

## THE FRACTIONAL REPRESENTATION APPROACH TO SYNTHESIS PROBLEMS: AN ALGEBRAIC ANALYSIS VIEWPOINT PART I: (WEAKLY) DOUBLY COPRIME FACTORIZATIONS\*

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**Abstract.** In this paper, we show how to reformulate the fractional representation approach to analysis and synthesis problems within an algebraic analysis framework. In terms of modules, we give necessary and sufficient conditions so that a system admits (weakly) left/right/doubly coprime factorizations. Moreover, we explicitly characterize the integral domains  $A$  such that every plant—defined by means of a transfer matrix whose entries belong to the quotient field of  $A$ —admits (weakly) doubly coprime factorizations. Finally, we show that this algebraic analysis approach allows us to recover, on the one hand, the approach developed in [M. C. Smith, *IEEE Trans. Automat. Control*, 34 (1989), pp. 1005–1007] and, on the other hand, the ones developed in [K. Mori and K. Abe, *SIAM J. Control Optim.*, 39 (2001), pp. 1952–1973; V. R. Sule, *SIAM J. Control Optim.*, 32 (1994), pp. 1675–1695 and 36 (1998), pp. 2194–2195; M. Vidyasagar, H. Schneider, and B. A. Francis, *IEEE Trans. Automat. Control*, 27 (1982), pp. 880–894; M. Vidyasagar, *Control System Synthesis: A Factorization Approach*, MIT Press, Cambridge, MA, 1985].

**Key words.** fractional representation approach to synthesis problems, (weakly) left/right/doubly coprime factorizations, coherent rings and modules, coherent Sylvester domains,  $H_\infty(\mathbb{C}_+)$ , Bézout domains, algebraic analysis, module theory, homological algebra

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**Introduction.** In the seventies, Vidyasagar and others introduced the idea of representing a class of transfer functions as the quotient field of a certain integral domain  $A$  of proper and stable transfer functions. Examples of such integral domains  $A$ , usually encountered in the literature, are the Banach algebra  $H_\infty(\mathbb{C}_+)$  of bounded analytic functions in the open right half-plane  $\mathbb{C}_+ = \{s \in \mathbb{C} \mid \operatorname{Re} s > 0\}$  [8], the algebra  $RH_\infty = \mathbb{R}(s) \cap H_\infty(\mathbb{C}_+)$  of proper stable real rational functions [49], and the Wiener algebras  $\mathcal{A}$ ,  $\hat{\mathcal{A}}$  [3, 8], and  $l_1(\mathbb{Z}_+)$  [49]. In the early eighties, this idea naturally led to the *fractional representation approach to synthesis problems*, principally developed in [3, 9, 48, 49]. The main outcome of this point of view is a reformulation of various questions of feedback stabilization of systems in terms of algebraic properties of some matrices whose entries belong to  $A$  (e.g., internal/strong/simultaneous/robust/optimal stabilization, parametrization of all the stabilizing controllers, graph topology, etc.).

Unfortunately, questions seem to remain for some classes of (infinite-dimensional) systems, in particular, the following:

1. Do necessary and sufficient conditions exist for internal stabilizability?
2. Is it possible to characterize all the integral domains  $A$  such that every plant—defined by means of a transfer matrix whose entries belong to the quotient field of  $A$ —is internally stabilizable?
3. What are the links between internal stabilizability and the existence of a doubly coprime factorization for the transfer matrix?

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4. Is it always possible to parametrize all the stabilizing controllers of a stabilizing plant by means of the Youla–Kučera parametrization?

In order to solve certain of these open questions, the authors of [40] tried to revisit the fractional representation approach to stabilization problems of single-input single-output (SISO) systems using a more intrinsic framework than the one used in [3, 9, 48, 49]. A module approach has recently been developed in [44] and continued in [22]. In the same years as that last work, another approach was developed in [30, 31, 32] using the ideas of *algebraic analysis* (see [27, 28, 29] and the references therein). The purpose of this paper is to present this new mathematical framework and to explain how certain of the previous open questions can be solved using this algebraic analysis point of view.

In this paper, we first introduce the concepts of *weakly left/right/doubly coprime factorizations*, give necessary and sufficient conditions in terms of modules so that a transfer matrix admits such factorizations, and characterize all the integral domains  $A$  over which every transfer matrix admits weakly doubly coprime factorizations (namely, *coherent Sylvester domains*, e.g.,  $H_\infty(\mathbb{C}_+)$ ). Moreover, we also give necessary and sufficient conditions so that a transfer matrix admits left/right/doubly coprime factorizations, and we recover a result of Vidyasagar [49] describing all the integral domains  $A$  over which every transfer matrix admits doubly coprime factorizations (namely, *Bézout domains*). In particular, we recover and generalize some standard results of [22, 43, 44, 49]. In the second part of the paper [33], we shall use the same mathematical framework and the previous results to develop necessary and sufficient conditions for *internal stabilizability* [9, 48, 49]. Moreover, we shall characterize all the integral domains  $A$  over which every plant—defined by means of a transfer matrix whose entries belong to the quotient field of  $A$ —is internally stabilizable (namely, *Prüfer domains*). Hence, the algebraic analysis framework seems to solve the first three questions listed above. We refer the reader to [34] for a general answer to the fourth one. Let us note that all these results use the techniques of module theory and homological algebra, and they seem difficult to obtain using only a matrix approach.

If we want to develop some general algorithms (i.e., valid for a general integral domain  $A$ ) that check the existence of (weakly) left/right/doubly coprime factorizations and compute them, we then have to overcome the difficulty arising from the fact that most of the integral domains of SISO stable plants are *Banach algebras* (e.g.,  $H_\infty(\mathbb{C}_+)$ ,  $\mathcal{A}$ ,  $\hat{\mathcal{A}}$ ,  $l_1(\mathbb{Z}_+)$ ). Indeed, a result proves that *noetherian* Banach algebras are only finite-dimensional [41], and thus, most of the Banach algebras used in systems theory are not noetherian. Therefore, it seems that we cannot use the standard techniques of commutative algebra, module theory, and homological algebra developed for noetherian rings to study general (infinite-dimensional) linear systems. (Some modules may fail to be finitely generated.) We show that the only possibility for coping with this difficulty seems to require that the Banach algebras be *coherent rings*. This result could explain why coherent Sylvester domains, Prüfer and Bézout domains, which play important roles in the fractional representation approach (see above), are all coherent. Using the fact that a system is defined by means of a transfer matrix, we prove that, if  $A$  is a coherent domain, then every system defines a *coherent  $A$ -module*. Now, the (category of) coherent  $A$ -modules over a coherent ring  $A$  (is) are invariant under all the elementary algebraic manipulations (e.g., intersection, sum, quotient, tensor product, duality, etc.). Therefore, we can use homological algebra to develop general algorithms which check the existence of (weakly) left/right/doubly coprime factorizations (or internal stabilizability in [33]) of (infinite-dimensional) linear systems defined over a coherent domain  $A$ .

**Plan.** In the first part of this paper (section 1), we describe the framework of the fractional representation approach to analysis and synthesis problems and explain why and how it is possible to use module theory. We recall some definitions of module theory and homological algebra that will be constantly used in the rest of the paper and in [33]. The second part (section 2) is related to factorization problems. We first introduce the concept of a *weakly doubly coprime factorization* and show that it corresponds to the weakest coprimeness for transfer matrices. We give necessary and sufficient conditions so that a transfer matrix admits a weakly left/right/doubly coprime factorization. In the third part (section 3), we introduce the concept of coherent rings and modules. We prove that every transfer matrix, with entries in the quotient field of  $A$ , admits a weakly doubly coprime factorization iff  $A$  is a *coherent Sylvester domain*. We show that  $H_\infty(\mathbb{C}_+)$  is a coherent Sylvester domain. Finally, in the last part (section 4), we give necessary and sufficient conditions so that a transfer matrix admits left/right/doubly coprime factorizations.

**Notation.** In the course of the text,  $A$  denotes a commutative integral domain ( $ab = 0, a \neq 0 \Rightarrow b = 0$ ) with a unit,  $M_{q \times p}(A)$  (resp.,  $M_p(A)$ ) the set of  $q \times p$  (resp.,  $p \times p$ ) matrices whose entries belong to  $A$ , and  $I_p$  the identity matrix. If  $R \in M_{q \times p}(A)$ , then  $R^T$  is the transposed matrix. *By convention, every vector with entries in  $A$  is a row vector.* The positive integers  $p, q \in \mathbb{Z}_+$  will always satisfy  $p \geq q$ . If  $M$  and  $N$  are two  $A$ -modules, then  $M \cong N$  means that  $M$  and  $N$  are isomorphic as  $A$ -modules,  $\text{hom}_A(M, N)$  is the  $A$ -module of the  $A$ -morphisms (i.e.,  $A$ -linear maps) from  $M$  to  $N$ , and  $M^* = \text{hom}_A(M, A)$ . Finally,  $(a_1, \dots, a_n)$  denotes the ideal  $Aa_1 + \dots + Aa_n$ , and  $\triangleq$  means “by definition.”

## 1. The fractional representation approach to synthesis problems.

**1.1. Introduction.** Following ideas of Zames [51], a class of transfer functions needs to have the structure of a ring if we want to connect two systems in cascade (product) or in parallel (sum). In the fractional representation approach to analysis and synthesis problems, we start with an integral domain  $A$  of *SISO stable plants* [3, 8, 9, 48, 49]. Classical examples of integral domains of SISO stable plants are the Banach algebra  $H_\infty(\mathbb{C}_+)$  of the bounded analytic functions on the open right half-plane  $\mathbb{C}_+ = \{s \in \mathbb{C} \mid \text{Re } s > 0\}$  [8], the ring  $RH_\infty$  of proper stable real rational functions [49], or the Wiener algebras  $\mathcal{A}, \hat{\mathcal{A}}, l_1(\mathbb{Z}_+)$  [8, 49]. Then, the class of (unstable) SISO plants considered is defined by the field of fractions of  $A$ :

$$(1.1) \quad K = Q(A) = \{n/d \mid 0 \neq d, n \in A\}.$$

*Example 1.1.* Let us give some examples.

- Let us consider  $A = RH_\infty = \{p = n/d \mid \deg n \leq \deg d, d(\underline{s}) = 0 \Rightarrow \text{Re } \underline{s} < 0\}$  the integral domain of proper stable real rational functions. The transfer function  $p = 1/(s-1)$  (resp.,  $p = s$ ) does not belong to  $A$  because  $p$  has the unstable pole 1 in  $\mathbb{C}_+$  (resp.,  $p$  is not proper) but belongs to  $K = Q(A) = \mathbb{R}(s)$  because  $p$  can be represented as  $p = n/d$  with  $n = 1/(s+1) \in A$  and  $d = (s-1)/(s+1) \in A$  (resp.,  $n = s/(s+1) \in A$  and  $d = 1/(s+1) \in A$ ).
- Let us consider the following Wiener algebra [3, 8]:

$$\mathcal{A} = \left\{ f(t) + \sum_{i=0}^{\infty} a_i \delta(t - t_i) \mid f \in L_1(\mathbb{R}_+), (a_i)_{i \geq 0} \in l_1(\mathbb{Z}_+), 0 = t_0 \leq t_1 \dots \right\},$$

with the two operations  $+$  and the convolution  $\star$  and the Dirac distribution  $\delta$  as the unit. Endowed with the topology defined by the norm

$$\|g\|_{\mathcal{A}} = \|f\|_{L_1(\mathbb{R}_+)} + \|(a_i)_{i \geq 0}\|_{l_1(\mathbb{Z}_+)},$$

$\mathcal{A}$  becomes a *Banach algebra* and an integral domain [3, 8, 48, 49]. The same properties hold for  $\hat{\mathcal{A}} = \{\hat{f} \mid f \in \mathcal{A}\}$  ( $\hat{\cdot}$  is the Laplace transform) with the norm  $\|\hat{f}\|_{\hat{\mathcal{A}}} = \|f\|_{\mathcal{A}}$ . For instance, an example of a transfer function which belongs to  $K = Q(\mathcal{A})$  is the following:

$$p = e^t Y(t) \star \delta(t - 1) = (\delta(t) - 2e^{-t} Y(t))^{-1} \star (e^{-t} Y(t) \star \delta(t - 1)),$$

where  $Y(t)$  denotes the Heaviside distribution (i.e., 1 for  $t \geq 0$ , and 0 otherwise) and  $\delta(t) - 2e^{-t} Y(t)$ ,  $e^{-t} Y(t) \star \delta(t - 1) \in \mathcal{A}$ . Equivalently, in the frequency domain, the same plant is defined by the following transfer function:

$$p = \frac{e^{-s}}{s-1} = \frac{\left(\frac{e^{-s}}{s+1}\right)}{\left(\frac{s-1}{s+1}\right)} \in Q(\hat{\mathcal{A}}), \quad \frac{e^{-s}}{s+1}, \frac{s-1}{s+1} \in \hat{\mathcal{A}}.$$

If  $P \in M_{q \times (p-q)}(K)$ , then it is always possible to write it as  $P = D^{-1} N = \tilde{N} \tilde{D}^{-1}$ , where  $D \in M_q(A)$  and  $\tilde{D} \in M_{p-q}(A)$  are two invertible matrices and  $N \in M_{q \times (p-q)}(A)$ ,  $\tilde{N} \in M_{q \times (p-q)}(A)$ , i.e., all the entries of these four matrices are stable. For example, we can use  $D = dI_q$  and  $N = dP$ , where  $0 \neq d \in A$  is the product of the denominators of all the entries of  $P$ , and similarly for  $\tilde{D} = dI_{p-q}$  and  $\tilde{N} = Pd$ .

*Example 1.2.* Let  $A = H_\infty(\mathbb{C}_+)$ , and let us consider the plant defined by

$$(1.2) \quad P = \begin{pmatrix} \frac{e^{-s}}{s+1} & \frac{s-1}{s+1} \\ 0 & \frac{1}{s-1} \end{pmatrix} \in M_2(K), \quad K = Q(A).$$

Then,  $P$  can be written as  $P = D^{-1} N$  with

$$(1.3) \quad D = \begin{pmatrix} \frac{s-1}{s+1} & 0 \\ 0 & \frac{s-1}{s+1} \end{pmatrix} \in M_2(A), \quad N = \begin{pmatrix} \frac{(s-1)e^{-s}}{(s+1)^2} & \left(\frac{s-1}{s+1}\right)^2 \\ 0 & \frac{1}{s+1} \end{pmatrix} \in M_2(A).$$

Thus, instead of representing a plant by  $y = Pu$  with  $P \in M_{q \times (p-q)}(K)$ , the fractional representation approach studies the following two systems:

$$(D : -N) \begin{pmatrix} y \\ u \end{pmatrix} = 0, \quad \begin{pmatrix} y \\ u \end{pmatrix} = \begin{pmatrix} \tilde{N} \\ \tilde{D} \end{pmatrix} z,$$

with  $R = (D : -N) \in M_{q \times p}(A)$  and  $\tilde{R} = (\tilde{N}^T : \tilde{D}^T)^T \in M_{p \times (p-q)}(A)$ . Then, using linear algebra over the ring  $A$ , it is possible to study the structural properties of  $P$  by looking at the algebraic properties of the matrices  $R$  and  $\tilde{R}$  (left/right/doubly factorizations). See [3, 8, 9, 48, 49] for more information.

For linear finite-dimensional systems [49], the fractional representation approach gives necessary and sufficient conditions for internal stabilizability, existence of doubly coprime factorizations, or Youla–Kučera parametrizations of all the stabilizing controllers, etc. [49]. The possibility of generalizing these results to linear infinite-dimensional

systems (delay systems, partial differential equations such as the wave/heat/Euler–Bernoulli equations) has naturally been asked from theoretical and practical points of view [3, 8, 9, 48, 49]. In this framework, classes of linear infinite-dimensional systems are generally modeled by means of Banach algebras such as  $H_\infty(\mathbb{C}_+)$ ,  $\hat{\mathcal{A}}$ ,  $l_1(\mathbb{Z}_+)$ . These rings are algebraically and topologically more complex than the ring  $RH_\infty$  used for finite-dimensional systems. Hence, some questions, such as the ones given in the introduction, are still open for some classes of infinite-dimensional systems [8, 48, 49]. As we described it in the introduction, the purpose of this paper is to show that we can solve certain of these problems if we adopt an *algebraic analysis* framework rather than a matricial one. Here, we call “algebraic analysis” a mathematical framework which uses commutative algebra, module theory, and homological algebra combined with functional analysis (Banach algebras). This idea could seem natural if we notice that, in order to understand the structural properties of a plant, defined by the transfer matrix  $P \in M_{q \times (p-q)}(K)$ , we need to study the matrices  $R \in M_{q \times p}(A)$  and  $\tilde{R} \in M_{p \times (p-q)}(A)$ , whose entries belong to a certain algebra of functions (e.g., a Banach algebra), and linear algebra over a ring is just a part of module theory.

**1.2. Definitions.** In this section, we give some definitions that we shall need to characterize intrinsically the structural properties of systems.

Let  $R \in M_{q \times p}(A)$ , and let us define the  $A$ -morphism (i.e., an  $A$ -linear map)  $.R$  by

$$.R : \begin{array}{ccc} A^q & \longrightarrow & A^p, \\ (a_1 : \dots : a_q) & \longrightarrow & (a_1 : \dots : a_q)R. \end{array}$$

Then, the image  $\text{im } .R$  is the  $A$ -module generated by the  $A$ -linear combinations of the rows of  $R$ . This  $A$ -module is usually used in control theory [22, 44]. In *algebraic analysis* [27, 28, 29], one usually prefers to use the  $A$ -module  $M = \text{coker } .R = A^p/A^q R$ .

DEFINITION 1.1. *We have the following definitions (see [1, 2, 39]):*

- A complex is a sequence of  $A$ -modules  $F_i$  and  $A$ -morphisms  $d_i$ , denoted by  $\dots \longrightarrow F_{i+1} \xrightarrow{d_{i+1}} F_i \xrightarrow{d_i} F_{i-1} \longrightarrow \dots$ , such that  $d_i \circ d_{i+1} = 0$ , i.e.,

$$\text{im } d_{i+1} \subseteq \ker d_i.$$

- The  $i$ th  $A$ -module of homology of a complex is defined by

$$H(F_i) = \frac{\ker d_i}{\text{im } d_{i+1}}.$$

- A sequence is said to be exact at  $F_i$  if  $H(F_i) = 0$ , i.e.,  $\ker d_i = \text{im } d_{i+1}$ , and exact if we have  $H(F_i) = 0$  for all  $i$ .

Example 1.3. For instance, the exact sequence  $0 \longrightarrow M' \xrightarrow{f} M$  means that  $f$  is injective, whereas the exact sequence  $M \xrightarrow{g} M'' \longrightarrow 0$  means that  $g$  is surjective.

Let  $\pi$  be the  $A$ -morphism which associates to every row vector of  $A^p$  its class in the quotient  $A$ -module  $M = A^p/A^q R$ . We have the following exact sequence:

$$(1.4) \quad A^q \xrightarrow{.R} A^p \xrightarrow{\pi} M \longrightarrow 0.$$

Let  $\{e_1, \dots, e_p\}$  be the canonical basis of  $A^p$ , and  $\{f_1, \dots, f_q\}$  that of  $A^q$ . We define

$z_i = \pi(e_i)$ ,  $i = 1, \dots, p$ . Then, we have for  $i = 1, \dots, q$

$$(1.5) \quad \begin{aligned} f_i R = (R_{i1} : \dots : R_{ip}) &= \sum_{j=1}^p R_{ij} e_j \in A^q R \\ \Rightarrow \pi(f_i R) &= \sum_{j=1}^p R_{ij} \pi(e_j) = \sum_{j=1}^p R_{ij} z_j = 0. \end{aligned}$$

Hence,  $M$  is defined by the equations  $Rz = 0$ , where  $z = (z_1 : \dots : z_p)^T$ , and their  $A$ -linear combinations. Moreover, for all  $m \in M$ , there exists  $(a_1 : \dots : a_p) \in A^p$  such that

$$m = \pi((a_1 : \dots : a_p)) = \pi\left(\sum_{i=1}^p a_i e_i\right) = \sum_{i=1}^p a_i \pi(e_i) = \sum_{i=1}^p a_i z_i.$$

This means that every element  $m$  of  $M$  is an  $A$ -linear combination of the elements  $z_1, \dots, z_p$ , and the  $A$ -module  $M$  is said to be *finitely generated*. In fact, the module is defined by a finite number of equations ( $q$  equations) with a finite number of unknowns ( $p$  unknowns). In this case, we say that  $M$  is *finitely presented*, a fact which is equivalent to the existence of the exact sequence (1.4).

*Example 1.4.* Let us reconsider Example 1.2. We have  $A = H_\infty(\mathbb{C}_+)$ , and the matrix  $R = (D : -N) \in M_{2 \times 4}(A)$  is defined by

$$(1.6) \quad R = \begin{pmatrix} \frac{s-1}{s+1} & 0 & -\frac{(s-1)e^{-s}}{(s+1)^2} & -\left(\frac{s-1}{s+1}\right)^2 \\ 0 & \frac{s-1}{s+1} & 0 & -\frac{1}{s+1} \end{pmatrix}.$$

Then, the  $A$ -morphism  $.R$  is defined by

$$\begin{aligned} A^2 &\longrightarrow A^4, \\ (a_1 : a_2) &\longrightarrow \left( a_1 \left( \frac{s-1}{s+1} \right) : a_2 \left( \frac{s-1}{s+1} \right) : -a_1 \left( \frac{(s-1)e^{-s}}{(s+1)^2} \right) : -a_1 \left( \frac{s-1}{s+1} \right)^2 - a_2 \frac{1}{s+1} \right). \end{aligned}$$

Therefore, the  $A$ -module  $M = A^4/A^2 R$  is defined by the equations

$$\begin{cases} \frac{(s-1)}{(s+1)} y_1 - \frac{(s-1)e^{-s}}{(s+1)^2} u_1 - \left(\frac{s-1}{s+1}\right)^2 u_2 = 0, \\ \frac{(s-1)}{(s+1)} y_2 - \frac{1}{s+1} u_2 = 0 \end{cases}$$

and their  $A$ -linear combinations, where  $y_i = \pi(e_i)$ ,  $u_i = \pi(e_{i+2})$ ,  $i = 1, 2$ .

DEFINITION 1.2. We have the following definitions (see [1, 39]):

- An  $A$ -module  $M$  is finitely generated if there exists an  $A$ -module of the form  $F_0 \cong A^{r_0}$ ,  $r_0 \in \mathbb{Z}_+$ , such that we have the following exact sequence:

$$(1.7) \quad F_0 \xrightarrow{d_0} M \longrightarrow 0.$$

- An  $A$ -module  $M$  is finitely presented if there exist two  $A$ -modules  $F_i \cong A^{r_i}$ ,  $r_i \in \mathbb{Z}_+$ ,  $i = 0, 1$ , such that we have the following exact sequence:

$$(1.8) \quad F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} M \longrightarrow 0.$$

One of the main interests of associating an  $A$ -module with a matrix  $R \in M_{q \times p}(A)$  is that we can use the natural classification of the properties of the module to understand the structural properties of the system  $Rz = 0$ . The purpose of the next sections and [33] is to illustrate how some concepts of modules play interesting roles in characterizing the existence of a (weakly) left/right/doubly coprime factorization and internal stabilizability.

DEFINITION 1.3 (see [2, 39]). *If  $M$  is a finitely generated  $A$ -module, then*

- $M$  is free if  $M \cong A^r$  for  $r \in \mathbb{Z}_+$ .
- $M$  is stably free if there exist  $r, s \in \mathbb{Z}_+$  such that  $M \oplus A^s \cong A^r$ .
- $M$  is projective if there exist an  $A$ -module  $N$  and  $r \in \mathbb{Z}_+$  such that

$$M \oplus N \cong A^r.$$

- $M$  is reflexive if the  $A$ -morphism  $\epsilon : M \rightarrow \text{hom}_A(\text{hom}_A(M, A), A)$ —defined by  $\epsilon(m)(f) = f(m)$  for all  $m \in M$ , for all  $f \in \text{hom}_A(M, A)$ —is an isomorphism.
- $M$  is torsion-free if its torsion submodule

$$t(M) = \{m \in M \mid \exists a \in A \setminus \{0\} : am = 0\}$$

is reduced to 0.  $m \in t(M)$  is a torsion element. We have the exact sequence:

$$(1.9) \quad 0 \rightarrow t(M) \rightarrow M \rightarrow M/t(M) \rightarrow 0.$$

- $M$  is a torsion  $A$ -module if  $t(M) = M$ .
- $M$  is a flat  $A$ -module if, for every relation  $\sum_{i=1}^q a_i m_i = 0$ ,  $a_i \in A$ ,  $m_i \in M$ , there exist  $n_i \in M$  and  $b_{ij} \in A$  such that

$$\begin{cases} m_i = \sum_{j=1}^r b_{ij} n_j, & 1 \leq i \leq q, \\ \sum_{i=1}^q a_i b_{ij} = 0, & 1 \leq j \leq r. \end{cases}$$

PROPOSITION 1.4 (see [1, 2, 39]). *We have the following relations:*

1. Free  $\Rightarrow$  stably free  $\Rightarrow$  projective  $\Rightarrow$  reflexive  $\Rightarrow$  torsion-free.
2. Projective  $\Rightarrow$  flat  $\Rightarrow$  torsion-free.
3. A finitely generated flat  $A$ -module is projective.
4. If  $A$  is a Bézout domain—namely, a domain  $A$  such that every finitely generated ideal  $I$  of  $A$  has the form  $I = (a)$  for a certain  $a \in A$ —then every finitely generated torsion-free  $A$ -module is free and  $M \cong t(M) \oplus M/t(M)$ .

DEFINITION 1.5. *We have the following definitions (see [6, 20]):*

- $A$  is a Hermite ring if every finitely generated stably free  $A$ -module is free, or, equivalently, if every unimodular row  $(a_1 : \dots : a_n) \in A^n$ —namely a row such that there exists  $(b_1 : \dots : b_n)^T$  satisfying  $\sum_{i=1}^n a_i b_i = 1$ —can be completed to a unimodular matrix, i.e., to a matrix of  $GL_n(A)$ .
- $A$  is a projective-free ring if every finitely generated projective  $A$ -module is free.

In particular, a projective-free ring (e.g., a Bézout domain) is a Hermite ring. A difficult result, namely, the Quillen–Suslin theorem [20, 39], proves that the ring of polynomials  $k[\chi_1, \dots, \chi_n]$  in  $\chi_i$ , with coefficients in a field  $k$ , is projective-free.

**THEOREM 1.6.**  $RH_\infty$  and  $k[s]$ , where  $k$  is a field, are principal ideal domains (namely, a domain such that every ideal  $I$  of  $A$  is principal, i.e., has the form  $I = (a)$  for a certain  $a \in A$ ; see [49]). The domain of entire functions with coefficients in  $k = \mathbb{R}, \mathbb{C}$ ,

$$E(k) = \left\{ f(s) = \sum_{n=0}^{+\infty} a_n s^n \mid a_n \in k, \lim_{n \rightarrow +\infty} |a_n|^{1/n} = 0 \right\},$$

and  $\mathcal{E} = E(\mathbb{R}) \cap \mathbb{R}(s)[e^{-s}]$  are Bézout domains (see [18, 21]). Thus, all these rings are projective-free.

We introduce the concept of *localization* of modules. In the following sections, we shall show why this concept is interesting for the study of the links between a transfer matrix  $y = Pu$ ,  $P = D^{-1}N \in M_{q \times (p-q)}(K)$  and the system  $Dy = Nu$ , where  $R = (D : -N) \in M_{q \times p}(A)$ .

**DEFINITION 1.7** (see [1, 39]). *We have the following definitions:*

- A multiplicative set  $S$  of  $A$  is a subset of  $A$  such that for all  $a, b \in S \Rightarrow ab \in S$  and  $1 \in S$ .
- Let  $M$  be an  $A$ -module. We define an equivalence relation  $\sim$  on  $S \times M$  by  $(s, m) \sim (s', m')$  if there exists  $t \in S$  such that  $t(sm' - s'm) = 0$ . The localization of the  $A$ -module  $M$  with respect to  $S$  is the  $S^{-1}A = \{a/s \mid (s, a) \in S \times A\}$ -module

$$S^{-1}M = (S \times M) / \sim.$$

If we denote by  $m/s$  the equivalence class of  $(s, m)$  in  $S^{-1}M$ , then we have

$$S^{-1}M = \{(a/s)m \mid (s, a) \in S \times A, m \in M\}.$$

The localization of a module is just a way to extend the scalars of the  $A$ -module  $M$  from  $A$  to  $S^{-1}A$ . Moreover, we have the following canonical  $A$ -morphism:

$$\begin{aligned} i_S : M &\longrightarrow S^{-1}M, \\ m &\longrightarrow m/1, \end{aligned}$$

from which we obtain  $\ker i_S = \{m \in M \mid \exists 0 \neq a \in S, am = 0\}$ .

**DEFINITION 1.8.** *If  $S = A \setminus 0$ , then  $S$  is a multiplicative set of  $A$ , and the field of fractions of  $A$  is defined by  $S^{-1}A = Q(A) = \{a/b \mid 0 \neq b, a \in A\}$ . We have*

$$(1.10) \quad \ker i_S = t(M) = \{m \in M \mid \exists 0 \neq a \in A : am = 0\}.$$

*In the course of the paper, we shall denote by  $K$  the field of fractions  $Q(A)$  of  $A$ .*

**Example 1.5.** Let us reconsider the  $A$ -module  $M = A^4/A^2R$  defined in Example 1.4. We can check that the element  $z = y_1 - \frac{e^{-s}}{(s+1)}u_1 - \frac{(s-1)}{(s+1)}u_2$  satisfies  $\frac{(s-1)}{(s+1)}z = 0$ , i.e.,  $z$  is a torsion element of  $M$ . (See Example 3.4 for an explicit computation of the torsion elements of  $M$ .) If  $S = A \setminus 0$ , then, in the  $K = Q(A)$ -module  $S^{-1}M$ , we have

$$\frac{(s-1)}{(s+1)} \left( \frac{z}{1} \right) = \frac{\frac{(s-1)}{(s+1)}z}{1} = \frac{0}{1} \Rightarrow \frac{(s+1)}{(s-1)} \frac{(s-1)}{(s+1)} \left( \frac{z}{1} \right) = \frac{z}{1} = \frac{0}{1} \Rightarrow i_S(z) = \frac{0}{1}.$$

Moreover, we have the following isomorphism (see [1, 39]):

$$(1.11) \quad \begin{aligned} S^{-1}M &\cong S^{-1}A \otimes_A M, \\ (a/b)m &\longleftrightarrow a/b \otimes m, \end{aligned}$$



which shows that the localization corresponds to the tensor product  $S^{-1}A \otimes_A$ .

DEFINITION 1.9 (see [1, 39]). *The rank of an  $A$ -module is defined by*

$$\text{rank}_A(M) = \dim_K(K \otimes_A M),$$

where  $K \otimes_A M$  is a  $K$ -vector space and  $\dim_K(K \otimes_A M)$  is its dimension over  $K$ .

PROPOSITION 1.10 (see [1, 2, 39]). *If  $S$  is a multiplicative set of  $A$ , then  $S^{-1}A$  is a flat  $A$ -module, and, for every exact sequence  $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ , we have the following exact sequence:*

$$0 \rightarrow S^{-1}A \otimes_A M' \rightarrow S^{-1}A \otimes_A M \rightarrow S^{-1}A \otimes_A M'' \rightarrow 0.$$

Moreover, if  $M', M$ , and  $M''$  are finitely generated  $A$ -modules, then we have

$$(1.12) \quad \text{rank}_A(M) = \text{rank}_A(M') + \text{rank}_A(M'').$$

**2. Weakly doubly coprime factorizations.** Since the work of Rosenbrock [38] on coprime factorizations of rational matrices, this concept has played an increasing role in analysis and synthesis problems. This technique has been popularized by the book of Vidyasagar [49]. However, contrary to finite-dimensional systems, the transfer matrices of more general systems (delay systems, partial differential equations) do not generally admit doubly coprime factorizations. Intuitively, this comes from the fact that the algebraic properties of rings such as  $H_\infty(\mathbb{C}_+)$ ,  $\mathcal{A}$ ,  $\hat{\mathcal{A}}$ , and  $l_1(\mathbb{Z}_+)$  are more complex than the ones of  $RH_\infty$  or  $k[s]$  (with  $k$  a field), which are used for finite-dimensional systems. In the next section, we shall give a mathematical formulation of the term “more complex.” In order to achieve this goal, we shall need to introduce the concept of weakly doubly coprime factorizations of a transfer matrix.

**2.1. Weak primeness and torsion-free modules.** Let us introduce the concept of a weakly left-prime matrix.

DEFINITION 2.1. *Let  $A$  be an integral domain and  $K = Q(A)$  its field of fractions. The matrix  $R \in M_{q \times p}(A)$  is weakly left-prime if it satisfies*

$$K^q R \cap A^p = A^q R,$$

namely, if, for every  $\lambda \in A^p$  satisfying  $\lambda = \mu R$  for a certain  $\mu \in K^q$ , there exists  $\nu \in A^q$  such that  $\lambda = \nu R$ . The concept of weak right-primeness can be defined similarly. Let us notice that  $R$  is weakly right-prime iff  $R^T$  is weakly left-prime.

If  $R \in M_{q \times p}(A)$  has full row rank, namely, if the  $q$  rows of  $R$  are  $A$ -linearly independent, then  $R$  is weakly left-prime iff

$$\mu \in K^q, \quad \mu R \in A^p \Rightarrow \mu \in A^q.$$

*Example 2.1.* Let us consider the full row rank matrix  $R$  defined by (1.6). This matrix  $R$  is not weakly left-prime because  $(\frac{s+1}{s-1} : 0) \notin A^2$  and we have

$$\left(\frac{s+1}{s-1} : 0\right) \begin{pmatrix} \frac{s-1}{s+1} & 0 & -\frac{(s-1)e^{-s}}{(s+1)^2} & -\left(\frac{s-1}{s+1}\right)^2 \\ 0 & \frac{s-1}{s+1} & 0 & -\frac{1}{s+1} \end{pmatrix} = \left(1 : 0 : -\frac{e^{-s}}{s+1} : -\frac{s-1}{s+1}\right) \in A^4.$$

PROPOSITION 2.2 (see [43]). *If  $A$  is a greatest common divisor domain (GCDD), namely, every couple of elements of  $A$  has a greatest common divisor, then a full row*

rank matrix  $R \in M_{q \times p}(A)$  is weakly left-prime iff  $R$  is irreducible (or minor left-prime [27]), namely, 1 is the greatest common divisor of the  $q \times q$  minors of  $R$ .

Let us notice that if  $A$  is no longer a GCDD, then the previous result fails to be true [43]. In particular, it is not known whether or not  $\mathcal{A}$  or  $\hat{\mathcal{A}}$  are GCDD.

LEMMA 2.3. *Let  $A$  be an integral domain,  $K = Q(A)$  its field of fractions, and  $R \in M_{q \times p}(A)$ . Then, we have*

$$K^q R \cap A^p = \overline{A^q R},$$

where  $\overline{A^q R} = \{\lambda \in A^p \mid \exists 0 \neq a \in A : a\lambda \in A^q R\}$  is called the  $A$ -closure of the submodule  $A^q R$  in  $A^p$  (see [7]).

*Proof.* Let  $\lambda \in K^q R \cap A^p$ ; then  $\lambda \in A^p$ , and there exists  $\mu \in K^q$  such that  $\lambda = \mu R$ . Let us write  $\mu = d^{-1}\nu$  with  $\nu \in A^q$  and  $0 \neq d \in A$ . Then, we have  $d\lambda = \nu R$ , i.e.,  $\lambda \in \overline{A^q R}$ . Conversely, let  $\lambda \in \overline{A^q R}$ ; then  $\lambda \in A^p$ , and there exists  $0 \neq d \in A$  such that  $d\lambda \in A^q R$ . Thus, there exists  $\nu \in A^q$  such that  $d\lambda = \nu R$ ; i.e.,  $\lambda = (d^{-1}\nu)R \in K^q R$ , i.e.,  $\lambda \in K^q R \cap A^p$ .  $\square$

PROPOSITION 2.4. *Let  $A$  be an integral domain,  $K = Q(A)$  its field of fractions,  $R \in M_{q \times p}(A)$ , and  $M = A^p/A^q R$ . Then, we have*

$$\begin{cases} t(M) = (K^q R \cap A^p)/A^q R, \\ M/t(M) = A^p/(K^q R \cap A^p). \end{cases}$$

*Proof.* Let us note that we have  $A^q R \subseteq K^q R \cap A^p$ . Therefore, we have the following commutative exact diagram,

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A^q R & \longrightarrow & A^p & \longrightarrow & M & \longrightarrow 0 \\ & & \downarrow & & \parallel & & \downarrow & \\ 0 & \longrightarrow & K^q R \cap A^p & \longrightarrow & A^p & \longrightarrow & A^p/(K^q R \cap A^p) & \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow & \\ & & (K^q R \cap A^p)/A^q R & & 0 & & 0 & \\ & & \downarrow & & & & & \\ & & 0 & & & & & \end{array}$$

from which we deduce the following exact sequence (snake lemma [2, 39]):

$$(2.1) \quad 0 \longrightarrow (K^q R \cap A^p)/A^q R \longrightarrow M \longrightarrow A^p/(K^q R \cap A^p) \longrightarrow 0.$$

Using Lemma 2.3, we obtain

$$(K^q R \cap A^p)/A^q R = \overline{A^q R}/A^q R = \{m \in M \mid \exists 0 \neq a \in A : am = 0\} = t(M).$$

Then, we have  $A^p/(K^q R \cap A^p) = M/t(M)$  (see (1.9)), which proves the proposition.  $\square$

A direct consequence of Proposition 2.4 is the following corollary, which gives a module interpretation of the weak left-primeness.

COROLLARY 2.5. *Let  $A$  be an integral domain and  $K = Q(A)$  its field of fractions,  $R \in M_{q \times p}(A)$ , and the  $A$ -module  $M = A^p/A^q R$ . Then, we have the equivalences*

1.  $R$  is weakly left-prime, i.e.,  $\overline{A^q R} = K^q R \cap A^p = A^q R$ ;
2.  $M$  is a torsion-free  $A$ -module, i.e.,  $t(M) = 0$ .

*Example 2.2.* From Example 2.1, we know that the matrix  $R$  defined by (1.6) is not weakly left-prime. By Corollary 2.5, we deduce that the  $A$ -module  $M = A^4/A^2 R$  is not torsion-free. A torsion element is obtained by taking the class of the vector  $(1 : 0 : -\frac{e^{-s}}{s+1} : -\frac{s-1}{s+1})$  (see Example 2.1) in  $M$  to obtain  $z = y_1 - \frac{e^{-s}}{(s+1)} u_1 - \frac{(s-1)}{(s+1)} u_2$ . We recover the torsion element  $z$  obtained in Example 1.5. It satisfies  $\frac{(s-1)}{(s+1)} z = 0$ .

Dually, we can prove that  $\tilde{R} \in M_{p \times (p-q)}(A)$  is weakly right-prime iff the  $A$ -module  $A^p/A^{p-q} \tilde{R}^T$  is torsion-free.

**2.2. Transfer matrices.** The following lemma shows that if a transfer matrix  $P \in M_{q \times (p-q)}(K)$  is such that  $P = D^{-1} N$ , then the  $A$ -module  $\overline{A^q R}$  depends only on  $P$  and not on  $R = (D : -N) \in M_{q \times p}(A)$ .

LEMMA 2.6. *Let  $A$  be an integral domain,  $K = Q(A)$  its field of fractions, and  $P \in M_{q \times (p-q)}(K)$  a transfer matrix. If  $P$  can be written as  $P = D_1^{-1} N_1 = D_2^{-1} N_2$ , where  $R_1 = (D_1 : -N_1) \in M_{q \times p}(A)$  and  $R_2 = (D_2 : -N_2) \in M_{q \times p}(A)$ , then we have*

$$\begin{cases} A^q R_1 \subseteq \overline{A^q R_2}, \\ A^q R_2 \subseteq \overline{A^q R_1}, \end{cases} \Rightarrow \overline{A^q R_1} = \overline{A^q R_2},$$

and thus,  $\overline{A^q R_i}$  and  $M_i/t(M_i) = A^p/\overline{A^q R_i}$  depend only on  $P$  and not on  $R_i$ , where  $M_i = A^p/A^q R_i$ . In particular, if  $A^q R_1$  (resp.,  $A^q R_2$ ) is  $A$ -closed, then we have  $\overline{A^q R_2} = A^q R_1$  (resp.,  $\overline{A^q R_1} = A^q R_2$ ). The same result holds for  $P = \tilde{N}_i \tilde{D}_i^{-1}$ .

*Proof.* Clearing the denominators of  $P$ , we have  $P = d^{-1} H = H d^{-1}$ , where  $0 \neq d \in A$ , and  $H \in M_{q \times (p-q)}(A)$ . Let  $R = (d I_q : -H) \in M_{q \times p}(A)$ . Then, we have

$$\begin{cases} D_i H = d N_i, \\ (\det D_i) H = (D_i^c d) N_i, \end{cases} \Rightarrow \begin{cases} D_i R = d R_i, \\ (\det D_i) R = (D_i^c d) R_i, \end{cases} \quad i = 1, 2,$$

where  $D_i^c$  is the cofactors matrix of  $D_i$ , i.e., it satisfies  $D_i^c D_i = (\det D_i) I_q$ . Let  $\lambda \in \overline{A^q R_i}$ ; then there exists  $\mu \in A^q$  such that  $\lambda = \mu R_i$ , and thus,

$$d \lambda = \mu (d R_i) = \mu (D_i R) = (\mu D_i) R \Rightarrow \lambda \in \overline{A^q R} \Rightarrow A^q R_i \subseteq \overline{A^q R}.$$

Conversely, let  $\lambda \in A^q R$ ; then there exists  $\mu \in A^q$  such that  $\lambda = \mu R$ . Thus,

$$(\det D_i) \lambda = \mu ((\det D_i) R) = \mu (D_i^c d R_i) = (\mu D_i^c d) R_i \Rightarrow \lambda \in \overline{A^q R_i} \Rightarrow A^q R \subseteq \overline{A^q R_i}.$$

Using the fact that  $X \subseteq Y \Rightarrow \overline{X} \subseteq \overline{Y}$  for two submodules  $X$  and  $Y$  of a free  $A$ -module, we obtain

$$A^q R_i \subseteq \overline{A^q R} \subseteq \overline{A^q R_j} \Rightarrow \overline{A^q R_i} = \overline{A^q R_j}, \quad i, j = 1, 2.$$

Now, if  $A^q R_i$  is  $A$ -closed, then

$$A^q R_i \subseteq \overline{A^q R_j} \subseteq \overline{A^q R_i} = A^q R_i \Rightarrow \overline{A^q R_j} = A^q R_i. \quad \square$$

LEMMA 2.7. *If  $R \in M_{q \times p}(A)$  has full row rank and  $F$  is a free submodule of  $\ker .R^T$  of rank  $p - q$ , then  $\overline{F} = \ker .R^T$ , where  $\overline{F}$  is the  $A$ -closure of  $F$  in  $A^p$ .*

*Proof.* Let us note  $N \triangleq \text{coker} .R^T$ . We have the following exact sequence:

$$0 \longleftarrow N \longleftarrow A^q \xleftarrow{.R^T} A^p \longleftarrow \ker .R^T \longleftarrow 0.$$

The  $A$ -module  $N$  is defined by the  $A$ -linear combinations of the equations  $R^T z = 0$ , where  $z_i$  is the class of the  $i$ th vector of the canonical basis of  $A^q$  in  $N = A^q/A^p R^T$  (see (1.5)). Using the fact that  $R$  has full row rank, then there exist  $q$  equations of  $R^T z = 0$  which are  $A$ -linearly independent. If we denote by  $R_0^T \in M_q(A)$  the full rank matrix corresponding to these  $q$   $A$ -linearly independent equations, then we have  $R_0^T z = 0$ , and thus, by multiplying  $R_0^T$  by its cofactors matrix, we obtain  $(\det R_0^T) z = 0$  with  $0 \neq \det R_0^T \in A$ . This equation shows that  $N$  is a torsion  $A$ -module. Now, let us notice that we have  $K \otimes_A N = 0$  because  $N$  is a torsion  $A$ -module: for all  $n \in N$ , there exists  $0 \neq a \in A : an = 0$ , and thus,  $1 \otimes n = (a/a) \otimes n = (1/a) \otimes an = 0$ . Using the fact that  $K = Q(A)$  is a flat  $A$ -module (see Proposition 1.10), we have the following exact sequence:

$$0 = K \otimes_A N \longleftarrow K^q \xleftarrow{R^T} K^p \longleftarrow K \otimes_A \ker .R^T \longleftarrow 0.$$

Here  $K \otimes_A \ker .R^T$  is a subvector space of  $K^p$  of dimension  $p - q$ . As  $F$  is a free submodule of  $\ker .R^T$  of rank  $p - q$ , we have  $K \otimes_A F = K \otimes_A \ker .R^T \subset K^p$ , and thus,

$$\overline{F} = (K \otimes_A F) \cap A^p = (K \otimes_A \ker .R^T) \cap A^p = \overline{\ker .R^T} = \ker .R^T$$

because  $\ker .R^T$  is an  $A$ -closed submodule of  $A^p$ . Indeed, using the fact that  $A$  is an integral domain, we obtain

$$\begin{aligned} \lambda \in \overline{\ker .R^T} &\Rightarrow \exists 0 \neq a \in A : a\lambda \in \ker .R^T \Rightarrow a(\lambda R^T) = 0 \Rightarrow \lambda R^T = 0 \\ &\Rightarrow \lambda \in \ker .R^T. \quad \square \end{aligned}$$

**PROPOSITION 2.8.** *Let  $P \in M_{q \times (p-q)}(K)$  be such that  $P = D^{-1}N = \tilde{N}\tilde{D}^{-1}$ , where  $R = (D : -N) \in M_{q \times p}(A)$  and  $\tilde{R} = (\tilde{N}^T : \tilde{D}^T)^T \in M_{p \times (p-q)}(A)$ . If we define the  $A$ -modules  $M = A^p/A^q R$  and  $\tilde{M} = A^p/A^{p-q} \tilde{R}^T$ , then we have*

$$\begin{cases} \ker .R^T = \overline{A^{p-q} \tilde{R}^T}, & \begin{cases} \tilde{M}/t(\tilde{M}) \cong A^p R^T, \\ M/t(M) \cong A^p \tilde{R}. \end{cases} \\ \ker .\tilde{R} = \overline{A^q R}, \end{cases}$$

*Proof.* Using the fact that  $R\tilde{R} = 0$ , we obtain the following two complexes:

$$\begin{aligned} 0 \longrightarrow A^q \xrightarrow{.R} A^p \xrightarrow{.\tilde{R}} A^{p-q}, \\ A^q \xleftarrow{.R^T} A^p \xleftarrow{.\tilde{R}^T} A^{p-q} \longleftarrow 0. \end{aligned}$$

Thus,  $A^{p-q} \tilde{R}^T$  (resp.,  $A^q R$ ) is a free submodule of  $\ker .R^T$  (resp.,  $\ker .\tilde{R}$ ) of rank  $p - q$  (resp.,  $q$ ). By Lemma 2.7, Proposition 2.4, and Lemma 2.3, we obtain that

$$\begin{cases} \ker .R^T = \overline{A^{p-q} \tilde{R}^T}, \\ \ker .\tilde{R} = \overline{A^q R}, \end{cases} \Rightarrow \begin{cases} A^p R^T \cong A^p / \ker .R^T = A^p / \overline{A^{p-q} \tilde{R}^T} = \tilde{M}/t(\tilde{M}), \\ A^p \tilde{R} \cong A^p / \ker .\tilde{R} = A^p / \overline{A^q R} = M/t(M). \quad \square \end{cases}$$

Let us notice that Proposition 2.8 is close in its spirit to some results obtained in [26] for linear multidimensional systems in the behavioral approach.

From Proposition 2.8 and Lemma 2.6, we obtain that the  $A$ -modules  $A^p \tilde{R}$  and  $A^p R^T$  depend only, up to an isomorphism, on the transfer matrix  $P$ . This result was proved in [22] in a different way (without any references to torsion-free  $A$ -modules). Using the fact that the structural properties of  $P$  do not depend on the choice of the fractional representation of  $P$ , we obtain the following corollary.

**COROLLARY 2.9.** *Let  $P \in M_{q \times (p-q)}(K)$  be such that  $P = D^{-1}N = \tilde{N}\tilde{D}^{-1}$ , where  $R = (D : -N) \in M_{q \times p}(A)$  and  $\tilde{R} = (\tilde{N}^T : \tilde{D}^T)^T \in M_{p \times (p-q)}(A)$ . Then, the structural (intrinsic) properties of  $P$  depend only on the  $A$ -modules  $\overline{A^q R}$  and  $\overline{A^{p-q} \tilde{R}^T}$  or, up to an isomorphism, on the  $A$ -modules  $A^p \tilde{R}$  and  $A^p R^T$ .*

**2.3. Weakly doubly coprime factorizations.** Let us introduce the concepts of weakly left/right/doubly coprime factorizations.

**DEFINITION 2.10.** *Let  $A$  be an integral domain and  $K = Q(A)$ .*

- *A transfer matrix  $P \in M_{q \times (p-q)}(K)$  admits a weakly left-coprime factorization if there exists a weakly left-prime matrix  $R = (D : -N) \in M_{q \times p}(A)$ , with  $\det D \neq 0$ , such that  $P = D^{-1}N$ .*
- *A transfer matrix  $P \in M_{q \times (p-q)}(K)$  admits a weakly right-coprime factorization if there exists a weakly right-prime matrix  $\tilde{R} = (\tilde{N}^T : \tilde{D}^T)^T \in M_{p \times (p-q)}(A)$ , with  $\det \tilde{D} \neq 0$ , such that  $P = \tilde{N}\tilde{D}^{-1}$ .*
- *A transfer matrix  $P$  admits a weakly doubly coprime factorization if  $P$  admits weakly left- and right-coprime factorizations  $P = D^{-1}N = \tilde{N}\tilde{D}^{-1}$ .*

**THEOREM 2.11.** *Let  $A$  be an integral domain,  $K = Q(A)$  its quotient field,  $P = D^{-1}N = \tilde{N}\tilde{D}^{-1} \in M_{q \times (p-q)}(K)$  a transfer matrix,  $R = (D : -N) \in M_{q \times p}(A)$ , and  $\tilde{R} = (\tilde{N}^T : \tilde{D}^T)^T \in M_{p \times (p-q)}(A)$ . Then,  $P = D^{-1}N$  admits a weakly left-coprime factorization (resp., weakly right-coprime factorization) iff  $\overline{A^q R}$  (resp.,  $\overline{A^{p-q} \tilde{R}^T}$ ) is a free  $A$ -module of rank  $q$  (resp.,  $p - q$ ).*

*Proof.*  $\Rightarrow$  If  $P$  admits a weakly left-coprime factorization, then there exists a weakly left-prime matrix  $R' = (D' : -N') \in M_{q \times p}(A)$ , with  $\det D' \neq 0$ , such that we have  $P = D'^{-1}N'$ . Using Lemma 2.6, we deduce that  $\overline{A^q R} = A^q R'$ . Moreover,  $A^q R' \cong A^q$  because  $R'$  has full row rank, and thus,  $\overline{A^q R} \cong A^q$ .

$\Leftarrow$  If  $\overline{A^q R}$  is a finitely generated free  $A$ -module of rank  $q$ , then, choosing a basis for  $\overline{A^q R}$ , we obtain a full row rank matrix  $R' \in M_{q \times p}(A)$  such that  $\overline{A^q R} = A^q R'$ , and thus,  $R'$  is weakly left-prime. If  $R_i$  is the  $i$ th row of  $R$ , then  $R_i \in \overline{A^q R} = A^q R'$  because  $R_i \in A^q R \subseteq \overline{A^q R}$ . Therefore, there exists  $R'_i \in A^q$  such that  $R_i = R'_i R'$ , and then, there exists  $R'' \in M_q(A)$  such that  $R = R'' R'$ . Using the fact that  $R$  has full row rank, we deduce that  $R''$  also has full row rank. Finally, let  $R' = (D' : N')$ , where  $R' \in M_{q \times p}(A)$ ; then we have  $D = R'' D'$  and  $N = R'' N'$ , and thus,  $\det D' \neq 0$ . This proves the result because we have  $P = D^{-1}N = (R'' D')^{-1}(R'' N') = D'^{-1}N'$ .

The result for weak right-coprime factorizations can be proved similarly.  $\square$

**COROLLARY 2.12.** *A transfer matrix  $P = D^{-1}N = \tilde{N}\tilde{D}^{-1}$  admits a weakly doubly coprime factorization iff the  $A$ -modules  $\overline{A^q R}$  and  $\overline{A^{p-q} \tilde{R}^T}$  are two free  $A$ -modules of rank, respectively,  $q$  and  $p - q$ .*

Let us notice that, from Corollary 2.9, Corollary 2.5, and the fact that a coprime factorization is an intrinsic property of the transfer matrix, we deduce that weakly left/right/coprime factorizations are the weakest possible coprime factorizations.

**3. Coherent rings and modules.**

**3.1. Introduction.** Any mathematical model of a plant is only an approximation of the real system. Thus, the algebra of SISO stable plants needs to be endowed with a norm in order to take into account some errors in the modelization. For technical reasons, we usually prefer to ask this normed algebra to be complete. Therefore, we generally require an algebra of SISO stable systems to be a *Banach algebra* [16, 49] (e.g.,  $H_\infty(\mathbb{C}_+)$ ,  $\mathcal{A}$ ,  $\hat{\mathcal{A}}$ ,  $l_1(\mathbb{Z}_+)$  [3, 8, 9, 48, 49]).

However, it is known that all *noetherian  $k$ -Banach algebras* ( $k = \mathbb{R}, \mathbb{C}$ )—namely, Banach algebras such that every ideal is finitely generated—are  $k$ -finite-dimensional [41]. Hence, for instance,  $H_\infty(\mathbb{C}_+)$ ,  $\hat{\mathcal{A}}$ ,  $L_1(\mathbb{R}_+) + \mathbb{R}\delta$ ,  $l_1(\mathbb{Z}_+)$  are not noetherian rings, and thus, an ideal  $I$  of these algebras  $A$  generally does not have the form  $I = \sum_{i=1}^n A a_i$  for a finite set  $\{a_1, \dots, a_n\}$  of elements of  $A$ . A direct consequence is that most of the algebraic objects (kernel, image, quotient, sum, intersection, etc.) are generally not finitely generated. Hence, we cannot study the algebraic properties of systems, defined by matrices whose entries belong to Banach algebras, by means of the concepts and techniques developed for noetherian rings (i.e., the main part of commutative algebra).

The concept of a *coherent ring* was first introduced in 1960 by Chase [5], and the definition of a *coherent module* appeared in [1] in 1964 (see [17] for more details). Coherent rings form a general class of rings including noetherian rings, Boolean rings, Bézout domains, semihereditary rings, etc. [17, 39]. This concept is closely related to the one of a *coherent sheaf* introduced by Cartan [4] and Serre [42] in the study of analytic and algebraic geometries.

In this section, we show that one possible way to cope with the fact that most of the integral domains of SISO stable plants are not noetherian is to require that these domains be coherent rings. In particular, for coherent rings, we give algorithms which compute the  $A$ -closure  $\overline{A^q R}$  of an  $A$ -module of the form  $A^q R$  (see Theorem 2.11) and which check whether or not a finitely generated  $A$ -module is torsion-free, reflexive, or projective. Finally, we shall characterize explicitly the class of integral domains  $A$  such that every transfer matrix, with entries in  $K = Q(A)$ , admits a weakly doubly coprime factorization.

**3.2. Definitions and results.**

DEFINITION 3.1 (see [2, 15, 39]). *We have the following definitions:*

- An  $A$ -module  $M$  is coherent if  $M$  is a finitely generated  $A$ -module and every finitely generated submodule of  $M$  is finitely presented.
- A ring  $A$  is coherent if it is coherent as an  $A$ -module.

Hence,  $A$  is a coherent ring iff every finitely generated ideal  $I = \sum_{i=1}^n A a_i$  of  $A$  is finitely presented, i.e., the *module of relations* of  $I$  (or *syzygy* of  $I$ ), defined by

$$(3.1) \quad S(I) = \left\{ r = (r_1 : \dots : r_n) \in A^n \mid \sum_{i=1}^n r_i a_i = 0 \right\},$$

is finitely generated. In terms of equations,  $A$  is a coherent ring iff for every  $n \in \mathbb{Z}_+$  and  $a = (a_1 : \dots : a_n)^T \in A^n$  there exist  $m \in \mathbb{Z}_+$  and  $R \in M_{m \times n}(A)$  such that

$$\forall r = (r_1 : \dots : r_n) \in A^n : r a = 0 \Leftrightarrow \exists b = (b_1 : \dots : b_m) \in A^m : r = b R.$$

*Example 3.1.* Any *noetherian ring*, namely a ring where any ideal  $I$  is finitely generated, i.e., has the form  $I = \sum_{i=1}^n A a_i$  for a finite number of  $a_i \in A$ , is coherent [1, 39]. In particular,  $RH_\infty$  and  $k[s]$ , with  $k$  a field, are coherent domains. An example of a coherent ring which is not noetherian is given by the ring  $k[\chi_i, i \in \mathbb{N}]$  of polynomials in infinitely many variables  $\chi_i$  with coefficients in a field  $k$  (see [39]).

We give a few definitions which are related to the extension of (1.4) on the left.

DEFINITION 3.2. *We have the following definitions (see [1, 14, 24, 39]):*

- A projective (resp., free, flat) resolution of an  $A$ -module  $M$  is an exact sequence of the form

$$(3.2) \quad \dots \xrightarrow{d_3} F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \longrightarrow M \longrightarrow 0,$$

where  $F_i$  is a projective (resp., free, flat)  $A$ -module and  $d_i$  is an  $A$ -morphism.

- A finite free resolution of an  $A$ -module  $M$  is an exact sequence of the form (3.2), where  $F_i$  is a finite free  $A$ -module, i.e.,  $F_i \cong A^{r_i}$ ,  $r_i \in \mathbb{Z}_+$ , for  $i \geq 0$ .
- The projective (resp., flat) dimension  $\text{pd}_A(M)$  (resp.,  $\text{w.dim}_A(M)$ ) of an  $A$ -module  $M$  is the minimum  $n \in \mathbb{Z}_+ \cup \{+\infty\}$  such that there exists a projective (resp., flat) resolution of  $M$  of length  $n$ , i.e., of the form

$$0 \longrightarrow F_n \xrightarrow{d_n} F_{n-1} \xrightarrow{d_{n-1}} \dots \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \longrightarrow M \longrightarrow 0.$$

- The global dimension and weak global dimension of  $A$  are defined by

$$\begin{aligned} \text{gl.dim}(A) &= \sup \{ \text{pd}_A(M) \mid A\text{-module } M \} \in \mathbb{Z}_+ \cup \{+\infty\}, \\ \text{w.gl.dim}(A) &= \sup \{ \text{w.dim}_A(M) \mid A\text{-module } M \} \in \mathbb{Z}_+ \cup \{+\infty\}. \end{aligned}$$

For a general ring  $A$ , we have the inequality  $\text{w.gl.dim}(A) \leq \text{gl.dim}(A)$ . If  $A$  is a noetherian ring, then the equality holds [2, 39].

*Remark 3.1.* Using the canonical basis of the free  $A$ -module  $F_i \cong A^{r_i}$ , every finite free resolution of an  $A$ -module  $M$  has the form

$$(3.3) \quad \dots \xrightarrow{.R_2} A^{r_1} \xrightarrow{.R_1} A^{r_0} \longrightarrow M \longrightarrow 0,$$

where  $R_i$  is an  $(r_i \times r_{i-1})$ -matrix whose entries belong to  $A$ , and  $.R_i : A^{r_i} \rightarrow A^{r_{i-1}}$  is defined by letting operate a row vector of length  $r_i$  on the left of  $R_i$  to obtain a row vector of length  $r_{i-1}$ . Moreover,  $M$  is defined by the system  $R_1 z = 0$ , where  $z_i$  is the class of  $e_i$  in  $M$  and  $\{e_1, \dots, e_{r_0}\}$  is the canonical basis of  $A^{r_0}$  (see (1.5)).

DEFINITION 3.3. We have the following definitions (see [1, 39]):

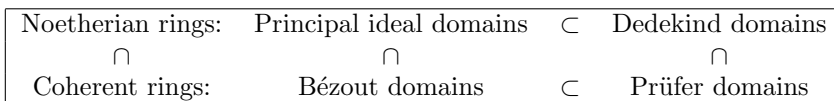
- A ring  $A$  is semihhereditary if every finitely generated ideal of  $A$  is a projective  $A$ -module.
- A semihhereditary integral domain is called a Prüfer domain.
- A ring  $A$  is a Bézout domain if every finitely generated ideal of  $A$  is a principal ideal, i.e., generated by a single element of  $A$ .
- A ring  $A$  is hereditary if every ideal of  $A$  is a projective  $A$ -module.
- A hereditary integral domain is called a Dedekind domain.
- A ring  $A$  is a principal ideal domain if every ideal of  $A$  is generated by a single element of  $A$ .

We shall give in [33] some examples of Prüfer and Dedekind domains, as they will play an important role in internal stabilizability. Coherent rings with small weak global dimensions have been studied and classified largely in algebra [17, 39, 47].

THEOREM 3.4 (see [2, 39]). We have the following results:

1. Semihhereditary rings and Prüfer domains are coherent rings.
2. Hereditary rings and Dedekind domains are noetherian and, thus, coherent rings.
3. If  $A$  is an integral domain, then  $\text{w.gl.dim}(A) \leq 1 \Leftrightarrow A$  is a Prüfer domain.
4. If  $A$  is an integral domain, then  $\text{gl.dim}(A) \leq 1 \Leftrightarrow A$  is a Dedekind domain.

We have the following inclusions of rings:



*Example 3.2.* The integral domains  $E(k)$ ,  $k = \mathbb{R}, \mathbb{C}$ , and  $\mathcal{E} = E(\mathbb{R}) \cap \mathbb{R}(s)[e^{-s}]$ , defined in Theorem 1.6, are Bézout domains and, thus, two coherent rings.

PROPOSITION 3.5 (see [15, 47]). *An integral domain with  $\text{gl.dim}(A) \leq 2$  is coherent.*

General rings with  $\text{w.gl.dim}(A) = 2$  are less understood [14, 17, 24, 47].

PROPOSITION 3.6 (see [1, 39]). *If  $A$  is a coherent ring, then an  $A$ -module  $M$  is coherent iff  $M$  is a finitely presented  $A$ -module.*

DEFINITION 3.7. *We call an  $A$ -system a system of the form  $Rz = 0$ , where  $z$  is a set of formal variables and  $R$  is a finite matrix whose entries belong to  $A$ .*

From Proposition 3.6, we have the following corollary.

COROLLARY 3.8. *If  $A$  is a coherent ring, then there is a one-to-one correspondence between coherent  $A$ -modules and  $A$ -systems.*

*Proof.*  $\Rightarrow$  Let  $\sum_{j=1}^p R_{ij} z_j = 0, R_{ij} \in A, i = 1, \dots, q$ , be an  $A$ -system and  $R = (R_{ij}) \in M_{q \times p}(A)$ . Let us define the following  $A$ -morphism:

$$\begin{aligned} .R = A^q &\longrightarrow A^p, \\ (a_1 : \dots : a_q) &\longrightarrow (a_1 : \dots : a_q) R. \end{aligned}$$

If we note  $M = \text{coker } .R = A^p/A^q R$ , then we have the following exact sequence,

$$(3.4) \quad A^q \xrightarrow{.R} A^p \xrightarrow{\pi} M \longrightarrow 0,$$

and, by Proposition 3.6,  $M$  is a coherent  $A$ -module because  $A$  is a coherent ring.

$\Leftarrow$  Let  $M$  be a coherent  $A$ -module. Using the fact that  $A$  is a coherent ring, by Proposition 3.6,  $M$  is a finitely presented  $A$ -module, there exists an exact sequence of the form (3.4), and, thus,  $M$  is defined by means of a system of equations of the form  $Rz = 0$ .  $\square$

**3.3. Elementary algebraic operations.** The next proposition shows that the class (category) of finitely presented  $A$ -modules over a coherent ring  $A$ , i.e., coherent modules, is invariant under elementary algebraic operations. First, let us notice that any finitely generated submodule of a coherent module is also coherent.

PROPOSITION 3.9 (see [1, 39]). *If two terms in the exact sequence*

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

*are coherent  $A$ -modules, so is the third one.*

COROLLARY 3.10 (see [1, 39]). *Let  $M, N, M' \subset M, M'' \subset M$  be coherent  $A$ -modules,  $\phi : M \longrightarrow N$  an  $A$ -morphism,  $I$  a coherent ideal, and  $S$  a multiplicative set of  $A$ . Then, we have the following:*

1.  $M/M', M \oplus N, M' \cap M'', M' + M''$  are coherent  $A$ -modules.
2.  $\ker \phi, \text{im } \phi, \text{coker } \phi$ , and  $\text{coim } \phi$  are coherent  $A$ -modules.
3.  $M \otimes_A N$  and  $\text{hom}_A(M, N)$  are coherent  $A$ -modules.
4.  $S^{-1} A$  is a coherent  $A$ -module.
5.  $IM = \{\sum_{i=1}^n a_i m_i \mid a_i \in I, m_i \in I\}$  is a coherent  $A$ -module.
6.  $\text{ann}(M) = \{a \in A \mid aM = 0\}$  is a coherent ideal of  $A$ .

COROLLARY 3.11. *Let  $A$  be a coherent ring and  $M$  a finitely presented  $A$ -module. Then there exists a finite free resolution of  $M$  of the form (3.3).*

*Proof.* Using Proposition 3.9, we prove by induction that every finite power  $A^{r_i}$  of  $A$  is a coherent  $A$ -module (take  $M = A^n, M' = A^{n-1}, M'' = A$ ). The kernel of a homomorphism  $d_i$  between two coherent  $A$ -modules is a coherent  $A$ -module and, by Proposition 3.6, is a finitely presented  $A$ -module. Then, the module of relations of  $R_i$  is finitely presented, and thus,  $M$  has a finite free resolution.  $\square$

DEFINITION 3.12. *Let  $M$  be an  $A$ -module with a projective resolution of the form (3.2) and  $N$  another  $A$ -module. Then we have the following definitions:*



- The defects of exactness of

$$(3.5) \quad \dots \xleftarrow{d_3^*} \text{hom}_A(F_2, N) \xleftarrow{d_2^*} \text{hom}_A(F_1, N) \xleftarrow{d_1^*} \text{hom}_A(F_0, N) \longleftarrow 0,$$

where  $d_i^*$  is defined by  $d_i^*(f) = f \circ d_i$  for all  $f \in \text{hom}_A(F_{i-1}, N)$ , depend only on  $M$  and  $N$  and not on (3.2) and are called  $\text{ext}_A^i(M, N)$  (see [2, 29, 39]). In particular, we have

$$\begin{cases} \text{ext}_A^0(M, N) = \ker d_1^* = \text{hom}_A(M, N), \\ \text{ext}_A^i(M, N) = \ker d_{i+1}^* / \text{im } d_i^*, \quad i \geq 1. \end{cases}$$

- The defects of exactness of

$$(3.6) \quad \dots \xrightarrow{\text{id}_N \otimes d_3} N \otimes_A F_2 \xrightarrow{\text{id}_N \otimes d_2} N \otimes_A F_1 \xrightarrow{\text{id}_N \otimes d_1} N \otimes_A F_0 \longrightarrow 0,$$

where  $\text{id}_N \otimes d_i$  is defined by  $(\text{id}_N \otimes d_i)(n \otimes m) = n \otimes d_i(m)$  for all  $n \in N$ , for all  $m \in F_i$ , depend only on  $M$  and  $N$  and not on (3.2) and are called  $\text{tor}_i^A(M, N)$  (see [2, 29, 39]). In particular, we have

$$\begin{cases} \text{tor}_0^A(M, N) = \text{coker}(\text{id}_N \otimes d_1) = N \otimes_A M, \\ \text{tor}_i^A(M, N) = \ker(\text{id}_N \otimes d_i) / \text{im}(\text{id}_N \otimes d_{i+1}), \quad i \geq 1. \end{cases}$$

*Remark 3.2.* If  $M$  has a finite free resolution of the form (3.3), then (3.5) is defined by  $\dots \xleftarrow{R_3} N^{r_2} \xleftarrow{R_2} N^{r_1} \xleftarrow{R_1} N^{r_0} \longleftarrow 0$ , where  $R_i : N^{r_{i-1}} \rightarrow N^{r_i}$  is defined by letting operate a column vector of length  $r_{i-1}$ , whose entries belong to  $N$  on the right of  $R_i$ , to obtain a column vector of length  $r_i$ , whose entries belong to  $N$ . We have

$$\text{ext}_A^i(M, N) = \ker_N(R_{i+1}) / \text{im}_N(R_i) \quad \forall i \geq 1.$$

Similarly, (3.6) becomes the complex  $\dots \xrightarrow{R_3} N^{r_2} \xrightarrow{R_2} N^{r_1} \xrightarrow{R_1} N^{r_0} \longrightarrow 0$ , where  $R_i : N^{r_i} \rightarrow N^{r_{i-1}}$  is defined by letting operate a row vector of length  $r_i$ , whose entries belong to  $N$  on the right of  $R_i$ , to obtain a row vector of length  $r_{i-1}$ , whose entries belong to  $N$  and

$$\text{tor}_i^A(M, N) = \ker_N .R_i / \text{im}_N .R_{i+1} \quad \forall i \geq 1.$$

**PROPOSITION 3.13** (see [2, 39]). *We have the following results:*

- $\text{ext}_A^i(M, N) = 0 \forall i \geq 1, \forall N$   $A$ -module  $\Leftrightarrow M$  is a projective  $A$ -module.
- $\text{tor}_i^A(M, N) = 0 \forall i \geq 1, \forall N$   $A$ -module  $\Leftrightarrow M$  is a flat  $A$ -module.

**COROLLARY 3.14.** *If  $A$  is a coherent ring, and  $M$  and  $N$  are two coherent  $A$ -modules, then  $\text{ext}_A^i(M, N)$  and  $\text{tor}_i^A(M, N)$  are coherent  $A$ -modules for  $i \geq 0$ . Moreover,  $\text{ext}_A^i(M, A)$  is a torsion  $A$ -module for  $i \geq 1$ .*

*Proof.* Using the fact that  $\text{ext}_A^i(M, N)$  (resp.,  $\text{tor}_i^A(M, N)$ ) does not depend on the projective resolution of  $M$ , by Proposition 3.6 and Corollary 3.11, we choose a finite free resolution (3.3) for  $M$ . By Proposition 3.9,  $\text{hom}_A(F_i, N)$  (resp.,  $N \otimes_A F_i$ ) is a coherent  $A$ -module, and thus,  $\ker d_i^*$  and  $\text{im } d_i^*$  (resp.,  $\ker(\text{id}_N \otimes d_i)$  and  $\text{im}(\text{id}_N \otimes d_i)$ ) are coherent  $A$ -modules. Finally,  $\text{ext}_A^i(M, N)$  (resp.,  $\text{tor}_i^A(M, N)$ ) is also a coherent  $A$ -module for  $i \geq 0$  as a quotient of two coherent  $A$ -modules. The proof of the fact that  $\text{ext}_A^i(M, A)$  is a torsion  $A$ -module is the same as that of Lemma 1 in [28].  $\square$

DEFINITION 3.15. *Let  $M$  be an  $A$ -module defined by a finite presentation:*

$$F_1 \xrightarrow{d_1} F_0 \longrightarrow M \longrightarrow 0.$$

*We call the transposed module<sup>1</sup> of  $M$ , the  $A$ -module  $T(M) = \text{coker } d_1^*$  defined by*

$$0 \longleftarrow T(M) \longleftarrow F_1^* \xleftarrow{d_1^*} F_0^*.$$

Hence, if  $M = A^p/A^q R$ , then the transposed module is  $T(M) = A^q/R A^p$ , where the vectors of  $A^q$  and  $A^p$  are now column ones (duality). Using the fact that  $A$  is commutative, we finally have  $T(M) = A^q/A^p R^T$ , where we use only row vectors.

If  $A$  is a coherent ring and  $M$  a coherent  $A$ -module, i.e., finitely presented  $A$ -module, then  $T(M)$  is also a coherent  $A$ -module because it is finitely presented.

*Remark 3.3.* We commit a little abuse of notation in denoting the transposed  $A$ -module of  $M$  by  $T(M)$ :  $\text{coker } d_1^*$  depends on the particular choice of  $d_1$ , i.e., on the particular form of the system of equations chosen to represent the module. However, we have (see [29]):

1. If  $R$  has full row rank, then  $T(M)$  depends only on  $M$  and not on  $R$ .
2. If  $R$  does not have full rank, i.e.,  $\ker .R \neq 0$ , then  $\text{coker } d_1^*$  depends only on  $M$  up to a projective equivalence [39], a fact which shows that  $\text{ext}_A^i(T(M), N)$  depends only on  $M$  and  $N$  for  $i \geq 1$ .

The next theorem shows how to characterize the module properties in terms of the extension and torsion functors.

THEOREM 3.16. *Let  $A$  be a coherent ring with  $\text{w.gl.dim}(A) \leq n$ ,  $M$  a finitely presented  $A$ -module, and  $T(M)$  its transposed  $A$ -module. Then, we have*

1.  $t(M) \cong \text{ext}_A^1(T(M), A)$ ,
2.  $t(M) \cong \text{tor}_1^A(K/A, M)$ ,
3.  $M$  is torsion-free iff  $\text{ext}_A^1(T(M), A) = 0$ ,
4.  $M$  is reflexive iff  $\text{ext}_A^i(T(M), A) = 0$ ,  $i = 1, 2$ ,
5.  $M$  is projective iff  $\text{ext}_A^i(T(M), A) = 0$ ,  $i = 1, \dots, n$ .

*Proof.* The proofs of 1, 3, 4, 5 are the same as those given in [27, 28] for noetherian rings: we just need to change finitely generated modules (resp., noetherian rings) into finitely presented (resp., coherent) ones. See also the proof of Proposition 3.4 of [33]. For a proof of 2, see [39].  $\square$

Using Proposition 2.4, Lemma 2.3, and Theorem 3.16, we obtain an algorithm which computes the closure  $\overline{A^q R}$  of an  $A$ -module of the form  $A^q R$ .

ALGORITHM. Input: A coherent ring  $A$  and  $R \in M_{q \times p}(A)$ . Output:  $R' \in M_{r \times p}(A)$  such that  $\overline{A^q R} = A^r R'$ .

1. Start with  $R \in M_{q \times p}(A)$ .
2. Transpose  $R$  to obtain  $R^T \in M_{p \times q}(A)$ .
3. Find a family of generators of  $\ker .R^T = \{\lambda \in A^p \mid \lambda R^T = 0\}$ . If  $\{\lambda_1, \dots, \lambda_m\}$  is a family of generators of  $\ker .R^T$ , then denote by  $R_{-1}^T \in M_{m \times p}(A)$  the matrix whose  $i$ th row is  $\lambda_i$ .
4. Transpose  $R_{-1}^T$  to obtain  $R_{-1} \in M_{p \times m}(A)$ .
5. Find a family of generators of  $\ker .R_{-1} = \{\eta \in A^p \mid \eta R_{-1} = 0\}$ . If  $\{\eta_1, \dots, \eta_r\}$  is a family of generators of  $\ker .R_{-1}$ , then denote by  $R' \in M_{r \times p}(A)$  the matrix whose  $i$ th row is  $\eta_i$ . We have  $\overline{A^q R} = A^r R'$ ,  $A^p/\overline{A^q R} \cong A^p R_{-1}$ .

<sup>1</sup>Do not confuse the notation of the transposed module  $T(M)$  of an  $A$ -module  $M$  with the torsion submodule  $t(M)$  of  $M$ .

If  $R'$  has full row rank, then  $\overline{A^q R} = A^r R'$  is a free  $A$ -module. (See Example 3.4 for the explicit computations of the  $A$ -closure  $\overline{A^q R}$  of a certain  $A$ -module  $A^q R$ .) To finish, let us note that we can use the previous algorithm to check whether a transfer matrix admits a weakly left/right/doubly coprime factorization (see Theorem 2.11). Indeed, the previous algorithm allows us to have a precise description of the  $A$ -closure of an  $A$ -module of the form  $A^q R$ . However, checking whether such an  $A$ -closure is free can be a very difficult algebraic problem (see, e.g., the proof of the Quillen–Suslin theorem in [39]).

### 3.4. Coherent Sylvester domains.

DEFINITION 3.17 (see [6, 10]). *A projective-free coherent domain with*

$$\text{w.gl.dim}(A) \leq 2$$

*is called a coherent Sylvester domain.*

*Example 3.3.* A Bézout domain (e.g.,  $E(k)$ ,  $\mathcal{E}$  by Theorem 1.6) and thus, a principal ideal domain (e.g.,  $RH_\infty$  by Theorem 1.6,  $k[s]$ , with  $k$  a field) are coherent Sylvester domains. More generally,  $A = B[x]$  is a coherent Sylvester domain iff  $B$  is a Bézout domain [11] (e.g.,  $A = \mathbb{Z}[x]$ ,  $A = k[s][z] = k[s, z]$ , with  $k$  a field, or  $A = B[x]$ , where  $B$  is the ring of all algebraic integers, i.e., the integral closure of  $\mathbb{Z}$  in  $\mathbb{C}$ ; see [39]).

DEFINITION 3.18. *A ring  $A$  is regular if every finitely generated ideal of  $A$  has a finite projective dimension.*

THEOREM 3.19 (see [50]). *A coherent regular domain  $A$  is a GCDD—every  $a$  and  $b$  of  $A$  have a greatest common divisor  $[a, b]$ —iff every finitely generated projective ideal of  $A$  is principal.*

COROLLARY 3.20<sup>2</sup> *A coherent Sylvester domain is a GCDD.*

*Proof.* A coherent Sylvester domain is a projective-free coherent domain with  $\text{w.gl.dim}(A) \leq 2$  and, thus, a regular ring which satisfies that every finitely generated projective ideal is free, i.e., is principal, because  $A$  is an integral domain. Then, the result follows directly from Theorem 3.19.  $\square$

PROPOSITION 3.21. *If  $A$  is a coherent Sylvester domain, then, for every  $A$ -module  $M$  defined by a finite free resolution,*

$$F_1 \xrightarrow{d_1} F_0 \longrightarrow M \longrightarrow 0,$$

*there exist a free  $A$ -module  $F'_1$  and two  $A$ -morphisms  $d'_1 : F'_1 \rightarrow F_0$  and  $d''_1 : F_1 \rightarrow F'_1$  such that  $d_1 = d'_1 \circ d''_1$  and we have the following exact sequences:*

$$(3.7) \quad 0 \longrightarrow F'_1 \xrightarrow{d'_1} F_0 \longrightarrow M/t(M) \longrightarrow 0,$$

$$(3.8) \quad 0 \longrightarrow \ker d_1 \longrightarrow F_1 \xrightarrow{d''_1} F'_1 \longrightarrow t(M) \longrightarrow 0.$$

<sup>2</sup>We would like to thank Prof. W. Dicks for pointing out to us that this result is already contained in Lemma 4.1 of [11].

*Proof.* We have the following commutative exact diagram:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 & & \ker d_1'' & & 0 & & t(M) \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & \ker d_1 & \longrightarrow & F_1 & \xrightarrow{d_1} & F_0 & \xrightarrow{\pi} & M & \longrightarrow & 0 \\
 & & \downarrow d_1'' & & \parallel & & \parallel & & \downarrow \pi' & & \\
 & & \ker \phi & \longrightarrow & F_0 & \xrightarrow{\phi} & M/t(M) & \longrightarrow & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \\
 & & \text{coker } d_1'' & & 0 & & 0 & & & & \\
 & & \downarrow & & & & & & & & \\
 & & 0 & & & & & & & & 
 \end{array}$$

where  $\phi = \pi' \circ \pi$  and  $d_1'' : F_1 \rightarrow \ker \phi$  is induced by the identity homomorphism from  $F_0$  to  $F_0$  and  $\pi' : M \rightarrow M/t(M)$ . An easy chase in the diagram shows that  $\ker d_1'' \cong \ker d_1$  and  $\text{coker } d_1'' \cong t(M)$ . Now, let us prove that  $\ker \phi$  is a finite free  $A$ -module.  $M/t(M)$  is a coherent  $A$ -module over a coherent ring  $A$  [15] and, in particular,  $M/t(M)$  is a finitely generated  $A$ -module. It is well known that  $M/t(M)$  can be imbedded into a finitely generated free  $A$ -module  $F_{-1}$  (see, e.g., [39] or [15]), and we have the following exact sequence:

$$0 \longrightarrow M/t(M) \longrightarrow F_{-1} \longrightarrow F_{-1}/(M/t(M)) \longrightarrow 0.$$

Hence, we have the following exact sequence:

$$0 \longrightarrow \ker \phi \longrightarrow F_0 \xrightarrow{\phi} F_{-1} \longrightarrow F_{-1}/(M/t(M)) \longrightarrow 0.$$

Using the fact that  $\text{w.gl.dim}(A) = 2$ , we then have  $\text{pd}_A(F_{-1}/(M/t(M))) \leq 2$ , and thus,  $\text{pd}_A(\ker \phi) = 0$  [2, 39], i.e.,  $\ker \phi$  is a projective  $A$ -module, and thus, a free  $A$ -module because  $A$  is a projective-free ring.  $\ker \phi$  is a finitely generated  $A$ -module because  $\ker \phi$  is a coherent  $A$ -module. ( $\ker \phi$  is the kernel of an  $A$ -morphism between two finite free  $A$ -modules.) Thus,  $\ker \phi \cong F_1^r \cong A^r$ ,  $r \in \mathbb{Z}_+$ , which gives (3.7) and (3.8).  $\square$

In the next corollary, we reformulate Proposition 3.21 in terms of weakly doubly coprime factorizations. It generalizes to coherent Sylvester domains a result obtained by Smith in [43] for  $H_\infty(\mathbb{C}_+)$  (see section 3.5).

**COROLLARY 3.22.** *If  $A$  is a coherent Sylvester domain, then every matrix  $P$ , whose entries belong to  $K = Q(A)$ , admits a weakly doubly coprime factorization.*

*Proof.* If  $P \in M_{q \times (p-q)}(K)$ , then we can define  $R = (dI_q : -H) \in M_{q \times p}(A)$ , where  $d$  is the product of the denominators of the entries of  $P$  and  $H = dP \in M_{q \times (p-q)}(A)$ . By Proposition 3.21, there exist a full rank matrix  $R'' \in M_q(A)$  and a weakly left-prime full row rank matrix  $R' = (D : -N) \in M_{q \times p}(A)$  such that  $R = R'' R'$ . Thus, we have  $(dI_q : -H) = R''(D : -N)$  and  $\det R'' \neq 0$ . Then, we have

$$\begin{cases} dI_q = R'' D & \Rightarrow \det D \neq 0, \\ H = R'' N, \end{cases} \Rightarrow P = (dI_q)^{-1} H = (R'' D)^{-1} (R'' N) = D^{-1} N.$$

Dually, we have  $P = G(dI_{p-q})^{-1}$ , and thus, there exists a weakly right-prime matrix  $\tilde{R}' = (\tilde{N}^T : \tilde{D}^T)^T$  such that  $\tilde{R} = (G^T : (dI_{p-q})^T)^T = (\tilde{N}^T : \tilde{D}^T)^T \tilde{R}'$ .

Therefore,  $P = G(dI_{p-q})^{-1} = (\tilde{N} \tilde{R}'')^{-1} (\tilde{D} \tilde{R}'')^{-1} = \tilde{N} \tilde{D}^{-1}$  is a weakly right-coprime factorization.  $\square$

To prove the next theorem, we shall need the next proposition due to Dicks.

**PROPOSITION 3.23** (based on [12]). *Let  $A$  be an integral domain. If, for every finitely generated free  $A$ -module  $F_0$  and every finitely generated free  $A$ -submodule  $F_1$  of  $F_0$ , the  $A$ -closure of  $F_1$  in  $F_0$  is a finitely generated free  $A$ -module, then  $A$  is a coherent Sylvester domain.*

*Proof.* Let  $K = Q(A)$ ,  $p, q \in \mathbb{Z}_+$ ,  $F_0 = A^p$ , and  $R$  be any matrix belonging to  $M_{p \times q}(A)$ . We have the exact sequence  $A^q \xleftarrow{\cdot R} A^p \longleftarrow \ker \cdot R \longleftarrow 0$ . Applying the tensor product  $K \otimes_A$  to the previous exact sequence, we obtain the exact sequence ( $K$  is a flat  $A$ -module)  $K^q \xleftarrow{\cdot R} K^p \longleftarrow K \otimes_A \ker \cdot R \longleftarrow 0$ . Therefore,  $K \otimes_A \ker \cdot R$  is a  $K$ -subvector space of  $K^p$ , and thus, there exists a finite basis  $\{e_1, \dots, e_m\}$  of  $K \otimes_A \ker \cdot R$ , where  $m = \dim_K(K \otimes_A \ker \cdot R) \leq p$ . Let us note  $e_i = f_i/d_i$ , with  $f_i \in A^p$  and  $0 \neq d_i \in A$ , and let  $F_1$  be the  $A$ -submodule of  $K^p$  generated by  $\{f_1, \dots, f_m\}$ . Then,  $F_1$  is a free  $A$ -submodule of  $\ker \cdot R$ . Thus,  $\overline{F_1} \subseteq \overline{\ker \cdot R} = \ker \cdot R$ , because  $\ker \cdot R$  is an  $A$ -closed submodule of  $A^q$ . Moreover, for every  $\lambda \in \ker \cdot R$ , we have  $\lambda = \sum_{i=1}^m a_i e_i$ , with  $a_i \in K$ , and clearing the denominators of  $a_i$  and  $e_i = f_i/d_i$ , there exists  $0 \neq a \in A$  such that  $a \lambda \in F_1$ , i.e.,  $\lambda \in \overline{F_1}$ , and thus  $\overline{F_1} = \ker \cdot R \subseteq A^p = F_0$ . Then, by hypothesis,  $\ker \cdot R$  is a finitely generated free  $A$ -module. Using the implication (v)  $\Rightarrow$  (i) of Theorem 10 of [10] (namely, the annihilator of every matrix is free  $\Rightarrow A$  is a coherent Sylvester domain), we obtain that  $A$  is a coherent Sylvester domain.  $\square$

The next theorem characterizes the integral domains over which every transfer matrix admits a weakly doubly coprime factorization.

**THEOREM 3.24.** *We have the following equivalences:*

1. *Every multi-input multi-output (MIMO) plant admits a weakly doubly coprime factorization.*
2.  *$A$  is a coherent Sylvester domain.*

*Proof.*  $1 \Rightarrow 2$ . Let  $F_0$  be any finitely generated free  $A$ -module, and suppose that  $F_0 = A^p$  for a certain positive integer  $p$ . Let  $F_1$  be any finitely generated free  $A$ -submodule of  $F_0$ , and suppose that  $F_1$  has rank  $q$ . Taking a basis for  $F_1$ , then there exists a full row rank matrix  $R \in M_{q \times p}(A)$  such that we have  $F_1 = A^q R$ . We can always suppose that  $R$  can be written as  $R = (D : -N)$ , where  $D$  is a full rank matrix. Then, by hypothesis,  $P = D^{-1} N$  has a weakly doubly coprime factorization, i.e., there exists a weakly left-prime matrix  $R' = (D' : -N') \in M_{q \times p}(A)$  such that  $\det D' \neq 0$  and  $P = D'^{-1} N'$ . Then, by Lemma 2.6 and Theorem 2.11, we have  $\overline{A^q R} = \overline{A^q R'}$  and, using the fact that  $A^q R'$  is a free  $A$ -module of rank  $q$ , we obtain that  $\overline{F_1} = \overline{A^q R}$  is a finite free  $A$ -submodule of  $F_0$ . From Proposition 3.23, it follows that  $A$  is a coherent Sylvester domain.

$2 \Rightarrow 1$  was already proved in Corollary 3.22.  $\square$

**3.5. An example:  $H_\infty(\mathbb{C}_+)$ .**

**THEOREM 3.25** (see [23, 37]). *If  $D$  is a finitely connected domain of  $\mathbb{C}$ , then  $H_\infty(D)$  is a coherent domain. In particular, if we denote the open right half-plane by  $\mathbb{C}_+ = \{s \in \mathbb{C} \mid \operatorname{Re} s > 0\}$  and the open unit disc by  $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$ , then  $H_\infty(\mathbb{C}_+)$  and  $H_\infty(\mathbb{D})$  are two coherent integral domains.*

In the rest of this paper, we shall consider only the case  $D = \mathbb{D}$  and, by extension,  $D = \mathbb{C}_+$ . The proof of the coherence of  $H_\infty(D)$  is based on the following theorem, which is a weak- $*$  version of the Beurling–Lax theorem [25]. The condition on  $m$  is given by point 2 of the *lemma on the local rank* (p. 44) and Remark (p. 45) of [25].

THEOREM 3.26. Let  $R \in M_{q \times p}(H_\infty(D))$ , and let us define the  $H_\infty(D)$ -morphism:

$$R: \begin{array}{ccc} H_\infty(D)^p & \longrightarrow & H_\infty(D)^q, \\ (a_1 : \cdots : a_p)^T & \longrightarrow & R(a_1 : \cdots : a_p)^T. \end{array}$$

Then, there exists  $R_{-1} \in M_{p \times m}(H_\infty(D))$  such that

$$(3.9) \quad \ker R = R_{-1} H_\infty(D)^m,$$

$$(3.10) \quad (R_{-1}(e^{i\theta}))^* R_{-1}(e^{i\theta}) = I_m \quad \text{for almost every } \theta \in [0, 2\pi),$$

$$(3.11) \quad m = p - \text{rank } R,$$

where  $\text{rank } R$  is the number of  $H_\infty(D)$ -linearly independent rows of  $R$ .

COROLLARY 3.27. Let  $A = H_\infty(D)$ . If  $M$  is a finitely presented  $A$ -module, then

$$\text{pd}_A(M) \leq 2.$$

*Proof.* Let  $A = H_\infty(D)$  and  $A^q \xrightarrow{.R} A^p \longrightarrow M \longrightarrow 0$  be a finite presentation of  $M$ . Using Theorem 3.26, up to a transposition, there exists an  $r \times q$ -matrix  $R_1$  whose entries belong to  $A$  such that we have the following exact sequence:

$$(3.12) \quad 0 \longrightarrow \ker .R_1 \longrightarrow A^r \xrightarrow{.R_1} A^q \xrightarrow{.R} A^p \longrightarrow M \longrightarrow 0.$$

From the exactness of (3.12), we obtain (see Definition 1.9 and Proposition 1.10)

$$\text{rank}(\ker .R_1) + \text{rank } M = r + p - q.$$

From the exact sequence  $0 \longrightarrow \text{im}.R \longrightarrow A^p \longrightarrow M \longrightarrow 0$ , we obtain  $\text{rank } M = p - \text{rank } R$ . From (3.11), we have  $r = q - \text{rank } R$ , and thus,  $\text{rank}(\ker .R_1) = 0$ , i.e.,  $\ker .R_1$  is a torsion  $A$ -module. However,  $\ker .R_1$  is a submodule of the free  $A$ -module  $A^q$ , and thus,  $\ker .R_1 = 0$  because a free module is torsion-free. Hence, every finitely presented  $A$ -module  $M$  has a finite free resolution of length at most 2, i.e.,  $\text{pd}_A(M) \leq 2$ .  $\square$

COROLLARY 3.28.  $H_\infty(D)$  has a weak global dimension 2, i.e.,

$$(3.13) \quad \text{w.gl.dim}(H_\infty(D)) = 2.$$

*Proof.* Let  $A = H_\infty(D)$ . Using Corollary 3.27 and the fact that every finitely presented flat module is projective (see 3 of Proposition 1.4), then every finitely presented  $A$ -module  $M$  has a finite flat resolution of length at most 2, i.e.,  $\text{w.dim}_A(M) = \text{pd}_A(M) \leq 2$ . Moreover,  $\text{w.gl.dim}(A)$  is attained by taking the supremum of the weak dimension of finitely presented modules [14, 24], and thus,  $\text{w.gl.dim}(A) \leq 2$ . In Example 4.3, we shall give a finitely presented torsion-free  $H_\infty(\mathbb{C}_+)$ -module which is not projective (similar examples can be exhibited for  $D = \mathbb{D}$ ). Thus, we have  $\text{w.gl.dim}(A) = 2$ .  $\square$

The next corollary follows directly from the fact that  $\text{w.gl.dim}(H_\infty(D)) = 2$ .

COROLLARY 3.29.  $H_\infty(D)$  is a regular ring.

The following corollary was first proved in [46] for full row rank matrices.

COROLLARY 3.30.  $H_\infty(D)$  is a projective-free integral domain.

*Proof.* Let  $A = H_\infty(D)$ . Every finitely generated projective module is finitely presented [2]. Hence, let us suppose that  $M$  is a finitely presented projective  $A$ -module defined by a finite free resolution  $F_1 \xrightarrow{d_1} F_0 \longrightarrow M \longrightarrow 0$ . Then,  $T(M)$  is a coherent

$A$ -module and, by Theorem 3.26,  $T(M)$  has a finite free resolution of length 2 of the form  $0 \leftarrow T(M) \leftarrow F_1^* \xleftarrow{d_1^*} F_0^* \xleftarrow{d_2^*} F_{-1}^* \leftarrow 0$ . The fact that  $M$  is a projective  $A$ -module implies that  $\text{ext}_A^i(T(M), A) = 0, i \geq 1$  (see Theorem 3.16), and thus,  $F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_2} F_{-1} \rightarrow 0$  is an exact sequence; i.e.,  $M = \text{coker } d_1 \cong \text{im } d_2 = F_{-1}$  is a free  $A$ -module.  $\square$

COROLLARY 3.31.  $H_\infty(D)$  is a coherent Sylvester domain and, thus, a GCDD.

The fact that  $H_\infty(D)$  is a GCDD was first proved in [36] (see also [43]).

Corollary 3.28 shows that, for any finitely presented  $A = H_\infty(D)$ -module  $M$ , we have  $\text{ext}_A^i(T(M), A) = 0$  for all  $i \geq 3$ . Hence, by Theorem 3.16, every finitely presented  $A$ -module  $M$  satisfies only one of the three following possibilities:  $M$  has a nontrivial torsion submodule,  $M$  is torsion-free but not free, or  $M$  is free.

Example 3.4. In Example 2.1, we proved that the factorization  $P = D^{-1}N$ , defined by (1.3), of the transfer matrix (1.2) was not weakly left-coprime. Let us notice that  $R = (D : -N)$  was obtained by clearing the denominators of  $P$  once all its entries were written as quotients of (stable) elements of  $A = H_\infty(\mathbb{C}_+)$ . Hence, clearing the denominators of  $P$  does not generally lead to weakly doubly coprime factorizations. In general, we need to use the algorithm developed at the end of section 3.3 to compute a weakly doubly coprime factorization of a transfer matrix. Let us compute a weakly left-coprime factorization of the transfer matrix (1.2).

1. Let us reconsider the  $A$ -module  $M = A^4/A^2 R$  defined in Example 1.4. The matrix  $R \in M_{2 \times 4}(A)$  has full row rank, and thus, we have the finite free presentation

$$0 \rightarrow A^2 \xrightarrow{.R} A^4 \rightarrow M \rightarrow 0.$$

2. The transposed  $A$ -module  $T(M)$  is defined by the exact sequence

$$0 \leftarrow T(M) \leftarrow A^2 \xleftarrow{.R^T} A^4 \leftarrow \ker .R^T \leftarrow 0.$$

3. Let  $\lambda = (\lambda_1 : \lambda_2 : \lambda_3 : \lambda_4)^T \in \ker .R^T$ ; then we have

$$(3.14) \quad \begin{cases} \frac{(s-1)}{(s+1)} \lambda_1 - \frac{(s-1)e^{-s}}{(s+1)^2} \lambda_3 - \left(\frac{s-1}{s+1}\right)^2 \lambda_4 = 0, \\ \frac{(s-1)}{(s+1)} \lambda_2 - \frac{1}{(s+1)} \lambda_4 = 0. \end{cases}$$

By Corollary 3.31,  $A$  is a GCDD. The greatest common factor of  $\frac{s-1}{s+1}$  and  $\frac{1}{s+1}$  is 1; thus, from the second equation of (3.14), we have

$$\begin{cases} \lambda_2 = \frac{1}{(s+1)} \mu_1, \\ \lambda_4 = \frac{(s-1)}{(s+1)} \mu_1, \end{cases} \quad \mu_1 \in A.$$

Substituting  $\lambda_4$  in the first equation of (3.14), we obtain

$$\frac{(s-1)}{(s+1)} \left( \lambda_1 - \frac{e^{-s}}{(s+1)} \lambda_3 - \left(\frac{s-1}{s+1}\right)^2 \mu_1 \right) = 0 \Rightarrow \lambda_1 = \frac{e^{-s}}{(s+1)} \lambda_3 + \left(\frac{s-1}{s+1}\right)^2 \mu_1$$

because  $A$  is an integral domain and  $\lambda_i, \mu_1 \in A$ . Finally, we have

$$\begin{cases} \lambda_1 = \left(\frac{s-1}{s+1}\right)^2 \mu_1 + \frac{e^{-s}}{(s+1)} \mu_2, \\ \lambda_2 = \frac{1}{(s+1)} \mu_1, \\ \lambda_3 = \mu_2, \\ \lambda_4 = \frac{(s-1)}{(s+1)} \mu_1, \end{cases} \Leftrightarrow (\lambda_1 : \lambda_2 : \lambda_3 : \lambda_4) = (\mu_1 : \mu_2) \begin{pmatrix} \left(\frac{s-1}{s+1}\right)^2 & \frac{1}{s+1} & 0 & \frac{s-1}{s+1} \\ \frac{e^{-s}}{s+1} & 0 & 1 & 0 \end{pmatrix}.$$

If we call the matrix in the second member  $R_{-1}^T$ , then we have the following exact sequence:

$$(3.15) \quad 0 \longleftarrow T(M) \longleftarrow A^2 \xleftarrow{R^T} A^4 \xleftarrow{R_{-1}^T} A^2 \longleftarrow 0.$$

Moreover, if we note  $\mu = (\mu_1 : \mu_2)$ , then, from  $\lambda = \mu R_{-1}^T$ , we have

$$\begin{cases} \mu_1 = 2\lambda_2 + \lambda_4, \\ \mu_2 = \lambda_3, \end{cases} \Rightarrow S_{-1}^T \triangleq \begin{pmatrix} 0 & 0 \\ 2 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} : R_{-1}^T S_{-1}^T = I_2 \Rightarrow S_{-1} R_{-1} = I_2.$$

4. Dualizing (3.15), we obtain the following complex:

$$0 \longrightarrow A^2 \xrightarrow{.R} A^4 \xrightarrow{.R_{-1}} A^2 \longrightarrow 0.$$

Therefore, we have

$$\begin{cases} \text{ext}_A^1(T(M), A) = \ker .R_{-1}/A^2 R, \\ \text{ext}_A^2(T(M), A) = A^2/A^4 R_{-1}. \end{cases}$$

From  $S_{-1} R_{-1} = I_2$ , we deduce that for all  $\xi \in A^2$ , the element  $\eta = \xi S_{-1} \in A^4$  is such that  $\xi = \eta R_{-1}$ , i.e.,  $A^4 R_{-1} = A^2$ , and thus,  $\text{ext}_A^2(T(M), A) = A^2/A^4 R_{-1} = 0$ .

5. If  $\eta = (\eta_1 : \eta_2 : \eta_3 : \eta_4) \in \ker .R_{-1}$ , then we have

$$(3.16) \quad \begin{cases} \left(\frac{s-1}{s+1}\right)^2 \eta_1 + \frac{1}{s+1} \eta_2 + \frac{s-1}{s+1} \eta_4 = 0, \\ \frac{e^{-s}}{s+1} \eta_1 + \eta_3 = 0, \end{cases} \Leftrightarrow \begin{cases} \eta_3 = -\frac{e^{-s}}{s+1} \eta_1, \\ \frac{s-1}{s+1} \left(\frac{s-1}{s+1} \eta_1 + \eta_4\right) = -\frac{1}{s+1} \eta_2. \end{cases}$$

Using the fact that the greatest common factor of  $\frac{s-1}{s+1}$  and  $\frac{1}{s+1}$  is 1, we then have:

$$(3.16) \Leftrightarrow \begin{cases} \eta_3 = -\frac{e^{-s}}{s+1} \eta_1, \\ \eta_2 = \frac{s-1}{s+1} \zeta_2, \zeta_2 \in A, \\ \frac{s-1}{s+1} \eta_1 + \eta_4 = -\frac{1}{s+1} \zeta_2, \end{cases} \Leftrightarrow \begin{cases} \eta_1 = \zeta_1, \zeta_1 \in A, \\ \eta_2 = \frac{s-1}{s+1} \zeta_2, \zeta_2 \in A, \\ \eta_3 = -\frac{e^{-s}}{s+1} \zeta_1, \\ \eta_4 = -\frac{s-1}{s+1} \zeta_1 - \frac{1}{s+1} \zeta_2, \end{cases} \Leftrightarrow \eta = \zeta R',$$

where  $\zeta = (\zeta_1 : \zeta_2)$  and the matrix  $R' \in M_{2 \times 4}(A)$  is defined by

$$(3.17) \quad R' = \begin{pmatrix} 1 & 0 & -\frac{e^{-s}}{s+1} & -\frac{s-1}{s+1} \\ 0 & \frac{s-1}{s+1} & 0 & -\frac{1}{s+1} \end{pmatrix}.$$

Thus, we have  $\overline{A^2 R} = A^2 R'$ , and  $R'$  has full row rank. Hence,  $\overline{A^2 R}$  is a free  $A$ -module of rank 2, and, by Theorem 2.11, the transfer matrix  $P$  defined by (1.2) admits the following weakly left-coprime factorization:

$$(3.18) \quad P = \begin{pmatrix} \frac{e^{-s}}{s+1} & \frac{s-1}{s+1} \\ 0 & \frac{1}{s-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{s-1}{s+1} \end{pmatrix}^{-1} \begin{pmatrix} \frac{e^{-s}}{s+1} & \frac{s-1}{s+1} \\ 0 & \frac{1}{s-1} \end{pmatrix}.$$



Moreover, using the fact that  $R'$  has full row rank, we have the exact sequence

$$(3.19) \quad 0 \longrightarrow A^2 \xrightarrow{\cdot R'} A^4 \xrightarrow{\cdot R_{-1}} A^2 \longrightarrow 0$$

and

$$\begin{cases} t(M) \cong \text{ext}_A^1(T(M), A) = A^2 R' / A^2 R, \\ M/t(M) = A^4 / A^2 \cdot R' \cong A^4 R_{-1} = A^2. \end{cases}$$

From  $t(M) \cong A^2 R' / A^2 R$ , we obtain that the class of  $(1 : 0) R'$  in  $t(M)$  is the torsion element  $z_1 = y_1 - \frac{e^{-s}}{(s+1)} u_1 - \frac{(s-1)}{(s+1)} u_2$ , which satisfies  $\frac{(s-1)}{(s+1)} z_1 = 0$ . Similarly, the class  $z_2$  of  $(0 : 1) R'$  in  $t(M)$  is the trivial torsion element 0 because we have  $z_2 = \frac{(s-1)}{(s+1)} y_2 - \frac{1}{(s+1)} u_2 = 0$ . Thus,  $M$  is not a torsion-free  $A$ -module and  $M/t(M)$  is a free  $A$ -module of rank 2. Finally, we have

$$\begin{pmatrix} \frac{s-1}{s+1} & 0 & -\frac{(s-1)e^{-s}}{(s+1)^2} & -\left(\frac{s-1}{s+1}\right)^2 \\ 0 & \frac{s-1}{s+1} & 0 & -\frac{1}{s+1} \end{pmatrix} = \begin{pmatrix} \frac{s-1}{s+1} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -\frac{e^{-s}}{(s+1)} & -\frac{s-1}{s+1} \\ 0 & \frac{s-1}{s+1} & 0 & -\frac{1}{s+1} \end{pmatrix}.$$

*Remark 3.4.* For the sake of simplicity, we have treated here just a simple example. Simple computations, which do not require the algorithm developed in section 3.3, can easily give the weakly left-prime matrix (3.17) and, thus, the weakly left-coprime factorization (3.18) of (1.2). However, for more general systems (see, e.g.,  $P = \left(\frac{e^{-s}}{(s-1)} : \frac{e^{-s}}{(s-1)^2}\right)^T$  [32, 35]), it becomes more difficult to guess a weakly left-coprime factorization, and thus, we really need the algorithm to obtain weakly left/right/doubly coprime factorizations.

**4. Doubly coprime factorizations.**

**4.1. Left-coprime factorizations and stably free modules.** Let us introduce the concept of a splitting exact sequence.

DEFINITION 4.1 (see [2, 39]). *An exact sequence  $0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$  is a splitting exact sequence if one of the following equivalent assertions is satisfied:*

- *there exists an  $A$ -morphism  $h : M'' \longrightarrow M$  such that  $g \circ h = \text{id}_{M''}$ ,*
- *there exists an  $A$ -morphism  $k : M \longrightarrow M'$  such that  $k \circ f = \text{id}_{M'}$ ,*
- *there exist  $\phi = \begin{pmatrix} k \\ g \end{pmatrix} : M \longrightarrow M' \oplus M''$  and  $\psi = (f : h) : M' \oplus M'' \longrightarrow M$  such that  $\phi \circ \psi = \text{id}_{M' \oplus M''}$  and  $\psi \circ \phi = \text{id}_M$ , where  $\text{id}_M(m) = m$ , for all  $m \in M$ .*

PROPOSITION 4.2. *We have the following results:*

1. *(see [2, 39]) Let  $R \in M_{q \times p}(A)$  be a full row rank matrix. Then, the  $A$ -module  $M = A^p / A^q R$  is stably free iff the exact sequence*

$$(4.1) \quad 0 \longrightarrow A^q \xrightarrow{\cdot R} A^p \longrightarrow M \longrightarrow 0$$

*is a splitting exact sequence, i.e., iff there exists  $S \in M_{p \times q}(A)$  such that*

$$(4.2) \quad R S = I_q.$$

2. *(see [19]) Let  $R \in M_{q \times p}(A)$  be a full row rank matrix and  $M = A^p / A^q R$  the corresponding  $A$ -module. Then,  $M$  is stably-free iff*

$$T(M) = A^q / A^p R^T = 0.$$

*Example 4.1.* Let us consider the full row rank matrix  $R' \in M_{2 \times 4}(A)$  defined by (3.17) and  $A = H_\infty(\mathbb{C}_+)$ . By point 1 of Proposition 4.2,  $R'$  admits a right-inverse  $S' \in M_{4 \times 2}(A)$  iff the  $A$ -module  $M' = A^4/A^2 R'$  is stably free, i.e., iff the  $A$ -module  $T(M') = A^2/A^4 R'^T = 0$  by point 2 of Proposition 4.2. The  $A$ -module  $T(M') = A^2/A^4 R'^T$  is defined by the following equations:

$$(4.3) \quad \begin{cases} \lambda_1 = 0, \\ \frac{(s-1)}{(s+1)} \lambda_2 = 0, \\ -\frac{e^{-s}}{(s+1)} \lambda_1 = 0, \\ -\frac{(s-1)}{(s+1)} \lambda_1 - \frac{1}{(s+1)} \lambda_2 = 0. \end{cases}$$

If we put a second member  $\mu = (\mu_1 : \mu_2 : \mu_3 : \mu_4)^T$  in (4.3), then we have

$$\begin{cases} \lambda_1 = \mu_1, \\ \lambda_2 = -2 \frac{(s-1)}{(s+1)} \mu_1 + \mu_2 - 2 \mu_4, \end{cases}$$

which proves that, from (4.3), we can deduce that  $\lambda_1 = \lambda_2 = 0$ , i.e.,  $T(M') = 0$  and  $M'$  is a stably free  $A$ -module. A right-inverse  $S'$  of  $R'$ , i.e.,  $R' S' = I_2$ , is defined by

$$(4.4) \quad S = \begin{pmatrix} 1 & -2 \frac{(s-1)}{(s+1)} \\ 0 & 1 \\ 0 & 0 \\ 0 & -2 \end{pmatrix} \in M_{4 \times 2}(A).$$

Let us give the definition of the *fitting ideals* of a finitely presented  $A$ -module  $M$ .

**DEFINITION 4.3** (see [13]). *Let  $d : F_1 \rightarrow F_0$  be an  $A$ -morphism between two finite free  $A$ -modules  $F_0$  and  $F_1$ . If we choose bases for  $F_0$  and  $F_1$  ( $F_0 \cong A^p$ ,  $F_1 \cong A^q$ ), then  $d$  is defined by a matrix  $R \in M_{q \times p}(A)$ .*

- We denote by  $I_i(R)$  the ideal of  $A$  defined by
  - all the  $i \times i$  minors of  $R$  if  $1 \leq i \leq \min\{p, q\}$ ,
  - $I_i(R) = 0$  if  $i > \min\{p, q\}$ ,
  - $I_i(R) = A$  if  $i \leq 0$ .
- Let us define the  $A$ -module  $M = \text{coker } d$ , i.e.,  $M = A^p/A^q R$ . The  $i$ th fitting ideal  $\text{Fitt}_i(M)$  is the ideal of  $A$  defined by  $I_{p-i}(R)$ .  $\text{Fitt}_i(M)$  does not depend on the choice of the finite free presentation of  $M$ .
- We denote by  $I(M)$  the first nonzero fitting ideal  $\text{Fitt}_i(M)$  of  $M$ .

**PROPOSITION 4.4** (see [13]). *Let  $M$  be a finitely presented  $A$ -module. Then, we have the following:*

- $M$  is a projective  $A$ -module of rank  $r$  iff  $\text{Fitt}_r(M) = A$  and  $\text{Fitt}_{r-1}(M) = 0$ .
- $M$  is a projective  $A$ -module of a certain rank iff  $I(M) = A$ .

*Example 4.2.* Let us reconsider Example 4.1. We have

$$\text{Fitt}_0(M') = \text{Fitt}_1(M') = 0, \quad \text{Fitt}_2(M') = \left( \frac{s-1}{s+1}, \frac{1}{s+1}, \frac{e^{-s}}{(s+1)^2}, \frac{(s-1)e^{-s}}{(s+1)^2} \right).$$

We can check that  $1 = \frac{s-1}{s+1} + \frac{2}{s+1} \in \text{Fitt}_2(M')$ , and thus,  $\text{Fitt}_2(M') = A$ ; i.e.,  $M'$  is a projective  $A$ -module of rank 2 by Proposition 4.4.

The next proposition characterizes the projective modules over Banach algebras.

PROPOSITION 4.5. *If  $A$  is a Banach algebra which is an integral domain without radical, i.e.,  $\sqrt{A} = \{a \in A \mid \lim_{n \rightarrow +\infty} \|a^n\|_A^{1/n} = 0\} = 0$ , then a full row rank matrix  $R \in M_{q \times p}(A)$  defines a projective  $A$ -module  $M = A^p/A^q R$  iff  $\text{rk}(\hat{R}(\chi))$  is a constant function on the maximal ideal space  $X(A)$  of  $A$  (see [16, 49]), where  $\hat{R}$  denotes the Gelfand transform of  $R$  (see [16, 49]), or, equivalently, iff*

$$\inf_{\chi \in X(A)} \sum_{i \in I} |\hat{R}_i(\chi)| \geq \delta > 0,$$

where  $(\hat{R}_i)_{i \in I}$  is the family of the  $q \times q$  minors of  $R$ .

*Proof.* Using the fact that the maximal ideal space  $X(A)$  of a Banach algebra is a Hausdorff compact set and  $A$  has only two idempotent elements 1 and 0, then, by the *Shilov theorem* [16],  $X(A)$  is a connected space. By the *Swan theorem* [45], any vector bundle over  $X(A)$  is in one-to-one correspondence to a projective module over the ring of continuous functions  $C(X(A))$  on  $X(A)$ . The fact that  $X(A)$  is a connected space implies that the rank of any vector bundle over  $X(A)$  is globally constant. Finally, using the fact that  $A$  is without radical, by the Gelfand transform [16], any matrix whose entries belong to  $A$  can be seen as a matrix whose entries belong to  $C(X(A))$ . Hence, we find that  $M$  is a projective  $A$ -module iff  $\text{rk}(\hat{R}(\chi))$  is a constant function on  $X(A)$ .  $\square$

*Example 4.3.*  $H_\infty(\mathbb{C}_+)$  and  $\hat{A}$  are two integral domains which are Banach algebras without radical. We can use Proposition 4.5 to check whether or not an  $A$ -module is projective. For  $A = H_\infty(\mathbb{C}_+)$ , we can use the fact that  $\mathbb{C}_+$  is dense in  $X(A)$  (by the *Corona theorem*; see [25]) in order to take  $\chi$  only in  $\mathbb{C}_+$  instead of the whole  $X(A)$ . Similarly, for  $A = \hat{A}$ , we can restrict the evaluation of  $\inf_{\chi \in X(A)} \sum_{i \in I} |\hat{R}_i(\chi)|$  to  $\chi \in \overline{\mathbb{C}_+}$ , where  $\overline{\mathbb{C}_+} = \{s \in \mathbb{C} \mid \text{Re } s \geq 0\}$  (see [3, 8]).

- Let  $A = H_\infty(\mathbb{C}_+)$ . Let  $R = \begin{pmatrix} s-1 & e^{-s} \\ s+1 & s+1 \end{pmatrix} \in M_{1 \times 2}(A)$  and the  $A$ -module  $M = A^2/AR$ . Then,  $M$  is a projective  $A$ -module (i.e., free because  $A$  is a coherent Sylvester domain) because we have

$$\inf_{s \in \mathbb{C}_+} \left( \left| \frac{s-1}{s+1} \right| + \left| \frac{e^{-s}}{s+1} \right| \right) > 0.$$

We can check that we have the following Bézout identity:

$$\left( \frac{s-1}{s+1} \right) \left( 1 + 2 \left( \frac{1 - e^{-(s-1)}}{s-1} \right) \right) + 2e \left( \frac{e^{-s}}{s+1} \right) = 1.$$

- Let  $A = H_\infty(\mathbb{C}_+)$ . The matrix  $R = \begin{pmatrix} 1 & e^{-s} \\ s+1 & s+1 \end{pmatrix} \in M_{1 \times 2}(A)$  does not define a projective  $A$ -module  $M = A^2/AR$  because we have

$$\inf_{s \in \mathbb{C}_+} \left( \left| \frac{1}{s+1} \right| + |e^{-s}| \right) = 0.$$

Indeed, if  $(x_n)_{n \in \mathbb{Z}_+}$  is a sequence of strictly positive real numbers tending to  $+\infty$ , we check that  $\lim_{n \rightarrow +\infty} \left| \frac{1}{x_n+1} \right| = 0$  and  $\lim_{n \rightarrow +\infty} |e^{-x_n}| = 0$ . However, the greatest common divisor of  $\frac{1}{s+1}$  and  $e^{-s}$  is 1, and thus,  $R$  is a weakly left-prime matrix by Proposition 2.2 and Corollary 3.31; i.e.,  $M$  is a torsion-free (see Corollary 2.5) but not free  $A$ -module.

DEFINITION 4.6. *Let  $A$  be an integral domain and  $K = Q(A)$  its field of fractions.*

- A transfer matrix  $P \in M_{q \times (p-q)}(K)$  admits a left-coprime factorization if there exists a matrix  $R = (D : -N) \in M_{q \times p}(A)$ , with  $\det D \neq 0$ , such that  $P = D^{-1}N$  and  $R$  has a right-inverse  $S = (X^T : Y^T)^T \in M_{p \times q}(A)$ , i.e.,

$$RS = DX - NY = I_q.$$

- A transfer matrix  $P \in M_{q \times (p-q)}(K)$  admits a right-coprime factorization if there exists a matrix  $\tilde{R} = (\tilde{N}^T : \tilde{D}^T)^T \in M_{p \times (p-q)}(A)$ , with  $\det \tilde{D} \neq 0$ , such that  $P = \tilde{N} \tilde{D}^{-1}$  and  $\tilde{R}$  has a left-inverse  $\tilde{S} = (-\tilde{Y} : \tilde{X}) \in M_{(p-q) \times p}(A)$ , i.e.,

$$\tilde{S} \tilde{R} = -\tilde{Y} \tilde{N} + \tilde{X} \tilde{D} = I_{p-q}.$$

PROPOSITION 4.7. Let  $P = D^{-1}N = \tilde{N} \tilde{D}^{-1} \in M_{q \times (p-q)}(K)$  be a transfer matrix, where  $R = (D : -N) \in M_{q \times p}(A)$  and  $\tilde{R} = (\tilde{N}^T : \tilde{D}^T)^T \in M_{p \times (p-q)}(A)$ . Let us define the  $A$ -modules  $M = A^p/A^q R$  and  $\tilde{M} = A^p/A^{p-q} \tilde{R}^T$ . Then, we have

1.  $P$  admits a left-coprime factorization iff  $\overline{A^q R}$  is a free  $A$ -module of rank  $q$  and  $M/t(M) = A^p/\overline{A^q R}$  is a stably free  $A$ -module.
2.  $P$  admits a right-coprime factorization iff  $A^{p-q} \tilde{R}^T$  is a free  $A$ -module of rank  $p - q$  and  $\tilde{M}/t(\tilde{M}) = A^p/A^{p-q} \tilde{R}^T$  is a stably free  $A$ -module.

Proof. 1.  $\Rightarrow$  Let us suppose that  $P$  admits a left-coprime factorization of the form  $P = D'^{-1}N'$ , where the matrix  $R' = (D' : -N') \in M_{q \times p}(A)$  has right-inverse  $S' = (X'^T : Y'^T)^T \in M_{p \times q}(A)$ . In particular,  $R'$  is weakly left-prime and, by Lemma 2.6, we have  $\overline{A^q R} = A^q R'$ . Moreover,  $R'$  is a full row rank matrix, and thus,  $A^q R' = \overline{A^q R}$  is a free  $A$ -module of rank  $q$ . We have the exact sequence

$$(4.5) \quad 0 \longrightarrow A^q \xrightarrow{R'} A^p \longrightarrow M/t(M) = A^p/\overline{A^q R} \longrightarrow 0.$$

Using the fact that  $R'$  has a right-inverse  $S'$ , we obtain that (4.5) splits, and thus, we have  $A^p \cong A^q \oplus M/t(M)$ ; i.e.,  $M/t(M)$  is a stably free  $A$ -module.

$\Leftarrow$  Let  $P = D^{-1}N$  be such that  $\overline{A^q R}$  is a free  $A$ -module of rank  $q$  and the  $A$ -module  $M/t(M) = A^p/\overline{A^q R}$  is stably free. Using the fact that  $\overline{A^q R}$  is a free  $A$ -module of rank  $q$ , there then exists a weakly left-prime matrix  $R' = (D' : -N') \in M_{q \times p}(A)$  such that  $\overline{A^q R} = A^q R'$  and  $P = D'^{-1}N'$ . Then, we have the exact sequence (4.5), which splits because  $M/t(M) = A^p/A^q R'$  is a stably free  $A$ -module, and thus, there exists  $S' = (X'^T : Y'^T)^T \in M_{p \times q}(A)$  such that  $D'X' - N'Y' = I_q$ , i.e.,  $P = D'^{-1}N'$  is a left-coprime factorization of  $P$ . Part 2 can be proved similarly.  $\square$

Example 4.4. In Example 3.4, we proved that  $\overline{A^2 R} = A^2 R'$ , where  $R$  (resp.,  $R'$ ) is defined by (1.6) (resp., (3.17)), is a free  $A$ -module of rank 2. Moreover, in Example 4.1, we proved that  $M/t(M) = A^4/A^2 R'$  is a stably free  $A$ -module. Hence, from point 1 of Proposition 4.7, we deduce that (3.18) is a left-coprime factorization of the transfer matrix (1.2). Finally, using  $S'$  obtained in Example 4.1, we obtain

$$\left\{ \begin{aligned} P &= \begin{pmatrix} \frac{e^{-s}}{s+1} & \frac{s-1}{s+1} \\ 0 & \frac{1}{s-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{s-1}{s+1} \end{pmatrix}^{-1} \begin{pmatrix} \frac{e^{-s}}{s+1} & \frac{s-1}{s+1} \\ 0 & \frac{1}{s+1} \end{pmatrix}, \\ \begin{pmatrix} 1 & 0 \\ 0 & \frac{s-1}{s+1} \end{pmatrix} \begin{pmatrix} 1 & -2\frac{(s-1)}{(s+1)} \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} \frac{e^{-s}}{s+1} & \frac{s-1}{s+1} \\ 0 & \frac{1}{s+1} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix} &= I_2. \end{aligned} \right.$$

*Example 4.5.* Let us consider the example defined in [49, p. 349]. Let us consider the integral domain  $A = \mathbb{R}[t_0, t_1, t_2]/(t_0^2 + t_1^2 + t_2^2 - 1)$  of polynomials on the unit sphere of  $\mathbb{R}^3$ . Let  $x_i$  be the class of  $t_i$  in  $A$ . We have  $A = \mathbb{R}[x_0, x_1, x_2]$  with the relation  $x_0^2 + x_1^2 + x_2^2 = 1$ . It is shown in [20, p. 32] that  $A$  is a *unique factorization domain* (UFD) [39], and thus,  $A$  is a GCDD.

Let us consider  $P = -(x_1/x_0 : x_2/x_0) \in M_{1 \times 2}(K)$  with  $K = \mathbb{R}(x_0, x_1, x_2)$ . We have  $P = -x_0^{-1}(x_1 : x_2)$  and, if we define  $R = (x_0 : x_1 : x_2) \in A^3$ , then we have  $RR^T = 1$ . Thus,  $\overline{AR} = AR$  is a free  $A$ -submodule of rank 1, and  $M = A^3/AR$  is a stably free  $A$ -module, which proves that  $P$  admits a (normalized) left-coprime factorization. Moreover, we have  $P = -(x_1 : x_2)(x_0^{-1} I_2)$ . Let us define the matrix

$$\tilde{R} = \begin{pmatrix} -x_1 & -x_2 \\ x_0 & 0 \\ 0 & x_0 \end{pmatrix} \in M_{3 \times 2}(A)$$

and the corresponding  $A$ -module  $\tilde{M} = A^3/A^2\tilde{R}^T$ . We easily check that  $\text{Fitt}_0(\tilde{M}) = 0$  and  $\text{Fitt}_1(\tilde{M}) = (x_0 x_1, x_0 x_2, x_0^2)$ . Thus,  $x_0$  is a greatest common factor of all the  $2 \times 2$  minors, which, by Proposition 2.2, proves that  $\tilde{R}^T$  is not weakly left-prime, i.e., the  $A$ -module  $\tilde{M}$ , defined by the equations

$$\begin{cases} -x_1 y_0 + x_0 y_1 = 0, \\ -x_2 y_0 + x_0 y_2 = 0, \end{cases}$$

has a nonzero torsion submodule. We easily check that  $z = -x_2 y_1 + x_1 y_2$ , satisfying  $x_0 z = 0$ , defines the torsion submodule of  $\tilde{M}$ . Therefore,  $A^2\tilde{R}^T$  is not  $A$ -closed, and we have  $\tilde{M}/t(\tilde{M}) = A^3/A^3\tilde{R}'^T$ , where  $\tilde{R}'^T$  is defined by

$$\tilde{R}'^T = \begin{pmatrix} -x_1 & x_0 & 0 \\ -x_2 & 0 & x_0 \\ 0 & -x_2 & x_1 \end{pmatrix} \in M_3(A).$$

We have  $\text{Fitt}_0(\tilde{M}/t(\tilde{M})) = 0$  and  $x_0^2, x_1^2, x_2^2 \in \text{Fitt}_1(\tilde{M}/t(\tilde{M})) \Rightarrow 1 \in \text{Fitt}_1(\tilde{M}/t(\tilde{M}))$ , and thus, by Proposition 4.4,  $\tilde{M}/t(\tilde{M})$  is a projective  $A$ -module of rank 1. However, a projective module of rank 1 over a UFD is free (see [20, 45]), and thus,  $\tilde{M}/t(\tilde{M})$  is a free  $A$ -module of rank 1:  $u = x_0 y_0 + x_1 y_1 + x_2 y_2$  is a basis of  $\tilde{M}/t(\tilde{M})$  because we have  $y_i = x_i u$  for  $i = 1, \dots, 3$ . Thus, we obtain that  $\tilde{M}/t(\tilde{M}) \cong A^3\tilde{R}'^T \cong A$ . Moreover, by Proposition 2.8, we know that  $\ker .R^T = \overline{A^2\tilde{R}^T} = A^3\tilde{R}'^T$ . However, it is well known that  $\ker .R^T$  is a stably free but not a free  $A$ -module [20, 49]. By Proposition 4.7,  $P$  does not admit a right-coprime factorization.

**COROLLARY 4.8.** *Let  $P = D^{-1}N = \tilde{N}\tilde{D}^{-1} \in M_{q \times (p-q)}(K)$  be a transfer matrix, where  $R = (D : -N) \in M_{q \times p}(A)$  and  $\tilde{R} = (\tilde{N}^T : \tilde{D}^T)^T \in M_{p \times (p-q)}(A)$ . Let us define the  $A$ -modules  $M = A^p/A^q R$  and  $\tilde{M} = A^p/A^{p-q}\tilde{R}^T$ . Then, we have*

1.  $P$  admits a left-coprime factorization iff  $\tilde{M}/t(\tilde{M}) = A^p/A^{p-q}\tilde{R}'^T$  is a free  $A$ -module of rank  $q$ .
2.  $P$  admits a right-coprime factorization iff  $M/t(M) = A^p/\overline{A^q R}$  is a free  $A$ -module of rank  $p - q$ .

*Proof.* 1.  $\Rightarrow$  Let us suppose that  $P$  admits the left-coprime factorization  $P = D'^{-1}N'$ , where  $R' = (D' : -N') \in M_{q \times p}(A)$  has a right-inverse  $S'$ . Then,  $\overline{A^q R} = A^q R'$  is a free  $A$ -module of rank  $q$ , and thus,  $M/t(M) = A^p/A^q R'$ , i.e., we have the

following exact sequence:

$$0 \longrightarrow A^q \xrightarrow{\cdot R'} A^p \longrightarrow M/t(M) \longrightarrow 0.$$

By Proposition 4.2, this exact sequence splits, and thus,  $A^p R'^T \cong A^q$ . Finally, by Proposition 2.8, we now have  $\tilde{M}/t(\tilde{M}) \cong A^p R'^T \cong A^q$ .

⇐ Let us suppose that the  $A$ -module  $\tilde{M}/t(\tilde{M})$  is a free  $A$ -module of rank  $q$ .  $t(\tilde{M})$  is a torsion  $A$ -module, and thus, we have  $(t(\tilde{M}))^* = 0$ . Hence, dualizing the exact sequence  $0 \longrightarrow t(\tilde{M}) \longrightarrow \tilde{M} \longrightarrow \tilde{M}/t(\tilde{M}) \longrightarrow 0$ , we obtain  $\tilde{M}^* \cong (\tilde{M}/t(\tilde{M}))^* \cong A^q$ . By Proposition 2.8, we know that  $\tilde{M}^* \cong \ker \cdot \tilde{R} = \overline{A^q \tilde{R}}$ , and thus,  $\overline{A^q \tilde{R}}$  is a free  $A$ -module of rank  $q$ . Moreover, by Proposition 1.9 of [33],  $A^p \tilde{R}$  is a projective  $A$ -module because so is  $\tilde{M}/t(\tilde{M})$ . Thus, the exact sequence  $0 \longrightarrow \ker \cdot \tilde{R} \longrightarrow A^p \longrightarrow A^p \tilde{R} \longrightarrow 0$  splits, and we obtain that  $A^p \cong A^p \tilde{R} \oplus \ker \cdot \tilde{R}$ . However, by Proposition 2.8,  $A^p \tilde{R} \cong M/t(M)$ . Thus, we have  $A^p \cong M/t(M) \oplus A^q$ ; i.e.,  $M/t(M)$  is a stably-free  $A$ -module. Then, by Proposition 4.7,  $P$  admits a left-coprime factorization. Point 2 can be proved similarly. □

*Example 4.6.* Let us reconsider the system defined in Example 4.5. We proved that the  $A$ -module  $\tilde{M}/t(\tilde{M})$  is a free  $A$ -module of rank 1, and thus, by Corollary 4.8,  $P$  admits a left-coprime factorization. Moreover, it is known that  $M/t(M) = M$  is a stably free but not a free  $A$ -module [20, 45], i.e.,  $P$  does not admit right-coprime factorizations.

**4.2. Doubly coprime factorizations and free modules.** The following result characterizes generalized Bézout identities in terms of free  $A$ -modules.

**PROPOSITION 4.9.** *Let  $M = A^p/A^q R$  be an  $A$ -module defined by a full row rank matrix  $R \in M_{q \times p}(A)$ , i.e., by the following finite free resolution:*

$$(4.6) \quad 0 \longrightarrow A^q \xrightarrow{\cdot R} A^p \longrightarrow M \longrightarrow 0.$$

*Then,  $M$  is a free  $A$ -module iff there exist three matrices  $R_{-1}, S_{-1}$ , and  $S$  such that we have the following splitting exact sequence,*

$$(4.7) \quad 0 \longrightarrow A^q \xrightarrow{\cdot R} A^p \xrightarrow{\cdot R_{-1}} A^{p-q} \longrightarrow 0, \\ \xleftarrow{\cdot S} \qquad \qquad \qquad \xleftarrow{\cdot S_{-1}}$$

*or equivalently, iff we have the following generalized Bézout identities:*

$$(i) \quad ( S \ R_{-1} ) \begin{pmatrix} R \\ S_{-1} \end{pmatrix} = I_p, \\ (ii) \quad \begin{pmatrix} R \\ S_{-1} \end{pmatrix} ( S \ R_{-1} ) = \begin{pmatrix} I_q & 0 \\ 0 & I_{p-q} \end{pmatrix} = I_p.$$

*Proof.* ⇒ The  $A$ -module  $M$  is free, and thus, there exists a  $p \times (p - q)$  matrix  $R_{-1}$  with entries in  $A$  such that the exact sequence (4.6) has the form

$$0 \longrightarrow A^q \xrightarrow{\cdot R} A^p \xrightarrow{\cdot R_{-1}} A^{p-q} \longrightarrow 0.$$

This exact sequence finishes by the free  $A$ -module  $A^{p-q}$ , and thus, by Proposition 4.2, it splits; i.e., there exists a  $(p - q) \times p$  matrix  $S_{-1}$  such that  $R_{-1} S_{-1} = I_{p-q}$ . By the equivalences of Definition 4.1, we have the Bézout identities (i) and (ii).

$\Leftarrow$  If we have the splitting exact sequence (4.7) or, equivalently, the Bézout identities (i) and (ii), then  $M \cong A^p R_{-1} = A^{p-q}$ , i.e.,  $M$  is a free  $A$ -module of rank  $p - q$ .  $\square$

DEFINITION 4.10. A transfer matrix  $P \in M_{q \times (p-q)}(K)$  admits a doubly coprime factorization if there exist  $(D : -N) \in M_{q \times p}(A)$ ,  $(\tilde{N}^T : \tilde{D}^T)^T \in M_{p \times (p-q)}(A)$ ,  $(X^T : Y^T)^T \in M_{p \times q}(A)$ , and  $(-\tilde{Y} : \tilde{X}) \in M_{(p-q) \times p}(A)$  such that

- $P = D^{-1} N = \tilde{N} \tilde{D}^{-1}$ ,
- $\begin{pmatrix} D & -N \\ -\tilde{Y} & \tilde{X} \end{pmatrix} \begin{pmatrix} X & \tilde{N} \\ Y & \tilde{D} \end{pmatrix} = I_p$ ,
- $\begin{pmatrix} X & \tilde{N} \\ Y & \tilde{D} \end{pmatrix} \begin{pmatrix} D & -N \\ -\tilde{Y} & \tilde{X} \end{pmatrix} = I_p$ .

THEOREM 4.11. Let  $P = D^{-1} N = \tilde{N} \tilde{D}^{-1}$ ,  $R = (D : -N)$ ,  $\tilde{R} = (\tilde{N}^T : \tilde{D}^T)^T$ , and the  $A$ -modules  $M = A^p/A^q R$  and  $\tilde{M} = A^p/A^{p-q} \tilde{R}^T$ . Then,  $P$  admits a doubly coprime factorization iff  $M/t(M)$  and  $\tilde{M}/t(\tilde{M})$  are free  $A$ -modules of rank  $p - q$  and  $q$ .

*Proof.*  $\Rightarrow$  If  $P$  admits a doubly coprime factorization, then  $P$  admits left and right-coprime factorizations, and thus, by Proposition 4.8, the  $A$ -modules  $M/t(M)$  and  $\tilde{M}/t(\tilde{M})$  are free  $A$ -modules of rank, respectively,  $p - q$  and  $q$ .

$\Leftarrow$  By Proposition 4.8, there exist a left and a right-coprime factorization of  $P$ :

$$P = D'^{-1} N' = \tilde{N}' \tilde{D}'^{-1}, \quad \begin{cases} D' X - N' Y = I_q, \\ -\tilde{Y}' \tilde{N}' + \tilde{X}' \tilde{D}' = I_{p-q}. \end{cases}$$

From  $P = D'^{-1} N' = \tilde{N}' \tilde{D}'^{-1}$ , we deduce that  $(D' : -N') \begin{pmatrix} \tilde{N}' \\ \tilde{D}' \end{pmatrix} = 0$ . If we take

$$\begin{cases} X' = X + \tilde{N}' (\tilde{Y}' X - \tilde{X}' Y), \\ Y' = Y + \tilde{D}' (\tilde{Y}' X - \tilde{X}' Y), \end{cases}$$

we can easily check that  $P = D'^{-1} N' = \tilde{N}' \tilde{D}'^{-1}$  is a doubly coprime factorization:

$$\begin{pmatrix} D' & -N' \\ -\tilde{Y}' & \tilde{X}' \end{pmatrix} \begin{pmatrix} X' & \tilde{N}' \\ Y' & \tilde{D}' \end{pmatrix} = I_p, \quad \begin{pmatrix} X' & \tilde{N}' \\ Y' & \tilde{D}' \end{pmatrix} \begin{pmatrix} D' & -N' \\ -\tilde{Y}' & \tilde{X}' \end{pmatrix} = I_p. \quad \square$$

Using Proposition 2.8, we obtain the following corollary of Theorem 4.11.

COROLLARY 4.12. Let  $P = D^{-1} N = \tilde{N} \tilde{D}^{-1} \in M_{q \times (p-q)}(A)$  be a transfer matrix,  $R = (D : -N) \in M_{q \times p}(A)$ , and  $\tilde{R} = (\tilde{N}^T : \tilde{D}^T)^T \in M_{p \times (p-q)}(A)$ . Then,  $P$  admits a doubly coprime factorization iff the  $A$ -modules  $A^p \tilde{R}$  and  $A^p R^T$  are two free  $A$ -modules of rank, respectively,  $p - q$  and  $q$ .

This corollary was first proved in [44]. We have the following corollary of Proposition 4.12, which was first obtained in [49].

COROLLARY 4.13. A SISO plant, defined by  $p = n/d$  ( $0 \neq d, n \in A$ ), admits a coprime factorization iff the ideal  $I = (n, d)$  of  $A$  is principal.

*Proof.* By Proposition 4.12,  $p = n/d$  has a coprime factorization iff the  $A$ -module  $I = A^2 R^T = (d, n)$  is free of rank 1, where  $R = (d : -n) \in M_{1 \times 2}(A)$ . Using the fact that  $A$  is an integral domain,  $I$  is a free  $A$ -module iff  $I$  is a principal ideal.  $\square$

The next corollary of Proposition 4.7 was first proved in [49].

COROLLARY 4.14. If  $A$  is a Hermite ring, then every transfer matrix  $P$  with a left-coprime (resp., right-coprime) factorization admits a doubly coprime factorization.

*Proof.* Let  $P = D^{-1} N$  be a left-coprime factorization of the transfer matrix  $P$ , where  $R = (D : -N) \in M_{q \times p}(A)$ . By Proposition 4.7, the  $A$ -module  $M = A^p/A^q R$

is stably free. Using the fact that  $A$  is a Hermite ring, then  $M$  is free, and the result follows directly from Corollary 4.8, and similarly for right-coprime factorizations.  $\square$

*Example 4.7.* In Example 4.4, we proved that the transfer matrix  $P$  defined by (1.2) admits a left-coprime factorization. Using the fact that  $A = H_\infty(\mathbb{C}_+)$  is a coherent Sylvester domain and, in particular, a Hermite ring, by Corollary 4.14, we know that  $P$  admits a doubly coprime factorization. In fact, we have already done all the computations to obtain a right-coprime factorization of  $P$ . Indeed, we proved that (3.19) is an exact sequence, and thus it splits. Hence, using the matrices  $R_{-1} = (\tilde{N}^T : \tilde{D}^T)^T \in M_{4 \times 2}(A)$  and  $S_{-1} = (-\tilde{Y} : \tilde{X}) \in M_{2 \times 4}(A)$ , defined in Example 3.4, we obtain the following right-coprime factorization of  $P$ :

$$\left\{ \begin{array}{l} P = \tilde{N} \tilde{D}^{-1} = \begin{pmatrix} \left(\frac{s-1}{s+1}\right)^2 & \frac{e^{-s}}{s+1} \\ \frac{1}{s+1} & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ \frac{s-1}{s+1} & 0 \end{pmatrix}^{-1}, \\ - \begin{pmatrix} 0 & -2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \left(\frac{s-1}{s+1}\right)^2 & \frac{e^{-s}}{s+1} \\ \frac{1}{s+1} & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ \frac{s-1}{s+1} & 0 \end{pmatrix} = I_2. \end{array} \right.$$

**THEOREM 4.15** (see [49]). *The following assertions are equivalent:*

1. every MIMO plant admits doubly coprime factorizations,
2. every SISO plant admits coprime factorizations,
3.  $A$  is a Bézout domain.

*Proof.* 1  $\Rightarrow$  2 is trivial. 2  $\Rightarrow$  3 is given by Lemma 4.13.

3  $\Rightarrow$  1. If  $A$  is a Bézout domain, then every  $A$ -module  $M = A^p/A^q R$ , defined by a full row rank matrix  $R = (D : -N) \in M_{q \times p}(A)$ , is such that  $M/t(M)$  is a free  $A$ -module. Moreover, a Bézout domain  $A$  is a coherent Sylvester domain, and thus, by Proposition 3.21, there exists a full row rank matrix  $R' = (D' : -N') \in M_{q \times p}(A)$  such that  $M/t(M) = A^p/A^q R'$  and  $P = D^{-1} N = D'^{-1} N'$ . Finally, using Proposition 4.9, we obtain that  $P$  admits a doubly coprime factorization.  $\square$

**Conclusion.** We hope we have convinced the reader that the reformulation of the fractional representation approach to analysis problems within the algebraic analysis framework allows us to obtain some new results. These results will be used in the second part of this work [33] to obtain necessary and sufficient conditions for internal stabilizability and to determine the class of rings  $A$  over which every plant is internally stabilizable. For the sake of simplicity, we have treated only the case of integral domains, but all the results are still valid for general rings: we need only to slightly change some definitions (e.g.,  $K = Q(A) = \{a/b \mid a \in A, b \in A \setminus Z(A)\}$ , where  $Z(A)$  is the set of the nonzero divisors of  $A$ ,  $t(M) = \{m \in M \mid \exists a \in A \setminus Z(A) : am = 0\}$ , etc.).

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