“Stabilizing” the stabilizing controllers

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Abstract

The main purpose of this paper is to revisit the internal/simultaneous/robust stabilization problems without assuming the existence of doubly coprime factorizations for the transfer matrices. Indeed, it has been recently shown in the literature that an internally stabilizable does not generally admit doubly coprime factorizations. Firstly, we give new necessary and sufficient conditions for internal stabilizability by means of matrix equalities. On these characterizations, the fact that internal stabilizability does not imply the existence of coprime factorizations becomes obvious. Secondly, we recall that a necessary condition for strong stabilizability is the existence of a doubly coprime factorization for the transfer matrix. Then, using the concept of *stable range* $\mathrm{sr}(A)$ of a ring A, introduced in (algebraic, topological) K-theory, we prove that $\mathrm{sr}(A) = 1$ implies that every transfer matrix defined over the quotient field of A and which admits a left- or a right-coprime factorization is strongly stabilizable. In particular, this result holds for $A = H_\infty(\mathbb{D})$, $H_\infty(\mathbb{C}_+)$, W_+ and $A(\mathbb{D})$, solving a question asked by A. Feintuch in [8]. Thirdly, we point out that the simultaneous stabilization problem is not equivalent to the strong stabilization problem if the plants do not admit doubly coprime factorizations. Using the fractional ideal approach to stabilization problems, we give a necessary and sufficient condition for a pair of SISO plants to be simultaneously stabilizable without assuming the existence of coprime factorizations. Finally, using the parametrization of all stabilizing controllers of an internally stabilizable SISO plant (which does not necessarily admit coprime factorizations), we show how to transform the non-linear sensitivity minimization problem into an affine, and thus, convex minimization problem.

Keywords Internal/strong/simultaneous/robust stabilization problems, parametrization of all stabilizing controllers, Bass stable range, topological stable range, module theory, theory of fractional ideals, K-theory.

1 Internal stabilization problem

In what follows, we shall use the *fractional representation approach to analysis and synthesis problems* developed in [5, 27]. In this theory, the set of all SISO (proper) stable time-invariant linear systems is defined by means of a commutative integral domain A and the class of all the (stable and unstable) systems is defined by the quotient field $K = Q(A) = \{n/d \,|\, 0 \neq d,\, n \in A\}$ of A. For instance, $A = RH_\infty$, $H_\infty(\mathbb{C}_+)$, W_+, $\hat{\mathcal{A}}$ are some examples of integral domains of SISO stable plants [14, 27]. Let us recall the definitions of internal, strong and simultaneous stabilizations [2, 7, 27].

Definition 1. • A plant, defined by a transfer matrix $P \in K^{q\times r}$, is *internally stabilizable* if there exists a controller $C \in K^{r\times q}$ such that the entries of the transfer matrix $H(P, C)$, defined by $(e_1^T : e_2^T)^T = H(P, C)\,(u_1^T : u_2^T)^T$ (see the figure below), belong to A or, in other words, if we have:

$$\begin{pmatrix} I_q & -P \\ -C & I_r \end{pmatrix}^{-1} = \begin{pmatrix} (I_q - P\,C)^{-1} & (I_q - P\,C)^{-1}\,P \\ C\,(I_q - P\,C)^{-1} & I_r + C\,(I_q - P\,C)^{-1}\,P \end{pmatrix} \in A^{(q+r)\times(q+r)} \quad (1)$$

$$= \begin{pmatrix} I_q + P\,(I_r - C\,P)^{-1}\,C & P\,(I_r - C\,P)^{-1} \\ (I_r - C\,P)^{-1}\,C & (I_r - C\,P)^{-1} \end{pmatrix} \in A^{(q+r)\times(q+r)}. \quad (2)$$

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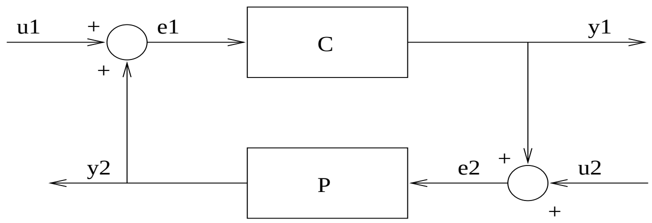

Then, C is called a *stabilizing controller* of P.

- A plant, defined by a transfer matrix $P \in K^{q \times r}$, is *strongly stabilizable* if there exists a stable stabilizing controller of P or, in other words, if $C \in A^{r \times q}$ internally stabilizes P.
- Two plants, defined by two transfer matrices P_1, $P_2 \in K^{q \times r}$, are *simultaneously stabilizable* if there exists a controller $C \in K^{r \times q}$ which internally stabilizes P_1 and P_2.

Let us state the first main result of this paper. Up to our knowledge, this simple result is new.

Theorem 1. *1. Let $P = D^{-1} N \in K^{q \times r}$ be a fractional representation of the transfer matrix P and $R = (D \,:\, -N) \in A^{q \times (q+r)}$. Then, P is internally stabilizable iff there exists a matrix $S = (X^T \,:\, Y^T)^T \in K^{(q+r) \times q}$ which satisfies* $\det X \neq 0$ *and:*

(a) $R\, S = D\, X - N\, Y = I_q$,

(b) $S\, R = \begin{pmatrix} X\, D & -X\, N \\ Y\, D & -Y\, N \end{pmatrix} \in A^{(q+r) \times (q+r)}$.

Then, the controller $C = Y\, X^{-1}$ internally stabilizes the plant P.

2. Let $P = \tilde{N}\, \tilde{D}^{-1} \in K^{q \times r}$ be a fractional representation of P and $\tilde{R} = (\tilde{N}^T \,:\, \tilde{D}^T)^T \in A^{(q+r) \times r}$. Then, P is internally stabilizable iff there exists a matrix $T = (-\tilde{Y} \,:\, \tilde{X}) \in K^{r \times (q+r)}$ which satisfies $\det \tilde{X} \neq 0$ *and:*

(a) $T\, \tilde{R} = -\tilde{Y}\, \tilde{N} + \tilde{X}\, \tilde{D} = I_r$,

(b) $\tilde{R}\, T = \begin{pmatrix} -\tilde{N}\, \tilde{Y} & \tilde{N}\, \tilde{X} \\ -\tilde{D}\, \tilde{Y} & \tilde{D}\, \tilde{X} \end{pmatrix} \in A^{(q+r) \times (q+r)}$.

Then, the controller $C = \tilde{X}^{-1}\, \tilde{Y}$ internally stabilizes the plant P.

Proof. $1 \Rightarrow$ Let us suppose that there exists a stabilizing controller $C \in K^{r \times q}$ of the plant P. Thus, we have:

$$\begin{cases} A_1 = (I_q - P\, C)^{-1} \in A^{q \times q}, \\ A_2 = (I_q - P\, C)^{-1}\, P \in A^{q \times r}, \\ A_3 = C\, (I_q - P\, C)^{-1} \in A^{r \times q}, \\ A_4 = I_r + C\, (I_q - P\, C)^{-1}\, P \in A^{r \times r}. \end{cases}$$

Let us define the following matrices:

$$X = A_1\, D^{-1}, \quad Y = A_3\, D^{-1}, \quad S = (X^T \,:\, Y^T)^T \in K^{(q+r) \times q}.$$

Then, we have $C = A_3\, A_1^{-1} = Y\, X^{-1}$. Moreover, we have

$$\begin{aligned} R\, S = (D :\, -N)\, S &= D\, A_1\, D^{-1} - N\, A_3\, D^{-1} = (D\, A_1 - N\, A_3)\, D^{-1} \\ &= (D - N\, C)\, (I_q - P\, C)^{-1}\, D^{-1} = D\, (I_q - P\, C)\, (I_q - P\, C)^{-1}\, D^{-1} = I_q, \end{aligned}$$

and thus, we obtain $R\, S = D\, X - N\, Y = I_q$, which proves 1.a. Using this identity, we obtain

$$A_1^{-1} = I_q - P\, C = I_q - (D^{-1}\, N)\, (Y\, X^{-1}) = D^{-1}\, (D\, X - N\, Y)\, X^{-1} = (X\, D)^{-1},$$

and thus, we have $A_1 = X\, D$, $A_2 = X\, N$, $A_3 = Y\, D$ and $A_4 = I_r + Y\, N$. Finally, we obtain

$$S\, R = \begin{pmatrix} X\, D & -X\, N \\ Y\, D & -Y\, N \end{pmatrix} = \begin{pmatrix} A_1 & -A_2 \\ A_3 & I_r - A_4 \end{pmatrix} \in A^{(q+r) \times (q+r)},$$

which proves 1.b.

$1 \Leftarrow$ Let us suppose that there exists a matrix $S = (X^T : Y^T)^T \in K^{(q+r)\times q}$ satisfying 1.a, 1.b and $\det X \neq 0$. Let us define $C = Y\,X^{-1}$. Then, using point 1.a, we obtain

$$I_q - P\,C = I_q - (D^{-1}\,N)\,(Y\,X^{-1}) = D^{-1}\,(D\,X - N\,Y)\,X^{-1} = (X\,D)^{-1},$$

and thus, we have:

$$(I_q - P\,C)^{-1} = X\,D,\ (I_q - P\,C)^{-1}\,P = X\,N,\ C\,(I_q - P\,C)^{-1} = Y\,D,\ C\,(I_q - P\,C)^{-1}\,P = Y\,N.$$

Hence, using 1.b, we obtain

$$H(P,C) = \begin{pmatrix} (I_q - P\,C)^{-1} & (I_q - P\,C)^{-1}\,P \\ C\,(I_q - P\,C)^{-1} & I_r + C\,(I_q - P\,C)^{-1}\,P \end{pmatrix} = \begin{pmatrix} X\,D & X\,N \\ Y\,D & I_r + Y\,N \end{pmatrix} \in A^{(q+r)\times(q+r)},$$

and thus, $C = Y\,X^{-1}$ internally stabilizes the plant $P = D^{-1}\,N$.

$2 \Rightarrow$ Let us suppose that there exists a stabilizing controller $C \in K^{r\times q}$ of the plant P. Thus, we have:

$$\begin{cases} B_1 = I_q + P\,(I_r - C\,P)^{-1} \in A^{q\times q}, \\ B_2 = P\,(I_r - C\,P)^{-1} \in A^{q\times r}, \\ B_3 = (I_r - C\,P)^{-1}\,C \in A^{r\times q}, \\ B_4 = (I_r - C\,P)^{-1} \in A^{r\times r}. \end{cases}$$

Let us define the following matrices:

$$\tilde{Y} = \tilde{D}^{-1}\,B_3, \quad \tilde{X} = \tilde{D}^{-1}\,B_4, \quad T = (-\tilde{Y} : \tilde{X}) \in K^{r\times(q+r)}.$$

Then, we have $C = B_4^{-1}\,B_3 = \tilde{X}^{-1}\,\tilde{Y}$. Moreover, we have

$$\begin{aligned} T\,\tilde{R} &= -\tilde{D}^{-1}\,B_3\,\tilde{N} + \tilde{D}^{-1}\,B_4\,\tilde{D} = \tilde{D}^{-1}\,(-B_3\,\tilde{N} + B_4\,\tilde{D}) \\ &= \tilde{D}^{-1}\,(I_r - C\,P)^{-1}\,(\tilde{D} - C\,\tilde{N}) = \tilde{D}^{-1}\,(I_r - C\,P)^{-1}\,(I_r - C\,P)\,\tilde{D} = I_r, \end{aligned}$$

and thus, we obtain $T\,\tilde{R} = -\tilde{Y}\,\tilde{N} + \tilde{X}\,\tilde{D} = I_r$, which proves 2.a. Using this identity, we obtain

$$B_4^{-1} = I_r - C\,P = I_r - (\tilde{X}^{-1}\,\tilde{Y})\,(\tilde{N}\,\tilde{D}^{-1}) = \tilde{X}^{-1}\,(\tilde{X}\,\tilde{D} - \tilde{Y}\,\tilde{N})\,\tilde{D}^{-1} = (\tilde{D}\,\tilde{X})^{-1},$$

and thus, we have $B_4 = \tilde{D}\,\tilde{X}$, $B_3 = \tilde{D}\,\tilde{Y}$, $B_2 = \tilde{N}\,\tilde{X}$ and $B_1 = I_q + \tilde{N}\,\tilde{Y}$. Finally, we have

$$\tilde{R}\,T = \begin{pmatrix} -\tilde{N}\,\tilde{Y} & \tilde{N}\,\tilde{X} \\ -\tilde{D}\,\tilde{Y} & \tilde{D}\,\tilde{X} \end{pmatrix} = \begin{pmatrix} I_q - B_1 & B_2 \\ -B_3 & B_4 \end{pmatrix} \in A^{(q+r)\times(q+r)},$$

which proves 2.b.

$2 \Leftarrow$ Let us suppose that there exists a matrix $T = (-\tilde{Y} : \tilde{X}) \in K^{r\times(q+r)}$ satisfying 2.a, 2.b and $\det \tilde{X} \neq 0$. Let us define $C = \tilde{X}^{-1}\,\tilde{Y} \in K^{r\times q}$. Then, using point 2.a, we obtain

$$I_q - C\,P = I_q - (\tilde{X}^{-1}\,\tilde{Y})\,(\tilde{N}\,\tilde{D}^{-1}) = \tilde{X}^{-1}\,(\tilde{X}\,\tilde{D} - \tilde{Y}\,\tilde{N})\,\tilde{D}^{-1} = (\tilde{D}\,\tilde{X})^{-1},$$

and thus, we obtain:

$$(I_q - C\,P)^{-1} = \tilde{D}\,\tilde{X},\ P\,(I_q - C\,P)^{-1} = \tilde{N}\,\tilde{X},\ (I_q - C\,P)^{-1}\,C = \tilde{D}\,\tilde{Y},\ P\,(I_q - C\,P)^{-1}\,C = \tilde{N}\,\tilde{Y}.$$

Hence, using 2.b, we obtain

$$H(P,C) = \begin{pmatrix} I_q + P\,(I_r - C\,P)^{-1}\,C & P\,(I_r - C\,P)^{-1} \\ (I_r - C\,P)^{-1}\,C & (I_r - C\,P)^{-1} \end{pmatrix} = \begin{pmatrix} I_q + \tilde{N}\,\tilde{Y} & \tilde{N}\,\tilde{X} \\ \tilde{D}\,\tilde{Y} & \tilde{D}\,\tilde{X} \end{pmatrix} \in A^{(q+r)\times(q+r)},$$

and thus, $C = \tilde{X}^{-1}\,\tilde{Y}$ internally stabilizes the plant $P = \tilde{N}\,\tilde{D}^{-1}$. □

The following corollary shows that every stabilizing controller C of P can be written in the form $C = Y\,X^{-1}$, where the matrix $S = (X^T : Y^T)^T$ satisfies 1.a and 1.b.

Corollary 1. *Let $C \in K^{r\times q}$ be a stabilizing controller of $P \in K^{q\times r}$.*

1. *Let $P = D^{-1} N$ be a fractional representation of the plant P and let us define the matrices $X = (D - N\,C)^{-1} \in K^{q \times q}$ and $Y = C\,(D - N\,C)^{-1} \in K^{r \times q}$. Then, $S = (X^T : Y^T)^T$ satisfies $C = Y\,X^{-1}$, 1.a and 1.b of Theorem 1.*
2. *Let $P = \tilde{N}\,\tilde{D}^{-1}$ be a fractional representation of the plant P and let us define the matrices $\tilde{X} = (\tilde{D} - C\,\tilde{N})^{-1} \in K^{r \times r}$ and $\tilde{Y} = (\tilde{D} - C\,\tilde{N})^{-1}\,C \in K^{r \times q}$. Then, $T = (-\tilde{Y} : \tilde{X})$ satisfies $C = \tilde{X}^{-1}\,\tilde{Y}$, 2.a and 2.b of Theorem 1.*

Proof. 1. Let C be a stabilizing controller of $P = D^{-1} N$. Then, using the same notations as in the proof $1 \Rightarrow$ of Theorem 1, let us define

$$\begin{cases} X = A_1\,D^{-1} = (I_q - P\,C)^{-1}\,D^{-1} = (D - N\,C)^{-1}, \\ Y = A_3\,D^{-1} = C\,(I_q - P\,C)^{-1}\,D^{-1} = C\,(D - N\,C)^{-1}, \end{cases}$$

and $S = (X^T : Y^T)^T \in K^{(q+r) \times q}$. Then, we have $C = A_3\,A_1^{-1} = Y\,X^{-1}$ and the result directly follows from the end of the proof $1 \Rightarrow$ of Theorem 1.

2. Let C be a stabilizing controller of $P = \tilde{N}\,\tilde{D}^{-1}$. Then, using the same notations as in the proof $2 \Rightarrow$ of Theorem 1, let us define

$$\begin{cases} \tilde{Y} = \tilde{D}^{-1}\,B_3 = \tilde{D}^{-1}\,(I_r - C\,P)^{-1}\,C = (\tilde{D} - C\,\tilde{N})^{-1}\,C, \\ \tilde{X} = \tilde{D}^{-1}\,B_4 = \tilde{D}^{-1}\,(I_r - C\,P)^{-1} = (\tilde{D} - C\,\tilde{N})^{-1}, \end{cases}$$

and $T = (-\tilde{Y} : \tilde{X}) \in K^{r \times (q+r)}$. Then, we have $C = B_4^{-1}\,B_3 = \tilde{X}^{-1}\,\tilde{Y}$ and the result directly follows from the end of the proof $2 \Rightarrow$ of Theorem 1. $\square$

Corollary 2. *Let $P \in K^{q \times r}$ be a plant and $C \in K^{r \times q}$ a controller.*

1. *Let $P = D^{-1} N$ be a fractional representation of P, $R = (D : -N) \in A^{q \times (q+r)}$ and let us define the following matrices $X = (D - N\,C)^{-1}$ and $Y = C\,(D - N\,C)^{-1}$. Then, P is internally stabilized by the controller C iff the matrix*

$$\Pi_1 = \begin{pmatrix} X\,D & -X\,N \\ Y\,D & -Y\,N \end{pmatrix}$$

is an idempotent *(*projector*) of $A^{(q+r) \times (q+r)}$, namely $\Pi_1 \in A^{(q+r) \times (q+r)}$ satisfies $\Pi_1^2 = \Pi_1$.*

2. *Let $P = \tilde{N}\,\tilde{D}^{-1} \in K^{q \times r}$ be a fractional representation of P, $\tilde{R} = (\tilde{N}^T : \tilde{D}^T)^T \in A^{(q+r) \times r}$ and let us define the following matrices $\tilde{X} = (\tilde{D} - C\,\tilde{N})^{-1}$ and $\tilde{Y} = (\tilde{D} - C\,\tilde{N})^{-1}\,C$. Then, P is internally stabilized by the controller C iff the matrix*

$$\Pi_2 = \begin{pmatrix} -\tilde{N}\,\tilde{Y} & \tilde{N}\,\tilde{X} \\ -\tilde{D}\,\tilde{Y} & \tilde{D}\,\tilde{X} \end{pmatrix}$$

is an idempotent *(*projector*) of $A^{(q+r) \times (q+r)}$, namely $\Pi_2 \in A^{(q+r) \times (q+r)}$ satisfies $\Pi_2^2 = \Pi_2$.*

Finally, the two idempotents Π_1 and Π_2 of $A^{(q+r) \times (q+r)}$ satisfy the relation $\Pi_1 + \Pi_2 = I_{q+r}$.

Proof. 1. $\Rightarrow$ Let us suppose that C internally stabilizes P. Then, by 1 of Corollary 1, we know that $S = (X^T : Y^T)^T$ satisfies 1.a and 1.b of Theorem 1. Therefore, using 1.a and 1.b of Theorem 1, we obtain

$$\begin{cases} \Pi_1 = \begin{pmatrix} X \\ Y \end{pmatrix} (D : -N) = S\,R \in A^{(q+r) \times (q+r)}, \\ \Pi_1^2 = S\,R\,S\,R = S\,(R\,S)\,R = S\,R = \Pi_1, \end{cases}$$

which proves that Π_1 is an idempotent of $A^{(q+r) \times (q+r)}$.

1. $\Leftarrow$ Let us suppose that Π_1 is an idempotent of $A^{(q+r) \times (q+r)}$. Let us define the matrix $S = (X^T : Y^T)^T \in K^{(q+r) \times q}$. Then, we have $\Pi_1 = S\,R \in A^{(q+r) \times (q+r)}$, i.e., condition 1.b is satisfied. Moreover, we have $R\,S = D\,(D - N\,C)^{-1} - N\,C\,(D - N\,C)^{-1} = (D - N\,C)\,(D - N\,C)^{-1} = I_q$, which proves 1.a.

2 can be proved similarly. Finally, using the following standard matrix equalities [27]

$$\left\{\begin{array}{l} (I_q - P\,C)^{-1} = P\,(I_r - C\,P)^{-1}\,C + I_q, \\ (I_q - P\,C)^{-1}\,P = P\,(I_r - C\,P)^{-1}, \\ C\,(I_q - P\,C)^{-1} = (I_r - C\,P)^{-1}\,C, \\ C\,(I_q - P\,C)^{-1}\,P = (I_r - C\,P)^{-1} - I_r, \end{array}\right.$$

we obtain:

$$\begin{array}{rl} \Pi_1 + \Pi_2 & = \left(\begin{array}{cc} X\,D - \tilde{N}\,\tilde{Y} & -X\,N + \tilde{N}\,\tilde{X} \\ Y\,D - \tilde{D}\,\tilde{Y} & -Y\,N + \tilde{D}\,\tilde{X} \end{array}\right) \\ & = \left(\begin{array}{cc} (I_q - P\,C)^{-1} - P\,(I_r - C\,P)^{-1}\,C & -(I_q - P\,C)^{-1}\,P + P\,(I_r - C\,P)^{-1} \\ C\,(I_q - P\,C)^{-1} - (I_r - C\,P)^{-1}\,C & -C\,(I_q - P\,C)^{-1}\,P + (I_r - C\,P)^{-1} \end{array}\right) = I_{q+r}. \end{array}$$

□

Corollary 2 can be reinterpreted in a more intrinsic way using module theory [21]. First of all, let us introduce a few definitions. We refer to [3, 21] for more details.

Definition 2. Let M be a finitely generated A-module [3, 21].

- The A-module M is a called *free* if M is isomorphic to a finite number of copies of A, i.e., there exists $m \in \mathbb{Z}_+$ such that $M \cong A^m$, where $\cong$ denotes the isomorphism of A-modules.
- The A-module M is a called *stably free* if there exist m and $n \in \mathbb{Z}_+$ such that $M \oplus A^n \cong A^m$, where $\oplus$ denotes the direct sum of A-modules.
- The A-module M is a called *projective* if there exist an A-module N and $m \in \mathbb{Z}_+$ such that $M \oplus N \cong A^m$.

We easily check that a free A-module is stably free (take $n = 0$) and a stably free A-module is projective (take $N = A^n$). Let us give an intrinsic formulation of Theorem 1 and Corollary 2.

Corollary 3. *The plant $P \in K^{q\times r}$ is internally stabilizable iff one of the following A-modules $(D :\, -N)\,A^{q+r}$ and $A^{1\times(q+r)}\,(\tilde{N}^T :\, \tilde{D}^T)^T$ is projective, where $P = D^{-1}\,N = \tilde{N}\,\tilde{D}^{-1}$ is a fractional representation of P, $R = (D :\, -N) \in A^{q\times(q+r)}$ and $\tilde{R} = (\tilde{N}^T :\, \tilde{D}^T)^T \in A^{(q+r)\times r}$.*

Proof. Let $P = D^{-1}\,N$ be a fractional representation of P with $R = (D :\, -N) \in A^{q\times(q+r)}$. We have the following exact sequence

$$\begin{array}{ccccccc} 0 \longrightarrow \ker(R.) & \xrightarrow{i} & A^{q+r} & \xrightarrow{R.} & R\,A^{q+r} \longrightarrow 0 \\ & & \lambda & \longmapsto & (D :\, -N)\,\lambda, \end{array} \quad (3)$$

i.e., the A-morphism $(R.) : A^{q+r} \longrightarrow R\,A^{q+r}$ (A-linear map) is surjective and i is the canonical injection of $\ker(R.)$ in A^{q+r}.

⇒ By Theorem 1, if P is internally stabilizable, then there exists $S = (X^T :\, Y^T)^T \in K^{(q+r)\times q}$ satisfying 1.a and 1.b. From 1.a, we can define the A-morphism $(S.)$ defined by:

$$\begin{array}{ccc} R\,A^{q+r} & \xrightarrow{S.} & A^{q+r} \\ \mu & \longmapsto & \left(\begin{array}{c} X \\ Y \end{array}\right)\mu. \end{array}$$

Indeed, $(S.)$ is trivially A-linear and it is well-defined as, for all $\mu \in R\,A^{q+r}$, there exists a certain $\nu \in A^{q+r}$ such that $\mu = R\,\nu$, and thus, we have $S\,\mu = (S\,R)\,\nu \in A^{q+r}$. Moreover, 1.b means that $(R.) \circ (S.) = (R\,S). = (I_q.)$, i.e., the A-morphism $(S.)$ is a right-inverse of the A-morphism $(R.)$. Then, let us prove that we have the following direct sum:

$$R\,A^{q+r} \oplus \ker(R.) \cong A^{q+r}. \quad (4)$$

For all $\lambda \in A^{q+r}$, let us define $\mu = R\,\lambda \in R\,A^{q+r}$ and $\kappa = (I_{q+r} - S\,R)\,\lambda$. We have $\kappa \in \ker(R.)$ as, by condition 1.a, we have $R\,\kappa = R\,\lambda - (R\,S)\,R\,\lambda = R\,\lambda - R\,\lambda = 0$. Moreover, we have $\lambda = \kappa + \mu$ and the result follows from the fact that $\ker(R.) \cap R\,A^{q+r} = \ker(R.) \cap \mathrm{im}(R.) = 0$. Finally, using (4), we obtain that $R\,A^{q+r}$ is a direct summand of the free A-module A^{q+r}, i.e., $R\,A^{q+r}$ is a projective A-module.

$\Leftarrow$ Let us suppose that $R\,A^{q+r}$ is a projective A-module. Then, the exact sequence (3) *splits* [21], namely, there exists an A-morphism $s : R\,A^{q+r} \to A^{q+r}$ such that $(R.) \circ s = id_{R\,A^{q+r}}$. The A-morphism s can be written by means of $S \in K^{(q+r)\times q}$, i.e., we have $s(\mu) = S\,\mu$ for $\mu \in R\,A^{q+r}$ [16]. Hence, we have $s(R\,A^{q+r}) = (S\,R)\,A^{q+r}$, i.e., we have $S\,R \in A^{(q+r)\times(q+r)}$. Moreover, we have $(R.) \circ s = (R.) \circ (S.) = (R\,S). = id_{R\,A^{q+r}}$, and thus, we obtain $(R\,S)(R\,A^{q+r}) = R\,A^{q+r}$, i.e., $R\,S\,R = R$. Finally, we have $(R\,S - I_q)\,R = 0$ and using the fact that $R = (D\,:\,-N)$ has full row rank, we obtain $R\,S = I_q$. □

Corollary 3 was first obtained in [24] with a different proof. See also [14, 16] for different proofs and equivalent results. Let us also recall some well-known definitions of coprime factorizations.

Definition 3. We have the following definitions [4, 5, 27]:

- A transfer matrix $P \in K^{q\times r}$ admits a *left-coprime factorization* if there exist two matrices $R = (D\,:\,-N) \in A^{q\times(q+r)}$ and $S = (X^T\,:\,Y^T)^T \in A^{(q+r)\times q}$ such that $\det D \neq 0$ and:

$$\begin{cases} P = D^{-1}\,N, \\ R\,S = D\,X - N\,Y = I_q. \end{cases}$$

- A transfer matrix $P \in K^{q\times r}$ admits a *right-coprime factorization* if there exist two matrices $\tilde{R} = (\tilde{N}^T\,:\,\tilde{D}^T)^T \in A^{(q+r)\times r}$ and $T = (-\tilde{Y}\,:\,\tilde{X}) \in A^{r\times(q+r)}$ such that $\det \tilde{D} \neq 0$ and:

$$\begin{cases} P = \tilde{N}\,\tilde{D}^{-1}, \\ T\,\tilde{R} = -\tilde{Y}\,\tilde{N} - \tilde{X}\,\tilde{D} = I_r. \end{cases}$$

- A transfer matrix $P \in K^{q\times r}$ admits a *doubly coprime factorization* if P admits both a left- and right-coprime factorization.

We have the following trivial corollary of Theorem 1.

Corollary 4. *1. Let the transfer matrix $P \in K^{q\times r}$ admit the following left-coprime factorization $P = D^{-1}\,N$, where $R = (D\,:\,-N) \in A^{q\times(q+r)}$, $S = (X^T\,:\,Y^T)^T \in A^{(q+r)\times q}$, $R\,S = I_q$ and $\det X \neq 0$. Then, the controller $C = Y\,X^{-1} \in K^{r\times q}$ internally stabilizes P.*

2. Let the transfer matrix $P \in K^{q\times r}$ admit the following right-coprime factorization $P = \tilde{N}\,\tilde{D}^{-1}$, where $\tilde{R} = (\tilde{N}^T\,:\,\tilde{D}^T)^T \in A^{(q+r)\times r}$, $T = (-\tilde{Y}\,:\,\tilde{X}) \in A^{r\times(q+r)}$, $T\,\tilde{R} = I_r$ and $\det \tilde{X} \neq 0$. Then, the controller $C = \tilde{X}^{-1}\,\tilde{Y} \in K^{r\times q}$ internally stabilizes P.

Proof. By hypothesis, 1.a of Theorem 1 is satisfied and using the fact that the entries of R and S are in A, then we have $S\,R \in A^{(q+r)\times(q+r)}$, and thus, 1.b is satisfied. Then, the result follows from 1 of Theorem 1. 2 can be proved similarly. □

See [27] for a proof of this well-known result by means of direct computations. From Corollary 4, we deduce that the existence of a left- or a right-coprime factorization for the transfer matrix P is a sufficient but not a necessary condition for internal stabilizability. Let us notice that this equivalence is still open for some classes of systems (e.g., $\hat{\mathcal{A}}$ (resp. W_+) the integral domain of BIBO stable continuous (resp. discrete) infinite-dimensional linear systems [4, 27]).

We recall the following necessary and sufficient conditions for the existence of left- or right-coprime factorization for a transfer matrix.

Theorem 2. *1. Let $P = D^{-1}\,N \in K^{q\times r}$ be a fractional representation of the transfer matrix P and $R = (D\,:\,-N) \in A^{q\times(q+r)}$. Then, $P \in K$ admits a left-coprime factorization iff $R\,A^{q+r}$ is a free A-module of rank q, namely, we have $R\,A^{q+r} \cong A^q$.*

2. Let $P = \tilde{N}\,\tilde{D}^{-1} \in K^{q\times r}$ be a fractional representation of P and $\tilde{R} = (\tilde{N}^T\,:\,\tilde{D}^T)^T \in A^{(q+r)\times r}$. Then, P admits a right-coprime factorization iff $A^{1\times(q+r)}\,(\tilde{N}^T\,:\,\tilde{D}^T)^T$ is a free A-module of rank r, namely, we have $A^{1\times(q+r)}\,(\tilde{N}^T\,:\,\tilde{D}^T)^T \cong A^{1\times r}$.

This result has been firstly obtained in [24]. See [14, 16] for different proofs and equivalent results. Hence, the concept of internal stabilizability is equivalent to the fact that certain A-modules are projective, whereas the existence of a doubly coprime factorization corresponds to the freeness of the same modules.

Definition 4. *[13] A ring A is called a* projective-free ring *(resp.* Hermite ring*) if every finitely generated projective (resp. stably free) A-module is free.*

In particular, a projective-free ring is a stably free ring.

Theorem 3. 1. *[14, 17] Let us denote by $\mathbb{D} = \{z \in \mathbb{C} \,|\, |z| < 1\}$ the open unit disc of $\mathbb{C}$, by $\mathbb{C}_+ = \{s \in \mathbb{C} \,|\, \mathrm{Re}\, s > 0\}$ the open right half plane of $\mathbb{C}$ and by $\overline{\mathbb{D}}^n = \{z \in \mathbb{C}^n \,|\, |z| \leq 1\}$ the closed unit polydisc of $\mathbb{C}^n$. Then, the integral domains RH_∞, $H_\infty(\mathbb{D})$, $H_\infty(\mathbb{C}_+)$ and $M_{\mathbb{D}^n} = \{a/b \,|\, 0 \neq b,\, a \in \mathbb{R}[z_1, \ldots, z_n],\, b(z) = 0 \Rightarrow z \in \mathbb{C}^n \backslash \overline{\mathbb{D}}^n\}$ are projective-free. In particular, every internally stabilizable plant admits a doubly coprime factorization.*

2. *[27] A is a Hermite ring iff every transfer which admits a left-coprime (resp. right-coprime) factorization admits a doubly coprime factorization (see also [14]).*

In (commutative) algebra, the differences between projective, stably free and free modules over a given ring A have been largely studied since few decades (e.g., Serre's conjecture (1955), Quillen-Suslin's theorem (1976) [13, 21]). A large literature on this subject exists and this problem is still active nowadays. In particular, (algebraic, topological) K-theory focuses on computing some invariants such as $K_0(A)$, $\tilde{K}_0(A)$, $C(A) \ldots$ of the ring A in order to check whether or not a finitely generated stably free (resp. projective) A-module is free [20]. Hence, K-theory develops some interesting mathematical tools in order to study the difference between internal stabilizable plants and plants admitting left- or right-coprime factorizations. For instance, see [17] for the recent proof of Z. Lin's conjecture on the existence of doubly coprime factorizations for internal stabilizable (in the sense of the structural stability) multidimensional linear systems. Moreover, see [15] for an application of the *Picard group* $C(A)$ in the problem of parametrizing all stabilizing controllers of an internally stabilizable SISO plant which does admit a coprime factorization. Hence, we believe that the mathematical tools developed in K-theory should be studied in details in the future.

2 Strong stabilization problem

One of the main purpose of K-theory is also to compute the group $K_1(A)$ (resp. $SK_1(A)$) defined as the quotient of the *general group* $GL(A)$ of invertible matrices with entries in A (resp. *special group* of matrices of determinant 1) by its normal subgroup $EL(A)$ of elementary matrices with entries in A. In order to "stabilize" the computation of $K_1(A)$, H. Bass introduced in [1] the concept of *stable range* of a (commutative) ring A. While we were studying K-theory in order to apply it to stabilization problems, we develop the idea of using the concept of stable range for the study of the Bézout identities appearing in internal stabilizability and specially in the strong stabilization problem. In the same period, a private communication with V. Blondel (University of Louvain-la-Neuve) showed me that he already had the idea to use such a concept in the study of the strong stabilization problem. More recently, while reading [7, 8], it appeared to us that A. Feintuch (Ben-Gurion University) had also already used this concept in the study of strong stabilizability.

In this section, we mainly recall some result on the strong stabilization problem recently obtained in [18] for MIMO plants. First of all, let us introduce some definitions [1, 9, 26].

Definition 5.
- A matrix $R \in A^{q \times p}$ is *unimodular* if there exists a matrix $S \in A^{p \times q}$ such that $R\,S = I_q$. In particular, we denote the set of all the unimodular vectors of $A^{1 \times n}$ by $\mathrm{U}_n(A)$.
- Let us denote the matrix formed by the columns $R_1, \ldots, R_p \in A^q$ by $\mathrm{col}(R_1, \ldots, R_p) \in A^{q \times p}$ and let k be a positive integer satisfying $1 \leq k \leq r = p - q$. Then, a unimodular matrix $\mathrm{col}(R_1, \ldots, R_p) \in A^{q \times p}$ is called *k-stable* if there exists a matrix $T_k \in A^{k \times (p-k)}$ such that the $q \times (p-k)$-matrix defined by $R_k = \mathrm{col}(R_1 : \ldots : R_{p-k}) + \mathrm{col}(R_{p-k+1} : \ldots : R_p)\, T_k$ is unimodular.

In particular, a vector $a = (a_1 : \ldots : a_n) \in \mathrm{U}_n(A)$ is 1-stable, or simply *stable*, if there exists a $(n-1)$-tuple $b = (b_1 : \ldots : b_{n-1}) \in A^{n-1}$ such that $(a_1 + a_n\, b_1 : \ldots : a_{n-1} + a_n\, b_{n-1}) \in \mathrm{U}_{n-1}(A)$, i.e., there exists $(c_1 : \ldots : c_{n-1}) \in A^{n-1}$ such that we have $\sum_{i=1}^{n-1} (a_i + a_n\, b_i)\, c_i = 1$.

Definition 6. *[1, 9, 26]* The *(Bass) stable range* $\mathrm{sr}(A)$ of A is the least integer n such that every vector of $\mathrm{U}_{n+1}(A)$ is stable. If no such integer exists, we set $\mathrm{sr}(A) = \infty$.

The following alternative definition of the base stable range was obtained by R. B. Warfield.

Proposition 1. *[19] The stable range $\mathrm{sr}(A)$ of A is the least integer n such that for every vector $(a_1 : \ldots : a_{n+1}) \in \mathrm{U}_{n+1}(A)$, there exists $(c_i)_{1 \leq i \leq n+1} \in A^{n+1}$ satisfying:*

- $\sum_{i=1}^{n+1} a_i\, c_i = 1$,
- $(c_1 : \ldots : c_n) \in \mathrm{U}_n(A)$.

Theorem 4. *$\mathrm{sr}(H_\infty(\mathbb{D})) = 1$ [25], $\mathrm{sr}(RH_\infty) = 2$ [27], $\mathrm{sr}(A(\mathbb{D})) = 1$ [12], $\mathrm{sr}(W_+) = 1$ [22], $\mathrm{sr}(L_\infty(i\,\mathbb{R})) = 1$ [10], $\mathrm{sr}(B(H)) = \infty$, where H is a separable Hilbert space (e.g. $L_2(\mathbb{R}_+)$, $l_2(\mathbb{Z}_+)$) [6].*

Remark 1. We have also $\mathrm{sr}(H_\infty(\mathbb{C}_+) = 1$ [18]. Let us notice that the stable range of the $\mathbb{R}$-subalgebra

$$A = \{f \in H_\infty(\mathbb{C}_+) \,|\, \overline{f(\overline{s})} = f(s)\}$$

of $H_\infty(\mathbb{C}_+)$ is still not known. We believe that $\mathrm{sr}(A) = 2$ but we do not know any proof.

Let us introduce some definitions. See [3] for more details.

Definition 7.
- A topological space X is *noetherian* if every decreasing chain

$$\mathcal{F}_1 \supset \mathcal{F}_2 \supset \mathcal{F}_3 \supset \mathcal{F}_4 \supset \ldots$$

  of distinct closed subsets $\mathcal{F}_i$ in X is stationary, i.e., there exists $n \in \mathbb{Z}_+$ such that, for all $m \geq n$, we have $\mathcal{F}_m = \mathcal{F}_n$.
- A non-empty topological set X is *irreductible* if X is not the union of two proper closed subsets of X.
- The *dimension* of X is the supremum of the lengths of all chains $\mathcal{F}_1 \supset \mathcal{F}_2 \supset \ldots \supset \mathcal{F}_d$ of distinct non-empty irreductible closed subsets of X.
- An ideal $\mathcal{P}$ of A is called *prime* if $x\,y \in \mathcal{P}$ and $x \notin \mathcal{P}$, then $y \in \mathcal{P}$ and *maximal* if A is the only ideal strictly containing $\mathcal{P}$. We denote by $\mathrm{spec}\,(A)$ (resp. $\max(A)$) the set of prime (resp. maximal) ideals in A.
- $\mathrm{spec}\,(A)$ becomes a topological space when it is equipped with the *Zariski topology*, namely the topology defined by the closed set of $\mathrm{spec}(A)$, parametrized by the ideals I of A, of the form $C_I = \{\mathcal{P} \in \mathrm{spec}\,(A) \,|\, I \subseteq \mathcal{P}\}$. By restriction, $\max(A) \subseteq \mathrm{spec}\,(A)$ can also be equipped with the Zariski topology.
- The *Krull dimension* of a commutative ring A is the supremum of the lengths of chains $\mathcal{P}_0 \subset \mathcal{P}_1 \subset \mathcal{P}_2 \ldots \subset \mathcal{P}_d$ of distinct proper prime ideals in A.

If A is a noetherian ring [3, 21], then $\max(A)$ is a noetherian topological space (see [3]).

Theorem 5.
- *[1] Let A be a ring such that $\max(A)$ is a noetherian topological space of dimension d. Then, we have $\mathrm{sr}(A) \leq d + 1$.*
- *[11] If A is a ring (resp. an integral domain) of Krull dimension d, then $\mathrm{sr}(A) \leq d+2$ (resp. $\mathrm{sr}(A) \leq d + 1$).*

We have the following corollary of Theorem 5.

Corollary 5. *If A is a* Dedekind domain*, namely an integral domain over which every ideal is projective (see Definition 2), then $\mathrm{sr}(A) \leq 2$. This result holds if A is a principal ideal domain.*

Let us recall that if $P = D_1^{-1}\, N_1 = D_2^{-1}\, N_2$ are two left-coprime factorizations of P and $R_i = (D_i \,:\, -N_i)$ for $i = 1, 2$, then there exists an invertible matrix $U \in \mathrm{GL}_q(A)$ such that $R_2 = U\, R_1$ [27]. Therefore, we obtain R_1 is k-stable iff R_2 is also k-stable.

Definition 8. Let $P \in K^{q\times r}$ be a transfer matrix which admits a left-coprime (resp. right-coprime) factorization $P = D^{-1}\, N$ (resp. $P = \tilde{N}\, \tilde{D}^{-1}$). Then, the transfer matrix P is called *left* (resp. *right*) *k-stable* if the matrix $(D \,:\, -N) \in A^{(q+r)\times r}$ (resp. $(\tilde{D}^T \,:\, \tilde{N}^T) \in A^{r\times(q+r)}$) is k-stable.

Hence, we have the following proposition.

Proposition 2. *[18] A plant defined by a transfer matrix $P \in K^{q\times r}$ is strongly stabilizable iff P admits a doubly coprime factorization $P = D^{-1}\, N = \tilde{N}\, \tilde{D}^{-1}$ such that the following matrices $(D \,:\, -N) \in A^{q\times(q+r)}$ and $(\tilde{D}^T \,:\, \tilde{N}^T) \in A^{r\times(q+r)}$ are respectively r and q-stable.*

Therefore, the existence of doubly coprime factorization for the transfer matrix is a necessary condition for strong stabilizability. We have the following interesting result.

Theorem 6. *[18] Let $P = D^{-1} N$ be a left-coprime factorization of P and $R = (D : \; -N) \in A^{q\times(q+r)}$. If R is k-stable and $s = r - k \geq 0$, then there exist two stable matrices $T_1 \in A^{k\times q}$ and $T_2 \in A^{k\times s}$ such that the matrix $R_k = (D - \Lambda\, T_1 : \, -(N_s + \Lambda\, T_2)) \in A^{q\times(q+s)}$ admits a right-inverse with entries in A, with the notations:*

$$R = (D : \, -N) = (\underset{\overset{\leftrightarrow}{q}}{D} \quad : \underset{\overset{\leftrightarrow}{s}}{-N_s} \quad : \underset{\overset{\leftrightarrow}{k}}{-\Lambda}) \quad \in A^{q\times(q+r)}.$$

Let us define by $S_k = (U^T : \, V^T)^T \in A^{(q+s)\times q}$, $U \in A^{q\times q}$, $V \in A^{s\times q}$, a right-inverse of R_k such that $\det U \neq 0$. Then, the controller $C \in K^{r\times q}$, defined by

$$C = \begin{pmatrix} V\, U^{-1} \\ T_1 + T_2\,(V\, U^{-1}) \end{pmatrix}, \qquad \begin{matrix} \updownarrow s = r - k \\ \updownarrow k \end{matrix}$$

internally stabilizes P. Moreover, if $\det(D - \Lambda\, T_1) \neq 0$, then the controller $C_s = V\, U^{-1} \in K^{s\times q}$ internally stabilizes $P_s = (D - \Lambda\, T_1)^{-1}\,(N_r + \Lambda\, T_2) \in K^{q\times s}$. Hence, the unstable part of C is only contained in the transfer matrix $C_s = V\, U^{-1}$ and its dimension is equal to $s \times q$.

Similar results also hold for a transfer matrix P admitting a right-coprime factorization.

Up to our knowledge, there is no general algorithm checking whether or not a matrix R is k-stable. However, we can prove that any matrix $R \in A^{q\times(q+r)}$ satisfying $r \geq \mathrm{sr}(A)$ is $r - \mathrm{sr}(A) + 1$-stable (see [18] for more details). Hence, Theorem 6 is always true for $s = \mathrm{sr}(A) - 1$, i.e., the unstable part of C can be reduced to a $(\mathrm{sr}(A) - 1) \times q$-matrix $C_{\mathrm{sr}(A)-1}$. Let us point out that this bound only depends on the ring A. Then, we have the following corollary of Theorem 6.

Corollary 6. *[18] If $\mathrm{sr}(A) = 1$, then every transfer matrix which admits a left- or a right-coprime factorization is strongly stabilizable. In particular, this result holds for $A = W_+$ or $A(\mathbb{D})$.*

Moreover, every internally stabilizable plant, defined by a transfer matrix P with entries in the quotient field of $A = H_\infty(\mathbb{D})$ or $H_\infty(\mathbb{C}_+)$, is strongly stabilizable.

Let us point out that Corollary 6 solves a question asked by A. Feintuch in [8] on the generalization of S. Treil's result [25] for MIMO systems defined over $A = H_\infty(\mathbb{D})$ or $H_\infty(\mathbb{C}_+)$.

Corollary 7. *If $\mathrm{sr}(A) = 1$, then A is a Hermite ring (see Definition 4). In particular, this result holds for the rings $H_\infty(\mathbb{D})$, $H_\infty(\mathbb{C}_+)$, $A(\mathbb{D})$, W_+ and $L_\infty(i\,\mathbb{R})$. Therefore, if a transfer matrix P with entries in $K = Q(A)$ admits a left- or a right-coprime factorization, then P admits a doubly coprime factorization.*

Proof. Let P be a transfer matrix which admits a left-coprime factorization. Then, using the fact $\mathrm{sr}(A) = 1$, by Corollary 6, P is strongly stabilizable. But, by Proposition 2, a necessary condition for strong stabilizability is the existence of a doubly coprime factorization for P. Hence, every transfer matrix which admits a left-coprime factorization admits a doubly coprime factorization. The result directly follows from 2 of Theorem 3. □

Up to our knowledge, it is not known whether or not $A = A(\mathbb{D})$ or W_+ are projective-free rings, and thus, whether or not every internally stabilizable plant over A admits doubly coprime factorizations (it is even not known whether or not $\hat{\mathcal{A}}$ [4] is a Hermite ring).

Corollary 8. *[18] If $\mathrm{sr}(A) = 1$, then every pair of plants, defined by two transfer matrices P_0 and P_1 with entries in $K = Q(A)$, with the same dimensions, and admitting doubly coprime factorizations is simultaneously stabilized by a controller. In particular, this result holds for $A = W_+$ or $A(\mathbb{D})$.*

Moreover, if $A = H_\infty(\mathbb{D})$ or $H_\infty(\mathbb{C}_+)$ and P_0, P_1 are two internally stabilizable plants with entries in K, then there exists a stabilizing controller which simultaneously stabilizes P_0 and P_1.

To finish this section, let us introduce the concept of *topological stable range*.

Definition 9. *[19]* If A is a *Banach algebra* [27], then the *topological stable range* $\mathrm{tsr}(A)$ of A is the least integer n such that $\mathrm{U}_n(A)$ is dense in A^n for the product topology. If no such integer exists, we set $\mathrm{tsr}(A) = \infty$.

Theorem 7. $\mathrm{tsr}(H_\infty(\mathbb{D})) = 2$ *[23]*, $\mathrm{tsr}(A(\mathbb{D})) = 2$ *[19]*.

Proposition 3. *[19] If A is a Banach algebra, then we have $\mathrm{sr}(A) \leq \mathrm{tsr}(A)$.*

We have the following application of the topological stable range in stabilization problems.

Proposition 4. *[18] If A is a Banach algebra such that $\operatorname{tsr}(A)=2$, then every SISO plant, defined by the transfer function $p=n/d$, $0 \neq d, n \in A$, satisfies:*

$$\forall\, \epsilon>0,\ \exists\, (d_\epsilon\,:\ n_\epsilon) \in \mathrm{U}_2(A):\quad \left\{\begin{array}{l} \parallel n-n_\epsilon \parallel_A \leq \epsilon, \\ \parallel d-d_\epsilon \parallel_A \leq \epsilon. \end{array}\right.$$

If $d_\epsilon \neq 0$, then, in the product topology, p is as close as we want to a transfer function $p_\epsilon = n_\epsilon/d_\epsilon$ which admits a coprime factorization. In particular, this result holds for $A = H_\infty(\mathbb{D})$ or $A(\mathbb{D})$.

Finally, different concepts of stable ranges have been recently introduced in the literature of Banach algebras as, for instance, the *Bass* and *(dual) topological stable ranges* for (projective) modules [19]. Their applications for stabilization problems will be studied in forthcoming publications.

3 Simultaneous stabilization problem

The main purpose of this section is to point out that the simultaneous stabilization problem is not equivalent to the strong stabilization problem if the plants do not admit doubly coprime factorizations. For simplicity reasons, we shall only consider here the case of SISO systems. The MIMO case can be handled similarly using the *lattice approach* developed in [16, 17].

We have recently shown in [15] how the *theory of fractional ideals* [3] developed in commutative algebra was a natural mathematical framework for the study of the fractional representation approach to analysis and synthesis problems. Let us introduce some definitions [3, 15].

Definition 10.

- A *fractional ideal* J of A is an A-submodule of $K = Q(A)$ such that there exists $0 \neq a \in A$ satisfying $(a)\, J = \{a\, j \mid j \in J\} \subseteq A$. The set of non-zero fractional ideals of A is denoted by $\mathcal{F}(A)$.
- If $I,\ J \in \mathcal{F}(A)$, then their sums, products, intersections and residuals, namely

$$\begin{array}{ll} I+J=\{a+b \mid a \in I,\, b \in J\}, & I\,J=\{\sum_{i=1}^n a_i\, b_i \mid a_i \in I,\, b_i \in J, n \in \mathbb{Z}_+\}, \\ I \cap J = \{a \in I,\, a \in J\}, & I : J = \{k \in K \mid (k)\, J = \{k\, j \mid j \in J\} \subseteq I\}, \end{array}$$

  belong to $\mathcal{F}(A)$.
- A fractional ideal J of A is called *principal* if there exists $k \in K$ such that $J = (k)$.
- A fractional ideal J of A is *invertible* if there exists $I \in \mathcal{F}(A)$ such that $I\, J = A$. In that case, J has a unique inverse denoted by J^{-1} and $J^{-1} = A : J$. We have $(J^{-1})^{-1} = J$.

A principal fractional ideal $J = (k)$, $0 \neq k \in K$, is invertible and its inverse is $J^{-1} = (1/k)$.

Example 1. If $p \in K = Q(A)$, then $J = (1,\, p) = A + A\, p = \{\lambda + \mu\, p \,|\, \lambda,\, \mu \in A\}$ is a fractional ideal of A because there exist $0 \neq d,\, n \in A$ such that $p = n/d$, and thus, $(d)\, J = (d,\, n) \subseteq A$. Then, the ideal $A : J = \{k \in K \mid k,\, k\, p \in A\} = \{d \in A \mid d\, p \in A\}$ of A is usually called the *ideal of the denominators* of p whereas $(p)\,(A : J)$ is the *ideal of the numerators* of p.

Let us recall necessary and sufficient conditions for internal stabilizability or for the existence of a coprime factorization recently obtained in terms of fractional ideals [15].

Theorem 8. *[15] Let A be an integral domain of SISO stable systems and $K = Q(A)$ its quotient field. Then, we have:*

1. *$p \in K$ admits a coprime factorization iff the fractional ideal $J = (1,\, p)$ is principal of the form $J = (1/d)$, where $0 \neq d \in A$. Then, $p = n/d$, where $n = d\, p \in A$, is a coprime factorization.*
2. *$p \in K$ is internally stabilizable iff the fractional ideal $J = (1,\, p)$ is* invertible*, namely we have*

$$J\,(A : J) = \{a + b\, p \mid a,\, b \in A:\ a\, p,\ b\, p \in A\} = A,$$

   or, equivalently, iff there exist $a,\, b \in A$ which satisfy:

$$\left\{\begin{array}{l} a - b\, p = 1, \\ a\, p \in A. \end{array}\right. \tag{5}$$

   Then, if $a \neq 0$ (resp. $a = 0$), $c = b/a \in K$ (resp. $c = 1 - b \in A$) is a stabilizing controller of p, $J^{-1} = (a,\, b)$ and $a = 1/(1 - p\, c)$, $b = c/(1 - p\, c)$.

3. *$c \in K$ internally stabilizes $p \in K$ iff the following equality holds:*

$$(1,\, p)\,(1,\, c) = (1 - p\,c). \tag{6}$$

In the following easy lemma, we firstly recall a well-known result (see e.g., [2]) and we secondly give a necessary and sufficient condition for *external stabilizability*.

Lemma 1. 1. *$c \in K$ internally stabilizes 0 iff $c \in A$.*

2. *$c \in K$ externally stabilizes p, namely $(p\,c)/(1 - p\,c) \in A$, iff $(1,\, p\,c) = (1 - p\,c)$.*

Proof. 1. Using 3 of Theorem 8, $c \in K$ internally stabilizes 0 iff $(1,\, 0)\,(1,\, c) = (1)$, i.e., iff $(1,\, c) = A$. But, we trivially have $(1,\, c) = A$ iff $c \in A$, which proves the result.

2. $\Rightarrow$ Let us suppose that c externally stabilizes p, namely $(p\,c)/(1 - p\,c) \in A$ or, equivalently, $1/(1 - p\,c) = 1 + (p\,c)/(1 - p\,c) \in A$. Therefore, we have:

$$\begin{cases} 1 = \left(\frac{1}{1 - p\,c}\right)(1 - p\,c), \\ p\,c = \left(\frac{p\,c}{1 - p\,c}\right)(1 - p\,c), \end{cases} \Rightarrow (1,\, p\,c) \subseteq (1 - p\,c).$$

Moreover, we trivially have the inclusion $(1 - p\,c) \subseteq (1,\, p\,c)$, which proves the first implication.

$\Leftarrow$ Let us suppose that we have $(1,\, p\,c) = (1 - p\,c)$. Thus, we obtain $\left(\frac{1}{(1 - p\,c)},\, \frac{p\,c}{1 - p\,c}\right) = A$, which shows that $(p\,c)/(1 - p\,c) \in A$, i.e., c externally stabilizes p. □

Let us give necessary and sufficient conditions for strong and bistably stabilizabilities.

Proposition 5. *The following assertions are equivalent:*

1. *$p \in K = Q(A)$ is strongly (resp. bistably) stabilizable,*
2. *there exists $c \in A$ (resp. $c \in \mathrm{U}(A)$) such that $(1,\, p) = (1 - p\,c)$,*
3. *there exists $c \in A$ (resp. $c \in \mathrm{U}(A)$) such that $p/(1 - p\,c) \in A$.*

Proof. Let us suppose that we have 1. Then, there exists a stable (resp. bistable) stabilizing controller c of p, i.e., $c \in A$ (resp. $c \in \mathrm{U}(A)$). Then, using (6), we obtain $(1,\, p) = (1 - p\,c)$ because $c \in A$ implies $(1,\, c) = A$, and thus, we obtain 2. Now, let us suppose that we have 2, i.e., there exists $c \in A$ such that $(1,\, p) = (1 - p\,c)$. Trivially, 2 is equivalent to $(1/(1 - p\,c),\, p/(1 - p\,c)) = A$ which, in particular, shows that $p/(1 - p\,c) \in A$, i.e., we obtain 3. Finally, let us suppose that we have 3. Therefore, there exists $c \in A$ such that $p/(1 - p\,c) \in A$. Hence, using the fact that $c \in A$, we obtain $(p\,c)/(1 - p\,c) \in A$ and $1/(1 - p\,c) = 1 + (p\,c)/(1 - p\,c) \in A$, which shows that $H(p, c)$, defined by (1), is A-stable, and thus, $c \in A$ internally stabilizable p, i.e., we obtain 1. □

Firstly, using 1 of Theorem 8, we obtain again the fact that a necessary condition for strong stabilizability is the existence of coprime factorization of the transfer function (see Proposition 2). Secondly, if $p = n/d$ is a fractional representation of p, with $0 \neq d,\, n \in A$, let us notice that 2 of Proposition 5 is equivalent to the existence of $c \in A$ (resp. $c \in \mathrm{U}(A)$) such that:

$$\left(1,\, \frac{n}{d}\right) = \left(1 - \frac{n\,c}{d}\right) \Leftrightarrow \left(\frac{1}{d}\right)(d,\, n) = \left(\frac{1}{d}\right)(d - n\,c) \Leftrightarrow (d,\, n) = (d - n\,c). \tag{7}$$

In particular, if $p = n/d$ is a coprime factorization of p, with $d\,x - n\,y = 1$ for certain $x,\, y \in A$, then we have $1 = d\,x - n\,y \in (d,\, n)$, i.e., $(d,\, n) = A$, and thus, $(d - n\,c) = A$. But, $(d - n\,c) = A$ is equivalent to $d - n\,c \in \mathrm{U}(A)$. Thus, we obtain the following corollary.

Corollary 9. *[2, 27] Let $p = n/d$ be a coprime factorization of $p \in K$. Then, p is strongly (resp. bistably) stabilizable iff there exists $c \in A$ (resp. $c \in \mathrm{U}(A)$) such that $d - n\,c \in \mathrm{U}(A)$.*

Corollary 10. *If $c \in \mathrm{U}(A)$ externally stabilizes $p \in K$, then c bistably stabilizes p.*

Proof. Using 2 of Lemma 1, we obtain that $(1,\, p\,c) = (1 - p\,c)$. Now, the fact that c belongs to $\mathrm{U}(A)$ implies that we have $(1,\, p\,c) = (c)\,(1/c,\, p) = (1,\, p)$, and thus, we obtain $(1,\, p) = (1 - p\,c)$, showing that c bistably stabilizes p by 2 of Proposition 5. □

Example 2. Let $p = (1 + e^{-2\,s})/(1 - e^{-2\,s}) \in Q(H_\infty(\mathbb{C}_+))$ be the transfer function of a wave equation (see [4, 15] for more details). Then, if we take $c = -1 \in \mathrm{U}(A)$, then $a = 1/(1 - p\,c) = (1 - e^{-2\,s})/2 \in A$, i.e., c externally stabilizes p. By Corollary 10, we obtain that c bistably stabilizes p. Similarly for $p = 1/(s-1) \in \mathbb{R}(s)$ and $c = -1 \in RH_\infty$.

Using the fractional ideal approach, we were able in [15] to parametrize all stabilizing controllers for an internally stabilizable plant which does not necessarily admit a coprime factorization.

Theorem 9. *[4, 15] Let $p \in K = Q(A)$ be an internally stabilizable plant and $c_\star$ a particular stabilizing controller of p. Then, we have:*

1. *All stabilizing controllers of p have the form*

$$c(q_1, q_2) = \frac{b + q_1\,a^2 + q_2\,b^2}{a + q_1\,a^2\,p + q_2\,b^2\,p} = \frac{(1 - p\,c_\star)\,c_\star + q_1 + q_2\,c_\star^2}{(1 - p\,c_\star) + q_1\,p + q_2\,p\,c_\star^2}, \tag{8}$$

   where $a = 1/(1 - p\,c_\star)$ and $b = c_\star/(1 - p\,c_\star)$ satisfy (5), and q_1, q_2 are any element of A satisfying $a + q_1\,a^2\,p + q_2\,b^2\,p \neq 0$.

2. *The parametrization (8) has only one arbitrary parameter iff p^2 admits a coprime factorization. If $p^2 = s/r$ is a coprime factorization of p, then the parametrization (8) becomes*

$$c(q) = \frac{b + q\,r}{a + q\,r\,p} = \frac{c_\star + q\,r\,(1 - p\,c_\star)}{1 + q\,r\,(1 - p\,c_\star)\,p},$$

   where q is any element of A such that $a + q\,p\,r \neq 0$.

3. *If p admits a coprime factorization $p = n/d$, $d\,x - n\,y = 1$ for some x, $y \in A$, then we have $a = d\,x$, $b = d\,y$ and $r = d^2$, and thus, (8) becomes*

$$c(q) = \frac{y + q\,d}{x + q\,n}, \tag{9}$$

   where q is any element of A such that $x + q\,n \neq 0$.

We recover the *Youla-Kučera parametrization* (9) of all stabilizing controllers [5, 27] as a particular case of the general parametrization (8). Let us point out that the issue of recognizing when there exist Youla-Kučera parametrizations or generalized parametrizations (8) with one or two arbitrary parameters is deeply related to the computation of the so-called Picard group $C(A)$ of A. See [15] for more details and the end of Section 1. Let us state the following useful lemma.

Lemma 2. *Let $c_1 \in K$ be stabilizing controller of $p_1 \in K$, $p_2 \in K$ and*

$$c(k) = \frac{b_1 + k}{a_1 + k\,p_1}, \quad k \in (a_1^2,\, b_1^2) : \; a_1 + k\,p_1 \neq 0, \tag{10}$$

be the parametrization of all stabilizing controllers (8) of p_1, where:

$$a_1 = 1/(1 - p_1\,c_1), \quad b_1 = c_1/(1 - p_1\,c_1).$$

Then, we have:

$$\frac{(1 - p_2\,c(k))}{(1 - p_1\,c(k))} = \frac{(1 - p_2\,c_1)}{(1 - p_1\,c_1)} - (p_2 - p_1)\,k \tag{11}$$

Proof. Let us point out that (10) is a compact way to rewrite the parametrization (8) of all stabilizing controllers of p_1. The lemma directly follows from the following computations:

$$\frac{(1 - p_2\,c(k))}{(1 - p_1\,c(k))} = (a_1 + p_1\,k) - p_2\,(b_1 + k) = \frac{(1 - p_2\,c_1)}{(1 - p_1\,c_1)} - (p_2 - p_1)\,k.$$

□

Let us give a necessary and sufficient condition for simultaneous stabilizability without requiring that the transfer functions p_1 and $p_2 \in K$ admit coprime factorizations.

Theorem 10. *Let $c_1 \in K$ be a stabilizing controller of p_1. Then, p_1 and p_2 are simultaneously stabilizable iff there exist α and $\beta \in A$ such that $k = \frac{\alpha}{(1-p_1\,c_1)^2} + \frac{\beta\,c_1^2}{(1-p_1\,c_1)^2}$ satisfies:*

$$\left(\frac{(1-p_2\,c_1)}{(1-p_1\,c_1)} - (p_2-p_1)\,k\right) = \left(\frac{1}{1-p_1\,c_1}, \frac{c_1}{1-p_1\,c_1}, \frac{p_2}{1-p_1\,c_1}, \frac{p_2\,c_1}{1-p_1\,c_1}\right). \tag{12}$$

Then, the controller

$$c(k) = \frac{c_1 + k\,(1-p_1\,c_1)}{1+k\,p_1\,(1-p_1\,c_1)}$$

simultaneously internally stabilizes p_1 and p_2.

Proof. p_1 and p_2 are simultaneously stabilizable if there exists a controller $c \in K$ which internally stabilizes p_1 and p_2. By 3 of Theorem 8, this is equivalent to:

$$\left\{\begin{array}{l} (1,\,p_1)\,(1,\,c) = (1-p_1\,c), \\ (1,\,p_2)\,(1,\,c) = (1-p_2\,c), \end{array}\right. \Leftrightarrow \left\{\begin{array}{l} (1,\,p_1)\,(1,\,c) = (1-p_1\,c), \\ (1,\,p_2)\,(1,\,p_1)^{-1} = \left(\frac{1-p_2\,c}{1-p_1\,c}\right). \end{array}\right. \tag{13}$$

Let (10) be the parametrization of all stabilizing controllers of p_1 (which is nothing else than (8) written in a more compact way) . By substituting $c(k)$ in (13), then this first equation of (13) is trivially fulfilled whereas, using the identity defined by (11), the second equation becomes

$$(1,\,p_2)\left(\frac{1}{1-p_1\,c_1}, \frac{1}{1-p_1\,c_1}\right) = \left(\frac{(1-p_2\,c_1)}{(1-p_1\,c_1)} - (p_2-p_1)\,k\right),$$

and thus, we obtain (12). Therefore, p_1 and p_2 are simultaneously stabilizable iff there exists $k \in (a_1^2,\,b_1^2) = (1/(1-p_1\,c_1)^2,\,c_1^2/(1-p_1\,c_1)^2)$ which satisfies (12) or, in other words, iff there exist $\alpha, \beta \in A$ such that $k = \frac{\alpha}{(1-p_1\,c_1)^2} + \frac{\beta\,c_1^2}{(1-p_1\,c_1)^2}$ satisfies (12). Then, the controller $c(k)$ which both internally stabilizes p_1 and p_2 is defined by:

$$c(k) = \frac{b_1+k}{a_1+k\,p_1} = \frac{c_1 + k\,(1-p_1\,c_1)}{1+k\,p_1\,(1-p_1\,c_1)}.$$

□

It seems that the simultaneous stabilization problem has not been yet investigated when the plants do not admit coprime factorizations. The next corollary allows us to find again different well-known results when the existence of coprime factorizations for the plants is assumed [2].

Corollary 11.

1. *Let $p_1 = n_1/d_1$, $p_2 = n_2/d_2 \in K$ be two coprime factorizations with:*

$$d_1\,x_1 - n_1\,y_1 = 1.$$

Then, p_1 and p_2 are simultaneously stabilizable iff the plant defined by the transfer function

$$p_3 = \frac{(d_1\,n_2 - n_1\,d_2)}{(d_2\,x_1 - n_2\,y_1)}$$

is strongly stabilizable.

2. *Let $p_1 \in A$ and $p_2 \in K$. Then, p_1 and p_2 are simultaneously stabilizable iff the plant defined by the transfer function $p_2 - p_1$ is strongly stabilizable.*

Proof. 1. The fact that $p_1 = n_1/d_1$ is a coprime factorization with $d_1\,x_1 - n_1\,y_1 = 0$ implies that $c_1 = y_1/x_1$ is a stabilizing controller of p_1 and $1/(1-p_1\,c_1) = d_1\,x_1$, $c_1/(1-p_1\,c_1) = d_1\,y_1$. By 1 of Theorem 8, we have $(1,\,p_1) = (1/d_1)$ and $(1,\,p_2) = (1/d_2)$, and thus, $(1,\,p_1)^{-1} = (d_1)$ and $(a_1^2,\,b_1^2) = (1,\,p_1)^{-2} = (d_1^2)$ (see [15] for more details). Moreover, we easily check that:

$$\left(\frac{1-p_2\,c_1}{1-p_1\,c_1}\right) = (d_2\,x_1 - n_2\,y_1)\left(\frac{d_1}{d_2}\right).$$

Now, by Theorem 10, p_1 and p_2 are simultaneously stabilizable iff there exists $l \in A$ such that:

$$\left(\frac{d_1}{d_2}\right)((d_2\,x_1 - n_2\,y_1) - l\,(d_1\,n_2 - n_1\,d_2)) = \left(\frac{d_1}{d_2}\right) \Leftrightarrow ((d_2\,x_1 - n_2\,y_1) - l\,(d_1\,n_2 - n_1\,d_2)) = A.$$

The last equality is equivalent to $(d_2\, x_1 - n_2\, y_1) - l\,(d_1\, n_2 - n_1\, d_2) \in \mathrm{U}(A)$. Therefore, by Corollary 9, p_1 and p_2 are simultaneously stabilizable iff $p_3 = (n_2\, d_1 - n_1\, d_2)/(d_1\, n_2 - n_2\, y_1)$ is strongly stabilizable.

2. By 1 of Lemma 1, the fact that p_1 belongs to A implies that $c_1 = 0$ is a stabilizing controller of p_1. Therefore, by Theorem 10, we obtain that $p_1 \in A$ and $p_2 \in K$ are simultaneously stabilizable iff there exists $k = \alpha \in A$ such that $(1 - (p_2 - p_1)\, k) = (1,\, p_2)$, and thus, by 2 of Proposition 5, iff $p_2 - p_1$ is strongly stabilizable. We can also use 1 with $n_1 = p_1$, $d_1 = 1$, $x_1 = 1$ and $y_1 = 0$. □

Finally, let us show how to find again some known results about simultaneous stabilization when the assumption of the existence of coprime factorizations is made. See [2] for different proves.

Proposition 6. *1. Let $p_1 = n_1/d_1, \ldots, p_k = n_k/d_k \in K$ be k plants which admit coprime factorizations and $d_1\, x_1 - n_1\, y_1 = 1$. Then, $p_1, \ldots, p_k$ are simultaneously stabilizable iff the plants $p_{k+1}, \ldots, p_{2\,k-1}$, defined by*

$$p_{k+i-1} = \frac{d_i\, n_1 - n_i\, d_1}{d_i\, x_1 - n_i\, y_1}, \quad i = 2, \ldots, k,$$

are simultaneously stabilized by a stable controller.

2. Let $p_1 \in A$ and $p_2, \ldots, p_k \in K$. Then, c simultaneously stabilizes $p_1, \ldots, p_k$ iff $c/(1-p_1\, c) \in A$ simultaneously stabilizes the plants $p_2 - p_1, \ldots, p_k - p_1$.

Proof. 1. By 3 of Theorem 8, c internally stabilizes p_i, $i = 1, \ldots, k$, iff we have the equalities:

$$(1,\, p_i)\,(1,\, c) = (1 - p_i\, c), \quad i = 1, \ldots, k. \tag{14}$$

Using 3 of Theorem 9, we obtain that $c(q) = (y_1 + q\, d_1)/(x_1 + q\, n_1)$ is the Youla-Kučera parametrization of all stabilizing controllers of p_1. Hence, taking $c(q)$ for c, (14) becomes:

$$(1,\, p_i)\,(1,\, c(q)) = (1 - p_i\, c(q)), \quad i = 2, \ldots, k. \tag{15}$$

By 1 of Theorem 8, $(1,\, p_i) = (1/d_i)$ and $(1,\, c(q)) = (1/(x_1 + q\, n_1))$ because $p_i = n_i/d_i$ (resp. $c(q) = (y_1 + q\, d_1)/(x_1 + q\, n_1)$) is a coprime factorization of p_i (resp. $c(q)$). Then, we have

$$\begin{aligned}
(15) \quad &\Leftrightarrow \left(\frac{1}{d_i}\right)\left(\frac{1}{x_1 + q\, n_1}\right) = \left(\frac{d_i\,(x_1 + q\, n_1) - n_i\,(y_1 + q\, d_1)}{d_i\,(x_1 + q\, n_1)}\right), \quad i = 2, \ldots, k,\\
&\Leftrightarrow (d_i\,(x_1 + q\, n_1) - n_i\,(y_1 + q\, d_1)) = A,\\
&\Leftrightarrow \exists\, u_i \in \mathrm{U}(A) : \ (d_i\, x_1 - n_i\, y_1) + q\,(d_i\, n_1 - n_i\, d_1) = u_i, \quad i = 2, \ldots, k,
\end{aligned}$$

and thus, by Corollary 9, we obtain that c internally stabilizes p_i, $i = 1, \ldots, k$ iff $-q \in A$ internally stabilizes the $k-1$ plants defined by $p_{k+i-1} = (d_i\, n_1 - n_i\, d_1)/(d_i\, x_1 - n_i\, y_1)$, for $i = 2, \ldots, k$.

2. First of all, let us notice that $p_1 \in A$ implies the following equalities:

$$\left\{\begin{array}{l} (1,\, p_1) = A,\\ (1,\, p_i) = (1,\, p_i - p_1), \quad 2 \leq i \leq k,\\ (1,\, c) = (1 - p_1\, c,\, c). \end{array}\right.$$

Moreover, using 3 of Theorem 8, we obtain that c simultaneously stabilizes $p_1 \in A$ and $p_i \in K$, $i = 2, \ldots, k$, iff we have (14). But, (14) is equivalent to

$$\left\{\begin{array}{l} (1,\, c) = (1 - p_1\, c),\\ (1,\, p_i)\,(1,\, c) = (1 - p_i\, c),\ 2 \leq i \leq k, \end{array}\right. \Leftrightarrow \left\{\begin{array}{l} (1,\, c) = (1 - p_1\, c),\\ (1,\, p_i - p_1)\,(1 - p_1\, c,\, c) = (1 - p_i\, c),\ 2 \leq i \leq k, \end{array}\right.$$

$$\Leftrightarrow \left\{\begin{array}{l} (1,\, c) = (1 - p_1\, c),\\ (1,\, p_i - p_1)\,(1 - p_1\, c)\left(1,\, \frac{c}{1 - p_1\, c}\right) = (1 - p_i\, c),\ 2 \leq i \leq k, \end{array}\right.$$

$$\Leftrightarrow \left\{\begin{array}{l} (1,\, c) = (1 - p_1\, c),\\ (1,\, p_i - p_1)\left(1,\, \frac{c}{1 - p_1\, c}\right) = \left(\frac{1 - p_i\, c}{1 - p_1\, c}\right),\ 2 \leq i \leq k, \end{array}\right.$$

$$\Leftrightarrow \left\{\begin{array}{l} (1,\, c) = (1 - p_1\, c),\\ (1,\, p_i - p_1)\left(1,\, \frac{c}{1 - p_1\, c}\right) = \left(1 - \frac{(p_i - p_1)\, c}{(1 - p_1\, c)}\right),\ 2 \leq i \leq k. \end{array}\right.$$

By interchanging p and c in Proposition 5, i.e., $(1,\, c) = (1 - p_1\, c) \Leftrightarrow c/(1 - p_1\, c) \in A$, the first equation of the last system is equivalent to $c/(1 - p_1\, c) \in A$. Therefore, using 3 of Theorem 8, the last system means that $c/(1-p_1\, c) \in A$ simultaneously stabilizes the plants $p_2 - p_1, \ldots, p_k - p_1$. $\square$

Let us notice that, using 1 of Lemma 1, 2 of Proposition 6 can be reformulated as c simultaneously stabilizing $p_1, \ldots, p_k$ iff $c/(1-p_1\, c)$ simultaneously stabilizes $p_1 - p_1 = 0$, $p_2 - p_1, \ldots, p_k - p_1$.

4 Robust stabilization problem

Let us start by recalling the *small gain theorem* [4, 27] for SISO plants.

Lemma 3. *Let $p,\, c \in A$. Then, c internally stabilizes p iff $1/(1 - p\, c) \in A$ or, equivalently, iff c externally stabilizes p. In particular, if A is a Banach algebra and $\| c \|_A < 1/ \| p \|_A$, then $c \in A$ internally stabilizes p.*

Proof. The fact that $p,\, c \in A$ implies that $(1,\, p) = (1,\, c) = A$. Hence, using 3 of Theorem 8, we obtain that c internally stabilizes p iff we have $(1,\, p)\,(1,\, c) = (1 - p\, c)$, and thus, iff $(1 - p\, c) = A$, i.e., $1 - p\, c \in \mathrm{U}(A)$, which proves the first part of the result. Now, if A is a Banach algebra, then we have $\| 1 - a \|_A < 1 \Rightarrow a \in \mathrm{U}(A)$. Hence, if we have $\| c \|_A < 1/ \| p \|_A$, then $\| p\, c \|_A \leq \| p \|_A \, \| c \|_A < 1$, and thus, $1 - p\, c \in \mathrm{U}(A)$, showing that c internally stabilizes p by the first part of the result. $\square$

Now, let us try to investigate how a stabilizing controller c of a plant p is transformed when we apply some particular linear fractional transformations on p (see Proposition 3 [15]).

Lemma 4. *1. $0 \neq c \in K$ internally stabilizes $0 \neq p \in K$ iff $1/c$ internally stabilizes $1/p$.*

2. Let $\delta \in A$. Then, c internally stabilizes $p \in K$ iff $c/(1 + \delta\, c)$ internally stabilizes $p + \delta$.

3. Let $\delta \in A$. Then, c internally stabilizes $p \in K$ iff $c + \delta$ internally stabilizes $p/(1 + \delta\, p)$.

Proof. 1. Let us suppose that $c \neq 0$ internally stabilizes $p \neq 0$. Thus, by 3 of Theorem 8, we obtain (6). But, we have

$$\begin{aligned} (6) \quad & \Leftrightarrow (p)\ (1,\, 1/p)\ (c)\ (1,\, 1/c) = (1 - p\, c), \\ & \Leftrightarrow (1,\, 1/p)\ (1,\, 1/c) = ((1 - p\, c)/(p\, c)) = (1 - 1/(p\, c)), \end{aligned}$$

which shows that $1/c$ internally stabilizes $1/p$ by 3 of Theorem 8.

2. Let us suppose that c internally stabilizes p and $\delta \in A$. Thus, by 3 of Theorem 8, we obtain (6). Moreover, using the fact that $\delta \in A$, we obtain $(1,\, p) = (1,\, p + \delta)$ and $(1,\, c) = (1 + \delta\, c,\, c)$. But, we have

$$\begin{aligned} (6) \quad & \Leftrightarrow (1,\, p + \delta)\, (1 + \delta\, c,\, c) = (1 - p\, c), \\ & \Leftrightarrow (1,\, p + \delta)\, (1 + \delta\, c) \left(1,\, \tfrac{c}{1 + \delta\, c}\right) = (1 - p\, c), \\ & \Leftrightarrow (1,\, p + \delta) \left(1,\, \tfrac{c}{1+\delta\, c}\right) = \left(\tfrac{1 - p\, c}{1 + \delta\, c}\right) = \left(1 - (p + \delta)\, \tfrac{c}{(1 + \delta\, c)}\right), \end{aligned}$$

which shows that $c/(1 + \delta\, c)$ internally stabilizes $p + \delta$ by 3 of Theorem 8.

3. The result directly follows from interchanging p and c in point 2. $\square$

In the next proposition, we give new proofs of well-known results on robust stabilization [27].

Proposition 7. *1. Let $\delta \in A$ and $c \in K$ be a stabilizing controller of $p \in K$. Then,*

(a) c internally stabilizes $p + \delta$ iff $1 - (\delta\, c/(1 - p\, c)) \in \mathrm{U}(A)$,

(b) c internally stabilizes $p/(1 + \delta\, p)$ iff $1 + (\delta\, p/(1 - p\, c)) \in \mathrm{U}(A)$.

2. Let A be a Banach algebra and c a stabilizing controller of $p \in K$. Then,

(a) for all perturbation $\delta \in A$ satisfying

$$\| \delta \|_A < \frac{1}{\| c/(1 - p\, c) \|_A}, \tag{16}$$

c internally stabilizes $p + \delta$,

(b) *for all perturbation* $\delta \in A$ *satisfying*

$$\| \, \delta \, \|_A < \frac{1}{\| \, p/(1 - p\,c) \, \|_A}, \tag{17}$$

c *internally stabilizes* $p/(1+\delta\,p)$.

Proof. 1.a. Let c be a stabilizing controller of p. Thus, we have (6). Moreover, the fact that $\delta \in A$ implies that $(1,\, p) = (1,\, p+\delta)$. Hence, by 3 of Theorem 8, c internally stabilizes $p+\delta$ iff we have:

$$\begin{aligned}(1,\, p+\delta)\,(1,\, c) = (1 - (p+\delta)\,c) \quad &\Leftrightarrow (1,\, p)\,(1,\, c) = (1 - (p+\delta)\,c),\\ &\Leftrightarrow (1 - p\,c) = (1 - (p+\delta)\,c),\\ &\Leftrightarrow \left(\tfrac{1-(p+\delta)\,c}{1-p\,c}\right) = A,\\ &\Leftrightarrow \tfrac{1-(p+\delta)\,c}{1-p\,c} = 1 - \tfrac{\delta\,c}{1-p\,c} \in \mathrm{U}(A).\end{aligned}$$

1.b. Let c be a stabilizing controller of p. Thus, we have (6). Moreover, the fact that $\delta \in A$ implies that $(1,\, p) = (1+\delta\,p,\, p)$. Hence, by 3 of Theorem 8, c internally stabilizes $p/(1+\delta\,p)$ iff we have:

$$\begin{aligned}\left(1,\, \tfrac{p}{1+\delta\,p}\right)(1,\, c) = \left(1 - \tfrac{p\,c}{1+\delta\,p}\right) \quad &\Leftrightarrow (1+\delta\,p,\, p)\,(1,\, c) = (1 - p\,c + \delta\,p),\\ &\Leftrightarrow (1,\, p)\,(1,\, c) = (1 - p\,c + \delta\,p),\\ &\Leftrightarrow (1 - p\,c) = (1 - p\,c + \delta\,p),\\ &\Leftrightarrow \left(\tfrac{1-p\,c+\delta\,p}{1-p\,c}\right) = A,\\ &\Leftrightarrow \tfrac{1-p\,c+\delta\,p}{1-p\,c} = 1 + \tfrac{\delta\,p}{1-p\,c} \in \mathrm{U}(A).\end{aligned}$$

2.a. Let us suppose that $\delta \in A$ satisfies (16). Then, we have

$$\| \, (\delta\,c)/(1-p\,c) \, \|_A \leq \| \, \delta \, \|_A \, \| \, c/(1-p\,c) \, \|_A < 1,$$

and thus, $1 - (\delta\,c)/(1-p\,c) \in \mathrm{U}(A)$, i.e., c internally stabilizes $p+\delta$ by Lemma 3.

2.b Let us suppose that $\delta \in A$ satisfies (17). Then, we have

$$\| \, (\delta\,p)/(1-p\,c) \, \|_A \leq \| \, \delta \, \|_A \, \| \, p/(1-p\,c) \, \|_A < 1,$$

and thus, $1 + (\delta\,p)/(1-p\,c) \in \mathrm{U}(A)$, i.e., c internally stabilizes $p/(1+\delta\,p)$ by Lemma 3. □

The extension to unstable perturbation δ ($\delta \in K$) will be studied in forthcoming publications.

To finish, let us extend some standard results on robust stabilization for internally stabilizable plants which do not necessarily admit coprime factorizations. In particular, the following results show that the main interests of the Youla-Kučera parametrization for robust stabilization problems [27] are still valid if we use the new parametrization (8) instead of the Youla-Kučera one.

Proposition 8. *Let* $p \in K = Q(A)$ *be a stabilizable plant and* $w \in A$ *a weighted transfer function.*

1. *Let* $c_\star$ *be a stabilizing controller of* p. *Then, all the stable transfer matrices* $H(p,\, c(q_1,\, q_2))$ *of the closed-loop system (1) have the form*

$$H(p,\, c(q_1,\, q_2)) = \begin{pmatrix} \frac{1}{(1-p\,c_\star)} + q_1\,\frac{p}{(1-p\,c_\star)^2} + q_2\,\frac{p\,{c_\star}^2}{(1-p\,c_\star)^2} & \frac{c_\star}{(1-p\,c_\star)} + q_1\,\frac{1}{(1-p\,c_\star)^2} + q_2\,\frac{{c_\star}^2}{(1-p\,c_\star)^2} \\ \frac{p}{(1-p\,c_\star)} + q_1\,\frac{p^2}{(1-p\,c_\star)^2} + q_2\,\frac{(p\,c_\star)^2}{(1-p\,c_\star)^2} & \frac{1}{(1-p\,c_\star)} + q_1\,\frac{p}{(1-p\,c_\star)^2} + q_2\,\frac{p\,{c_\star}^2}{(1-p\,c_\star)^2} \end{pmatrix}, \tag{18}$$

for all $q_1,\, q_2 \in A$. *Hence, all the stable transfer matrices* $H(p,\, c(q_1,\, q_2))$ *of the closed-loop system (1) are affine, and thus, convex in the free parameters* q_1 *and* $q_2 \in A$, *namely for all* $\lambda,\, q_1,\, q_1',\, q_2,\, q_2' \in A$, *we have:*

$$H(p,\, c(\lambda\,q_1 + (1-\lambda)\,q_1',\, \lambda\,q_2 + (1-\lambda)\,q_2')) = \lambda\,H(p,\, c(q_1,\, q_2)) + (1-\lambda)\,H(p,\, c(q_1',\, q_2')). \tag{19}$$

2. *Let A be a Banach algebra with the norm $\| \cdot \|_A$. Then, we have the following equality*

$$\inf_{c \in \mathrm{Stab}(p)} \| w/(1 - p\,c) \|_A \tag{20}$$

$$= \inf_{q_1,\, q_2 \in A} \left\| w \left(\frac{1}{(1 - p\,c_\star)} + p \left(\frac{q_1}{(1 - p\,c_\star)^2} + \frac{q_2\, c_\star^2}{(1 - p\,c_\star)^2} \right) \right) \right\|_A, \tag{21}$$

where $\mathrm{Stab}(\mathrm{p})$ *denotes the set of all stabilizing controllers of p and $c_\star \in$* $\mathrm{Stab}(\mathrm{p})$.

3. *If $p = n/d$ is a coprime factorization of p, $d\,x - n\,y = 1$ for some $x,\, y \in A$, then we have:*

$$\inf_{c \in \mathrm{Stab}(p)} \| w/(1 - p\,c) \|_A = \inf_{q \in A} \| w\, d\,(x + n\,q) \|_A \,.$$

Proof. 1. If $c(q_1, q_2)$ is the parametrization of all stabilizing controllers of p defined (8), then $H(p, c(q_1, q_2))$ becomes

$$H(c(q_1, q_2)) = \begin{pmatrix} a + q_1\, a^2\, p + q_2\, b^2\, p & b + q_1\, a^2 + q_2\, b^2 \\ a\,p + q_1\, a^2\, p^2 + q_2\, b^2\, p^2 & b\,p + q_1\, a^2\, p + q_2\, b^2\, p \end{pmatrix} \in A^{2\times 2},$$

where $a = 1/(1 - p\,c_\star)$ and $b = c_\star/(1 - p\,c_\star)$ and $c_\star$ is a particular stabilizing controller of p. Now, by substituting $a = 1/(1 - p\,c_\star)$ and $b = c_\star/(1 - p\,c_\star)$ in the previous matrix, we obtain (18). Finally, using the fact that (18) is affine in the arbitrary parameters q_1 and q_2, we easily check (19).

2. Using the parametrization (8) of all stabilizing controllers of p, we obtain that all the stable sensitivity transfer function are defined by

$$\frac{1}{(1 - p\,c(q_1, q_2))} = \frac{1}{(1 - p\,c_\star)} + p \left(\frac{q_1}{(1 - p\,c_\star)^2} + \frac{q_2\, c_\star^2}{(1 - p\,c_\star)^2} \right),$$

where $c_\star$ is a particular stabilizing controller of p. From this result, we easily prove that the non-linear optimization problem (20) becomes the affine, and thus, convex optimization problem (21). □

Up to our knowledge, the results stated in Proposition 8 are new. We refer the reader to [17] for an extension of Proposition 8 to MIMO systems. To finish, let us point out that using the general parametrization (8) of all stabilizing controllers, any convex optimization problem on the stable closed-loop system $H(p, c)$ can be rewritten as a convex optimization problem in $q_1,\, q_2 \in A$.

5 Conclusion

In this paper, we introduced different mathematical concepts coming from module theory, K-theory and theory fractional ideals in order to investigate the internal/strong/simultaneous/robust stabilization problems. Some new results have been obtained within this new mathematical framework.

In particular, using module theory, we first emphasized the fact that an internally stabilizable did not generally imply the existence of doubly coprime factorizations for the transfer matrix. Indeed, the first property is characterized in terms of projective modules whereas the second one is characterized in terms of free modules. In commutative algebra, it is well-known that projective modules (resp. invertible fractional ideals) are not generally free (resp. principal) and the study of projective but not free modules has played a fundamental role in the development of the modern commutative algebra. It can be tracked back to the work of R. Dedekind on algebraic numbers theory (in part related to the Last Fermat's Theorem) and algebraic geometry (algebraization of B. Riemann's ideas on complex manifolds). Nowadays, the problem of recognizing whether or not a stably free (resp. projective) module is free constitutes one of the main parts of the so-called (algebraic, topological) K-theory. Using the K-theoretic tools (e.g., stable range, topological stable range), we obtained new results on the strong stabilization problem. Finally, we revisited some well-known results of stabilization problems without assuming the existence of (doubly) coprime factorizations. We also refer to [14, 15, 16, 17] for more results in this direction.

Finally, the first main philosophy of R. Dedekind's work on the theory of ideals is that an ideal contained more information than just the elements that generated it. The second one is that the standard algebraic operations can be extended from numbers to ideals. We hope to have convinced the reader that this natural "algebraic calculus on fractional ideals" can also be useful for the study of stabilization problems and, in particular, for the internal, strong, simultaneous and robust stabilization problems.

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