligned} \Omega &= \{L \in K^{(p-q)\times q} \mid \\ &\quad L\,(I_q \,:\, -P)\,A^p \subseteq \{\lambda \in A^{p-q} \mid P\,\lambda \in A^q\}\} \\ &= \{L \in K^{(p-q)\times q} \mid \\ &\quad L\,A^q,\, L\,P\,A^{p-q} \subseteq \{\lambda \in A^{p-q} \mid P\,\lambda \in A^q\}\} \\ &= \{L \in K^{(p-q)\times q} \mid L\,A^q \subseteq A^{p-q}, \\ &\quad L\,P\,A^{p-q} \subseteq A^{p-q},\, P\,L\,A^q \subseteq A^q, \\ &\quad P\,L\,P\,A^{p-q} \subseteq A^q\} \\ &= \{L \in A^{(p-q)\times q} \mid L\,P \in A^{(p-q)\times(p-q)}, \\ &\quad P\,L \in A^{q\times q},\; P\,L\,P \in A^{q\times(p-q)}\}. \end{aligned}$$

*Corollary 3.* If $P \in K^{q\times(p-q)}$ admits a doubly coprime factorization $P = D^{-1}\,N = \tilde{N}\,\tilde{D}^{-1}$,

$$\begin{pmatrix} D & -N \\ -\tilde{Y} & \tilde{X} \end{pmatrix} \begin{pmatrix} X & \tilde{N} \\ Y & \tilde{D} \end{pmatrix} = I_p, \tag{17}$$

then all stabilizing controllers of $P$ are of the form

$$\begin{aligned} C(\Lambda) &= (Y + \tilde{D}\,\Lambda)\,(X + \tilde{N}\,\Lambda)^{-1} \\ &= (\tilde{X} + \Lambda\,N)^{-1}\,(\tilde{Y} + \Lambda\,D), \end{aligned}$$

where $\Lambda \in A^{(p-q)\times q}$ is every matrix such that $\det(X + \tilde{N}\,\Lambda) \neq 0$ and $\det(\tilde{X} + \Lambda\,N) \neq 0$.

*Proof.* If $P$ admits a doubly coprime factorization $P = D^{-1}\,N = \tilde{N}\,\tilde{D}^{-1}$, then, by 1 and 2 of Theorem 1, we have

$$\begin{cases} (I_q \,:\, -P)\,A^p = D^{-1}\,A^q, \\ A^{1\times p}\begin{pmatrix} P \\ I_{p-q}\end{pmatrix} = A^{1\times(p-q)}\,\tilde{D}^{-1}, \end{cases}$$

and thus, $A : \left(A^{1\times p}\begin{pmatrix} P \\ I_{p-q}\end{pmatrix}\right) = \tilde{D}\,A^{p-q}$ and:

$$\begin{aligned} \Omega &= \tilde{D}\,A^{p-q} : D^{-1}\,A^q \\ &= \{T \in K^{(p-q)\times q} \mid T\,D^{-1}\,A^p \subseteq \tilde{D}\,A^{p-q}\}. \end{aligned}$$

Let us denote $D^{-1} = (D_1^{-1} \,:\, \ldots \,:\, D_q^{-1})$, where $D_i^{-1} \in A^q$. If $T \in \Omega$, then $T\,D_i^{-1} \in \tilde{D}\,A^{p-q}$, i.e. there exists $\lambda_i \in A^{p-q}$ such that $T\,D_i^{-1} = \tilde{D}\,\lambda_i$, $1 \leq i \leq q$. Now, if we denote

$$\Lambda = (\lambda_1 : \ldots : \lambda_q) \in A^{(p-q)\times q},$$

then we have $T\,D^{-1} = \tilde{D}\,\Lambda \;\Rightarrow\; T = \tilde{D}\,\Lambda\,D$.

Conversely, if $T = \tilde{D}\,\Lambda\,D$, with $\Lambda \in A^{(p-q)\times q}$, then $T\,D^{-1} = \tilde{D}\,\Lambda$, and thus, we have:

$$T\,D^{-1}\,A^q = \tilde{D}\,\Lambda\,A^q \subseteq \tilde{D}\,A^{p-q} \Rightarrow T \in \Omega,$$

$$\Rightarrow \Omega = \{\tilde{D}\,\Lambda\,D \mid \Lambda \in A^{(p-q)\times q}\} = \tilde{D}\,A^{(p-q)\times q}\,D.$$

By Corollary 2, $S = ((X\,D)^T \,:\, (Y\,D)^T)^T$ satisfies 1.*a* and 1.*b* of Theorem 2, and thus, by 1 of Theorem 2, $C = (Y\,D)\,(X\,D)^{-1} = Y\,X^{-1}$ is a stabilizing controller of $P$. Moreover, by Corollary 2, $T = (-\tilde{D}\,\tilde{Y} \,:\, \tilde{D}\,\tilde{X})$ satisfies 2.*a* and 2.*b* of Theorem 2, and thus, by 2 of Theorem 2,

$$C' = (\tilde{D}\,\tilde{X})^{-1}\,(\tilde{D}\,\tilde{Y}) = \tilde{X}^{-1}\,\tilde{Y}$$

is a stabilizing controller of $P$. Using (17), we obtain that $-\tilde{Y}\,X + \tilde{X}\,Y = 0$, and thus, we have $C' = C$. By Theorem 3, we obtain that all the stabilizing controllers of $P$ are defined by

$$\begin{aligned} C(\Lambda) &= (Y\,D + \tilde{D}\,\Lambda\,D)\,(X\,D + P\,\tilde{D}\,\Lambda\,D)^{-1} \\ &= (Y + \tilde{D}\,\Lambda)\,D\,D^{-1}\,(X + \tilde{N}\,\Lambda)^{-1} \\ &= (Y + \tilde{D}\,\Lambda)\,(X + \tilde{N}\,\Lambda)^{-1}, \\ C(\Lambda) &= (\tilde{D}\,\tilde{X} + \tilde{D}\,\Lambda\,D\,P)^{-1}\,(\tilde{D}\,\tilde{Y} + \tilde{D}\,\Lambda\,D) \\ &= (\tilde{X} + \Lambda\,N)^{-1}\,\tilde{D}^{-1}\,\tilde{D}\,(\tilde{Y} + \Lambda\,D) \\ &= (\tilde{X} + \Lambda\,N)^{-1}\,(\tilde{Y} + \Lambda\,D), \end{aligned}$$

where $\Lambda \in A^{(p-q)\times q}$ is every matrix which satisfies $\det(X + \tilde{N}\,\Lambda) \neq 0$ and $\det(\tilde{X} + \Lambda\,N) \neq 0$.

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